

A HOPF GLOBAL BIFURCATION THEOREM FOR RETARDED FUNCTIONAL DIFFERENTIAL EQUATIONS

BY

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ABSTRACT. We prove a result concerning the global nature of the set of periodic solutions of certain retarded functional differential equations. Our main theorem is an analogue, for retarded F.D.E.'s, of a result by J. Alexander and J. Yorke for ordinary differential equations.

Introduction. In the past ten or fifteen years there has been considerable interest in the global nature of the set of periodic solutions of certain parametrized families of F.D.E.'s. These equations arise in a variety of applications, for example, mathematical biology [19]. References at the end of this paper give some guidance to the relevant literature.

For those equations to which it is applicable, the global bifurcation theorem in [21] appears to provide the sharpest global information. However, there are simple-looking F.D.E.'s for which the results of [21] are not easily applicable. We mention one example; consider the equation

$$(1) \quad x'(t) = [-\alpha x(t-1) - c\alpha x(t-\gamma)][1 - x^2(t)],$$

where c and γ are positive constants, $1 < \gamma \leq 2$ and $\alpha > 0$. Let α_0 denote the smallest positive α such that the equation

$$(2) \quad z = -\alpha e^{-z} - c\alpha e^{-\gamma z}$$

has a pair of pure imaginary solutions. For a variety of reasons, it is reasonable to conjecture that for every $\alpha > \alpha_0$, (1) has a "slowly oscillating" (a term we leave undefined) nonconstant periodic solution. Despite remarks made in [15] for the case $\gamma = 2$, this modest conjecture has still not been proved in general. The cases $c = 0$ and $c = 1$ (for $\gamma = 2$) treated in [15] are atypical.

Thus it seems reasonable to try to obtain a global bifurcation theorem for periodic solutions which would perhaps provide less detailed information than the one in [21] but which would be more broadly applicable. J. Alexander and J. Yorke have established a generalization of the classical Hopf bifurcation theorem [1], and J. Ize [12], [13] has given a considerable

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simplification of the original proof. We shall prove here an analogue of the Yorke-Alexander theorem for retarded F.D.E.'s. The proof follows the general outlines of Ize's simplification, but the infinite dimensional nature of the problem and, more importantly, the lack of compactness of certain maps introduce considerable difficulties; and it is the treatment of these difficulties we shall emphasize. The proof we give here can be abstracted to certain evolution equations in Banach spaces, though we do not pursue this. We should remark that if the operation of translation along trajectories (for the evolution equation) is compact, the use of finite dimensional projections given here can be avoided and the proof considerably simplified. The techniques we give can also be used to study the global nature of nonconstant periodic solutions of integral equations like those in [4]. Although our primary interest in the theorem here is its application to specific equations, we defer these applications to [25] because of considerations of length. We hope to show in [25] how a variety of techniques (including Theorem 4 below) can be used to study periodic solutions of, for example, equation (1).

After this paper was written we received a preprint of a paper by Chow and Mallet-Paret in which they outline a proof of a result like Theorem 4 below (for the case $\text{Mult}(i\beta) = \{i\beta\}$ in our later notation). The proof involves approximation of retarded F.D.E.'s by Kupka-Smale systems (as in [18]) and generalizations of Fuller's index [6] to retarded FDE's; presumably such extensions would also be necessary in applying their ideas to other kinds of equations. Chow and Mallet-Paret also give an interesting application to (1) (but only for γ an integer) in order to obtain "rapidly oscillating" periodic solutions. However, the existence of slowly oscillating periodic solutions for $\alpha > \alpha_0$ does not follow, and it is the slowly oscillating periodic solutions which have been studied numerically and which are of greater interest.

An outline of this paper may be in order. In the first section we prove that the operator of translation along trajectories for retarded F.D.E.'s is strongly approximation proper (strongly A -proper) with respect to a natural set of projections $\{P_m\}$, although the operator is not, in general, compact. This observation provides a means of passing from finite dimensional to infinite dimensional results and is extensively used. The second section reviews the linear theory of retarded F.D.E.'s and derives some simple consequences of known results. The third section shows, in Theorem 3, that a certain element of the first homotopy group of $GL^+(R^p)$ is nonzero for large p , where $GL^+(R^p)$ denotes a connected component of the general linear group on R^p . The main result of the paper is Theorem 4 of §4, which is an exact analogue of the Yorke-Alexander result.

1. A class of strongly A -proper mappings. In this section we shall prove the A -properness (see [26]) of a new class of mappings. For technical reasons, this

result will be crucial for the remainder of the paper.

First we need some notation. Let γ be a fixed positive constant and define X to be the Banach space of continuous maps $x: [-\gamma, 0] \rightarrow K^n$, where K denotes either the reals or the complexes. The norm is the usual sup norm. For each $m \geq 1$ let $t_0 = -\gamma < t_1 < t_2 < \dots < t_m = 0$ be a partition of $[-\gamma, 0]$ into m intervals $\Delta_j = [t_{j-1}, t_j]$ of equal length and define a finite dimensional linear projection $P_m: X \rightarrow X$ by $P_m x = y$, where

$$(3) \quad y(t) = \left(\frac{t_j - t}{t_j - t_{j-1}} \right) x(t_{j-1}) + \left(\frac{t - t_{j-1}}{t_j - t_{j-1}} \right) x(t_j)$$

for $t \in \Delta_j$. It is easy to check that $\|P_m\| = 1$ and that $\lim_{m \rightarrow \infty} P_m x = x$ for each $x \in X$. We shall adhere to the above use of X, Δ_j, t_j and P_m throughout this section.

Next suppose that Z is a Banach space and that $\{Q_m: m \geq 1\}$ is a sequence of continuous linear projections with the property that $\lim_{m \rightarrow \infty} Q_m z = z$ for every $z \in Z$. Let B be a compact metric space, A a closed subset of $Z \times B$ and $\Phi: Z \rightarrow Z$ a continuous map. Define $\Pi: Z \times B \rightarrow Z$ by $\Pi(z, b) = z$.

DEFINITION 1. The map $\Pi - \Phi$ is "strongly A -proper with respect to $\{Q_m\}$ " if for every subsequence $\{m_i\}$ of the integers and every bounded sequence $(a_{m_i}, b_{m_i}) \in A$ such that $a_{m_i} - Q_{m_i} \Phi(a_{m_i}, b_{m_i})$ is convergent, there exists a further subsequence (a_{m_j}, b_{m_j}) which is convergent.

The above definition strengthens the usual notion of A -properness in that it is not assumed that $a_{m_i} \in Q_{m_i}(Z)$.

We also need a notion of restricted equicontinuity.

DEFINITION. If $S \subset X$ is a family of functions and $J \subset [-\gamma, 0]$ is a closed subinterval, " S is equicontinuous on J " if the restriction of elements of S to J gives an equicontinuous family on J .

THEOREM 1. Let A be a closed subset of $X \times B, B$ a compact metric space, $\Phi: A \rightarrow X$ a continuous map and $\Pi: X \times B \rightarrow X$ the standard projection. Assume that there exists $\epsilon > 0$ such that whenever $A_1 \subset A$ is closed, bounded and $\Pi(A_1)$ is equicontinuous on an interval $[-\rho_1, 0], 0 \leq \rho_1 \leq \gamma$, then $A_2 = \{\Phi(a, b) : (a, b) \in A_1 \times B\}$ is closed, bounded and equicontinuous on $[-\rho_2, 0]$, where $\rho_2 = \min(\rho_1 + \epsilon, \gamma)$. Then it follows that $\Pi - \Phi$ is strongly A -proper with respect to $\{P_m\}$.

PROOF. Let $(a_{m_i}, b_{m_i}) \in A$ be a bounded sequence such that

$$(4) \quad a_{m_i} - P_{m_i} \Phi(a_{m_i}, b_{m_i}) = z_{m_i} \rightarrow z.$$

By relabelling the projections and using the compactness of B , we can write $m_i = m$ and assume $b_m \rightarrow b$. According to the Ascoli-Arzelà theorem, it suffices to show $\{a_m: m \geq 1\}$ is equicontinuous on $[-\gamma, 0]$. The latter will

follow by a bootstrap argument if we can prove that $\{a_m\}$ equicontinuous on $[-\rho, 0]$, $0 < \rho < \gamma$, implies that $\{a_m\}$ is equicontinuous on $[-\rho_1, 0]$, where $\rho_1 = \min(\gamma, \rho + \varepsilon/2)$.

Thus suppose we have shown that $\{a_m: m \geq 1\}$ is equicontinuous on $[-\rho, 0]$. Since $\{z_m\}$ is convergent and, hence, equicontinuous on $[-\gamma, 0]$, (4) shows that it suffices to prove $P_m \Phi(a_m, b_m)$ is equicontinuous on $[-\rho_1, 0]$. By assumption, $w_m = \Phi(a_m, b_m)$ is equicontinuous on $[-\rho_2, 0]$, where $\rho_2 = \min(\gamma, \rho + \varepsilon)$. If $\rho_2 = \gamma$, we claim that $\{P_m w_m: m \geq 1\}$ is equicontinuous on $[-\gamma, 0]$. To see this, given $\eta > 0$ select $\delta > 0$ such that $|t - s| < \delta$ implies that $|w_m(t) - w_m(s)| < \eta/3$ for all $m \geq 1$. An easy argument using the definition of P_m shows that if $m \geq N$, where $\gamma/N < \delta$, and $|t - s| < \delta$, then $|P_m w_m(t) - P_m w_m(s)| < \eta$. This shows that $\{P_m w_m: m \geq N\}$ is equicontinuous on $[-\gamma, 0]$, and, consequently, $\{P_m w_m: m \geq 1\}$ is equicontinuous on $[-\gamma, 0]$. In the case that $\rho_2 = \rho + \varepsilon < \gamma$, if one takes η and δ as above and N such that $\gamma/N < \min(\delta, \varepsilon/2)$, then the same sort of argument used above works to show $\{P_m w_m: m \geq 1\}$ is equicontinuous on $[-\rho_1, 0]$. \square

Our interest in Theorem 1 stems from its applicability to the operator of translation along trajectories for retarded functional differential equations (F.D.E.'s). Specifically, let X be as usual, with scalar field the reals, and let Λ denote an open interval of real numbers. We shall henceforth denote by $f: X \times \Lambda \rightarrow R^n$ a map such that:

H1. $f: X \times \Lambda \rightarrow R^n$ is continuous and takes bounded sets in $X \times \Lambda$ to bounded sets in R^n .

Following the notation in [10], we are interested in nonconstant periodic solutions of

$$(5) \quad x'(t) = f(x_t, \lambda).$$

For each $\phi \in X$, we can consider the initial value problem

$$(6) \quad x'(t) = f(x_t, \lambda) \quad \text{for } t \geq 0, \quad x[-\gamma, 0] = \phi.$$

We must assume that:

H2. For each $\phi \in X$ and $\lambda \in \Lambda$ equations (6) have a unique solution $x(t) = x(t; \phi, \lambda)$ defined and continuous on $[-\gamma, \delta)$ for some positive δ and C^1 on $[0, \delta)$.

