TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 240, June 1978

INFINITESIMAL CALCULUS ON LOCALLY CONVEX SPACES: 1. FUNDAMENTALS

BY

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ABSTRACT. Differential calculus on nonnormed locally convex spaces suffers from technical difficulties (and the subsequent plethora of different definitions) partly because the families of multilinear maps over the spaces do not inherit a suitable topology. In this note we give the elementary ingredients of a strong differentiation based on Abraham Robinson's theory of infinitesimals.

Though nontopologizable, our theory is a natural generalization of standard infinitesimal calculus (finite dimensional or Banach space), see Robinson [1966], Keisler [1976], or Stroyan and Luxemburg [1976]. It is simpler than many recent developments, e.g., Yamamuro [1974] and Keller [1974].

The technical improvement of our approach should lead to advances in a variety of subjects.

1. Infinitesimal vectors and maps. Let $\mathbf{E}$ be a complete real locally convex (Hausdorff) linear space whose topology is defined by a family of seminorms P. Each $p \in P$ is a function $p\colon \mathbf{E} \to [0, \infty)$ satisfying:

$$p(\lambda x) = |\lambda| p(x) \quad \text{for } \lambda \in \mathbf{R}, x \in \mathbf{E}, \tag{H}$$

and

$$p(x + y) \leqslant p(x) + p(y) \quad \text{for } x, y \in \mathbf{E}. \tag{A}$$

Let $\mathbf{F}$ be a complete real locally convex (Hausdorff) linear space with a gauge of seminorms Q. We denote the set of *continuous* m-linear maps from $\mathbf{E} \times \ldots \times \mathbf{E} = \mathbf{E}^m$ into $\mathbf{F}$ by $\mathrm{Lin}^m(\mathbf{E}; \mathbf{F})$.

Now consider a full type-theoretic model or a superstructure $\mathfrak{X}$ which contains $\mathbf{R}$, $\mathbf{E}$, and $\mathbf{F}$. We shall deal with a κ-saturated elementary extension ${}^*\mathfrak{X}$ of $\mathfrak{X}$ where $\kappa = \mathrm{card}(\mathfrak{X})$ and henceforth refer to this as a polysaturated extension of $\mathfrak{X}$. Inside ${}^*\mathfrak{X}$ live the extensions ${}^*\mathbf{R}$, ${}^*\mathbf{E}$, ${}^*\mathbf{F}$ and an extended function ${}^*p$, or just $p\colon {}^*\mathbf{E} \to {}^*\mathbf{R}$ for each seminorm $p \in P$. The *finite scalars* are denoted by $\mathcal{O}$, they are the numbers in ${}^*\mathbf{R}$ bounded by the extension of some ordinary $r \in \mathbf{R}$. For specificity we denote $\{ {}^*r\colon r \in \mathbf{R}\}$, the standard numbers embedded one at a time in the extension by ${}^\sigma\mathbf{R} \subset {}^*\mathbf{R}$. We use $a \approx b$ for "a is infinitesimally close to b" in the scalars ${}^*\mathbf{R}$. The reader is referred to

Received by the editors July 13, 1976 and, in revised form, January 31, 1977.
AMS (MOS) subject classifications (1970). Primary 46A99, 26A96; Secondary 26A98.
Key words and phrases. Infinitesimal calculus, locally convex space.

one of the following introductions to model-theoretic analysis and an answer to the question "What are infinitesimals?": Robinson [1966], Luxemburg [1969], [1973], Machover and Hirschfeld [1969], Bernstein [1973], Barwise [1977], or Stroyan and Luxemburg [1976]. The main facts are that infinite sets have proper extensions, that is, ${}^*A \setminus {}^\sigma A \neq \varnothing$, and first order formal properties are preserved in the extension. Thus ${}^*\mathbf{R}$ is a totally ordered field and ${}^*\mathbf{R} \setminus {}^\sigma\mathbf{R} \neq \varnothing$, in particular, there are infinite hyperreals (greater than each $r \in {}^\sigma\mathbf{R}^+$) and infinitesimals as well as finite numbers.

In studying the spaces ${}^*\mathbf{E}$ and ${}^*\mathbf{F}$, one measures distances with the families ${}^\sigma P = \{ {}^*p : p \in P\}$ and ${}^\sigma Q = \{ {}^*q : q \in Q\}$ rather than the whole extension ${}^*P$ or ${}^*Q$. Two fundamental ideas are the notion of finite and infinitesimal in ${}^*\mathbf{E}$ and ${}^*\mathbf{F}$. The *P-finite points* in $\mathbf{E}$ are:

$$\mathrm{fin}_P({}^*\mathbf{E}) = \{x \in {}^*\mathbf{E} : P(x) \in \mathcal{O} \text{ for each } p \in {}^\sigma P\}.$$

The *P-infinitesimal relation* on ${}^*\mathbf{E}$ is:

$$x \overset{P}{\approx} y \quad \text{if and only if } p(x - y) \approx 0 \text{ for each } p \in {}^\sigma P.$$

The set of points a P-infinitesimal from zero in ${}^*\mathbf{E}$:

$$\mathrm{inf}_P({}^*\mathbf{E}) = \{x \in {}^*\mathbf{E} : x \overset{P}{\approx} 0\}.$$

We use $x \approx y$ when P is clear from the contex. The infinitesimals are frequently denoted $\mu_P(0)$ and referred to as the *P-monad* of zero. The quotient

$$\hat{\mathbf{E}} = \mathrm{fin}({}^*\mathbf{E})/\mathrm{inf}({}^*\mathbf{E})$$

is the infinitesimal hull (or nonstandard hull) of ${}^*\mathbf{E}$ with the topology of $\{\hat{p} : p \in P\}$, where $\hat{p}(\hat{x}) = st(p(x))$, for $x \in \mathrm{fin}({}^*\mathbf{E})$ is a seminorm on $\hat{\mathbf{E}}$ defined by p. This has been extensively studied; see Henson and Moore [1972] for an introduction and Stroyan and Luxemburg [1976] for a survey. In the finite dimensional case the hull is isomorphic to the original space $\widehat{\mathbf{R}^n} \simeq \mathbf{R}^n$. Generally, though not always, an infinite dimensional hull is larger to reflect various kinds of limits. The original (Hausdorff) space is always embedded as ${}^\sigma\mathbf{E} \subseteq \widehat{\mathbf{E}}$, so we write $\mathbf{E}$ for ${}^\sigma\widehat{\mathbf{E}}$.

A subset $B \subseteq \mathbf{E}$ is bounded in the conventional sense if and only if its nonstandard extension consists entirely of finite points,

$${}^*B \subseteq \mathrm{fin}({}^*\mathbf{E}).$$

When $\mathbf{E}$ is not topologized by a finite set of seminorms the relation between bounded sets in $\mathbf{E}$ and finite points in ${}^*\mathbf{E}$ is not easily transferrable. We call the union of the standard bounded sets the *P-bounded points* of ${}^*\mathbf{E}$,

$$\mathrm{bd}_P({}^*\mathbf{E}) = \bigcup [\, {}^*B : B \text{ is bounded in } \mathbf{E}].$$

This is the largest union monad inside the finite points. We have $\mathrm{bd}_P({}^*\mathbf{E}) \subseteq$

$\mathrm{fin}_P(\,{}^*\mathbf{E})$ *with equality if and only if P is equivalent to a single seminorm.* Both the bounded points and finite points are modules over the finite scalars.

A linear map $L\colon \mathbf{E} \to \mathbf{F}$ has a nonstandard extension ${}^*L\colon\ {}^*\mathbf{E} \to {}^*\mathbf{F}$ which we denote by L also since it agrees with L on the embedded ${}^\sigma\mathbf{E}$. As one might expect, a linear map is continuous (at zero) if and only if whenever $x \overset{P}{\approx} 0$ in ${}^*\mathbf{E}$, then $L(x) \overset{Q}{\approx} 0$ in ${}^*\mathbf{F}$. As a result of homogeneity, a standard linear map is continuous also if and only if it maps finite points to finite points, $\mathrm{fin}_P({}^*\mathbf{E})$ into $\mathrm{fin}_Q({}^*\mathbf{F})$.

