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DIFFEOMORPHISMS ALMOST REGULARLY HOMOTOPIC TO THE IDENTITY

BY

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ABSTRACT. Let f: $M \to M$ be a self-map of a closed smooth n-manifold. Does there exist a diffeomorphism φ: $M \to M$ homotopic to f? Define φ to be almost regularly homotopic to the identity if $\varphi|M - \text{pt.}$ is regularly homotopic to the inclusion $M - \text{pt.} \subset M$. Let ψ: $M \to M \vee M$ be the result of collapsing the boundary of a smooth n-cell in M, and let $M \vee M \xrightarrow{\Delta'} M$ be the codiagonal. For $\xi \in \pi_n(M)$ define $\tau(\xi)$ to be the composition

$$M \xrightarrow{\psi} M \vee M \xrightarrow{1 \vee \xi} M \vee M \xrightarrow{\Delta'} M.$$

THEOREM. *If M is 2-connected, s-parallelizable, and $n = 2l > 5$ with $l \not\equiv 0$ mod (4), then $\tau(\xi)$ contains a diffeomorphism almost regularly homotopic to the identity iff ξ is in the kernel of the stabilization map $\pi_n(M) \to \pi_n^s(M)$.*

1. Introduction. The problem of describing the homotopy behavior of diffeomorphisms is obviously very complicated. For instance, if M is a smooth closed n-manifold and $f: M \to M$ is a map, one may ask, usually in vain, whether there exists a diffeomorphism φ: $M \xrightarrow{\cong} M$ homotopic to f. There are many necessary conditions, but not many sufficient conditions. In this paper we investigate a severely restricted class of diffeomorphisms, and corresponding homotopy classes, to obtain a sufficient condition in that class, and also a new necessary condition.

A diffeomorphism φ: $M \to M$ is *almost regularly homotopic to the identity* if the restriction φ: $M - \{\text{pt.}\} \to M$ is regularly homotopic to the inclusion $M - \{\text{pt.}\} \subset M$. The homotopy classes of such diffeomorphisms are represented by maps $M \xrightarrow{f} M$ such that $f|\overline{M - D^n}$ is the inclusion $\overline{M - D^n} \subset M$, where D^n is a smooth n-disk in M not depending on f. Such a map f defines a map $(f|D^n) \cup \text{id}$: $D^n \cup_{S^{n-1}} D^n \to M$, and so an element of $\pi_n(M)$. Thus we obtain a 1-1 correspondence between the elements of $\pi_n(M)$ and such maps f modulo homotopies fixing $\overline{M - D^n}$. If $\xi \in \pi_n(M)$, let $\tau'(\xi)$ be the corresponding class of such maps f; that class $\tau'(\xi)$ in turn determines a class $\tau(\xi)$ of

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maps $M \to M$ modulo unrestricted homotopies. The theorem we wish to prove is

THEOREM 1. *If M is stably parallelizable and 2-connected, and* $\dim M = 2l > 5$ *with* $l \not\equiv 0 \bmod 4$, *then there is in $\tau(\xi)$ a diffeomorphism almost regularly homotopic to the identity iff ξ is in the kernel of the stabilization $\pi_n(M) \to \pi_n^s(M)$.*

I wish to thank Tom Farrell for very useful discussions regarding this theorem. Also, I wish to thank the referee for pointing out the following consquence of Theorem 1:

THEOREM 2. *Suppose that M satisfies the hypothesis of Theorem* 1 *and that, in addition, M' has cells only in dimensions* $r \equiv 0, 1, 3, 7 \bmod (8)$. *Then $\tau(\xi)$ contains a diffeomorphism iff ξ is in the kernel of the stabilization map $\pi_n(M) \to \pi_n^s(M)$.*

PROOF OF THEOREM 2. Since M is 2-connected, by reversing a Morse function we see that M' has no cells in dimensions $n-2$, $n-1$, or n. Then $H^r(M'\colon \pi_r(SO)) = H^r(M'\colon \pi_r SO(n))$. But since M' is parallelizable, the obstructions to regularizing homotopies of immersions $M' \to M$ lie in these groups. The condition on the cells of M' makes these groups zero. Thus $\varphi|M'$ homotopic to the inclusion implies $\varphi|M'$ regularly homotopic to the inclusion, and the theorem follows.

2. Proof of sufficiency. The proof of sufficiency requires constructions made in [2]. We will recall those constructions here, but refer to [2] for details and proofs. We assume M is an oriented smooth closed, stably parallelizable n-manifold. Let the following diagram be the Moore-Postnikov factorization of $M \to \mathrm{pt.}$:

$$\begin{tikzcd}
 & \vdots \arrow[d] & \\
 & M_{\lambda+1} \arrow[r, "k_{\lambda+1}"] \arrow[dd] & K(\pi_{\lambda+2}(M), \lambda+3) \\
M \arrow[r, "\iota"] & \overline{M} \arrow[ur, "u_{\lambda+1}"] \arrow[dr, "u_\lambda"'] & \\
 & M_\lambda \arrow[r, "k_\lambda"] \arrow[d] & K(\pi_{\lambda+1}(M), \lambda+2) \\
 & \vdots &
\end{tikzcd}$$