A standard argument shows that $x(t; \phi, \lambda)$ can be extended to some maximal, half-open interval of definition $[-\gamma, t(\phi, \lambda))$. Furthermore, if $G = \{(\phi, \lambda, t) \in X \times \Lambda \times [0, \infty): x(t; \phi, \lambda) \text{ is defined}\}$ arguments like those for O.D.E.'s show that G is an open subset of $X \times \Lambda \times [0, \infty)$ and the map $(\phi, \lambda, t) \rightarrow x(t; \phi, \lambda)$ is continuous; see [10] for details. We shall reserve the letter G to denote the above set.

Unfortunately, an example of K. Hannsgen (see [10, p. 39]) shows that $x(t; \phi, \lambda)$ may not be bounded on closed, bounded subsets of G ; since we

shall need this boundedness we assume:

H3. If A is any closed, bounded subset of G , the function $x(t; \phi, \lambda)$ is bounded on A .

Assuming that H1, H2 and H3 hold, define a map

$$(7) \quad F: G \rightarrow X$$

by $F(\phi, \lambda, t) = x_t$, where $x(t) = x(t; \phi, \lambda)$ is the unique solution of (6). We shall always use F to denote this map. Let $\Pi: X \times \Lambda \times [0, \infty) \rightarrow X$ be projection onto X .

THEOREM 2. *Assume that H1, H2 and H3 hold and let A be a closed, bounded subset of G such that $\inf\{t: (\phi, \lambda, t) \in A \text{ for some } \phi \text{ and } \lambda\} = \epsilon > 0$. Then the map $\Pi - F|A$ is strongly A -proper with respect to $\{P_m\}$.*

PROOF. It suffices to show that $\Phi = F$ satisfies the hypothesis of Theorem 1. Let A_1 be a subset of A such that $\Pi(A_1)$ is equicontinuous on $[-\rho_1, 0]$. If $(\phi, \lambda, t) \in A_1$, consider $x(s; \phi, \lambda)$ for $0 \leq s \leq t$, and note that by H3 and the boundedness of A_1 there is a constant M (independent of $(\phi, \lambda, t) \in A_1$) such that

$$(8) \quad |x(s; \phi, \lambda)| \leq M$$

for $-\gamma < s \leq t$. The boundedness of f now implies that there is a constant M_1 such that

$$(9) \quad |x'(s; \phi, \lambda)| \leq M_1$$

for $0 \leq s \leq t$. Since $t \geq \epsilon$ for every $(\phi, \lambda, t) \in A$, it follows from (9) and the assumption that $\Pi(A_1)$ is equicontinuous on $[-\rho_1, 0]$ that $\{F(\phi, \lambda, t): (\phi, \lambda, t) \in A_1\}$ is equicontinuous on $[-\rho_2, 0]$, $\rho_2 = \min(\rho_1 + \epsilon, \gamma)$. \square

2. Linear theory of retarded functional differential equations. In this section we shall recall for the reader's convenience some basic facts about linear retarded F.D.E.'s (further details appear in [10]) and derive some simple consequences. As usual, let X (\tilde{X} respectively) denote the continuous functions $[-\gamma, 0]$ to \mathbf{R}^n (to \mathbf{C}^n respectively). Suppose that $L: \tilde{X} \rightarrow \mathbf{C}^n$ is a bounded linear map and consider

$$(10) \quad x'(t) = L(x_t) \text{ for } t \geq 0, \quad x|[-\gamma, 0] = \phi \in \tilde{X}.$$

For each $\phi \in \tilde{X}$, (10) has a unique solution $x(t; \phi)$ defined for $t \geq -\gamma$. The map $\phi \rightarrow x_t = T(t)(\phi)$ defines a bounded linear operator $T(t)$ and $\{T(t): t \geq 0\}$ gives a strongly continuous, linear semigroup on \tilde{X} . The infinitesimal generator A of the semigroup $T(t)$ is given by

$$(A\phi)(s) = \phi'(s) \text{ for } -\gamma \leq s \leq 0;$$

$$(11) \quad D(A) = \text{domain of } A = C^1 \text{ functions } \phi \in \tilde{X}$$

$$\text{such that } \phi'(0) = L(\phi).$$

For each complex number z define a linear map $\Delta(z): \mathbf{C}^n \rightarrow \mathbf{C}^n$ by the formula

$$\Delta(z)(b) = zb - L(e^{zb}),$$

where b denotes a vector in \mathbf{C}^n and e^{zb} denotes the map $s \in [-\gamma, 0] \rightarrow e^{zs}b$. One can check directly that the map $z \rightarrow \Delta(z)$ is complex analytic (and not identically zero), so the map $z \rightarrow (\det(\Delta(z)))^{-1}$ is meromorphic (\det denotes determinant). We shall need the fundamental facts that $\sigma(A)$ = the point spectrum of A and that $\sigma(A) = \{z \in \mathbf{C}: \det \Delta(z) = 0\}$. Furthermore, recall that since $T(t)^m$ is a compact linear operator for $mt \geq \gamma$, it follows that $\sigma(T(t))$ = the point spectrum of $T(t)$ and that (compare [11, p. 467] and [10, p. 112]) $\sigma(T(t)) - \{0\} = \{\exp(tz) : z \in \sigma(A)\}$.

We shall also need an explicit formula for $(z - A)^{-1}(\psi) = \phi$ (assuming $\det \Delta(z) \neq 0$). One can check that

$$(13) \quad \begin{aligned} \phi(t) &= e^{zt}b + \int_t^0 e^{z(t-s)}\psi(s) ds \\ \text{where } b &= \Delta(z)^{-1} \left[\psi(0) + L \left(\int_t^0 e^{z(t-s)} \psi(s) ds \right) \right]. \end{aligned}$$

Next suppose that Λ is an open interval of reals, and that for each $\lambda \in \Lambda$, $L_\lambda: \tilde{X} \rightarrow \mathbf{C}^n$ is a continuous map and $\lambda \rightarrow L_\lambda$ is continuous in the uniform operator topology. In the obvious notation we can consider the strongly continuous linear semigroup $T_\lambda(t)$ ($t \geq 0$) generated by solving

$$(14) \quad x'(t) = L_\lambda(x_t), \quad t \geq 0; \quad x|_{[-\gamma, 0]} = \phi,$$

the infinitesimal generator A_λ of $T_\lambda(t)$ and $\Delta_\lambda(z)$ defined by a formula like (12) with L_λ substituted for L . We shall maintain this notation for the rest of the paper.

It follows directly from (12) that if $\lambda \rightarrow L_\lambda$ is continuous, $\Delta_\lambda(z) \rightarrow \Delta_{\lambda_0}(z)$ as $\lambda \rightarrow \lambda_0$ uniformly for z in a compact set. Since $\sigma(A_\lambda) = \{z \in \mathbf{C}: \det \Delta_\lambda(z) = 0\}$, we see that if Γ is any compact subset of the resolvent set $\rho(A_{\lambda_0})$ of A_{λ_0} , then for $|\lambda - \lambda_0| < \epsilon$, Γ is in the resolvent set of A_λ . Furthermore, (13) implies that

$$(15) \quad \|(z - A_\lambda)^{-1} - (z - A_{\lambda_0})^{-1}\| \rightarrow 0$$

as $\lambda \rightarrow \lambda_0$, uniformly in $z \in \Gamma$.

We shall also need some elementary results from the functional calculus for linear operators. Let G be a bounded open set whose boundary consists of a finite number of simple, closed Jordan curves which lie in the resolvent set of a closed, densely defined linear operator B on a complex Banach space Z . One can consider a bounded linear operator

$$(16) \quad P = \frac{1}{2\pi i} \int_{\Gamma} (z - B)^{-1} dz.$$

The operator P is a projection whose range lies in the domain of B . If Γ contains only one point z_0 of $\sigma(B)$, and the Laurent expansion of $(z - B)^{-1}$ at z_0 has only finitely many terms with negative indices so that

$$(z - B)^{-1} = \sum_{j=-k}^{\infty} C_j(z - z_0)^j$$

with C_j bounded and $C_{-k} \neq 0$, then $R(P) = \text{range of } P$ is finite dimensional and $R(P)$ is the null space of $(z_0 - B)^k = N((z_0 - B)^k)$, which is the same as the null space of $(z_0 - B)^j$ for $j \geq k$ (see [17, p. 29] and [11, Chapter 5]). Let σ_1 denote the part of $\sigma(B)$ which lies inside Γ , $\sigma_2 = \sigma(B) - \sigma_1$, $Z_1 = R(P)$ and $Z_2 = R(Q)$, where $Q = I - P$; also denote by B_1 the restriction of B to Z_1 and by B_2 the restriction of B to Z_2 . Then it follows (see Theorem 6.17, p. 178 in [17]) that σ_j is the spectrum of B_j .

In our case we take A_λ to be B and Γ to be a union of curves as above in the resolvent of A_λ , and we define projections P_λ and Q_λ by

$$(17) \quad P_\lambda = \frac{1}{2\pi i} \int_{\Gamma} (z - A_\lambda)^{-1} dz, \quad Q_\lambda = I - P_\lambda,$$

and set $\tilde{X}_\lambda = P_\lambda(\tilde{X})$, $\tilde{Y}_\lambda = Q_\lambda(\tilde{X})$, $X_\lambda = X \cap \tilde{X}_\lambda$ and $Y_\lambda = X \cap \tilde{Y}_\lambda$. Again, we shall maintain this notation from now on. Let σ_1 denote the finite number of points in $\sigma(A_\lambda)$ which lie inside Γ and $\sigma_2 = \sigma(A_\lambda) - \sigma_1$.

REMARK 1. In general, Γ must be chosen to vary with λ , but the previous remarks show that if Γ is permissible for λ_0 , it is also permissible for $|\lambda - \lambda_0| < \epsilon$, $\epsilon > 0$. For these λ , (15) shows that $\lambda \rightarrow P_\lambda$ is continuous (assuming $\lambda \rightarrow L_\lambda$ is continuous).

Our next proposition can be found, for the most part, in [10, pp. 94–115]; we sketch a proof for completeness.

PROPOSITION 1. *Let notation be as above. Then it follows that $\tilde{X} = \tilde{X}_\lambda \oplus \tilde{Y}_\lambda$, \tilde{X}_λ is finite dimensional, $T_\lambda(t)(\tilde{X}_\lambda) \subset \tilde{X}_\lambda$ and $T_\lambda(t)(\tilde{Y}_\lambda) \subset \tilde{Y}_\lambda$ for $t > 0$. Furthermore, for $t > 0$ the spectrum of $T_\lambda(t)|_{\tilde{X}_\lambda}$ equals $\{\exp(ts) : s \in \sigma_1\}$, and the spectrum of $T_\lambda(t)|_{\tilde{Y}_\lambda}$ equals $\{\exp(ts) : s \in \sigma_2\}$.*

PROOF. The general theory of linear semigroups implies that $T_\lambda(t)$ and A_λ commute on the domain of A_λ ; from this one obtains that $T_\lambda(t)$ and $(z - A_\lambda)^{-1}$ commute and, hence, $T_\lambda(t)$ and P_λ commute. Thus $T_\lambda(t)$ maps \tilde{X}_λ and \tilde{Y}_λ into themselves. The general theory of the functional calculus for linear operators implies that $\tilde{X} = \tilde{X}_\lambda \otimes \tilde{Y}_\lambda$, P_λ is a projection and X_λ is in the domain of P_λ . Since G_j , the interior of Γ , is assumed bounded and $\sigma(A_\lambda)$ is discrete, σ_1 consists of a finite number of points. Thus to show X_λ is finite dimensional, it suffices to show $(z - A_\lambda)^{-1}$ is meromorphic at any point z_0

such that $\det \Delta(z_0) = 0$; in fact it follows from (13) that if z_0 is a zero of $\det \Delta(z)$ of order k , then $(z - A_\lambda)^{-1}$ has a pole of order k at z_0 .