(1.1). Definition. *Let* $\mathbf{E}$ *and* $\mathbf{F}$ *be complete standard (Hausdorff) locally convex* $\mathbf{R}$*-vector spaces.*

The finite m-linear internal maps from ${}^*\mathbf{E}$ *to* ${}^*\mathbf{F}$, $\mathrm{FLin}^m(\,{}^*\mathbf{E};\ {}^*\mathbf{F})$, *are those internal linear maps* $L\colon\ {}^*\mathbf{E}^m \to {}^*\mathbf{F}$, $L \in {}^*[\mathrm{Lin}^m(\mathbf{E};\mathbf{F})]$ *such that if* $x_1, \ldots, x_m \in \mathrm{fin}(\,{}^*\mathbf{E})$, *then* $L(x_1, \ldots, x_m) \in \mathrm{fin}(\,{}^*\mathbf{F})$.

The infinitesimal m-linear maps from ${}^*\mathbf{E}$ *to* ${}^*\mathbf{F}$, $\mathrm{ILin}^m(\,{}^*\mathbf{E};\ {}^*\mathbf{F})$, *are those internal m-linear maps such that if* $x_1, \ldots, x_m \in \mathrm{fin}(\,{}^*\mathbf{E})$ *then* $L(x_1, \ldots, x_m) \in \mathrm{inf}(\,{}^*\mathbf{F})$. *We shall write* $K \approx L$ *for* $(K - L) \in \mathrm{ILin}^m(\,{}^*\mathbf{E};\ {}^*\mathbf{F})$ *when* $\mathbf{E}$, P, $\mathbf{F}$, Q, *and* m *are clear from the context.*

We shall denote the space $\mathrm{FLin}^m(\,{}^*\mathbf{E};\ {}^*\mathbf{F})/\mathrm{ILin}^m({}^*\mathbf{E};\ {}^*\mathbf{F})$ *by* $\widehat{\mathrm{Lin}}^m(\hat{\mathbf{E}};\hat{\mathbf{F}})$.

Some simple observations are as follows.

(1.2). Proposition. $\mathrm{FLin}^m(\,{}^*\mathbf{E};\ {}^*\mathbf{F})$ *and* $\mathrm{ILin}^m(\,{}^*\mathbf{E};\ {}^*\mathbf{F})$ *are* $\mathcal{O}$*-modules.*

(1.3). Proposition. $\mathrm{FLin}^m({}^*\mathbf{E};\ {}^*\mathbf{F})$ *is precisely the space of m-linear maps nearly annihilated by infinitesimal scalars, that is,* $L \in \mathrm{FLin}^m({}^*\mathbf{E};\ {}^*\mathbf{F})$ *iff* $\varepsilon L \in \mathrm{ILin}^m({}^*\mathbf{E};\ {}^*\mathbf{F})$ *for each* ε *infinitesimal in* ${}^*\mathbf{R}$.

(1.4). Proposition. *Evaluation and the natural identifications*

$$\mathrm{FLin}^m(\,{}^*\mathbf{E};\ \mathrm{FLin}^n(\,{}^*\mathbf{E};\ {}^*\mathbf{F})) \cong \mathrm{FLin}^{(m+n)}(\,{}^*\mathbf{E};\ {}^*\mathbf{F})$$

as well as compositions

$$\mathrm{FLin}^m({}^*\mathbf{F};\ {}^*\mathbf{G}) \times \left[\mathrm{FLin}^n({}^*\mathbf{E};\ {}^*\mathbf{F})\right]^m \to \mathrm{FLin}^n(\,{}^*\mathbf{E};\ {}^*\mathbf{G})$$

are STANDARDLY-continuous or S-continuous, that is, infinitesimal change in the variable produces infinitesimal change in the answer.

This natural generalization of Robinson's notion of S-continuity applies to vector maps as well as multilinear maps despite the fact that it is nontopological in the map spaces. While intermediate stages of these 'continuities' are not necessarily topological, maps from vectors in ${}^*\mathbf{E}$ to vectors in ${}^*\mathbf{F}$ thru such compositions are continuous in the ordinary sense by a standard result of the theory of infinitesimals (e.g., Stroyan and Luxemburg [**1976**, 8.3.1]).

We shall need to talk about the intermediate stages nonetheless. One immediate reason for this machinery is that a map is "$\mathcal{C}^1$" if $[Df_a(\cdot) - Df_b(\cdot)]$ is an infinitesimal linear map when $a \approx b$; that is, if $x \mapsto DF_x$ is S-continuous.

In the case where $\mathbf{E}$ and $\mathbf{F}$ are normed, finite maps are those with finite norm and infinitesimal maps those with infinitesimal norm. In the finite dimensional case, finite (resp. infinitesimal) means representation by matrices with finite (resp. infinitesimal) entries for standard bases. Finally, "$\mathcal{C}^1$" in the infinitesimal map sense is the usual notion of "continuously differentiable" in the normed space case.

2. Differentiable and uniformly differentiable functions. Because of the distinction between bounded and finite points in a nonnormed space two notions of "$\mathcal{C}^1$" seem appropriate. We give the definitions and some elementary facts about them in this section. The two notions coincide in Banach and metrizable spaces. Other possibilities would be to require the remainder (η below) to be a bounded infinitesimal.

The weaker notion of derivative is:

(2.1). DEFINITION. *A standard map f defined on a subset of* $\mathbf{E}$, *taking values in* $\mathbf{F}$ *is boundedly differentiable at a point* $a \in \mathbf{E}$ *with bound derivative* $\partial f_a \in \mathrm{Lin}(\mathbf{E}, \mathbf{F})$ *if whenever* $x \in \mathrm{bd}(\,{}^*\mathbf{E})$ *and* δ *is a positive infinitesimal scalar in* ${}^*\mathbf{R}^+$, *we have*

$$\frac{1}{\delta}\left[f(a+\delta x)-f(a)\right]=\partial f_a(x)+\eta$$

for some infinitesimal vector $\eta \in \inf(\,{}^*\mathbf{F})$.

This is a bounded-open condition in $\mathbf{E}$ because it requires something (including being defined) of a standard map for every point a bounded infinitesimal from a. (See "Cauchy's principle" in Stroyan and Luxemburg [**1976**].)

Bounded derivatives exist if and only if the following condition is satisfied: "For every bounded set $B \subseteq \mathbf{E}$ and every positive tolerance θ and each seminorm q on $\mathbf{F}$, there exists a positive Δ such that whenever $\delta < \Delta$ and $x \in B$, then

$$q\left\{\frac{1}{\delta}\left[f(a+\delta x)-f(a)\right]-\partial f_a(x)\right\}<\theta."$$

This is a simple translation exercise which we shall not include.

A simple example which demonstrates a potential use for bounded differentiability of this kind is to take the function $f(x) = \|x\|^2$ on a Hilbert space with its weak topology. In this case "bounded" means finite norm since the norm and weak topologies are compatible with self-duality. There are weak infinitesimals of infinite norm (e.g., Stroyan and Luxemburg [**1976**, (10.2.5)]) so f cannot satisfy Definition (2.2) below while it does satisfy (2.1). Examples of this kind can also be constructed in function spaces with noncompact domain, but from the point of view of the nonstandard space the difficulty with infinite norm looks much the same.

(2.2). DEFINITION. *A standard map f is differentiable at $a \in \mathbf{E}$ with standard derivative $Df_a \in \mathrm{Lin}(\mathbf{E}; \mathbf{F})$ if whenever a finite vector $x \in \mathrm{fin}(^*\mathbf{E})$ and an infinitesimal scalar $\delta \in {}^*\mathbf{R}^+$ are given*

$$\frac{1}{\delta}\left[f(a + \delta x) - f(a)\right] = Df_a(x) + \eta$$

for some infinitesimal vector $\eta \in \mathrm{inf}(^*\mathbf{F})$.