That is, ι is a homotopy equivalence, each vertical map $M_{\lambda+1} \to M_\lambda$ is a fibration induced by the k-invariant k_λ, and $u_{\lambda *}\colon \pi_i(\overline{M}) \to \pi_i(M_\lambda)$ is an

isomorphism for $i < \lambda$, and $\pi_i(M_\lambda) = 0$ for $i > \lambda$. Let $E \to BSO$ be the principal SO bundle and let

$$\begin{tikzcd}
 & \vdots \arrow[d] & \\
 & BSO_{\lambda+1} \arrow[dr, "\xi_{\lambda+1}"] \arrow[dd] & \\
E \arrow[ur, "\eta_{\lambda+1}"] \arrow[dr, "\eta_\lambda"'] & & BSO \\
 & BSO_\lambda \arrow[ur, "\xi_\lambda"'] \arrow[d] & \\
 & \vdots &
\end{tikzcd}$$

be the Moore-Postnikov factorization of $E \to SO$. Then the Moore-Postnikov factorization of the (homotopically trivial) classifying map $M \to BSO$ of the normal bundle of M is given by

$$\begin{tikzcd}
 & \vdots \arrow[d] & \\
 & M_{\lambda+1} \times BSO_{\lambda+1} \arrow[dr, "V_{\lambda+1}"] \arrow[dd] & \\
M \xrightarrow{i'} \overline{M} \times E \arrow[ur, "W_{\lambda+1}"] \arrow[dr, "W_\lambda"'] & & BSO \\
 & M_\lambda \times BSO_\lambda \arrow[ur, "V_\lambda"'] \arrow[d] & \\
 & \vdots &
\end{tikzcd}$$

where we write $w_\lambda = u_\lambda \times \eta_\lambda$. We may assume $M \to BSO$ is a Gauss map–that is, induced by an immersion. And each of the fibrations v_λ determines a Lashof cobordism theory $\Omega_*(v_\lambda)$ as in [1].

Suppose φ: $M \xrightarrow{\cong} M$ is a diffeomorphism and h a regular homotopy from the inclusion $M' = M - \{\text{pt.}\} \subset M$ to the restriction $\varphi|M'$: $M' \to M$. Let $S^1 \times_\varphi M$ be the mapping torus of φ; then h defines an immersion H: $S^1 \times M' \to S^1 \times_\varphi M$. We fix throughout an orientation $[S^1]$ for S^1; then $[S^1]$ and $[M]$ determine an orientation $[S^1 \times_\varphi M]$ of $S^1 \times_\varphi M$. Let n_φ: $S^1 \times_\varphi M \to BSO$ be any Gauss map extending $1 \times M \to M \to BSO$. Then we have a diagram

$$\begin{tikzcd}
(S^1 \times M') \cup (1 \times M) \arrow[r, "\text{incl.}"] \arrow[d, "H \cup \text{incl.}"] & S^1 \times M \arrow[r] & \overline{M} \times E \arrow[d] \\
S^1 \times_\varphi M \arrow[dr, "n_\varphi"] & & M_{n-1} \times BSO_{n-1} \arrow[dl, "V_{n-1}"] \\
& BSO &
\end{tikzcd}$$

which commutes on $1 \times M$ and up to homotopy mod $1 \times M$. There is a unique lift $t(\varphi, h)$: $S^1 \times \varphi M \to M_{n-1} \times BSO_{n-1}$ making the diagram

$$\begin{tikzcd}
S^1 \times M' \cup 1 \times M \arrow[r] \arrow[d] & \overline{M} \times E \arrow[d] \\
S^1 \times_\varphi M \arrow[r, "t(\varphi, h)"] & M_{n-1} \times BSO_{n-1}
\end{tikzcd}$$

commute up to homotopy mod $1 \times M$. The lift $t(\varphi, h)$ does depend on the choice of n_φ, but it represents an element $t(\varphi, h) \in \Omega_{n+1}(V_{n-1})$ which is independent of the choice of n_φ. We regard $t(\varphi, h)$ as the bordism mapping torus of (φ, h).

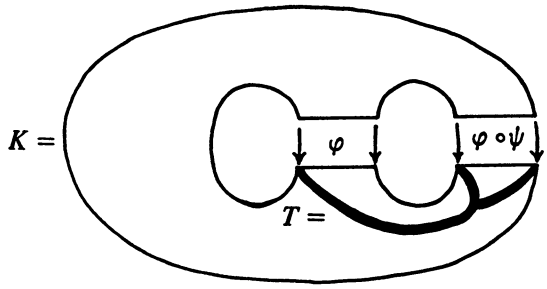
The element $t(\varphi, h)$ represents also an element of another group $D(M)$ which we now define. Let $\overline{D}(M) = \{(\varphi, t) | \varphi\colon M \to M$ is a diffeomorphism and $t\colon S^1 \times_\varphi M \to M_{n-1} \times BSO_{n-1}$ is a lift of a Gauss map $S^1 \times_\varphi M \to BSO$ such that $t|1 \times M = (1 \times M \to M \to \overline{M} \to BSO)\}$.

We introduce an equivalence relation $\approx$ in $\overline{D}(M)$ by setting $(\varphi_0, t_0) \approx (\varphi_1, t_1)$ if there exist:

(1) a concordance Φ: $M \times I \to M \times I$ from φ_0 to φ_1,

(2) a lifting L: $S^1 \times_\Phi (M \times I) \to M_{n-1} \times BSO_{n-1}$ of a Gauss map,

(3) diffeomorphisms α_i: $S^1 \times_{\varphi_i} M \xrightarrow{\cong} S^1 \times_\Phi (M \times i)$ fixed on $1 \times M$, such that $L \circ \alpha_0 = t_0$, $L \circ \alpha_1 = t_1$ and $L|1 \times M \times I = (1 \times M \times I \to M \to \overline{M} \to M_{n-1} \times BSO_{n-1})$. Denote the quotient $\overline{D}(M)/\approx$ by $D(M)$ and the equivalence class of (φ, t) by $[\varphi, t]$. We define a group structure on $D(M)$ by defining $[\varphi, t] \cdot [\psi, s] \in D(M)$ as follows: Let K be the displayed subset of the plane