It is clear that $T_\lambda(t)|\tilde{X}_\lambda$ and $T_\lambda(t)|\tilde{Y}_\lambda$ are strongly continuous semigroups with generators $A_\lambda|\tilde{X}_\lambda$ and $A_\lambda|\tilde{Y}_\lambda$. Theorem 6.17 [17, p. 178] implies that $\sigma(A_\lambda|\tilde{X}_\lambda) = \sigma_1$ and $\sigma(A_\lambda|\tilde{Y}_\lambda) = \sigma_2$; it follows from Theorem 16.7.2 [11, p. 467] that for $t > 0$, the spectrum of $T_\lambda(t)|\tilde{X}_\lambda$ is $\{\exp(ts) : s \in \sigma_1\}$ plus possibly 0, and similarly for $T_\lambda(t)|\tilde{Y}_\lambda$. However, a strongly continuous semigroup on a finite dimensional space is necessarily continuous in the uniform operator topology. Thus $T_\lambda(t)|\tilde{X}_\lambda$ is invertible for t small, and, hence, $T_\lambda(t)|\tilde{X}_\lambda$ is invertible for all $t \geq 0$ and 0 is not in its spectrum. On the other hand, \tilde{Y}_λ is infinite dimensional and, for each $t > 0$, $T_\lambda(t)^N$ is compact for some N . It follows that 0 cannot be in the resolvent of $T_\lambda(t)|\tilde{Y}_\lambda$ (if it were, the unit ball in \tilde{Y}_λ would be compact). \square

If $L: \tilde{X} \rightarrow C^n$ is a bounded linear function, let A and $T(t)$ be defined as at the beginning of this section. Suppose that $L(X) \subset \mathbb{R}^n$ (where $X = C([-\gamma, 0]; \mathbb{R}^n)$); then it is easy to check that A and $T(t)$ map X into X . If z_j and \bar{z}_j , $1 \leq j \leq k$, are k pairs of conjugate complex numbers (we allow $z_j = \bar{z}_j$) such that z_j and \bar{z}_j are elements of $\sigma(A)$, let Γ_j and $\bar{\Gamma}_j$ be simple, closed curves containing, respectively, z_j and \bar{z}_j and no other points of $\sigma(A)$. Let Γ denote the union of these curves and define

$$\begin{aligned}
 (18) \quad P &= \frac{1}{2\pi i} \int_{\Gamma} (z - A)^{-1} dz \\
 &= \frac{1}{2\pi i} \sum_{j=1}^k \int_{\Gamma_j} (z - A)^{-1} dz + \int_{\bar{\Gamma}_j} (z - A)^{-1} dz.
 \end{aligned}$$

PROPOSITION 2. *Let assumptions be as in the preceding paragraph. Then it follows that $P(X) \subset X$.*

PROOF. It suffices by (18) to prove the proposition in the case of two points z_j and \bar{z}_j (possibly equal). We assume $z_j \neq \bar{z}_j$, since the proof in the case $z_j = \bar{z}_j$ is essentially the same. Since $(z - A)^{-1}$ is analytic on the interior of Γ_j ($\bar{\Gamma}_j$) except at z_j (\bar{z}_j), we can assume that Γ_j and $\bar{\Gamma}_j$ are circles C and \bar{C} , respectively, with radius r and centers at z_j and \bar{z}_j , respectively. If $\phi \in X$, a simple calculation gives that

$$\begin{aligned}
 (19) \quad P\phi &= \frac{r}{2\pi} \int_{\theta=0}^{2\pi} \left[(z_j + re^{i\theta} - A)^{-1}(e^{i\theta}\phi) \right. \\
 &\quad \left. + (\bar{z}_j + re^{-i\theta} - A)^{-1}(e^{-i\theta}\phi) \right] d\theta.
 \end{aligned}$$

(19) shows that to prove the proposition it suffices to prove that

$$(20) \quad (z - A)^{-1}(e^{i\theta}\phi) + (\bar{z} - A)^{-1}(e^{-i\theta}\phi)$$

is an \mathbf{R}^n -valued function whenever ϕ is. However, since $A(X) \subset X$ it is easy to check that if $x \in \tilde{X}$ and $y = (z - A)^{-1}(x)$, then $\bar{y} = (\bar{z} - A)^{-1}(\bar{x})$, which shows the expression in (20) is \mathbf{R}^n -valued. \square

If $L: \tilde{X} \rightarrow C^n$ is a bounded linear map and $z_0 \in \sigma(A)$, so $\det \Delta(z_0) = 0$, we shall need the idea of the algebraic multiplicity of z_0 .

DEFINITION 2. Let Γ be a circle containing z_0 and no other points of $\sigma(A)$ and define a projection $P: \tilde{X} \rightarrow \tilde{X}$ by $P = (2\pi i)^{-1} \int_{\Gamma} (z - A)^{-1} dz$. Then the algebraic multiplicity of z_0 is the dimension of the range of P .

Several comments are in order. If $\det \Delta(z) = (z - z_0)^k g(z)$, where $g(z_0) \neq 0$, then our previous comments show that the algebraic multiplicity of z_0 is the dimension of the null space of $(z_0 - A)^k$. If $L_{\lambda}: \tilde{X} \rightarrow C^n$ for $\lambda \in \Lambda$, $\lambda \rightarrow L_{\lambda}$ is continuous, z_0 is a zero of $\det \Delta_{\lambda_0}(z)$ of multiplicity k , and Γ is a circle about z_0 , then Rouché's theorem implies that for $|\lambda - \lambda_0| < \delta$ ($\delta > 0$), $\det \Delta_{\lambda}(z)$ will have a total of k zeros inside Γ (counting algebraic multiplicity).

Furthermore, if δ is so small that $P_{\lambda} = (2\pi i)^{-1} \int_{\Gamma} (z - A)^{-1} dz$ is defined and $\|P_{\lambda} - P_{\lambda_0}\| < 1$ for $|\lambda - \lambda_0| < \delta$, then the dimension of the range of P_{λ} will be the same as the dimension of the range of P_{λ_0} , in fact if P_1 and P_2 are projections on a general Banach space and $\|P_1 - P_2\| < 1$, then the range of P_1 and P_2 have the same dimension.

Finally, suppose $L: \tilde{X} \rightarrow C^n$ and $\det \Delta(z_j) = 0$ for $1 \leq j \leq k$. Let Γ be a circle about z_j which contains no other points of $\sigma(A)$, let $\Gamma = \Gamma_j$ and define

$$\begin{aligned} P &= \frac{1}{2\pi i} \int_{\Gamma} (z - A)^{-1} dz \\ &= \sum_{j=1}^k \frac{1}{2\pi i} \int_{\Gamma_j} (z - A)^{-1} dz = \sum_{j=1}^k P_j. \end{aligned}$$

Since $P_l P_m = P_m$ for $l = m$ and 0 otherwise, the dimension of the range of P is $\sum_{j=1}^k n_j$, where n_j is the algebraic multiplicity of z_j . If V is the range of P , $A|_V = B$ is a bounded linear operator, and it makes sense to talk about the characteristic polynomial $\det(zI - B) = p(z)$ of B . General facts about the functional calculus imply that $z_j, j = 1, \dots, k$, are the roots of $p(z)$ and that n_j is the algebraic multiplicity of root z_j .

3. A nontrivial element of $\Pi_1(GL_m^+)$. We return now to the study of (6) in §1. In addition to H1-H3 we suppose

H4. The function $f(\phi, \lambda)$ satisfies

$$f(\phi, \lambda) = L_{\lambda}(\phi) + R(\phi, \lambda)$$

where $L_{\lambda}: X \rightarrow X$ is a continuous linear map and $R(\phi, \lambda) = o(\|\phi\|)$ uniformly on compact λ -intervals. The map $\lambda \rightarrow L_{\lambda}$ is continuous in the uniform operator topology.

REMARK 2. The assumption that $\lambda \rightarrow L_{\lambda}$ is continuous in the uniform

operator topology is restrictive. For example, if $(L_\lambda \phi) = \phi(g(\lambda))$, where $g(\lambda)$ is a continuous function of λ such that $-\gamma \leq g(\lambda) \leq 0$, the map $\lambda \rightarrow L_\lambda$ is continuous in the strong operator topology, but not the uniform operator topology. Actually, the results of this paper carry over if $\lambda \rightarrow L_\lambda$ is only continuous in the strong operator topology, but for simplicity we restrict ourselves to H4.

Now assume that H1–H4 hold, let $F(\phi, \lambda, t)$ be defined by (7) and $T_\lambda(t)$ defined as in §2.

LEMMA 1. *Assume that H1–H4 hold and let M be a compact subset of $\Lambda \times \mathbf{R}^+$. Then given $\varepsilon > 0$, there exists $\delta = \delta(\varepsilon, M) > 0$ such that*

$$\|F(\phi, \lambda, t) - T_\lambda(t)(\phi)\| \leq \varepsilon \|\phi\|$$

whenever $(\lambda, t) \in M$ and $\|\phi\| \leq \delta$.

PROOF. This is an exercise in the variation of constants formula for F.D.E.'s. First, by the continuity of F , we can assume δ_1 chosen such that $F(\phi, \lambda, t)$ is defined for $\|\phi\| \leq \delta_1$ and $(\lambda, t) \in M$. By the variation of constants formula we have

$$(21) \quad F(\phi, \lambda, t) = T_\lambda(t)\phi + \int_0^t T_\lambda(t-s)R(x_s, \lambda) ds,$$

where $x(s; \phi, \lambda)$ is the solution of (6). It is known (and not hard to prove) that

$$\|T_\lambda(u)\| \leq \exp(\|L_\lambda\|u).$$

It follows that there is a constant C such that

$$\|T_\lambda(t-s)\| \leq C$$

for $(\lambda, t) \in M$ and $0 \leq s < t$. By the continuity of F and assumption H4 there exists $\delta_2 > 0$ such that for $\|\phi\| \leq \delta_2$ we can assume

$$\|R(x_s, \lambda)\| \leq \|x_s\|$$

for $0 \leq s < t$ and all $(\lambda, t) \in M$. (21) then implies that for $\|\phi\| \leq \delta_2$ and $(\lambda, t) \in M$ we have

$$(22) \quad \|x_t\| \leq C\|\phi\| + C \int_0^t \|x_s\| ds.$$

It follows by Gronwall's inequality that there is a constant C_1 such that for $\|\phi\| \leq \delta_2$, $0 \leq s < t$ and $(\lambda, t) \in M$ we have

$$(23) \quad \|x_s\| \leq C_1\|\phi\|.$$

Using (21) and (23) and H4, the result now follows easily. \square

Next we need to recall the basic result from §8 of [1]. Let V be a real finite dimensional vector space; denote by $GL(V)$ the invertible linear operators taking V to V . It is well known that $GL(V)$ comprises two connected

components: $GL^+(V)$, the component containing the identity I and $GL^-(V)$.