Although we have no immediate need for the conventional formulation, this is equivalent to the statement that "for every seminorm q on $\mathbf{F}$ and positive tolerance θ, there exists a neighborhood N of zero in $\mathbf{E}$ and there exists a Δ such that if $\delta < \Delta$ and $x \in N$, then $q\{(1/\delta)[f(a + \delta x) - f(a)] - Df_a(x)\} < \theta$." This condition was apparently introduced by Bastiani [**1964**]. The proof of the equivalence is as follows: The infinitesimal condition implies that whenever S is a *-finite extension of ${}^oP \subset S \subset {}^*P$ and Δ is infinitesimal, then $p(x) < 1$ for all $p \in S$ and $\delta < \Delta$ imply x is finite and the q-norm of the remainder is less than θ. The set of seminorms and Δ's is internal by virtue of our description (see the internal definition principle of Stroyan and Luxemburg [**1976**]) and consequently must contain a standard finite family of p's and a standard Δ (see Luxemburg's *-finite family lemma in Stroyan and Luxemburg [**1976**]).

Conversely, if the condition above holds, $x \in \mathrm{fin}(^*\mathbf{E})$ and δ is a positive infinitesimal, then $\sqrt{\delta} \cdot (x) \in {}^*N$, for each standard neighborhood, $\sqrt{\delta} < \Delta$, for each standard Δ, so that $q(\eta) < \theta$, for each standard positive θ.

(2.3). DEFINITION. *A standard map f: $U \to \mathbf{F}$, $U \subseteq \mathbf{E}$, is uniformly differentiable provided there is a standard map Df defined on U satisfying the condition that whenever a is near a standard point of $^*U$, $Df_a \in \mathrm{FLin}(^*\mathbf{E}; {}^*\mathbf{F})$ and when $x \in \mathrm{fin}(^*\mathbf{E})$ is a finite vector and δ is a positive infinitesimal,*

$$\frac{1}{\delta}\left[f(a + \delta x) - f(a)\right] = Df_a(x) + \eta$$

for some infinitesimal $\eta \in \mathrm{inf}(^*\mathbf{F})$.

Observe that our definition is an open condition in $\mathbf{E}$ since we require something for all infinitesimals on an internal f (Cauchy's principle). It agrees with the conventional definition of $\mathcal{C}^1$ when $\mathbf{E}$ and $\mathbf{F}$ are Banach spaces. Our formula for Df_z is only unique up to an infinitesimal map, but since Df is a standard map it is unique.

Uniform bound differentiability is defined by restricting x to bounded vectors of $\mathrm{bd}(^*\mathbf{E})$, rather than allowing it to run over all finite points.

The reader might observe that $f(x) = x^2 \sin(\pi/x)$ is not uniformly differentiable in any neighborhood of zero. (The condition fails when a is a positive infinitesimal.)

(2.4). PROPOSITION. *A uniformly differentiable function is S-continuously differentiable at near-standard points.*

PROOF. Suppose f is uniformly differentiable on a set containing near standard points a and b with $a \approx b$. We must show that $Df_a \approx Df_b$ as linear maps.

A simple application of saturation in $^*\mathfrak{X}$ shows that there is an infinite scalar Ω such that $\Omega(b - a)$ is still infinitesimal (Henson and Moore [1972] or Stroyan and Luxemburg [1976, (10.1.15)]). Let $\delta = 1/\Omega$. Uniform-differentiability at b says:

$$f(a) - f(b) = Df_b(a - b) + \delta \cdot \eta_1 \tag{1}$$

with η_1 infinitesimal in $^*\mathbf{F}$. Let $\xi = \delta x + a$ for some finite x and apply uniform-differentiability at a:

$$f(\xi) - f(a) = \delta \cdot Df_a(x) + \delta \cdot \eta_2 \tag{2}$$

with η_2 infinitesimal in $^*\mathbf{F}$. We also have $\xi = \delta[x + \Omega(a - b)] + b$ so uniform-differentiability at b says:

$$f(\xi) - f(b) = \delta Df_b(x) + Df_b(a - b) + \delta \cdot \eta_3 \tag{3}$$

with η_3 infinitesimal in $^*\mathbf{F}$. Now (3)–(2)–(1) gives:

$$\delta\big[Df_b(x) - Df_a(x)\big] = \delta \cdot \eta$$

with η infinitesimal. Since $\delta \neq 0$, $Df_a(x) \approx Df_b(x)$.

(2.5). REMARK. Notice that the proof actually shows more, namely *if* $Df_a \in \mathrm{FLin}(^*\mathbf{E};\ ^*\mathbf{F})$ *for STANDARD values of* a, *then* $Df_b \in \mathrm{FLin}(^*\mathbf{E};\ ^*\mathbf{F})$ *for* $b \approx a$, in other words the map Df need only be assumed to take values in $^*\mathrm{Lin}(^*\mathbf{E};\ ^*\mathbf{F})$ rather than the external space $\mathrm{FLin}(^*\mathbf{E};\ ^*\mathbf{F})$.

(2.6). CHAIN RULE. *Let* $g\colon U \subseteq \mathbf{E} \to \mathbf{F}$ *and* $f\colon V \subseteq \mathbf{F} \to \mathbf{G}$ *be uniformly differentiable maps for which the composite can be formed. Then* $D(f \circ g)_x = Df_{g(x)} \circ Dg_x$ *is the uniform derivative of* $h = f \circ g$.

CALCULATION. Let $b = g(a)$.

$$\begin{aligned} h(a + \delta x) - h(a) &= f(g(a + \delta x)) - f(g(a)) \\ &= Df_b\big[\delta \cdot (Df_a(x) + \eta)\big] + \delta \cdot \zeta \\ &= \delta \cdot Df_{g(a)} \circ Dg_a(x) + \delta \cdot \theta, \end{aligned}$$

where η, ζ, θ are infinitesimal. Notice that $Df_b(\eta) \approx 0$ since $\Omega\eta \approx 0$ for some infinite scalar Ω. Hence $\Omega \cdot Df_b(\eta)$ is finite and $1/\Omega$ times a finite vector is infinitesimal.

In many concretely defined spaces, verification of uniform differentiability is easy because infinitesimals have a simple description and formal calculation lifts to $^*\mathbf{E}$. For example, on the space of real sequences the Tychonoff topology is described by $x \approx y$ if and only if $x_j \approx y_j$ for finite indices $j \in {}^\sigma\mathbf{N}$. The map

$$(x_1, x_2, \ldots) \overset{f}{\mapsto} (\sin(x_1), \sin(2x_2), \sin(3x_3), \ldots)$$

is uniformly differentiable at finite $a = (a_1, a_2, \dots)$ (i.e., such that a_j is finite in $^*\mathbf{R}$ for finite $j \in {}^\sigma\mathbf{N}$). We simply have

$$\frac{1}{\delta}\left[f(a+\delta x) - f(a)\right]_j = \sin(ja_j)\frac{\left[\cos(j\delta x_j) - 1\right]}{\delta} + \cos(ja_j)\frac{\sin(j\delta x_j)}{\delta}$$
$$= jx_j\cos(ja_j) + \delta\cdot z_j, \qquad z_j \text{ finite},$$

and

$$\left[Df_a(x)\right]_j = j\cos(ja_j)x_j.$$

On the other hand, as a map restricted to l^∞ the function is not differentiable at zero. Let $a = 0$, $x = (1, 1, 1, \dots)$ and $\delta = \pi/2\Omega$ for $\Omega \in {}^*\mathbf{N}_\infty$, so that

$$\frac{1}{\delta}\left[f(a+\delta x) - f(a)\right]_\Omega = \frac{2\Omega}{\pi}\left[\sin\left(\frac{\pi}{2}\right)\right] = \left(\frac{2\Omega}{\pi}\right).$$

Now, the l^∞-norm is described by $x \approx y$ if and only if $x_j \approx y_j$ for all $j \in {}^*\mathbf{N}$, finite or not, so the difference quotient is infinite in $^*l^\infty$.