We obtain a manifold with boundary by starting with $K \times M$ and identifying as the above picture suggests. Let X be the resulting manifold; then canonically,

$$\partial X = S^1 \times_\varphi M \amalg S^1 \times_\psi M \amalg S^1 \times_{\varphi \circ \psi} M.$$

Let T be the subset of K indicated by the heavy lines. Then $T \times M \subset X$ and $T \times M \cap S^1 \times_\varphi M =$ the copy of $1 \times M$ in $S^1 \times_\varphi M$: the same is true for $T \times M \cap S^1 \times_\psi M$ in $S^1 \times_\psi M$; and for $T \times M \cap S^1 \times_{\varphi \circ \psi} M$ in $S^1 \times_{\varphi \circ \psi} M$. Moreover, $S^1 \times_\varphi M \cup T \times M \cup S^1 \times_\psi M$ is a strong deformation retract of X. Thus we may extend the lift $t \amalg s$ over $T \times M$ by $(T \times M \to M \to \overline{M} \to M_{n-1} \times BSO_{n-1})$, and then over X by the retraction. Restricting to $S^1 \times_{\varphi \circ \psi} M$ the extended lift, we obtain a lift $t \cdot s$ which restricts to $(1 \times M \to M \to \overline{M} \to M_{n-1} \times BSO_{n-1})$ over $1 \times M$. Thus $[\varphi \circ \psi, t \cdot s] \in D(M)$. It is shown in [2] that the product $[\varphi, t] \cdot [\psi, s] = [\varphi \circ \psi, t \cdot s]$ is well defined and a group structure on $D(M)$.

Clearly, each $(\varphi, t) \in \overline{D}(M)$ determines an element $\beta[\varphi, t] \in \Omega_{n+1}(V_{n-1})$. This element depends only on $[\varphi, t]$, and the resulting map β: $D(M) \to \Omega_{n+1}(V_{n-1})$ is a homomorphism. Let π_0 Diff: M be the group of concordance classes of diffeomorphisms. The map $D(M) \to \pi_0$ Diff: M given by $[\varphi, t] \to [\varphi]$ is clearly a well-defined homomorphism. The mapping torus construction above assigned to each pair (φ, h) (with φ: $M \xrightarrow{\cong} M$ a diffeomorphism and h a regular homotopy from incl: $M' \subset M$ to $\varphi|M'$) a lift $t(\varphi, h)$ so that $[\varphi, t(\varphi, h)] \in D(M)$; then clearly $t(\varphi, h) = \beta[\varphi, t(\varphi, h)]$. Finally for each $[\varphi, t] \in D(M)$ there exists a pair (φ, h) such that $[\varphi, t] = [\varphi, t(\varphi, h)]$. Thus the image of the homomorphism $D(M) \to \pi_0$ Diff: M consists precisely of the concordance classes of diffeomorphisms almost regularly homotopic to the identity. Finally, in [2] is established also a surgery exact sequence $L_{n+2}(1) \to D(M) \xrightarrow{\beta} \Omega_{n+1}(V_{n-1}) \to L_{n+1}(1)$.

Since M is stably parallelizable we have for large r a degree one map α: $S^{n+r} \to S^r M$. Recall the track (abelian) group $\{X, Y\} = \lim\{S^i X, S^i Y\}$; composition turns $\{M, M\}$ into a ring. Let $E(M)$ be the H-space of homotopy equivalence $M \to M$. Then we have a canonical homomorphism $\pi_0 E(M) \to \{M, M\}^0$ where R^0 is the group of units of the ring R. Also α^*: $\{M, M\} \to \{S^n, M\} = \pi_n^s(M, \text{pt.})$ is a homomorphism with respect to track addition. Since α represents an element of $\pi_n^s(M, \text{pt.})$, we denote also by α both that element and the constant map to that element. We have a map $\alpha^* - \alpha$: $\{M, M\}^0 \to \pi_n^s(M, \text{pt.})$ which is not necessarily a homomorphism. To interpret the map, consider the cofibration $M' \xrightarrow{\iota} M \xrightarrow{P} S^n$. Then α^* splits the track exact sequence to give a short split exact sequence

$$0 \longrightarrow \{S^n, M\} \underset{P^*}{\overset{\longleftarrow}{\longrightarrow}} \{M, M\} \xrightarrow[\iota^*]{} \{M', M\} \longrightarrow 0$$

Thus, if $r > 2n + 1$ and f: $S'M \to S'M$ restricted to $S'M'$ is homotopic to the inclusion, $\alpha^*[f] - \alpha$ is the obstruction to homotoping f to the identity. Finally, we have a canonical homomorphism π_0 Diff: $M \to \pi_0 E(M)$. The composition of the above three maps is a map π_0Diff: $M \xrightarrow{\theta'} \pi_n^s(M, \text{pt.})$ which measures very coarsely the homotopy difference between a diffeomorphism and the identity. Let π_0 Diff^{n-1}: M be the subgroup of π_0Diff: M consisting of concordance classes of diffeomorphisms almost regularly homotopic to the identity. Set $\theta = \theta'|\pi_0\text{Diff}^{n-1}$: M. For $\varphi \in \pi_0\text{Diff}^{n-1}$: M we have $\theta(\varphi) = 0$ iff φ is stably homotopic to the identity.