Suppose that Λ is an open interval of real numbers and that for each $\lambda \in \Lambda$, $M_\lambda: V \rightarrow V$ is a linear map. Assume that $\lambda \rightarrow M_\lambda$ is continuous. Following [1], suppose that $i\beta$ ($\beta > 0$) is an eigenvalue of M_{λ_0} and denote by $\text{Mult}(i\beta)$ the set $\{im\beta: m = \text{a positive integer, } im\beta \text{ is an eigenvalue of } M_{\lambda_0}\}$. Assume that 0 is not an eigenvalue of M_{λ_0} . If we cover each element z of $\text{Mult}(i\beta)$ by a disc D_z which contains only the eigenvalue z of M_{λ_0} , then for λ close enough to λ_0 , each disc D_z contains an eigenvalue z_λ of M_λ . Furthermore, if the algebraic multiplicity of z as an eigenvalue of M_{λ_0} is k , then D_z will contain k eigenvalues of M_λ (counting algebraic multiplicities) for $|\lambda - \lambda_0|$ small. Denote by $\text{Mult}_\lambda(i\beta)$ the set of eigenvalues which lie in the union of the discs D_z for $z \in \text{Mult}(i\beta)$. Assume that there is an open interval Λ_0 containing λ_0 such that if $\lambda \in \Lambda_0 - \{\lambda_0\}$ and $z \in \text{Mult}(i\beta)$, then the real part of z is nonzero; by the above discussion we can assume that $\text{Mult}_\lambda(i\beta)$ and $\text{Mult}(i\beta)$ have the same number of elements (counting algebraic multiplicity) for $\lambda \in \Lambda_0$.

With the above assumptions, it makes sense to define r_+ to be the number of elements of $\text{Mult}(i\beta)$ (counting algebraic multiplicity) whose real part is positive for $\lambda > \lambda_0$ and $\lambda \in \Lambda_0$ and r_- to be the number of elements of $\text{Mult}_\lambda(i\beta)$ whose real part is positive for $\lambda < \lambda_0$. Define the index of $i\beta$ with respect to M_λ to be $r = r_+ - r_-$ and the parity of $i\beta$ to be the parity (even or odd) of r . Then Alexander and Yorke [1, §8] prove the following lemma.

LEMMA 2 (ALEXANDER-YORKE). *Let notation and assumptions be as in the above two paragraphs. Let $t_0 = 2\pi/\beta$, and for $\rho > 0$ define $S_\rho = \{(\lambda, t) : (\lambda - \lambda_0)^2 + (t - t_0)^2 = \rho^2\}$. Then there exists $\rho_0 > 0$ such that for $0 < \rho < \rho_0$, the map $(\lambda, t) \in S_\rho \rightarrow \exp(itM_\lambda) - I$ is a continuous map into $GL^\pm(V)$. Since S_ρ is homotopic to S^1 , the above map gives an element h of $\Pi_1(GL^\pm(V))$; and if the parity of $i\beta$ is k and g denotes a generator of $\Pi_1(GL^\pm(V))$, $h = g^k$. In particular, h is nonzero if k is odd and $h = g$ if k is even and $\dim(V) > 2$.*

We wish to make assumptions analogous to those of the Alexander-Yorke lemma. Thus suppose

H5. The equation $\det \Delta_{\lambda_0}(z) = 0$ has a solution $i\beta$ with $\beta > 0$, and no zero solution. There exist positive constants ϵ and M such that if $\det \Delta_\lambda(z) = 0$ for $|\lambda - \lambda_0| < \epsilon$, then $|\text{Im}(z)| \leq M$. Let $\text{Mult}(i\beta)$ denote the set $\{im\beta: m \text{ a positive integer and } \det \Delta_{\lambda_0}(im\beta) = 0\}$. Cover the elements z of $\text{Mult}(i\beta)$ by closed discs D_z which contain only the eigenvalue z of A_{λ_0} , denote by D the union of the discs D_z and let $\text{Mult}_\lambda(i\beta) = \{z: \det \Delta_\lambda(z) = 0 \text{ and } z \in D\}$. Assume that there is an open interval Λ_0 containing λ_0 such that the real part of z is nonzero if $z \in \text{Mult}_\lambda(i\beta)$ and $\lambda \in \Lambda_0 - \{\lambda_0\}$.

REMARK 3. If H5 holds, then for $|\lambda - \lambda_0|$ small, the number of elements in

$\text{Mult}_\lambda(i\beta)$ (counted with algebraic multiplicity) is the same as the number of elements in $\text{Mult}(i\beta)$ (counted with algebraic multiplicity).

DEFINITION 3. If H5 holds, the index of $i\beta$ with respect to L_λ is $r = r_+ - r_-$, where r_+ (r_- , respectively) is the number of elements of $\text{Mult}_\lambda(i\beta)$ (counting algebraic multiplicity in the sense of §2) whose real part is positive for $\lambda > \lambda_0$ ($\lambda < \lambda_0$, respectively). The parity of $i\beta$ is the parity (even or odd) of r .

In the statement of the next theorem, the projections P_m are as in §1, and we denote by X_m the $m(n + 1)$ dimensional range of P_m in X .

THEOREM 3. *Assume that H1–H5 hold and that the parity of $i\beta$ is odd. Let $t_0 = 2\pi/\beta$, and for $\rho > 0$ define $S_\rho = \{(\lambda, t) : (\lambda - \lambda_0)^2 + (t - t_0)^2 = \rho^2\}$. Then there exist $\rho_0 > 0$ and, for each positive $\varepsilon < \rho_0$, an integer $m_0 = m_0(\varepsilon)$ such that for $\varepsilon \leq \rho \leq \rho_0$ and for $m \geq m_0$, the map $(\lambda, t) \in S_\rho \rightarrow I - P_m T_\lambda(t)|X_m$ is a continuous map into $GL^\pm(X_m)$ and gives a nonzero element of $\Pi_1(GL^\pm(X_m))$.*

PROOF. The proof is long and we divide it into steps.

Step 1. First we show that there is $\rho_0 > 0$ such that $I - T_\lambda(t)$ is one-one for $0 < |\lambda - \lambda_0|^2 + |t - t_0|^2 < \rho_0^2$. It suffices to show that 1 is not in $\sigma(T_\lambda(t))$ for such (λ, t) . By the work of §2 (since we assume $\det \Delta_{\lambda_0}(0) \neq 0$) this is equivalent to showing $2\pi ij/t \notin \sigma(A_\lambda)$ for $(\lambda, t) \in S_\rho$ and positive integers j . If $(\lambda_0 - \rho_0, \lambda_0 + \rho_0) \subset \Lambda_0$, H5 assures that (with possibly a smaller ρ_0) $\sigma(A_\lambda)$ contains no points ir , r real, for $0 < |\lambda - \lambda_0| < \rho_0$. If $\lambda = \lambda_0$, $t = t_0 \pm \rho$, and one can see that if ρ_0 is taken small enough, $2\pi ij/t_0 \notin \sigma(A_{\lambda_0})$ for positive integers j and for $0 < \rho \leq \rho_0$. Note that if $0 \in \sigma(A_{\lambda_0})$, then $I \in \sigma(T_{\lambda_0}(t))$ for all $t \geq 0$, and the above argument fails.

Step 2. Take $\varepsilon > 0$ with $0 < \varepsilon < \rho_0$. We shall show that there exists m_0 such that for $m \geq m_0$ and $(\lambda, t) \in S_\rho$ with $\varepsilon \leq \rho \leq \rho_0$, the map $I - P_m T_\lambda(t)$ is one-one. Suppose not. Then there is a sequence $(\lambda_j, t_j) \in S_\rho$ and a sequence of integers $m_j \rightarrow \infty$ such that $I - P_{m_j} T_{\lambda_j}(t_j)$ is not one-one; in particular, there is a sequence of unit vectors ϕ_j with $\phi_j - P_{m_j} T_{\lambda_j}(t_j)(\phi_j) = 0$. We can assume $\rho_0 < t_0$ and apply Theorem 2 to the linear system $x'(t) = L_\lambda(x_t)$; according to Theorem 2 we can assume, by taking a subsequence, that $\phi_j \rightarrow \phi$, $\lambda_j \rightarrow \lambda$ and $t_j \rightarrow t$. It follows by continuity that $\phi - T_\lambda(t)(\phi) = 0$, which contradicts the fact that $I - T_\lambda(t)$ is one-one.

Since $(\lambda, t) \in S_\rho \rightarrow I - P_m T(\lambda, t)|X_m$ is continuous in the norm topology for operators on X_m (because X_m is finite dimensional), it determines an element of $\Pi_1(GL^\pm(X_m))$ for $m \geq m_0$. We now wish to show that this element of the homotopy group is nonzero.

Step 3. As a preliminary step we shall show that the map $(\lambda, t) \rightarrow I - P_m T_\lambda(t)|X_m$ is homotopic in $GL^\pm(X_m)$ to a map of the form

$$(\lambda, t) \rightarrow I - P_m B_\lambda^{-1} T_\lambda(t) B_\lambda P_{\lambda_0} - P_m C_\lambda^{-1} T_\lambda(t) C_\lambda Q_{\lambda_0}.$$

For each element $z_j = ik_j \beta \in \text{Mult}(i\beta)$ let Γ_j be a circle about z_j which contains no other points of $\sigma(A_{\lambda_0})$. Let $\bar{\Gamma}_j$ be a circle about $-ik_j \beta$ and denote by Γ the union of these circles. As in §2, for λ near λ_0 define

$$P_\lambda = \frac{1}{2\pi i} \int_\Gamma (z - A_\lambda)^{-1} dz.$$

By Proposition 2, §2, P_λ maps the real B -space X into X . We follow the notation of §2, so X_λ is the range of P_λ , etc. For $|\lambda - \lambda_0| < \rho_0$ we can assume that $\|P_\lambda - P_{\lambda_0}\| < 1$, and it follows that for such λ that

$$B_\lambda = I + (P_\lambda - P_{\lambda_0}) \quad \text{and} \quad C_\lambda = I + (Q_\lambda - Q_{\lambda_0})$$

are one-one and onto X . Since $B_\lambda(X_{\lambda_0}) \subset X_\lambda$, and since X_λ and X_{λ_0} have the same dimension, we get $B_\lambda(X_{\lambda_0}) = X_\lambda$ and $B_\lambda^{-1}(X_\lambda) = X_{\lambda_0}$. A slightly more involved argument (which we omit) shows that $C_\lambda(Y_{\lambda_0}) = Y_\lambda$ and $C_\lambda^{-1}(Y_\lambda) = Y_{\lambda_0}$.