Yamamuro [1974] uses the function

$$(x_1, x_2, \dots) \overset{g}{\mapsto} x_1 \sum_{k=1}^{\infty} \frac{|x_k|}{[1 + |x_k|]\cdot 2^k}$$

to distinguish several conventional notions of differentiability. If $a_j \approx b_j$ for finite j, then Robinson's sequential lemma says $a_j \approx b_j$ for $1 < j < \Omega \in {}^*\mathbf{N}_\infty$, so

$$\sum_{k=1}^{\Omega} \frac{|a_k|}{[1 + |a_k|]\cdot 2^k} \approx \sum_{k=1}^{\Omega} \frac{|b_k|}{[1 + |b_k|]\cdot 2^k}$$

and $\sum_{k=\Omega+1}^{\infty}(1/2^k) \approx 0$, therefore

$$\frac{1}{\delta}\left[g(a+\delta x) - g(a)\right] = x_1 \sum_{k=1}^{\infty} \frac{|a_k|}{[1 + |a_k|]\cdot 2^k} + \varepsilon,$$

with $\varepsilon \approx 0$.

(2.7). *Riemann integrals for vectors and maps.* A continuous standard image of a compact interval is compact, so the Riemann integral of a continuous standard function f: $[a, b] \to \mathbf{F}$,

$$\int_a^b f(t)\,dt \in \mathbf{F}$$

exists. The infinitesimal Riemann sums are simply $(b - a)$ times a *-convex combination of values of f. Since $\mathbf{F}$ is complete and the closed absolutely convex envelope of a compact set is totally bounded, the common standard part of all the Riemann sums exists in $\mathbf{F}$.

Next, suppose $t \mapsto L_t$ is a standard S-continuous map with values in $\mathrm{Lin}^m(\mathbf{E}, \mathbf{F})$ for $a < t < b$. Given standard vectors $x_1, \dots, x_m$ in $\mathbf{E}$, the

function $f(t) = L_t(x_1, \ldots, x_m)$ is continuous, so the integral $\int_a^b L_t(x_1, \ldots, x_m)\, dt$ exists in $\mathbf{F}$. Hence $x_1, \ldots, x_m \mapsto \int_a^b L_t(x_1, \ldots, x_m)\, dt$ is a standard m-linear map denoted by the integral and $\int_a^b L_t(\cdots)\, dt$ is in $\mathrm{Lin}^m(\mathbf{E}, \mathbf{F})$. To prove the continuity one simply takes finite vectors $x_1, \ldots, x_m$ in fin(*E), approximates the integral infinitesimally (using transfer of the "epsilon-delta" integral definition)

$$\int_a^b L_t\, dt(x_1, \ldots, x_m) \approx \sum_{j=1}^{\omega} L_{t_j}(x_1, \ldots, x_m)\Delta t_j$$

and estimates a q-norm of the sum by $(b - a) \cdot \max_j(q[L_{t_j}(x_1, \ldots, x_m)]) = (b - a)L_{t_\lambda}(x_1, \ldots, x_m)$ for some λ between 1 and ω. Now $L_{t_\lambda} \approx L_s$ as maps, where $s = st(t_\lambda)$, and L_s is finite since it is standard and continuous. Thus $\int_a^b L_t\, dt$ is finite and therefore continuous since it is standard. The Riemann integral of an S-continuous standard map with values in $\mathrm{Lin}^m(\mathbf{E}; \mathbf{F})$ is directly well defined by taking

$$\int_a^b L_t\, dt = \left(\left[\sum_{j=1}^{\omega} L_{t_j}(\cdots)\Delta t_j\right] / \approx\right),$$

the infinitesimal hull of an infinitesimal Riemann sum. The point of the above remarks is that this hull is *standard* and *the Riemann sums approximate the integral* (*even*) *at* (*nonstandard*) *finite points* of evaluation, and in particular, uniformly on bounded sets.

(2.8.a). FUNDAMENTAL THEOREM. *When*

$$F(x) = \int_a^x f(t)\, dt,$$

where $f\colon [a, b] \to \mathbf{F}$ *is a continuous standard function, then* $DF_x(h) = h \cdot f(x)$ *is the uniform derivative of* F, $DF_x \in F\,\mathrm{Lin}(\mathbf{R}; \mathbf{F})$, $x \in (a, b)$.

PROOF. For each seminorm q on $\mathbf{F}$, x in $^*[a, b]$ and infinitesimal δ, with $x + \delta$ in $[a, b]$,

$$\begin{aligned} q\{[F(x + \delta) - F(x)] - \delta \cdot f(x)\} &= q\left\{\int_x^{x+\delta} [f(t) - f(x)]\, dt\right\} \\ &\leq \int_x^{x+\delta} q\{f(t) - f(x)\}\, dt \\ &\leq \delta \cdot \max[q\{f(t) - f(x)\} : x \leq t \leq x + \delta] \end{aligned}$$

and the *-maximum is attained at $c \approx x$, hence is infinitesimal by S-continuity of f at x. Notice that $f(x)$ is near-standard by compactness, so $DF_x \in \mathrm{FLin}(\mathbf{R}; \mathbf{F})$.

This is one-half of the Fundamental Theorem.

(2.8.b). FUNDAMENTAL THEOREM. *Suppose f is a standard uniformly-differentiable function on a segment $[a, b]$ in* **E**. *Let* $(df/dt)(t) = Df_{[a+t(b-a)]}(b-a)$. *Then* $f(b) - f(a) = \int_0^1 (df/dt)(t)\,dt$.

PROOF. $(df/dt)(t)$ is continuous by (2.4). We know the integral exists by (2.7) so we select an equal infinitesimal partition to calculate it with left evaluation. By definition of uniformly-differentiable,

$$f\left(a + \frac{k+1}{\Omega}(b-a)\right) - f\left(a + \frac{k}{\Omega}(b-a)\right)$$
$$= \frac{df}{dt}\left(a + \frac{k}{\Omega}(b-a)\right) + \frac{1}{\Omega}\eta_k,$$

with η_k infinitesimal, so a Riemann sum equals the telescoping sum

$$\sum_{k=1}^{\Omega} \Delta f(k) = f(b) - f(a)$$

plus a sum of infinitesimals in $^*\mathbf{F}$,

$$\sum_{k=1}^{\Omega} \eta_k \cdot \frac{1}{\Omega}.$$

In any S-continuous norm q on $^*\mathbf{F}$,

$$q\left(\sum_{k=1}^{\Omega} \eta_k \frac{1}{\Omega}\right) < \frac{1}{\Omega} \sum_{k=1}^{\Omega} \max(q(\eta_k): 1 < k < \Omega) = \max(q(\eta_k)) \approx 0,$$

since the internal maximum is attained. Therefore, the difference is infinitesimal in $^*\mathbf{F}$ and our theorem is proved.

3. Higher order derivatives. The second derivative of a standard function f at a standard point "a" can be described as follows. Suppose first that f has a first derivative Df in a finite neighborhood of a. If there is a standard finite (continuous) bilinear map $B(\cdot, \cdot)$ satisfying the condition that whenever $x \in \text{fin}(^*\mathbf{E})$ and δ is an infinitesimal scalar, then

$$\frac{1}{\delta}\left[Df_{a+\delta x}(\cdot) - Df_a(\cdot)\right] = B(x, \cdot) + \eta(\cdot)$$

for some infinitesimal map $\eta \in \text{ILin}(\mathbf{E}; \mathbf{F})$. When such a B exists we shall denote it by $D^2f_a(\cdot, \cdot)$.