To see that θ is a homomorphism, we introduce a similar map Θ: $D(M) \to \pi_n(M)$. If $[\varphi, t(\varphi, h)] \in D(M)$, we let

$$\begin{aligned} c[\varphi, t(\varphi, h)] &\in H^n(M, M' : \pi_n(M)) \\ &= H^{n+1}(M \times I, M \times 0 \cup M' \times I \cup M \times 1 : \pi_n(M)) \end{aligned}$$

be the obstruction to extending the regular homotopy h to a homotopy $\bar{h}$ from the identity to φ. Let $[M]$ be the orientation of M; we set $\Theta[\varphi, t(\varphi, h)] = c[\varphi, t(\varphi, h)] \cap [M]$. It follows from the details in [2] of the construction of $D(M)$ that Θ is well defined; in that case the diagram

$$\begin{array}{ccc} D(M) & \xrightarrow{\Theta} & \pi_n(M) \\ \downarrow & & \downarrow S \\ \pi_0\,\text{Diff}^{n-1} : M & \xrightarrow{\theta} & \pi_n^s(M, \text{pt.}) \end{array}$$

clearly commutes, where the left vertical is the canonical epimorphism, and the right vertical, stabilization.

PROPOSITION 1. *Θ is a homomorphism.*

PROOF. We have the diagram, whose right square commutes, whose left square homotopy commutes, and whose left square commutes when restricted to $1 \times M$:

$$\begin{array}{ccccc} 1 \times M \cup S^1 \times M & \longrightarrow & M_n \times BSO_{n-1} & \longrightarrow & \text{Path}(K(\pi_n(M)), n+1) \\ \downarrow & \overline{t(\varphi, h)} & \downarrow & \bar{k}_{n-1} & \downarrow \\ S^1 \times_\varphi M & \longrightarrow & M_{n-1} \times BSO_{n-1} & \longrightarrow & K(\pi_n(M), n+1), \end{array}$$

in which the square, i.e. 'homotopy maps of pairs', are denoted by barred letters; the unbarred letters denote the bottom maps.

The obstruction $\overline{t(\varphi, h)}^* \bar{k}_{n-1}$ to a relative lift $S^1 \times_\varphi M \to M_n \times BSO_{n-1}$ is equal to the obstruction

$$c[\varphi, t(\varphi, h)] \in H^{n+1}(M \times I, M \times 0 \cup M' \times I \cup M \times 1 : \pi_n(M))$$

to extending the regular homotopy h to a homotopy from the identity to φ. Then

$$\Theta[\varphi, t(\varphi, h)] = t(\varphi, h)^* k_{n-1} \cap [S^1 \times_\varphi M].$$

Thus we obtain the commutative diagram

$$\begin{tikzcd} D(M) \arrow[d, "\beta"] \arrow[dr, "\Theta"] & \\ \Omega_{n+1}(V_{n-1}) \arrow[r, "k_{n-1}"'] & \pi_n(M) \end{tikzcd}$$

where k_{n-1} is the characteristic class homomorphism. But then Θ is factored into homomorphisms, itself must be a homomorphism, and the proposition is proved.

COROLLARY. *$\theta|\pi_0\mathrm{Diff}^{n-1}$: M is a homomorphism.*

Theorem 1 is a corollary of the following, slightly stronger theorem.

THEOREM 1′. *If $n = 2l > 5$ with $l \not\equiv 0 \bmod 4$ and M is 2-connected, then the sequence $D(M) \xrightarrow{\Theta} \pi_{2l}(M) \xrightarrow{S} \pi^s_{2l}(M, \mathrm{pt.})$ is exact.*

THEOREM 1′ $\Rightarrow$ *sufficiency*. We have a commutative diagram with left vertical an epimorphism and top row exact if $n = 2l$ with $l \not\equiv 0 \bmod 4$ and M is 2-connected:

$$\begin{tikzcd} D(M) \arrow[r, "\Theta"] \arrow[d] & \pi_n(M) \arrow[r, "S"] \arrow[d, "\tau"] & \pi^s_n(M, \mathrm{pt.}) \\ \pi_0\,\mathrm{Diff}^{n-1}: M \arrow[r] & \pi_0\,\mathrm{Map}(M, M) & \end{tikzcd}$$

If $\xi \in \ker S$, then $\xi = \Theta[\varphi, t]$ and $\varphi \in \tau(\xi)$, so sufficiency in the theorem is proved modulo Theorem 1′.

PROOF OF THEOREM 1′. Let $T(\xi_\mu)$ be the spectrum obtained by taking the Thom spaces of finite dimensional vector bundles, over finite subcomplexes of BSO_μ, which stabilize to represent restrictions of ξ_μ. Let $\Omega_*(V_\lambda)$ be the Lashof cobordism theory of the fibration $M_\lambda \times BSO_\lambda \xrightarrow{V_\lambda} BSO$. The Thom Isomorphism Theorem determines canonical isomorphisms $\Omega_{n+1}(V_\lambda) \xrightarrow{\cong} \pi^s_{n+1}(M_\lambda^+ \wedge T(\xi_\lambda)) \cong H_{n+1}(M_\lambda : T(\xi_\lambda))$ for all λ. Now $(M_\lambda, M_{\lambda+1}) \to (M_\lambda, M_\lambda)$ may be regarded as a fibration pair with fiber the pair (contractible, $K(\pi_{\lambda+1}(M), \lambda + 1)$). We apply the Whitehead Spectral