Now assume that ρ_0 is small, and for $0 < \rho \leq \rho_0$ and each real number μ with $0 \leq \mu \leq 1$ define $q_\mu: S_\rho \rightarrow L(X, X) =$ bounded linear operators on X ,

$$(24) \quad q_\mu(\lambda, t) = I - B_{\mu(\lambda)}^{-1} T_\lambda(t) B_{\mu(\lambda)} P_{\lambda_0} - C_{\mu(\lambda)}^{-1} T_\lambda(t) C_{\mu(\lambda)} Q_{\lambda_0},$$

where $\mu(\lambda) = (1 - \mu)\lambda + \mu\lambda_0$. Note that for $\mu = 1$ we obtain $I - T_\lambda(t)$. We claim that the linear operator $q_\mu(\lambda, t)$ is one-one for μ, λ, t as above. To see this suppose

$$(25) \quad q_\mu(\lambda, t)(\phi) = 0$$

for a nonzero ϕ . Then composing on the left with P_{λ_0} and Q_{λ_0} , respectively, gives

$$(26) \quad \begin{aligned} P_{\lambda_0} \phi - B_\mu^{-1} T_\lambda(t) B_\mu P_{\lambda_0} \phi &= 0, \\ Q_{\lambda_0} \phi - C_\mu^{-1} T_\lambda(t) C_\mu Q_{\lambda_0} \phi &= 0. \end{aligned}$$

From (26) we derive that

$$(27) \quad (I - T_\lambda(t)) B_\mu P_{\lambda_0} \phi = 0, \quad (I - T_\lambda(t)) C_\mu Q_{\lambda_0} \phi = 0.$$

Since $I - T_\lambda(t)$ is one-one for $(\lambda, t) \in S_\rho$, and because B_μ and C_μ are one-one, equations (27) imply that $\phi = 0$.

Next suppose that ϵ is any positive number with $\epsilon < \rho_0$. We claim that there is an integer $N = N(\epsilon)$ such that for $m \geq N$, $\epsilon \leq \rho \leq \rho_0$ and $(\lambda, t) \in S_\rho$ the linear operator

$$(28) \quad I + P_m(q_\mu(\lambda, t) - I)$$

is one-one.

As in previous cases, the proof is by contradiction. Suppose not. Then there is a sequence of integers $m(j) \rightarrow \infty$, a sequence of unit vectors $\phi_j \in X$ and

sequences $(\lambda_j, t_j) \in S_{\rho_j}$ and $\mu_j \in [0, 1]$ for which, when $\mu = \mu_j, \lambda = \lambda_j, t = t_j$ and $m = m(j)$, the operator in (28) vanishes when applied to ϕ_j . We can assume by taking subsequences that $(\lambda_j, t_j) \rightarrow (\lambda, t) \in S_{\rho}$ and $\mu_j \rightarrow \mu$. Notice that $B_{\mu_j(\lambda_j)}^{-1}$ approaches $B_{\mu(\lambda)}^{-1}$ in norm and that $B_{\mu(\lambda)}^{-1} = I + K$, where K is a compact linear operator; analogous statements hold for $C_{\mu_j(\lambda_j)}^{-1}$. It follows from the compactness of K that by taking a further subsequence we can assume

$$(29) \quad q_{\mu_j}(\lambda_j, t_j)(\phi_j) - (I - T_{\lambda_j}(t_j)B_{\mu_j}P_{\lambda_0} - T_{\lambda_j}(t_j)C_{\mu_j}Q_{\lambda_0})(\phi_j)$$

converges in norm to some element ψ . Using this fact, we can assume that

$$(30) \quad \phi_j - P_{m(j)}T_{\lambda_j}(t_j)B_{\mu_j}P_{\lambda_0}(\phi_j) - P_{m(j)}T_{\lambda_j}(t_j)C_{\mu_j}Q_{\lambda_0}(\phi_j)$$

converges in norm as $j \rightarrow \infty$. Since B_{μ_j} and C_{μ_j} approach B_{μ} and C_{μ} in norm and B_{μ} and C_{μ} are compact perturbations of the identity, we can assume by taking a further subsequence that

$$(31) \quad \phi_j - P_{m(j)}T_{\lambda_j}(t_j)\phi_j$$

converges in norm as $j \rightarrow \infty$. It now follows from Theorem 2 that by taking a still further subsequence we can assume that

$$(32) \quad \phi_j \rightarrow \phi.$$

It now follows from (28) by taking limits that

$$(33) \quad (q_{\mu}(\lambda, t))(\phi) = 0,$$

and this contradicts the previous results.

We have shown that for $m \geq N$, (28) provides a permissible homotopy in $GL^{\pm}(X_m)$ between $I - P_m T_{\lambda}(t)|X_m$ and $P_m q_1(\lambda, t)|X_m$.

Step 4. We shall now show that for m large enough there is a homotopy in $GL^{\pm}(X_m)$ between the map $(\lambda, t) \rightarrow P_m q_1(\lambda, t)|X_m$ and the map

$$(\lambda, t) \rightarrow I - P_m B_{\lambda}^{-1} T_{\lambda}(t) B_{\lambda} P_{\lambda_0} - P_m T_{\lambda_0}(t_0) Q_{\lambda_0} | X_m.$$

If we recall that $I - T_{\lambda}(t)|Y_{\lambda_0}$ (where Y_{λ_0} is the range of Q_{λ_0}) is one-one and onto Y_{λ_0} for $(\lambda - \lambda_0)^2 + (t - t_0)^2 \leq \rho_0^2$, it is not hard to see that if we define $s(\lambda) = (1 - s)\lambda + s\lambda_0$ and $s(t) = (1 - s)t + st_0$ for $0 \leq s \leq 1$, then

$$(34) \quad H_s(\lambda, t) = I - B_{\lambda}^{-1} T_{\lambda}(t) B_{\lambda} P_{\lambda_0} - C_{s(\lambda)}^{-1} T_{s(\lambda)}(s(t)) C_{s(\lambda)} Q_{\lambda_0}$$

is one-one for $0 \leq s \leq 1$ and for $0 < \varepsilon \leq (\lambda - \lambda_0)^2 + (t - t_0)^2 \leq \rho_0^2$. Arguments similar to ones previously used show that for $m \geq N_1 = N_1(\varepsilon)$,

$$(35) \quad P_m H_s(\lambda, t) | X_m$$

is one-one for $0 \leq s \leq 1$ and (λ, t) as above, and this establishes the assertion in Step 4.

The homotopies of Steps 3 and 4 reduce the problem to showing that

$$(36) \quad (\lambda, t) \rightarrow I - P_m B_{\lambda}^{-1} T_{\lambda}(t) B_{\lambda} P_{\lambda_0} - P_m T_{\lambda_0}(t_0) Q_{\lambda_0} | X_m$$

gives a nonzero element of $\Pi_1(GL^\pm(X_m))$ for m large.

Step 5. The form of (36) is still not quite suitable. In this step we shall show that for $(\lambda, t) \in S_\rho$, $0 < \varepsilon \leq \rho \leq \rho_0$, $0 \leq s \leq 1$, and m large enough the linear operator

$$I - (sP_m + (1 - s)I)B_\lambda^{-1}T_\lambda(t)B_\lambda P_{\lambda_0} - P_m T_{\lambda_0}(t_0)Q_{\lambda_0}$$

is one-one as a map of X to X .

To prove this, first observe that

$$(37) \quad G(\lambda, t) = I - B_\lambda^{-1}T_\lambda(t)B_\lambda P_{\lambda_0} - T_{\lambda_0}(t_0)Q_{\lambda_0}$$

is one-one and onto X for $(\lambda, t) \in S_\rho$ and $0 < \rho \leq \rho_0$. Furthermore, because P_{λ_0} has finite dimensional range, the map

$$(38) \quad (\lambda, t) \rightarrow G(\lambda, t)$$

is continuous in the norm operator topology, not just the strong operator topology. It follows that there exists a positive constant $c = c(\varepsilon)$ such that for any ρ with $0 < \varepsilon \leq \rho \leq \rho_0$ and any unit vector ϕ in X ,

$$(39) \quad \|\phi - B_\lambda^{-1}T_\lambda(t)B_\lambda P_{\lambda_0}(\phi) - T_{\lambda_0}(t_0)Q_{\lambda_0}(\phi)\| > c.$$

An argument using the strong A -properness of $I - T_{\lambda_0}(t_0)$ now shows that for m large enough and ϕ and (λ, t) as above, we have

$$(40) \quad \|\phi - P_m B_\lambda^{-1}T_\lambda(t)B_\lambda P_{\lambda_0}(\phi) - P_m T_{\lambda_0}(t_0)Q_{\lambda_0}(\phi)\| > \frac{c}{2}.$$

Using the fact that P_{λ_0} is a compact linear operator, it is not hard to show that for m large enough and $(\lambda - \lambda_0)^2 + (t - t_0)^2 \leq \rho_0^2$, we have

$$(41) \quad \|P_m B_\lambda^{-1}T_\lambda(t)B_\lambda P_{\lambda_0} - B_\lambda^{-1}T_\lambda(t)B_\lambda P_{\lambda_0}\| \leq c/4.$$

It follows that for m large, $0 < \varepsilon \leq \rho \leq \rho_0$, $(\lambda, t) \in S_\rho$ and $0 \leq s \leq 1$, the following operator is one-one:

$$(42) \quad I - (sP_m + (1 - s)I)B_\lambda^{-1}T_\lambda(t)B_\lambda P_{\lambda_0} - P_m T_{\lambda_0}(t_0)Q_{\lambda_0}.$$

Step 6. Now let Y_m denote the linear subspace spanned by X_m and X_{λ_0} . To show that the map given by (36) yields a nonzero element of $\Pi_1(GL^\pm(X_m))$, it suffices to show that

$$(43) \quad (\lambda, t) \rightarrow I - P_m B_\lambda^{-1}T_\lambda(t)B_\lambda P_{\lambda_0} - P_m T_{\lambda_0}(t_0)Q_{\lambda_0}|Y_m$$

gives a nonzero element of $\Pi_1(GL^\pm(Y_m))$. (42) gives a homotopy in $GL^\pm(Y_m)$ for $0 \leq s \leq 1$, so it suffices to show that

$$(44) \quad (\lambda, t) \rightarrow I - B_\lambda^{-1}T_\lambda(t)B_\lambda P_{\lambda_0} - P_m T_{\lambda_0}(t_0)Q_{\lambda_0}|Y_m$$

gives a nonzero element of $\Pi_1(GL^\pm(Y_m))$. If we note that $Q_{\lambda_0}(Y_m) \subset Y_m$, and if we write

$$(45) \quad Y_m = X_{\lambda_0} \oplus Q_{\lambda_0}(Y_m),$$

then the map in (44) is reduced by X_{λ_0} and $Q_{\lambda_0}(Y_m)$. On X_{λ_0} the map is given by

$$(46) \quad (\lambda, t) \rightarrow I - B_\lambda^{-1}T_\lambda(t)B_\lambda|X_{\lambda_0},$$

and on $(Q_{\lambda_0}(Y_m))$ it is given by

$$(47) \quad (\lambda, t) \rightarrow I - P_m T_{\lambda_0}(t_0)|Q_{\lambda_0}(Y_m),$$

which is just a constant map. It follows that to show (44) gives a nonzero element, it suffices to show the map (46) is g^r in $\Pi_1(GL^\pm(X_{\lambda_0}))$, where r is the index of $i\beta$ and g is a generator of $\Pi_1(GL^\pm(X_{\lambda_0}))$. Notice that A_λ , the infinitesimal generator of $T_\lambda(t)$, is everywhere defined on X_λ , so that we have

$$(48) \quad T_\lambda(t)|X_\lambda = \sum_{j=0}^{\infty} \frac{(tA_\lambda)^j}{j!}.$$