A standard function f: $U \subseteq \mathbf{E} \to \mathbf{F}$ is twice uniformly-differentiable provided first that the standard map $D^2f_a(\cdot, \cdot)$ exists for standard values of a, but also whenever a is near a standard value of $^*U$, $x \in \text{fin}(^*\mathbf{E})$, and δ is an infinitesimal scalar,

$$\frac{1}{\delta}\left[Df_{a+\delta x}(\cdot) - Df_a(\cdot)\right] = D^2f_a(x, \cdot) + \eta(\cdot)$$

for some infinitesimal map $\eta \in \text{ILin}(\mathbf{E}; \mathbf{F})$.

A twice uniformly differentiable function is twice continuously differentiable (in the sense of S-continuity): when $b \approx a$, near-standard in $^*U$, then $[D^2f_b - D^2f_a] \in \mathrm{ILin}^2(\mathbf{E}; \mathbf{F})$. The proof of (2.4) can be carried over to this setting simply by applying the condition above to Df and taking the η_j in $\mathrm{ILin}(\mathbf{E}; \mathbf{F})$.

In this section we generalize these remarks to k-times uniformly differentiable functions. These functions have the same basic properties as ordinary Banach-space-$\mathcal{C}^k$-functions; Leibniz' rule, the higher order chain rule, Taylor's small oh formula and its converse. The proofs in the locally convex setting are nearly the same as standard infinitesimal proofs (compare Stroyan and Luxemburg [**1976**], for example).

(3.1). Definition. *Let f: $U \subseteq \mathbf{E} \to \mathbf{F}$ be a standard map. Suppose f is k-uniformly differentiable, where the case $k = 1$ is as in* (2.3). *We say f is $(k+1)$-uniformly differentiable provided there is a standard map $D^{(k+1)}f$: $U \to \mathrm{Lin}^{(k+1)}(\mathbf{E}, \mathbf{F})$ such that whenever a is near a standard point of $^*U$, $D^{(k+1)}f_a \in \mathrm{FLin}^{(k+1)}(^*\mathbf{E}, {}^*\mathbf{F})$ and when x is a finite vector and δ is a positive infinitesimal,*

$$\frac{1}{\delta}\left[D^kf_{(a+\delta x)}(\cdot) - D^kf_a(\cdot)\right] = D^{(k+1)}f_a(x, \cdot) + \eta(\cdot)$$

for some infinitesimal map $\eta(\cdot) \in \mathrm{ILin}^k(^*\mathbf{E}; {}^*\mathbf{F})$.

More generally, if L: $U \to \mathrm{Lin}^k(\mathbf{E}; \mathbf{F})$ is a standard mapping we say L is uniformly differentiable if there is a standard map DL: $U \to \mathrm{Lin}^{(k+1)}(\mathbf{E}, \mathbf{F})$ satisfying the "small oh" formula above.

A simple modification of the proof of (2.4) yields the following:

(3.2). Proposition. *If f is a k-uniformly differentiable standard function on U, then $x \mapsto D^kf_x$ is S-continuous for x near-standard in $^*U$.*

Now we turn to the locally convex analog of the classical necessary and sufficient condition for a function to be a $\mathcal{C}^k$-map. Let $\mathrm{SFLin}^h(^*\mathbf{E}; {}^*\mathbf{F})$ denote the space of symmetric finite h-linear internal maps from $^*\mathrm{SLin}^h(^*\mathbf{E}; {}^*\mathbf{F})$, $L(x_1, \ldots, x_h) = L(x_{\theta(1)}, \ldots, x_{\theta(h)})$ for a permutation θ, where $\mathrm{SLin}^h(\mathbf{E}; \mathbf{F})$ denotes the symmetric continuous h-linear maps. A *standard* map is continuous if and only if its *-extension is finite, but $^*\mathrm{SLin}^h(^*\mathbf{E}; {}^*\mathbf{F})$ properly contains $\mathrm{SFLin}^h(^*\mathbf{E}; {}^*\mathbf{F})$ in general,

(3.3). Taylor's uniform small oh formula. *Let U be open in $\mathbf{E}$ and f: $U \to \mathbf{F}$ be a standard function. Then f is k-uniformly-differentiable on U if and only if there exist standard maps $L^h_{(\cdot)}$: $U \to \mathrm{SLin}^h(\mathbf{E}; \mathbf{F})$, for $1 \leq h \leq k$, such that whenever a is near-standard in $^*U$, x is a finite vector in $\mathrm{fin}(^*\mathbf{E})$ and δ is an infinitesimal scalar, then there is an infinitesimal vector $\eta \in \mathrm{inf}(^*\mathbf{F})$ satisfying*

$$f(a + \delta x) = \sum_{h=0}^{k} \frac{\delta^h}{h!} L_a^h(x)^{(h)} + \delta^k \cdot \eta.$$

The unique maps $L^h = D^h f$.

When f is k-uniformly differentiable we need to show symmetry of the higher order derivatives plus the formula. We know $D^h f$ maps into $\mathrm{Lin}^h(\mathbf{E}; \mathbf{F})$ by finiteness of $D^h f_a$ at standard a. For the converse we must show finiteness of $D^h f_a$ at near-standard a and that $L^h = D^h f$, $1 \leqslant h \leqslant k$.

PROOF. Suppose f is k-uniformly differentiable with $k \geqslant 2$ (since $k = 1$ follows from the definition). Define a standard function of two $\mathbf{E}$ variables,

$$g(x, y) = f(a + x + y) - f(a + x) - D^2 f_a(y, x),$$

with the derivative with respect to x is given by:

$$Dg_x(\cdot\,) = \left[Df_{(a+x+y)}(\cdot\,) - Df_{(a+x)}(\cdot\,) - D^2 f_a(y, \cdot\,) \right].$$

We may estimate a change in g, $[g(x, y) - g(0, y)]$ by the Fundamental Theorem (2.9.b) and the triangle inequality obtaining the result that "for every x and y such that $a + x$ and $a + x + y$ are in U, $q(g(x, y) - g(0, y)) \leqslant \max[q((dg/dt)(s)): 0 \leqslant s \leqslant 1]$," where q is a seminorm on $\mathbf{F}$. Moreover the maximum of the statement in quotes is attained.

Now we transform the property in quotes to the nonstandard model, let z, w be finite vectors and δ, ε be infinitesimal scalars and apply the property with $x = \delta z$ and $y = \varepsilon w$, then with $x = \varepsilon w$ and $y = \delta z$. The difference

$$\begin{aligned} &\left[g(\delta z, \varepsilon w) - g(0, \varepsilon w) \right] - \left[g(\varepsilon w, \delta z) - g(0, \delta z) \right] \\ &\qquad = D^2 f_a(\delta z, \varepsilon w) - D^2 f_a(\varepsilon w, \delta z). \end{aligned}$$

Substitution in the expression above for $Dg_x(\cdot)$ gives

$$\frac{dg}{dt}(t) = Df_{(a+t\delta z+\varepsilon w)}(\delta z) - Df_{(a+t\delta z)}(\delta z) - D^2 f_a(\varepsilon w, \delta z)$$

and U-differentiability of Df gives

$$\frac{dg}{dt}(t) = D^2 f_{(a+t\delta z)}(\varepsilon w, \delta z) - D^2 f_a(\varepsilon w, \delta z) + \eta(\varepsilon w, \delta z)$$

where η is an infinitesimal map. We know also by (3.2) that $D^2 f_{(a+t\delta z)} \approx D^2 f_a$, so $(dg/dt)(t) = \zeta(\varepsilon w, \delta z)$ for an infinitesimal map ζ. Consequently, the difference $D^2 f_a(w, z) - D^2 f_a(z, w)$ is an infinitesimal, forcing equality when a is standard. Thus $D^2 f$ has its range in $\mathrm{SLin}^2(\mathbf{E}; \mathbf{F})$ since it is a standard map.