Sequence to this fibration and the generalized homology theory $H_*(\ :\ T(\xi_\mu))$ to obtain

$$H_m(M_\lambda, M_{\lambda+1}\colon T(\xi_\mu)) \cong H_{m-\lambda-2}(M_\lambda\colon \pi_{\lambda+1}(M)),$$

where $H_*(\ :\ \pi_{\lambda+1}(M))$ means ordinary homology with coefficients $\pi_{\lambda+1}(M)$, with $m \leqslant 2\lambda + 2$ if $\mu \geqslant \lambda + 1$, and $m \leqslant \lambda + \mu + 1$ if $\mu \leqslant \lambda + 1$. Thus

$$H_{2l+2}(M_\lambda, M_{\lambda+1}\colon T(\xi_{2l+1})) = H_{2l+1}(M_\lambda, M_{\lambda+1}\colon T(\xi_{2l+1})) = 0 \quad \text{for } \lambda \geqslant 2l$$

and we have an isomorphism

$$H_{2l+1}(M\colon T(\xi_\lambda)) \xrightarrow{\cong} H_{2l+1}(M_{2l}\colon T(\xi_{2l+1})).$$

(Notice $T(\xi_\mu)$ is the same as the sphere spectrum in given dimensions above.) Then the exact sequence of $H_*(\ :\ T(\xi_{2l+1}))$ for the pair (M_{2l-1}, M_{2l}) and the two isomorphisms above yield the exact sequence

$$H_{2l+1}(M\colon T(\xi_{2l+1})) \to H_{2l+1}(M_{2l-1}\colon T(\xi_{2l+1})) \to H_0(M_{2l-1}\colon \pi_{2l}(M))$$

which, in turn, shows that this sequence is exact,

$$H_{2l+1}(M\colon T(\xi_{2l+1})) \to H_{2l+1}(M_{2l-1}\colon T(\xi_{2l+1})) \xrightarrow{k_{2l-1}\cap} \pi_{2l}(M),$$

where $k_{2l-1}\cap$ is the obvious characteristic class homomorphism.

Now we need a lemma.

LEMMA. *Suppose* $[m/2] + 1 \leqslant \lambda$. *Then the kernel of* $H_m(Y\colon T(\xi_{\lambda+1})) \to H_m(Y\colon T(\xi_\lambda))$ *is a subquotient of* $H_{m-\lambda}(Y \times BSO_\lambda\colon \pi_{\lambda+1}(BSO))$ *and the cokernel is a subquotient of* $H_{m-\lambda-1}(Y \times BSO_\lambda\colon \pi_{\lambda+1}(BSO))$.

PROOF OF LEMMA. The cofibration sequence of spectra $T(\xi_{\lambda+1}) \to T(\xi_\lambda) \to T(\xi_\lambda)/T(\xi_{\lambda+1})$ gives rise to a long exact sequence of homology theories

$$\ldots \to H_{m+1}(Y\colon T(\xi_\lambda)/T(\xi_{\lambda+1})) \to H_m(Y\colon T(\xi_{\lambda+1})) \to H_m(Y\colon T(\xi_\lambda)) \to \ldots.$$

We may interpret $H_{m+1}(Y\colon T(\xi_\lambda)/T(\xi_{\lambda+1}))$ as the lift bordism classes of lifts

$$\begin{tikzcd} \partial\Gamma \arrow[r] \arrow[d, phantom, "\cap"] & Y \times BSO_{\lambda+1} \arrow[d] \\ \Gamma \arrow[r, "\bar{f}"] & Y \times BSO_\lambda \end{tikzcd}$$

of Gauss maps $\Gamma \to BSO$, with $\dim \Gamma = m + 1$. We have the fibration map

$$\begin{tikzcd} Y \times BSO_{\lambda+1} \arrow[r, "\bar{k}"] \arrow[d] & \mathrm{Path}(K(\pi_{\lambda+1}(BSO), \lambda + 1)) \arrow[d] \\ Y \times BSO_\lambda \arrow[r] & K(\pi_{\lambda+1}(BSO), \lambda + 1). \end{tikzcd}$$

That is, $\bar{k} \in H^{\lambda+1}(Y \times BSO_\lambda, Y \times BSO_{\lambda+1}: \pi_{\lambda+1}(BSO))$, so

$$H_{m+1}(Y \times BSO_\lambda, Y \times BSO_{\lambda+1}) \xrightarrow{\cap \bar{k}} H_{m-\lambda}(Y \times BSO_\lambda: \pi_{\lambda+1}(BSO)).$$

Clearly $\bar{f}_*[\Gamma, \partial\Gamma] \cap \bar{k} \neq 0$ implies that the bordism class of $\bar{f}$ is nontrivial. On the other hand, suppose $\bar{f}_*[\Gamma, \partial\Gamma] \cap \bar{k} = 0$. We may assume that $\Gamma \to Y \times BSO_\lambda$ is homotopy equivalent to a map with $([(m+1)/2] - 1)$-connected fiber. Since $[m/2] + 1 \leqslant \lambda$, it follows that $H_{m-\lambda}(\Gamma) \to H_{m-\lambda}(Y \times BSO_\lambda)$ is an isomorphism with any coefficients. Now,

$$[\Gamma, \partial\Gamma] \cap \bar{f}^*\bar{k} \in H_{m-\lambda}(\Gamma: \pi_{\lambda+1}(BSO))$$

and we have $\bar{f}_*([\Gamma, \partial\Gamma] \cap \bar{f}^*\bar{k}) = \bar{f}_*[\Gamma, \partial\Gamma] \cap \bar{k} = 0$, so $[\Gamma, \partial\Gamma] \cap \bar{f}^*\bar{k} = 0$. Then by Poincaré duality, $\bar{f}^*\bar{k} = 0$. But then there is a relative lift $\Gamma \to Y \times BSO_\lambda$ which makes the bordism class of $\bar{f}$ trivial. Thus

$$\begin{array}{c} 0 \to H_{m+1}(Y: T(\xi_\lambda)/T(\xi_{\lambda+1})) \to H_{m+1}(Y \times BSO_\lambda, Y \times BSO_{\lambda+1}) \cap \bar{k} \\ \cap \\ H_{m-\lambda}(Y \times BSO_\lambda : \pi_{\lambda+1}(BSO)). \end{array}$$

It follows that the kernel is as described; the argument for the cokernel is identical, and the lemma is proved.