Thus we find that if we set $M_\lambda = B_\lambda^{-1}A_\lambda B_\lambda|X_{\lambda_0}$, we have

$$(49) \quad B_\lambda^{-1}T_\lambda(t)B_\lambda|X_{\lambda_0} = \exp(tM_\lambda).$$

If $z \in \text{Mult}_\lambda(i\beta)$, the results of §2 show that the algebraic multiplicity of z as an element of the spectrum of A_λ is the same as the multiplicity of z as a root of $\det(x - A_\lambda|X_\lambda)$, which in turn is the multiplicity of z as a root of $\det(x - M_\lambda)$. This shows that the index of $i\beta$ in our theorem is the same as the index of $i\beta$ with respect to M_λ . Thus the Alexander-Yorke theorem implies that (46) gives a nonzero element of $\Pi_1(GL^\pm(X_{\lambda_0}))$, and we are done. \square

4. Global bifurcation of periodic solutions of retarded functional differential equations. After the preliminaries of the previous three sections, we can begin to prove our main theorem. Assuming that H1–H3 hold, we write

$$(50) \quad S = \{(\phi, \lambda, t) \in X \times \Omega \times [0, \infty) : x(s; \phi, \lambda) \text{ is periodic of period } t > 0 \text{ (not necessarily least period) and } x(s) \text{ is not a constant function}\}.$$

$$\mathfrak{S} = \text{the closure of } S \text{ in } X \times \Lambda \times [0, \infty).$$

LEMMA 3. *Assume H1–H3. If (ϕ, λ, t) is an element of \mathfrak{S} but not of S , then ϕ is a constant function and $f(\phi, \lambda) = 0$; in particular, $x(s; \phi, \lambda)$ is a constant function.*

PROOF. By definition, there exists a sequence $(\phi_j, \lambda_j, t_j) \in S$ which converges to (ϕ, λ, t) . If we define

$$(51) \quad x_j(s) = x(s; \phi_j, \lambda_j),$$

$x_j(s)$ is periodic with period t_j , $x_j[-\gamma, 0] = \phi_j$ and

$$(52) \quad x_j(u) = \phi_j(0) + \int_0^u f((x_j)_s, \lambda_j) ds$$

for $u \geq 0$. H3 and the periodicity of x_j imply that there is a constant M_1 such that

$$(53) \quad |x_j(s)| \leq M_1$$

for $j \geq 1$ and $s \geq -\gamma$. (52) and the fact that f is bounded on bounded sets now imply that the functions x_j are Lipschitzian with uniform Lipschitz constant M_2 . The functions $x_j|[-\gamma, t + 1]$ form a bounded, equicontinuous family, so by taking a subsequence we can assume that x_j approaches some continuous function y uniformly on $[-\gamma, t + 1]$, and by continuity we see that y satisfies

$$(54) \quad \begin{aligned} y(u) &= \phi(0) + \int_0^u f(y_s, \lambda) ds \quad \text{for } u \geq 0, \\ y|[-\gamma, 0] &= \phi. \end{aligned}$$

It follows that $y(s) = x(s; \phi, \lambda)$ for $s \geq -\gamma$. There are two cases to consider.

(i) $t > 0$. We know by equicontinuity and the facts that $t_j \rightarrow t$ and x_j has period t_j that

$$(55) \quad \lim_{j \rightarrow \infty} \left(\sup_{-\gamma \leq s < 0} |x_j(t + s) - x_j(s)| \right) = 0.$$

(55) implies that $y(t + s) = y(s)$ for $-\gamma \leq s \leq 0$, so $y(s) = x(s; \phi, \lambda)$ is periodic of period $t > 0$. Since we assume that $(\phi, \lambda, t) \notin S$, we must have that $y(s)$ is a constant function.

(ii) $t = 0$. Since x_j is periodic of period t_j and Lipschitzian with Lipschitz constant M_2 (independent of j), we have for any numbers $s_1, s_2 \geq -\gamma$ that

$$(56) \quad |x_j(s_1) - x_j(s_2)| \leq M_2 t_j.$$

By taking limits (since $t_j \rightarrow 0$) we obtain that $x(s; \phi, \lambda)$ is a constant function.

□

Before proving our next lemma, we need to recall some facts from homotopy theory. Let notation be as in §3, so that for $\rho > 0$, $S_\rho \subset \mathbb{R}^2$ is a homeomorphic image of a circle. Suppose that $(\lambda, t) \in S_\rho \rightarrow A(\lambda, t) \in GL^+(n)$ is a continuous map. For $r > 0$ define (following [13]) $S_{r,\rho}$ by

$$\begin{aligned} S_{r,\rho} &= \{ (x, \lambda, t) : x \in \mathbb{R}^n, \lambda, t \in \mathbb{R}, \\ &\quad \|x\|^2 + |\lambda - \lambda_0|^2 + |t - t_0|^2 = r^2 + \rho^2 \}. \end{aligned}$$

Of course, $S_{r,\rho}$ is homeomorphic to S^{n+1} , the $n + 1$ sphere. Extend the map $(\lambda, t) \rightarrow A(\lambda, t)$ to all of \mathbb{R}^2 in any way such that $A(\lambda, t)$ is nonsingular for $(\lambda, t) \neq (\lambda_0, t_0)$. Then we can define a map from $S_{r,\rho}$ to $\mathbb{R}^{n+1} - \{0\}$ by

$$(57) \quad (x, \lambda, t) \rightarrow (A(\lambda, t)x, \|x\|^2 - r^2).$$

If we use the notation $[Y_1, Y_2]$ to denote the free homotopy classes of continuous maps from the topological space Y_1 to the topological space Y_2 , then (57) induces a map J from $[S^1, GL^+(n)]$ to $[S_{r,\rho}, \mathbf{R}^{n+1} - \{0\}]$, and the map J is a bijection. Further details and references can be found in [12].

In our case, if we define

$$(58) \quad S_{r,\rho}(m) = \{(x, \lambda, t) \in X_m \times \mathbf{R}^2: \|x\|^2 + (\lambda - \lambda_0)^2 + (t - t_0)^2 = r^2 + \rho^2\}$$

and assume that $0 < r, \rho$ and $\rho \leq \rho_0$, then Theorem 3 of §3 in conjunction with the above remarks shows that the map

$$(59) \quad (x, \lambda, t) \in S_{r,\rho}(m) \rightarrow (x - P_m T_\lambda(t)x, \|x\|^2 - r^2) \in (X_m \times \mathbf{R}) - \{0\}$$

defines (for m large enough) a nontrivial element of $[S_{r,\rho}(m), (X_m \times \mathbf{R}) - \{0\}]$ (that is, the map in (59) is not homotopic to the constant map).

With these preliminaries we can study the homotopy type of the map

$$(60) \quad (x, \lambda, t) \rightarrow (x - P_m F(x, \lambda, t), \|x\|^2 - r^2),$$

where F is defined by (7) in §1.

LEMMA 4. *Assume H1–H5 and that the parity of $i\beta$ is odd (see Definition 3 in §3). Let $\rho \leq \rho_0$ be a positive number. Then there exists a positive number $r(\rho)$ (depending continuously on ρ) such that if $0 < r < r(\rho)$ and $m \geq m(r)$, (60) defines a map from $S_{r,\rho}(m)$ to $(X_m \times \mathbf{R}) - \{0\}$ which is not homotopic to a constant map.*

PROOF. Since $I - T_\lambda(t)$ depends continuously on (λ, t) and is one-one and onto for $(\lambda - \lambda_0)^2 + (t - t_0)^2 = \rho^2$, $0 < \rho \leq \rho_0$, there is a continuous positive function $c(\rho)$ such that

$$\|x - T_\lambda(t)x\| \geq c(\rho)\|x\|$$

for (λ, t) as above. Furthermore, it is not hard to show that there is a continuous, strictly increasing function $\varepsilon(r)$ of r for $r \geq 0$ with $\varepsilon(0) = 0$ such that for all (λ, t) with $(\lambda - \lambda_0)^2 + (t - t_0)^2 \leq \rho_0^2$ we have

$$\|T_\lambda(t)(x) - F(x, \lambda, t)\| \leq \varepsilon(\|x\|)\|x\|.$$

If ε^{-1} denotes the inverse function and we take $0 < r < r(\rho) = \varepsilon^{-1}(\frac{1}{2}c(\rho))$, we find that for $(\lambda - \lambda_0)^2 + (t - t_0)^2 = \rho^2$, $0 \leq s \leq 1$ and $\|x\| \leq r$ we have

$$\|x - sT_\lambda(t)(x) - (1 - s)F(x, \lambda, t)\| \geq \frac{1}{2}c(\rho)\|x\|.$$

It follows that if we define, for $0 \leq s \leq 1$,

$$(61) \quad \Phi(x, \lambda, t, s) = sT_\lambda(t)(x) + (1 - s)F(x, \lambda, t),$$

and let

$$\Sigma_{r,\rho} = \left\{ (x, \lambda, t) \in X \times \Lambda \times \mathbf{R}^+ : \right. \\ \left. \|x\|^2 + (\lambda - \lambda_0)^2 + (t - t_0)^2 = r^2 + \rho^2 \right\}$$

(with r and ρ as above), then the map

$$(62) \quad (x, \lambda, t, s) \in \Sigma_{r,\rho} \times [0, 1] \rightarrow (x - \Phi(x, \lambda, t, s), \|x\|^2 - r^2)$$

does not have zero in its range.

We now claim that for $m \geq m(r)$ the map

$$(63) \quad (x, \lambda, t, s) \in \Sigma_{r,\rho} \times [0, 1] \rightarrow (x - P_m \Phi(x, \lambda, t, s), \|x\|^2 - r^2)$$

does not have zero in its range. In order to prove this, suppose not, so that there is a sequence $(x_m, \lambda_m, t_m, s_m) \in \Sigma_{r,\rho} \times [0, 1]$ for which the right-hand side of (63) vanishes. One can check that Φ satisfies the hypotheses of Theorem 1, so that by taking a subsequence we can assume that

$$(x_m, \lambda_m, t_m, s_m) \rightarrow (x, \lambda, t, s) \in \Sigma_{r,\rho} \times [0, 1].$$

By continuity we have that

$$(x - \Phi(x, \lambda, t, s), \|x\|^2 - r^2) = 0,$$

which is a contradiction.

To complete the proof recall that by the results of §3, the map

$$(\lambda, t) \in S_\rho \rightarrow I - P_m T_\lambda(t) \in GL^\pm(X_m)$$

gives a nonzero element of $\Pi_1(GL^\pm(X_m))$ for large m . By our previous remarks, it follows that for r, ρ as above,

$$(x, \lambda, t) \in S_{r,\rho}(m) \rightarrow (x - P_m T_\lambda(t)(x), \|x\|^2 - r^2) \\ \in (X_m \times \mathbf{R}) - \{0\}$$

is not homotopic to a constant in $(X_m \times \mathbf{R}) - \{0\}$. Since we have shown that this map is homotopic in $(X_m \times \mathbf{R}) - \{0\}$ to the map of (60), the lemma is proved.

Now let λ_0 and $t_0 = 2\pi/\beta$ be as in §3. We are interested in the connected component \mathfrak{S}_0 of \mathfrak{S} which contains $(0, \lambda_0, t_0) \in X \times \Lambda \times [0, \infty)$.