Symmetry of the higher order derivatives follows by induction applying this argument to $x \mapsto D^{h-2} f_x$, using the inductive hypothesis that $D^{h-1} f$ is symmetric.

The classical integral formula for the remainder (see Stroyan and Luxemburg [**1976**], for example) carries over to our locally convex setting giving

$$R^{h+1}(a, \delta x) = \int_0^1 \frac{(1-t)^h}{h!} \left[D^{h+1} f_{(a+t\delta x)}(\delta x)^{(h+1)} - D^{h+1} f_a (\delta x)^{(h+1)} \right] dt$$

and again the triangle inequality and the fact that $D^{(h+1)} f_{(a+t\delta x)} \approx D^{h+1} f_a$ yield Taylor's formula. Specifically, transfer the statement

$$q\left(R^{h+1}(a, \delta z)\right) \leqslant \frac{\delta^{(h+1)}}{(h+1)!} \, q\left(D^{h+1} f_{(a+s\delta x)}(x)^{(h+1)} - D^{h+1} f_a (x)^{(h+1)}\right)$$

at an s, $0 \leqslant s \leqslant 1$, and apply (3.2).

The converse claims first that if $b \approx a$, for a standard in ${}^{\sigma}U$, then $L_b^h \approx L_a^h$ and in particular L_b^h is finite. The case $h = 1$ is the Remark (2.5). When $k > 1$ we take δ a positive infinitesimal scalar and $x_0, x_1, \ldots, x_k$ finite vectors and form the differences:

$$\begin{aligned} & f(a + \delta x_0 + \delta x_1 + \cdots + \delta x_k) \\ & \quad - \sum_{j=1}^{k} f(a + \delta x_0 + \delta x_1 + \cdots + \widehat{\delta x_j} + \cdots + \delta x_k) \\ & \quad + \sum_{1 \leqslant i < j \leqslant k} f(a + \delta x_0 + \delta x_1 + \cdots + \widehat{\delta x_i} + \cdots + \widehat{\delta x_j} + \cdots + \delta x_k) \\ & \quad + \cdots + (-1)^{(k-1)} \sum_{j=1}^{k} f(a + \delta x_0 + \delta x_j) + (-1)^k f(a + \delta x_0). \end{aligned}$$

Expanding these differences in Taylor polynomials about $a + \delta x_0$ gives $L^k_{(a+\delta x_0)}(\delta x_1, \ldots, \delta x_k) + \delta^k \cdot \eta$, while expanding about a gives $L_a^k(\delta x_1, \ldots, \delta x_k) + \delta^k \cdot \zeta$ with η and ζ infinitesimal vectors. This shows that $L^k_{a+\delta x_0} \approx L_a^k$ and since L_a^k is a standard k-linear continuous map it is finite when a is standard. Finally, any $b \approx a$ can be written as $b = a + [\Omega \cdot (b - z)]/\Omega$ where Ω is an infinite scalar for which $\Omega \cdot (b - a)$ is still infinitesimal. Let $\delta = 1/\Omega$ and $x_0 = \Omega \cdot (b - a)$, so $b = a + \delta x_0$.

When $k = 1$, Taylor's formula is the definition of the uniform derivative on U. We assume for the induction that it is known that $L_x^h = D^h f_x$ when $h < k$ and that the function may be expanded by the formula. We also know now that $L_b^{k+1} \approx L_a^{k+1}$ when $b \approx a$. We conclude the proof by showing $D(D^k f) = L^{k+1}$ as follows. Let δ be a positive infinitesimal and let x and y be finite vectors. We expand $f(a + \delta x + \delta y)$ two ways:

$$\begin{aligned} f(a + \delta x + \delta y) &= \sum_{h=0}^{k} \frac{\delta^h}{h!} D^h f_{(a+\delta x)}(y)^{(h)} \\ &\quad + \frac{\delta^{(k+1)}}{(k+1)!} L^{(k+1)}_{(a+\delta x)}(y)^{(k+1)} + \delta^{(k+1)} \cdot \eta \\ &= \sum_{h=0}^{k} \frac{\delta^h}{h!} D^h f_a (x+y)^{(h)} + \frac{\delta^{(k+1)}}{(k+1)!} L_a^{(k+1)}(x+y)^{(k+1)} + \delta^{(k+1)} \cdot \zeta \end{aligned}$$

where η and ζ are infinitesimal in $^*\mathbf{F}$. Next, form the difference between the two expansions and rewrite the terms involving multilinear maps applied to $(x + y)$ as a polynomial in y.

$$\begin{aligned}\delta^{(k+1)} \cdot (\zeta - \eta) &= \sum_{h=0}^{k} \frac{\delta^h}{h!} \left[D^h f_{(a+\delta x)}(y)^{(h)} - D^h f_a (x+y)^{(h)} \right] \\ &\quad + \frac{\delta^{(k+1)}}{(k+1)!} \left[L^{(k+1)}_{(a+\delta x)}(y)^{(k+1)} - L^{(k+1)}_a (x+y)^{(k+1)} \right] \\ &= \sum_{h=0}^{k+1} P_h(x) y^{(h)}\end{aligned}$$

where

$$P_k(x) y^{(k)} = \frac{\delta^k}{k!} \left[D^k f_{(a+\delta x)}(y)^{(k)} - D^k f_a(y)^{(k)} - L^{(k+1)}_a(\delta x, y^{(k)}) \right]$$

and

$$P_{k+1}(x) y^{(k+1)} = \frac{\delta^{(k+1)}}{(k+1)!} \left[L^{(k+1)}_{(a+\delta x)}(y)^{(k+1)} - L^{(k+1)}_a (y)^{(k+1)} \right].$$

This term is in $\mathrm{ILin}^{k+1}(\,^*\mathbf{E};\ ^*\mathbf{F})$ by the paragraph above, $L^{k+1}_{a+\delta x} \approx L^{k+1}_a$. Hence, $\Sigma_{h=0}^{k} P_h(x) y^{(h)} = \delta^{k+1} \cdot \theta$, where θ is infinitesimal in $^*\mathbf{F}$. By virtue of the homogeneity of the polynomial we have $P_k(x) y^{(k)} = \delta^{k+1}\xi$, with ξ infinitesimal in $^*\mathbf{F}$, so by the expression above for P_k, $D_a(D^k f(y^{(k)}))(x)$ equals $L^{(k+1)}_a(x, y^{(k)})$.

The fact that $P_k(x) y^{(k)} = \delta^{(k+1)}\xi$ can be seen as follows. Let $a_0, \ldots, a_k$ be distinct standard scalars and form the Vandermonde matrix times the columns vector of P's:

$$\begin{bmatrix} 1 & a_0 & \cdots & a_0^k \\ 1 & a_1 & \cdots & a_1^k \\ \vdots & \vdots & & \vdots \\ 1 & a_k & \cdots & a_k^k \end{bmatrix} \begin{bmatrix} P_0(x) \\ P_1(x) y \\ \vdots \\ P_k(x) y^k \end{bmatrix} = \begin{bmatrix} \delta^{(k+1)}\xi_0 \\ \delta^{(k+1)}\xi_1 \\ \vdots \\ \delta^{(k+1)}\xi_k \end{bmatrix}$$

by the calculations above since $a_j y$ are all finite. The Vandermonde matrix has a standard inverse and ξ is the kth entry of the inverse applied to the column of ξ_j's.

Suppose M is a symmetric k-linear map and $y_1, \ldots, y_k$ are finite vectors. We can write $M(y_1, \ldots, y_k)$ in terms of $M([y_1 + \cdots + y_k]^k)$, $M([y_1 + \cdots + \hat{y}_j + \cdots + y_k]^k)$, $M([y_1 + \cdots + \hat{y}_i + \cdots + \hat{y}_j + \cdots + y_k]^k), \ldots, M(y_j^k)$ by using the multinomial theorem. For example, $2M(x, y) = M([x + y]^2) - M(x^2) - M(y^2)$ and $6M(x, y, z) = M([x + y + z]^3) - M([x + y]^3) - M([x + z]^3) - M([y + z]^3) + 2M(x^3) + 2M(y^3) + 2M(z^3)$

and so on. Applying this to both sides of our formula for $D_a(D^k f) = L_a^{k+1}$ shows that $D^{(k+1)}f_a = L_a^{(k+1)}$ in the general case of different y's.