It follows from the lemma, and the hypotheses that $l \not\equiv 0 \bmod 4$ and M is 2-connected, that

$$H_{2l+1}(M_{2l-1}: T(\xi_{2l+1})) \to H_{2l+1}(M_{2l-1}: T(\xi_{2l-1}))$$

is an isomorphism. Thus we obtain the following commutative diagram with top row exact:

$$\begin{tikzcd} H_{2l+1}(M: T(\xi_{2l+1})) \arrow[r] & H_{2l+1}(M_{2l-1}: T(\xi_{2l-1})) \arrow[r, "\cap k"] \arrow[d, equal] & \pi_{2l}(M) \\ & \Omega_{2l+1}(V_{2l-1}) \arrow[ur, "\Theta"'] & \\ & D(M) \arrow[u, "\beta"] & \end{tikzcd}$$

and it follows from the surgery exact sequence of [2],

$$L_{2l+2}(1) \to D(M) \xrightarrow{\beta} \Omega_{2l+1}(V_{2l-1}) \to L_{2l+1}(1),$$

that β is onto. Thus Image(Θ) = Image$(\cap k)$, and from now on we seek to show that Image$(\cap k)$ = kernel$(S: \pi_{2l}(M) \to \pi^s_{2l}(M, \text{pt.}))$.

Denoting the sphere spectrum by $T(\xi_\infty)$, the same methods as above, using $l \not\equiv 0 \bmod 4$ and the 2-connectivity of M, show that the following maps are isomorphisms:

$$H_{2l+1}(M\colon T(\xi_\infty)) \xrightarrow{\cong} H_{2l+1}(M\colon T(\xi_{2l+1})),$$

$$H_{2l+1}(M_{2l-1}\colon T(\xi_\infty)) \xrightarrow{\cong} H_{2l+1}(M_{2l-1}\colon T(\xi_{2l-1})).$$

Using again the Whitehead Spectral Sequence as before, we find that the obstruction characteristic class homomorphism is an isomorphism,

$$H_{2l+1}(M_{2l-1}, M\colon T(\xi_\infty)) \xrightarrow[\cong]{\cap \bar{k}} \pi_{2l}(M),$$

and we have the commutative diagram

$$\begin{tikzcd}
H_{2l+1}(M_{2l-1}\colon T(\xi_{2l-1})) \arrow[dr, "\cap k"] & \\
H_{2l+1}(M_{2l-1}\colon T(\xi_\infty)) \arrow[u, "\cong"'] \arrow[r, "\cap k"] \arrow[d] & \pi_{2l}(M) \\
H_{2l+1}(M_{2l-1}, M\colon T(\xi_\infty)) \arrow[ur, "\cap \bar{k}", "\cong"'] &
\end{tikzcd}$$

from which we see that Image(Θ) = Image($\cap k$), where $\cap k$ is now the middle horizontal in the diagram instead of the top.

Let $\tilde{H}$ denote reduced homology. The image of $\tilde{H}_{2l+1}(M_{2l-1}\colon T(\xi_\infty))$ in $H_{2l+1}(M_{2l-1}, M\colon T(\xi_\infty))$ is the same as that of $H_{2l+1}(M_{2l-1}\colon T(\xi_\infty))$, so we finally seek the image of $\tilde{H}_{2l+1}(M_{2l-1}\colon T(\xi_\infty)) \xrightarrow{\cap k} \pi_{2l}(M)$. We have the following commutative diagram with exact rows,

$$\begin{tikzcd}
\pi^s_{2l+1}(M, \text{pt.}) \arrow[r] \arrow[d, equal] & \pi^s_{2l+1}(M_{2l-1}, \text{pt.}) \arrow[r] \arrow[d, equal] & \pi^s_{2l+1}(M_{2l-1}, M) \arrow[r, "\partial"] \arrow[d, equal] & \pi^s_{2l}(M, \text{pt.}) \arrow[d, equal] \\
\tilde{H}_{2l+1}(M\colon T(\xi_\infty)) \arrow[r] & \tilde{H}_{2l+1}(M_{2l-1}\colon T(\xi_\infty)) \arrow[r] \arrow[dr, "\cap k"'] & H_{2l+1}(M_{2l-1}, M\colon T(\xi_\infty)) \arrow[r] \arrow[d, "\cong"', "\cap \bar{k}"] & \tilde{H}_{2l}(M\colon T(\xi_\infty)) \\
& & \pi_{2l}(M) &
\end{tikzcd}$$

Thus it will suffice to show that $\partial \circ (\cap \bar{k})^{-1} = S$.