LEMMA 5. *Assume that H1–H5 hold and that the parity of $i\beta$ is odd (see Definition 3). If \mathfrak{S}_0 is defined as above, \mathfrak{S}_0 is nonempty.*

PROOF. In the notation of the previous lemma, select a positive number r with $r < r_0 = r(\rho_0)$. According to the previous lemma, the map given by (60) is not homotopic to a constant in $(X_m \times \mathbf{R}) - \{0\}$. It follows (see [20, p. 1]) that for all m sufficiently large, there exists (x_m, λ_m, t_m) such that

$$(64) \quad \|x_m\|^2 + (\lambda_m - \lambda_0)^2 + (t_m - t_0)^2 < r^2 + \rho_0^2, \\ \|x_m\| = r, \quad x_m - P_m F(x_m, \lambda_m, t_m) = 0.$$

According to Theorem 2, we can assume by taking a subsequence that $(x_m, \lambda_m, t_m) \rightarrow (x, \lambda, t)$, and by continuity we obtain

$$\|x\| = r \quad \text{and} \quad x - F(x, \lambda, t) = 0.$$

This completes the proof. \square

We need to recall at this point the definition and basic properties of “compact vector fields”; those concepts have been extensively used by A. Granas [9]. If Y is a Banach space, A is a closed subset of Y and $g: A \rightarrow Y$ is a continuous map, g is a “compact vector field” if $g(x) = x - G(x)$, where G takes bounded sets to precompact sets. We work in the category of compact vector fields; in particular, two compact vector fields g_0 and g_1 are (compactly) homotopic if there is a compact map $G: A \times [0, 1] \rightarrow X$ such that $g_0(x) = x - G(x, 0)$ and $g_1(x) = x - G(x, 1)$.

If B is a closed set containing A and $g: A \rightarrow Y - \{0\} = P$ is a compact vector field, we say “ g is inessential with respect to B ” if g has an extension to a zero field $\bar{g}: B \rightarrow P$. If $g_0, g_1: A \rightarrow P$ are compact vector fields and if they are (compactly) homotopic on A by a zero-free homotopy $x - G(x, t) \in P$, then the basic lemma (see [9]) is that (assuming B is bounded) g_0 is essential with respect to B if and only if g_1 is. Of course, in the finite dimensional case more general versions of this lemma have long been known: see [2, Theorem 8.1, p. 94].

We shall need a simple extension of Granas’ idea. If Y is a Banach space, A is a closed subset of $Y \times \mathbb{R}^m$ and $g: A \rightarrow Y \times \mathbb{R}^k$ is a continuous map, we shall say g is a compact vector field if we have $g(y, u) = (y - G(y, u), h(y, u))$, where G takes bounded subsets of A into precompact sets and $h: A \rightarrow \mathbb{R}^k$ takes bounded sets to bounded sets. If B is a closed, bounded set containing A and $g: A \rightarrow P = (Y \times \mathbb{R}^k) - \{0\}$ is a compact vector field, we say g is inessential with respect to B if g has an extension to a compact vector field $\bar{g}: B \rightarrow P$; otherwise g is “essential with respect to B ”. Again, if B is bounded and $g_0, g_1: A \rightarrow P$ are compact vector fields which are homotopic (in the sense of compact vector fields) by a zero-free homotopy, then g_0 is essential with respect to B if and only if g_1 is. The proof is the same as in Granas’ case, and we omit it.

With these preliminaries we can prove an extension of Lemma 4.

LEMMA 6. Assume that H1–H5 hold and that the parity of $i\beta$ (see Definition 3) is odd. Let $\rho \leq \rho_0$ and $r \leq r(\rho)$ ($r(\rho)$ as in Lemma 4) be positive numbers and define sets A and B by

$$(65) \quad \begin{aligned} A &= \Sigma(r, \rho) = \{(x, \lambda, t) \in X \times \mathbb{R}^2: \\ &\quad \|x\|^2 + (\lambda - \lambda_0)^2 + (t - t_0)^2 = r^2 + \rho^2\}, \\ B &= B(r, \rho) = \{(x, \lambda, t) \in X \times \mathbb{R}^2: \\ &\quad \|x\|^2 + (\lambda - \lambda_0)^2 + (t - t_0)^2 \leq r^2 + \rho^2\}. \end{aligned}$$

Then for n large enough the map

$$(66) \quad (x, \lambda, t) \in A \rightarrow (x - P_n F(x, \lambda, t), \|x\|^2 - r^2) \in (X \times \mathbf{R}) - \{0\}$$

is essential with respect to B .

PROOF. Our first claim is that for $\|x\| = r, 0 < s < 1$ and $n > n_0$ we have

$$(67) \quad x - sF(x, \lambda, t) - (1 - s)P_n F(x, \lambda, t) \neq 0.$$

If not, there would be a sequence (x_n, λ_n, t_n) for which (67) would be zero. By arguments like those in §1, we could then assume (by taking a subsequence) that $(x_n, \lambda_n, t_n) \rightarrow (x, \lambda, t)$, and we would have $x - F(x, \lambda, t) = 0$, contrary to our previous work.

Using the above fact, it follows from Theorem 1 of §1 that for any given $n > n_0$, for $\|x\| = r$ and for $0 < s < 1$ we have

$$(68) \quad x - sP_m F(x, \lambda, t) - (1 - s)P_n F(x, \lambda, t) \neq 0$$

for $m \geq m_0$ (m_0 dependent on n).

To prove the lemma, suppose the contrary, so the map (66) has an extension

$$(69) \quad (x, \lambda, t) \in B \rightarrow (x - G(x, \lambda, t), h(x, \lambda, t)) \in (X \times \mathbf{R}) - \{0\}.$$

By the compactness of G and h it follows that for m large enough we have

$$(70) \quad (x - P_m G(x, \lambda, t), h(x, \lambda, t)) \in (X \times \mathbf{R}) - \{0\}.$$

If we take m also to be an integral multiple of n , so that $P_m P_n = P_n$, we see from (70) that

$$(71) \quad \begin{aligned} (x, \lambda, t) \in A \cap (X_m \times \mathbf{R}^2) &\rightarrow (x - P_n F(x, \lambda, t), \|x\|^2 - r^2) \\ &\in (X_m \times \mathbf{R}) - \{0\} \end{aligned}$$

is inessential with respect to $B \cap (X_m \times \mathbf{R}^2)$. However, our previous work shows that (71) is homotopic to

$$(72) \quad (x, \lambda, t) \in S_{r,\rho}(m) \rightarrow (x - P_m F(x, \lambda, t), \|x\|^2 - r^2)$$

(as maps into $(X_m \times \mathbf{R}) - \{0\}$). This shows that (72) is inessential with respect to $B \cap (X_m \times \mathbf{R}^2)$, which contradicts the fact that (72) is not homotopic to a constant map. \square

We can now state and begin to prove our main theorem.

THEOREM 4. *Assume that H1–H5 hold and that the parity of $i\beta$ is odd (see Definition 3, §3). Let \mathfrak{S} be defined by (50) and let \mathfrak{S}_0 denote the connected component of \mathfrak{S} which contains $(0, \lambda_0, t_0) \in X \times \Lambda \times [0, \infty)$ (Lemma 5 shows that \mathfrak{S}_0 is nonempty). Then it follows that either*

- (i) \mathfrak{S}_0 is not a compact subset of $X \times \Lambda \times [0, \infty)$, or
- (ii) \mathfrak{S}_0 contains an element of the form (ϕ_1, λ_1, t_1) , where ϕ_1 is a constant

function, $f(\phi_1, \lambda_1) = 0$ and $(\phi_1, \lambda_1, t_1) \neq (0, \lambda_0, t_0)$ (so that $x(s; \phi_1, \lambda_1)$ is a constant function).

PROOF. We suppose the theorem false and try to obtain a contradiction. Thus we can assume \mathcal{S}_0 is a compact subset of $X \times \Lambda \times [0, \infty)$, and that if we define \mathcal{C} by

$$(73) \quad \mathcal{C} = \{(\phi, \lambda, t) \in X \times \Lambda \times [0, \infty): \phi \text{ is a constant function, } f(\phi, \lambda) = 0 \text{ and } t \geq 0\},$$

then $\mathcal{S}_0 \cap \mathcal{C} = \{(0, \lambda_0, t_0)\}$. If $\Lambda = (a, b)$, then the compactness of \mathcal{S}_0 implies that there is a positive number δ_1 such that $a + \delta_1 \leq \lambda \leq b - \delta_1$ for all $(\phi, \lambda, t) \in \mathcal{S}_0$. Lemma 3, in conjunction with the assumptions on \mathcal{S}_0 , shows that there exists a positive number δ such that $t \geq 2\delta$ for all $(\phi, \lambda, t) \in \mathcal{S}_0$.

In the notation of §§2 and 3, we have assumed that $\det(\Delta_{\lambda_0}(0)) \neq 0$. It follows that $\det(\Delta_\lambda(0)) \neq 0$ for $|\lambda - \lambda_0| \leq \alpha_0$ (α_0 some positive number), and thus $L_\lambda(\phi) \neq 0$ for ϕ a nonzero constant function and $|\lambda - \lambda_0| \leq \alpha_0$. This in turn implies that there is a positive number μ_0 such that $f(\phi, \lambda) \neq 0$ for ϕ a constant function with $0 < \|\phi\| \leq \mu_0$ and $|\lambda - \lambda_0| \leq \alpha_0$. In particular, there is a bounded, open neighborhood U_0 in $X \times \Lambda \times [0, \infty)$ of $(0, \lambda_0, t_0)$ such that the only elements of $\mathcal{C} \cap \bar{U}_0$ are of the form $(0, \lambda, t)$. We can assume also that $t \geq \delta$ for $(\phi, \lambda, t) \in U_0$. The set $\mathcal{S}_0 - U_0$ is a compact set containing no elements of the closed set \mathcal{C} , so there exists a bounded open neighborhood Ω_1 of $\mathcal{S}_0 - U_0$ such that $\bar{\Omega}_1 \cap \mathcal{C}$ is empty. Again we can suppose that $t \geq \delta$ for $(\phi, \lambda, t) \in \Omega_1$. We define $\Omega_2 = \Omega_1 \cup U_0$, a bounded open neighborhood of \mathcal{S}_0 .

We now use an argument from [27]. Since $t \geq \delta$ for $(\phi, \lambda, t) \in \bar{\Omega}_2$, the results of §1 show that $\mathcal{S} \cap \bar{\Omega}_2 = K$ is a compact metric space (we define a norm on $X \times \mathbf{R}^2$ by $\|(\phi, \lambda, t)\|^2 = \|\phi\|^2 + |\lambda|^2 + |t|^2$). Define A to be \mathcal{S}_0 , a compact subset of K and write $B = \mathcal{S} \cap \partial\Omega_2$; note that A and B are disjoint. It follows from a theorem of Whyburn [28, Chapter 1] that there exist disjoint compact subsets K_A and K_B of K such that $K = K_A \cup K_B$, $A \subset K_A$ and $B \subset K_B$. If $N_\epsilon(K_A)$ denotes the ϵ neighborhood of K_A , there exists $\epsilon > 0$ such that $\overline{N_\epsilon(K_A)} \cap K_B$ is empty and $\overline{N_\epsilon(K_A)} \subset \Omega_2$. It follows that if we define $\Omega = N_\epsilon(K_A)$, then $\partial\Omega \cap \mathcal{S}$ is empty; and if $\phi = F(\phi, \lambda, t)$ for $(\phi, \lambda, t) \in \partial\Omega$ then $(\phi, \lambda, t) \in (\partial\Omega) \cap U_0$ and $\phi = 0$.