(3.4). REMARK. The Fundamental Theorem and integration by parts (needed in this proof) also work for map-valued integrals; in fact, the proof of (2.8) carries over using infinitesimal map errors instead of infinitesimal vector errors. We omit the details.

4. Equidifferentiable and equi-integrable families. In solving differential, integral, or functional equations one frequently devises a sequence, net, filter, or other family of approximate solutions, $\mathcal{Q}$. The approximations $f \in \mathcal{Q}$ themselves need not be differentiable, though one seeks a differentiable solution "in the limit." The conventional method applies differentiability equably or equably "at infinity." Robinson [**1966**] and others have shown how to replace such family conditions by the simpler standard conditions on single infinitesimal approximations $f \in {}^*\mathcal{Q}$. Hence we shall concentrate on STANDARD-differentiability conditions of internal functions in the non-standard model. The translation exercises into equidifferentiability of standard families "at infinity" are left to the reader.

The basic idea of the section is that standard infinitesimal conditions like Definition (2.3) when applied to internal functions mean the same thing in terms of "epsilons and deltas" on the infinitesimal hulls [$\hat{\mathbf{E}}$, $\hat{\mathbf{F}}$, $\widehat{\mathrm{Lin}}(\hat{\mathbf{E}}; \hat{\mathbf{F}})$, etc.] that the standard conditions mean for standard functions on the standard unextended spaces. (This has particularly powerful consequences in the (HM)-spaces.) The equivalent conventional equiconditions without benefit of infinitesimal hulls become excessively awkward in our opinion.

To shorten things a little we shall make our definition of uniform standard derivatives for internal functions taking values in ${}^*\mathrm{Lin}^k({}^*\mathbf{E}; {}^*\mathbf{F})$ where we let $\mathrm{Lin}^0(\mathbf{E}; \mathbf{F})$ equal $\mathbf{F}$. Notice that when f is standard our definition is the same as (2.3) on the set of near-standard points.

(4.1). DEFINITION. *An internal map f with values in* ${}^*\mathrm{Lin}^k({}^*\mathbf{E}; {}^*\mathbf{F})$ *is SU-differentiable on a subset W of its domain in* ${}^*\mathbf{E}$ *provided that there is an internal map Df whose domain includes W such that for $w \in W$, $f(w) \in \mathrm{FLin}^k({}^*\mathbf{E}; {}^*\mathbf{F})$, $Df_w \in \mathrm{FLin}^{k+1}({}^*\mathbf{E}; {}^*\mathbf{F})$, and whenever x is in* $\mathrm{fin}({}^*\mathbf{E})$ *and δ is a positive infinitesimal, $f(w + \delta x)$ is defined and*

$$\frac{1}{\delta}\left[f(w + \delta x) - f(w)\right] = Df_w(x) + \eta$$

where η is infinitesimal in $\mathrm{ILin}^k({}^*\mathbf{E}; {}^*\mathbf{F})$.

The internal derivative Df_w is not unique, but if L_w also satisfies the "small oh" condition, then $Df_w - L_w$ is an infinitesimal map. Thus $(Df_w/\approx)$ is unique.

The argument of (2.4) above yields:

(4.2). PROPOSITION. *If the internal function f is SU-differentiable on the infinitesimals around a in* $^*\mathbf{E}$, *then Df is S-continuous at a.*

As in the case of derivatives of standard functions, S-continuity of the internal map Df has the standard infinitesimal meaning in $^*\mathbf{E}$, $b \approx a$ implies $Df_b \approx Df_a$ as maps. The associated "epsilon-delta" continuity of $\hat{D}f$ in the hulls is nontopologizable in general.

A simple application of equable uniform differentiability is Leibniz' rule for differentiating under the integral. Notice that the hypotheses are fulfilled if the function is jointly uniformly differentiable.

(4.3). LEIBNIZ' RULE. *Let U be open in* $\mathbf{E}$, *let* $f\colon [a, b] \times U \to \mathrm{Lin}^k(\mathbf{E}; \mathbf{F})$ *be a standard S-continuous map and suppose a standard S-continuous partial derivative* $\partial f/\partial x\colon [a, b] \times U \to \mathrm{Lin}^{(k+1)}(\mathbf{E}; \mathbf{F})$ *exists satisfying the SU-derivative small oh equation at near-standard x for each t in* $^*[a, b]$. *Then*

$$D\int_a^b f(t, x)\, dt = \int_a^b \frac{\partial f}{\partial x}(t, x)\, dt.$$

PROOF. By transfer we may infinitesimally approximate

$$\begin{aligned}\int_a^b \frac{1}{\delta}\left[f(t, x+\delta z) - f(t, x)\right] dt &\approx \sum_{j=1}^{\omega} \frac{1}{\delta}\left[f(t_j, x+\delta z) - f(t_j, x)\right]\Delta t_j \\ &\approx \sum_{j=1}^{\omega} \frac{\partial f}{\partial x}(t_j, x)(z)\Delta t_j + \sum_{j=1}^{\omega} \eta_j \Delta t_j \\ &\approx \left[\int_a^b \frac{\partial f}{\partial x}(t, x)\, dt\right](z).\end{aligned}$$

This proves Leibniz' rule.

We shall give a general formulation of Peano's theorem for differential equations on the spaces $^*\mathrm{Lin}^k(\,^*\mathbf{E};\ ^*\mathbf{F}) = \mathbf{L}$. Specializing to the case $k = 0$ and standard f our hypotheses are equivalent to uniform continuity and bounded range.

(4.4). PEANO'S THEOREM. *Let* $f\colon [0, 1] \times \mathbf{L} \to \mathbf{L}$ *be an internal S-continuous map whose range consists entirely of finite points. For each finite $b \in \mathbf{L}$, there exists an internal SU-differentiable function* $v\colon [0, 1] \to \mathbf{L}$ *satisfying the initial value problem*

$$\hat{v}(0) = \hat{b} \quad \textit{and} \quad d\hat{v} = \hat{f}(t, \hat{v}(t))\, dt.$$

Notice that our theorem is new even for (B)-spaces and standard vector-valued maps, since the solution generally lies in $\hat{\mathbf{E}}$ and not $\mathbf{E}$. Cf. Godunov [1975]. Hence the following

(4.5). COROLLARY. *With f as in* (4.4), *if* $\mathbf{F}$ *is an (HM)-space in particular, if it is finite dimensional, (FM), nuclear, or Schwartz then the initial value*

problem is standard and $\hat{v}(t)$ is a standard solution.

PROOF. (HM)-spaces are those for which the hulls of finite points of $^*\mathbf{F}$ all lie in the standard completion; thus $\hat{v}(t)$ is a map into the completion, $\hat{f}$ takes values in the completion, and the differential equation in the hull holds on the completion. We assume in this paper that our spaces are complete, hence $\hat{\mathbf{F}} = \mathbf{F}$.

REMARK. This property is characteristic of (HM)-spaces. Let y be a finite point not near any standard point. Let x be a fixed standard vector with the seminorm q such that $q(x-y)$, $q(x)$, and $q(y)$ are noninfinitesimal. Let f be a linear extension of the "rotation" map $\left(\begin{smallmatrix}0 & -1\\ 1 & 0\end{smallmatrix}\right)\left(\begin{smallmatrix}x\\ y\end{smallmatrix}\right)$. The equation $d\hat{u} = \hat{f}(\hat{u})$ has only nonstandard solutions at $x = u(0)$, $\hat{u}(t) = \cos(t)\hat{x} + \sin(t)\hat{y}$.