To see that $\partial \circ (\cap \bar{k})^{-1} = S$, recall that we may interpret $H_{2l+1}(M_{2l-1}, M\colon T(\xi_\infty))$ as the lift bordism classes of lifts

$$\begin{tikzcd}
\partial\Gamma \arrow[r] \arrow[d, phantom, "\cap"] & M \times E \arrow[d] \\
\Gamma \arrow[r] \arrow[d] & M_{2l-1} \times E \\
BSO &
\end{tikzcd}$$

of Gauss maps, where $\dim \Gamma = 2l + 1$. Say $x \in \pi_{2l}(M)$. We choose a lift

$$\begin{array}{ccc} S^{2l} & \xrightarrow{f_1} & M \times E \\ \cap & \bar{f} & \downarrow \\ D^{2l+1} & \longrightarrow & M_{2l-1} \times E \end{array}$$

of a Gauss map, such that f_1 represents x. Then bordism class $[\bar{f}] = (\cap \bar{k})^{-1}(x)$, and $S^{2l} \xrightarrow{f_1} M$ represents $\partial \circ (\cap \bar{k})^{-1}(x)$ in $\pi^s_{2l}(M, \text{pt.})$. But that element is $S[f_1] = Sx$, and Theorem 1′ is proved.

3. Proof of necessity. For this part we have that M is a 2-connected smooth, closed, stably parallelizable manifold. We define a map τ: $(\pi_0\text{Map}(M, M)) \times \pi_n(M) \to \pi_0\text{Map}(M, M)$. Then the proof of the theorem follows immediately from simple properties of τ.

We define $\tau([h], [f])$ to be the homotopy class of the composition

$$M \xrightarrow{\varphi} M \vee S^n \xrightarrow{h \vee f} M \vee M \xrightarrow{\Delta'} M,$$

where φ is the map that pinches to a point the boundary of a smooth D^n in M, and Δ' is the folding map.

PROPOSITION 2. *If* degree$(f) = 0$, *then* $\tau(1, [f]) \in \pi_0 E(M)$. *The proof is clear.*

PROPOSITION 3. *If* degree$(f) = 0$ *and* degree$(g) = 0$, *then* $\tau(1, g) \cdot \tau(1, f) = \tau(\tau(1, g), f)$.

PROOF. We may assume that the diagram

$$\begin{array}{ccc} S^n & \xrightarrow{f} & M \\ \psi \downarrow & & \downarrow \varphi \\ S^n \vee S^n & \xrightarrow{f \vee 0} & M \vee S^n \end{array}$$

commutes, where ψ is a pinching of S^n defining an H' structure. Then the following diagram homotopy commutes:

$$\begin{array}{ccccccc} M \xrightarrow{\varphi} M \vee S^n & \xrightarrow{1 \vee f} & M \vee M & \xrightarrow{\Delta'} & M & & \\ \downarrow 1 \vee \psi & & \downarrow \varphi \vee \varphi & & \searrow \varphi & & \\ M \vee S^n \vee S^n & \xrightarrow{\varphi_1 \vee f_2 \vee 0_3} & (M \vee S^n) \vee (M \vee S^n) & \xrightarrow{\Delta'} & M \vee S^n & & \\ \downarrow ((1 \vee g) \circ \varphi)_1 \vee f_2' \vee 0_3' & & & & \downarrow 1 \vee g & & \\ (M \vee M) \vee M \vee M & \xrightarrow{\Delta'} & & & M \vee M & & \\ \downarrow 1 \vee \Delta' & & & & \downarrow \Delta' & & \\ (M \vee M) \vee M & \xrightarrow{\Delta''} & & & M & & \end{array}$$

where φ_1 is φ from the first term to the first term, f_2 is f from the second term to the first term in the second, and 0_3 is 1 from the third term to the second term in the second. Also, $((1 \vee g) \circ \varphi)_1$ is $(1 \vee g) \circ \varphi$ from the first term to the first, f_2' is f from the second to the second, and $0_3'$ is g from the third to the third. Finally Δ'' is the triple fold.

The composition around the upper right corner represents $\tau(1, [g]) \cdot \tau(1, [f])$. Therefore so does the composition around the lower left corner. This last composition is the composition around the upper right corner in the following commutative diagram; we abbreviate $H = (1 \vee g) \circ \varphi$ in the following commutative diagram:

$$\begin{tikzcd}
M \arrow[r, "\varphi"] \arrow[dd, bend right=60, "\tau(\tau(1,g),f+0)"'] \arrow[ddrr, bend right=40, "\tau(1,g) \vee (f+0)"'] & M \vee S^n \arrow[r, "1 \vee \psi"] \arrow[d, "H \vee 1"] \arrow[ddrr, bend right=40, "H \vee (\Delta' \circ (f \vee 0) \circ \psi)"'] & M \vee S^n \vee S^n \arrow[r, "H_1 \vee f_2' \vee 0_3"] \arrow[d, "H \vee 1"] & (M \vee M) \vee M \vee M \arrow[r, "1 \vee \Delta'"] & (M \vee M) \vee M \arrow[dd, "\Delta' \vee 1"] \\
 & M \vee M \vee S^n \arrow[r, "1 \vee \psi"] & (M \vee M) \vee S^n \vee S^n \arrow[ur, "1 \vee f_2' \vee 0_3"] & & \\
 & & & & M \vee M \arrow[d, "\Delta'"] \\
 & & & & M,
\end{tikzcd}$$

from which the proposition follows immediately.

PROPOSITION 4. $\tau(\tau(h, f), g) = \tau(h, f + g)$.

PROOF. The following diagram homotopy commutes:

$$\begin{tikzcd}
M \arrow[r, "\varphi"] \arrow[dr, "\varphi"] \arrow[ddrr, bend right=40, "\tau(h, f+g)"'] & M \vee S^n \arrow[r, "\varphi \vee 1"] & M \vee S^n \vee S^n \arrow[r, "h \vee f \vee 1"] \arrow[d, "h \vee f \vee g"] & M \vee M \vee S^n \arrow[r, "\Delta' \vee 1"] & M \vee S^n \arrow[d, "1 \vee g"] \\
 & M \vee S^n \arrow[ur, "1 \vee \varphi"] \arrow[d, "h \vee (f+g)"] & M \vee M \vee M \arrow[d, "1 \vee \Delta'"] & & M \vee M \arrow[d, "\Delta'"] \\
 & & M \vee M \arrow[rr, "\Delta'"] & & M
\end{tikzcd}$$

and the proposition follows immediately.