Now, following Ize [13], define $d(\phi, \lambda, t)$ to be the distance of (ϕ, λ, t) to $\bar{\Omega} \cap \mathcal{C}$ and define (for positive r)

$$(74) \quad H_r(\phi, \lambda, t) = (\phi - F(\phi, \lambda, t), (d(\phi, \lambda, t))^2 - r^2).$$

As in [13], we have that $H_r(\phi, \lambda, t) \neq 0$ for $(\phi, \lambda, t) \in \partial\Omega$ and $r > 0$. For if $\phi = F(\phi, \lambda, t)$ for $(\phi, \lambda, t) \in \partial\Omega$, the remarks above show that $(\phi, \lambda, t) \in \partial\Omega$

$\cap U_0$ and $\phi = 0$. This shows that $d(\phi, \lambda, t) = 0$, so that $H_r(\phi, \lambda, t) = (0, -r^2)$.

If r and ρ are positive numbers, define (for notational convenience)

$$B(r, \rho) = \{(\phi, \lambda, t) \in X \times \Lambda \times [0, \infty): \|\phi\|^2 + (\lambda - \lambda_0)^2 + (t - t_0)^2 \leq r^2 + \rho^2\}.$$

It is not hard to show (by using the fact $U_0 \cap \mathcal{C} = \{(0, \lambda, t) \in U_0\}$) that there exist positive numbers r_1 and ρ_1 such that $d(\phi, \lambda, t) = \|\phi\|$ for $(\phi, \lambda, t) \in B(r_1, \rho_1)$. We can also assume that $T_\lambda(t)$ is one-one for $0 < (\lambda - \lambda_0)^2 + (t - t_0)^2 \leq \rho_1^2 + r_1^2$. We can suppose that we originally took U_0 to be $B(r_1, \rho_1)$ in our construction of Ω_2 and Ω and that there is a positive number k such that $d(\phi, \lambda, t) \geq k$ for $(\phi, \lambda, t) \in \bar{\Omega} - B(r_1, \rho_1)$. Select positive numbers $r_2 \leq k$ and $\rho_2 \leq \rho_1$ such that $B(r_2, \rho_2) \subset \Omega$. Finally, take a positive number r_3 such that

$$(75) \quad \begin{aligned} \phi - F(\phi, \lambda, t) \neq 0 \quad \text{for } \|\phi\| = r_3 \text{ and} \\ \rho_2^2 \leq (\lambda - \lambda_0)^2 + (t - t_0)^2 \leq \rho_1^2 + r_1^2. \end{aligned}$$

We claim that $H_{r_3}(\phi, \lambda, t) \neq 0$ for $(\phi, \lambda, t) \in \bar{\Omega} - B(r_3, \rho_2)$. Since $r_3 \leq k$, it suffices to show that $H_{r_3}(\phi, \lambda, t) \neq 0$ for $(\phi, \lambda, t) \in B(r_1, \rho_1) - B(r_3, \rho_2)$. However, $d(\phi, \lambda, t) = \|\phi\|$ for $(\phi, \lambda, t) \in B(r_1, \rho_1)$, so that $\|\phi\| = r_3$ if $H_{r_3}(\phi, \lambda, t) = 0$, and this contradicts (75).

At this point we need to interrupt the proof of Theorem 4 to prove two lemmas.

LEMMA 7. Assume H1–H5 and let Ω be as above. Define $K_1 = \overline{\text{co}} F(\Omega)$ (where $\overline{\text{co}}$ denotes the closure of the convex hull of a set) and $\Omega_1 = \{(\phi, \lambda, t) \in \Omega: \phi \in K_1\}$; generally define $K_{j+1} = \overline{\text{co}} F(\Omega_j)$ and $\Omega_{j+1} = \{(\phi, \lambda, t) \in \Omega: \phi \in K_{j+1}\}$. Then the K_j and Ω_j form decreasing sequences of sets and there is an integer N such that K_j is compact for $j \geq N$. If P is a continuous retraction into K_N (P exists by results in [5]), the equation

$$(76) \quad \phi = sF(\phi, \lambda, t) + (1 - s)PF(\phi, \lambda, t)$$

is satisfied for $(\phi, \lambda, t) \in \Omega$ and some s with $0 \leq s \leq 1$ if and only if

$$(77) \quad \phi = F(\phi, \lambda, t).$$

PROOF. The fact that the K_j and Ω_j form decreasing sequences of sets is immediate. Since we know that $t \geq \delta$ if $(\phi, \lambda, t) \in \Omega$, the results of the first section show that $\{F(\phi, \lambda, t): (\phi, \lambda, t) \in \Omega\}$ is equicontinuous on $[-\delta, 0]$, and it follows immediately that K_1 is equicontinuous on $[-\delta, 0]$. Repeating this argument, we find that K_j is equicontinuous on $[-c, 0]$, where $c = \max(-\gamma, -j\delta)$ and γ is as in the definition of X . It follows that K_N is compact if $N\delta \geq \gamma$.

Finally, suppose that (76) holds for some s with $0 \leq s \leq 1$. Since the right-hand side of (76) is a convex combination of points in K_1 , it follows that $\phi \in K_1$ and $(\phi, \lambda, t) \in \Omega_1$. Repeating this argument, we find that $\phi \in K_2, \phi \in K_3$ and eventually that $\phi \in K_N$. However, if $\phi \in K_N$, then $F(\phi, \lambda, t) \in K_{N+1} \subset K_N$ and $PF(\phi, \lambda, t) = F(\phi, \lambda, t)$, so that (ϕ, λ, t) satisfies (77). \square

The next lemma is crucial for our work. In fact, our main reason for introducing the projections $\{P_n\}$ was to prove the following lemma. We should remark that if $t_0 > \gamma$, we can prove the lemma without introducing the projections $\{P_n\}$ and with a much simpler proof.

LEMMA 8. *Assume H1–H5 and suppose the parity of $i\beta$ is odd. Let Ω be as above, and let notation be as in Lemma 6. Then if P is the retraction onto K_N defined above, the map*

$$(x, \lambda, t) \in A \rightarrow (x - PF(x, \lambda, t), \|x\|^2 - r^2) \in (X \times \mathbb{R}) - \{0\}$$

is essential with respect to B .

PROOF. Consider, for $0 \leq s \leq 1$ and $(x, \lambda, t) \in A$, the homotopy

$$(78) \quad (x - (1 - s)P_n F(x, \lambda, t) - sPF(x, \lambda, t), \|x\|^2 - r^2).$$

According to Lemma 6 it suffices to show that (78) is nonzero for n large. If not, there will be a subsequence $(x_n, \lambda_n, t_n) \in A$ and $s_n \in [0, 1]$ for which (78) is zero. By taking a further subsequence we can assume (since P is compact) that $PF(x_n, \lambda_n, t_n) \rightarrow z$. If we define $\Phi(x, \lambda, t, s) = (1 - s)F(x, \lambda, t)$ and apply Theorem 1, we can suppose (by taking a further subsequence) that $(x_n, \lambda_n, t_n) \rightarrow (x, \lambda, t)$ and $s_n \rightarrow s$. Continuity now implies that

$$x - (1 - s)F(x, \lambda, t) - sPF(x, \lambda, t) = 0 \quad \text{and} \quad \|x\| = r.$$

It follows from Lemma 7 that $(x - F(x, \lambda, t), \|x\|^2 - r^2) = 0$, which is a contradiction.

We can now complete the proof of Theorem 4. Let r_3 and ρ_2 be as defined in that proof and suppose also that $0 < r_3 \leq r(\rho_2)$, where $r(\rho)$ is the function defined in Lemma 4. Modify the definition of $H_r, r > 0$, so as to make it a compact vector field.

$$\tilde{H}_r(\phi, \lambda, t) = (\phi - PF(\phi, \lambda, t), \|\phi\|^2 - r^2).$$

According to Lemma 7, the zeros of \tilde{H}_r and H_r are the same, so we know that $\tilde{H}_{r_2}(\phi, \lambda, t) \neq 0$ on $\bar{\Omega} - B(r_3, \rho_2)$.

The remainder of our proof (now that we have modified H_r properly) follows the outlines of [13]. Let C denote a closed ball in $X \times \mathbb{R}^2$ with center at $(0, \lambda_0, t_0)$ and containing Ω . For r large, say $r \geq r_4$, it is clear that $H_r|_{\bar{\Omega}}$ is inessential with respect to C . Since $H_r|_{\partial\Omega}$ is nonzero for all $r > 0$, it follows

that $\tilde{H}_{r_2}|_{\partial\Omega}$ has an extension to a compact vector field $\tilde{\tilde{H}}_{r_2}: C \rightarrow (X \times \mathbf{R})$ which is nonzero on C . Define $\hat{H}_{r_2}: C \rightarrow (X \times \mathbf{R})$ by

$$\hat{H}_{r_2}(\phi, \lambda, t) = \begin{cases} \tilde{\tilde{H}}_{r_2}(\phi, \lambda, t) & \text{for } (\phi, \lambda, t) \notin \Omega, \\ \tilde{H}_{r_2}(\phi, \lambda, t) & \text{for } (\phi, \lambda, t) \in \Omega. \end{cases}$$

By our construction we have that

$$\hat{H}_{r_2}(\phi, \lambda, t) \neq 0 \quad \text{for } (\phi, \lambda, t) \notin B(r_2, \rho_3) = B.$$

Now restrict attention to $A = \Sigma(r_2, \rho_3) = \partial B$, and for (ϕ, λ, t) let R denote the radius of C , and define a homotopy $g_\tau: A \rightarrow (X \times \mathbf{R}) - \{0\}$ through compact vector fields by

$$g_\tau(\phi, \lambda, t) = \hat{H}_{r_2}((0, \lambda_0, t_0) + \lambda(\tau)(\phi, \lambda - \lambda_0, t - t_0)),$$

where $\lambda(\tau) = (1 + \tau(R - 1))$ and $0 \leq \tau \leq 1$. By our construction we have that $g_\tau(\phi, \lambda, t) \neq 0$ for $(\phi, \lambda, t) \in A$ and $0 \leq \tau \leq 1$. Since we know that $\tilde{H}_{r_2}|_{\partial C}$ is inessential with respect to C , it follows that $g_1|_A$ is inessential with respect to B and, consequently, that $\hat{H}_{r_2}|_A$ is inessential with respect to B . However this contradicts Lemma 8 and completes the proof. \square

REMARK. Of course the difficulty in applying Theorem 4 is the same as for the O.D.E. case: one has little control over the t , or period variable, in $(\phi, \lambda, t) \in \mathfrak{S}_0$. The period need not be the minimal period, and it can happen that (ϕ, λ) is bounded for $(\phi, \lambda, t) \in \mathfrak{S}_0$ but \mathfrak{S}_0 is unbounded.

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