From the proposition above we obtain the formula

$$(*) \qquad \tau(1, x) \cdot \tau(1, y) = \tau(1, x + y)$$

for degree(x) = degree$(y) = 0$. This formula is what we need to prove necessity in the theorem.

PROOF OF NECESSITY. As before, let S: $\pi_n(M) \to \pi_n^s(M, \text{pt.})$ be the stabilization map. Let $K_n(M) = \{x \in \pi_n(M) | \deg(x) = 0\}$ and $K_n^s(M) = \{x \in \pi_n^s(M, \text{pt.}) | \deg(x) = 0\}$. These are subgroups and S: $K_n(M) \to K_n^s(M)$ is a homomorphism. Since M is s-parallelizable we have for large r the map α: $S^{n+r} \to S^r M$ of degree 1, and as before we have $\alpha^* - \alpha$: $\{M, M\} \to \{S^n, M\} = \pi_n^s(M, \text{pt.})$, not necessarily a homomorphism. Let p: $M \to S^n$ be obtained by pinching to a point the outside of a smooth D^n in M. Then $S^r p \circ \alpha$: $S^{n+r} \to S^{n+r}$ has degree 1, so it is homotopic to the identity, and we have $\alpha^* \circ p^* = \mathrm{id} | \pi_n^s(M, \text{pt.})$. Let $\{M, M\}^1$ be the multiplicative subgroup of $\{M, M\}^0$ consisting of degree 1 homotopy equivalence; then $\alpha^* - \alpha$: $\{M, M\}^1 \to K_n^s(M)$. Let $1 + p^*$: $K_n^s(M) \to \{M, M\}$ be the map $x \to 1 + p^*(x)$. Then $1 + p^*$: $K_n^s(M) \to \{M, M\}^1$, not necessarily a homomorphism, but

$$(\alpha^* - \alpha) \circ (1 + p^*) = \mathrm{id} | K_n^s(M).$$

Let $E^1(M)$ be the H-space of degree 1 homotopy equivalences of M; we have a canonical homomorphism $\pi_0 E^1(M) \to \{M, M\}^1$, and we define δ: $\pi_0 E^1(M) \to K_n^s(M)$ to be the composition

$$\pi_0 E^1(M) \to \{M, M\}^1 \xrightarrow{\alpha^* - \alpha} K_n^s(M).$$

Then δ is not necessarily a homomorphism, but $\delta(1) = 0$.

LEMMA. *Define* τ: $K_n(M) \to \pi_0 E^1(M)$ *by* $\tau(x) = \tau(1, x)$. *Then this diagram commutes*:

$$\begin{tikzcd} K_n(M) \arrow[rr, "\tau"] \arrow[dr, "S"'] & & \pi_0 E^1(M) \arrow[dl, "\delta"] \\ & K_n^s(M) & \end{tikzcd}$$

PROOF OF LEMMA. Let ψ: $SX \to SX \vee SX$ be the H' structure. Then

$$\psi^*: \{SX \vee SX: A\} = \{SX: A\} \times \{SX: A\} \to \{SX: A\}$$

is track addition. Then the homotopy commutativity of the diagram

$$\begin{tikzcd} & M & \\ M \arrow[ur, "1"] \arrow[rr, "\varphi"] \arrow[dr, "p"'] & & M \vee S^n \arrow[ul] \arrow[dl] \\ & S^n & \end{tikzcd}$$

that $[S\varphi] = [S\iota] + [Sp]$ where ι = inclusion $M \subset M \vee S^n$. Denote by $\tau'(x)$ the image of $\tau(x)$ under $\pi_0 E^1(M) \to \{M, M\}^1 \subset \{M, M\}$. Then $[S\varphi] = [S\iota] + [Sp]$ implies that $\tau(x) = 1 + p^*(Sx)$, where Sx is the image of x under S: $\pi_n(M) \to \pi_n^s(M, \text{pt.})$. But then

$$\delta(\tau(x)) = (\alpha^* - \alpha) \circ \tau'(x) = (\alpha^* - \alpha)(1 + p^*(Sx)) = Sx,$$

and the lemma is proved.

Finally to prove necessity, notice that the map τ: $\pi_n(M) \to_0 \mathrm{Map}(M, M)$ defined in the introduction is the same as that appearing in the lemma, i.e. $x \to \tau(1, x)$. Notice also that the kernel of suspension is contained in the kernel of the Hurewicz map so that we may extract from Theorem 1, the diagram below it, and the lemma, the following commutative diagram:

$$\begin{tikzcd}
D(M) \arrow[r, "\Theta"] \arrow[d] & K_n(M) \arrow[r, "S"] \arrow[d, "\tau"] & K_n^s M \\
\pi_0 \mathrm{Diff}^{n-1}: M \arrow[r] & \pi_0 E^1(M) \arrow[ur, "S"'] &
\end{tikzcd}$$

with the top row exact and the left vertical onto. Now a diagram chase using the fact that τ is a homomorphism establishes that if $\tau(x)$ is in the image of $\pi_0 \mathrm{Diff}^{n-1}$: $M \to \pi_0 E^1(M)$, then $Sx = 0$. The proof of necessity, and thus of the theorem, is complete.

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