Standard bounded continuous differential equations have standard solutions on the larger class of (M)-spaces; see Dubinsky **[1964]** and Corollary (4.6).

PROOF OF (4.4). Define an infinitesimal polygon:

$$v(0) = b, \qquad v\left(\frac{k+1}{\Omega}\right) = v\left(\frac{k}{\Omega}\right) + \frac{1}{\Omega} f\left(\frac{k}{\Omega}, v\left(\frac{k}{\Omega}\right)\right)$$

filling in linearly between k/Ω and $(k+1)/\Omega$, where Ω is an infinite hyperinteger. Let t and $t+\delta$ lie in $^*[0, 1]$ with δ infinitesimal. We claim

$$v(t+\delta) - v(t) = \delta \cdot f(t, v(t)) + \delta \cdot \eta$$

with η infinitesimal. This is automatic if $t = \lambda/\Omega$ and $t+\delta$ lies in the $1/\Omega$-interval. In general,

$$\begin{aligned} v(t+\delta) - v(t) = \varepsilon \cdot f\left(\frac{\lambda-1}{\Omega}, v\left(\frac{\lambda-1}{\Omega}\right)\right) \\ + \sum_{j=\lambda}^{\omega-1} f\left(\frac{j}{\Omega}, v\left(\frac{j}{\Omega}\right)\right)\frac{1}{\Omega} + \zeta \cdot f\left(\frac{\omega}{\Omega}, v\left(\frac{\omega}{\Omega}\right)\right) \end{aligned}$$

where

$$\varepsilon + \frac{(\omega - \lambda - 1)}{\Omega} + \zeta = \delta = (t+\delta) - t.$$

Since the range of f is finite and internal, whenever q is an S-continuous seminorm and $x_1, \ldots, x_m$ are finite in $^*\mathbf{E}$,

$$\begin{aligned} &q\{[v(t+\delta) - v(t)](x_1, \ldots, x_m)\} \\ &\qquad \leq \delta \max_{(\lambda-1) \leq j \leq \omega} q\left\{f\left(\frac{j}{\Omega}, v\left(\frac{j}{\Omega}\right)\right)(x_1, \ldots, x_m)\right\}, \end{aligned}$$

and the *-finite maximum is attained. Hence v is S-continuous in $\mathbf{L}$. Applying this and the S-continuity of f, we see that $f(t, v(t)) \approx f(j/\Omega,$

$v(j/\Omega))$ for $(\lambda - 1) \leqslant j \leqslant \omega$ Let $f(t, v(t)) + \eta_j = f(j/\Omega, v(j/\Omega))$, so that

$$v(t + \delta) - v(t) = \delta \cdot f(t, v(t)) + \varepsilon \cdot \eta_{\lambda-1} + \frac{1}{\Omega} \sum_{j=\lambda}^{\omega-1} \eta_j + \zeta \cdot \eta_\omega$$

$$= \delta \cdot f(t, v(t)) + \delta \cdot (\text{hyperconvex combination of infinitesimals}).$$

Since the infinitesimals are hyperconvex, we see that $f(t, v(t))$ is an SU-derivative of v.

(4.6). COROLLARY. *Let f be as in* (4.4) *and assume in addition that the range of f consists entirely of near-standard vectors or maps. Then for each standard b, $\hat{v}(t)$ is a standard solution to the standard initial value problem.*

PROOF. The closed convex envelope of a compact set in a complete space is compact; hence the *-convex combinations of $f(t, v(t))$-vectors defining $v(t)$ are all near-standard vectors and $\hat{v}(t) = st(v(t))$.

REMARK. When f is standard and $\mathbf{L} = \mathbf{F}$ $(k = 0)$ the additional hypothesis in (4.6) means f has relatively compact range (cf. Dubinsky [**1964**]). Variations on this idea are possible when a weak notion of compactness is present. Weak compactness can be used to show that the strong hull solution is standard; however weak continuity of f must be assumed. (E.g., see Knight [**1974**] for the (B)-space case.)

(4.7). DEFINITION. *An internal map f: $[a, b] \to \mathrm{Lin}^k(\,{}^*\mathbf{E};\ {}^*\mathbf{F})$ on an interval with $(b - a)$ finite is SR-integrable provided all the infinitesimal Riemann sums of f have the same hull,*

$$\int_a^b f(x)\,dx = \left(\left[\sum_{i=1}^{\Omega} f(y_i)(x_i - x_{i-1})\right]/\approx\right).$$

*An infinitesimal Riemann sum is a sum over a *-finite partition $a = x_0 < x_1 < \ldots < x_\Omega = b$; $x_i \approx x_{i-1}$ for $1 \leqslant i \leqslant \Omega$ with subordinate internal evaluations $x_{i-1} \leqslant y_i \leqslant x_i$ for $1 \leqslant i \leqslant \Omega$.*

This agrees with the standard notion of Riemann integrability when f is a standard vector valued function on a standard interval.

REMARKS. The hyperconvexity of the infinitesimals implies that S-continuous internal maps with finite range are SR-integrable. The proof of (4.3) carries over to an internal map with S-continuous partial SU-derivative. The proof of (4.4) could be interpreted in terms of an integral equation in the hull. The fundamental theorem carries over to internal functions satisfying the S-conditions by putting ^'s everywhere. More general vector and map integrals can be studied by generalized infinitesimals in $[a, b]$ (compare Stroyan and Luxemburg [**1976**, (9.2.3)]); we restrict ourselves to the classical setting here.

We can apply these ideas to obtain an extension of the theory of continuous linear semigroups. Internal finite semigroups include extensions of equicontinuous standard families and may also prove useful in *-finite probability and infinitesimal discrete partial differential equations. We plan to take up applications and a nonlinear theory elsewhere.

(4.8). DEFINITION. *A finite linear semigroup is an internal map* T: $^*[0, \infty) \to \mathrm{FLin}(^*\mathbf{E};\ ^*\mathbf{E})$ *such that* $T_0 = \mathrm{Id}$, *the identity, and* $T_{(t+s)} = T_t \circ T_s$ *for* t, $s \in [0, \infty)$. *The semigroup is (strongly) continuous at* $t \in {}^*[0, \infty)$ *if* $T_r x \approx T_t x$ *whenever* $r \approx t$ *and* $x \in {}^\sigma\mathbf{E}$. *It is S-continuous at* t *if* $(T_r - T_t) \in \mathrm{ILin}(^*\mathbf{E};\ ^*\mathbf{E})$ *when* $r \approx t$.

Note that continuity does not imply S-continuity except in (HM)-spaces (and that is characteristic of those spaces).

We shall denote the set of finite vectors x where $f(t) = T_t x$ is continuous for all finite t in $^*[0, \infty)$ by $\mathrm{Ctn}(T)$. When T is (strongly) continuous for positive finite time, $\mathrm{Ctn} \supseteq {}^\sigma\mathbf{E}$ and we shall assume that at least this much is true. When T is S-continuous on the finite numbers in $^*[0, \infty)$, then $\mathrm{Ctn} = \mathrm{fin}(^*\mathbf{E})$. Notice that the translation semigroup on $C[0, \infty]$ is not continuous at all finite vectors, for example, $\sin(\omega t) \not\approx \sin(\omega[t + \pi/2\omega])$.

When x is continuous the improper integrals defined as a sequential limit in the hull $\hat{\int}_0^\infty \exp(-\lambda t) T_t(x)\, dt$ exist in $\hat{\mathbf{E}}$ for all positive finite λ with $\lambda \not\approx 0$. Let $R_\lambda x$ be an (internal) infinitesimal Riemann sum for this integral, for example, $(1/\omega)\sum_{k=1}^{\omega\Omega} \exp(-\lambda\, k/\omega) T_{(k/\omega)} x$, where ω and Ω are infinite hyperintegers that approximate the hull integrals for all standard λ and x. The operator R_λ is finite, but different Riemann sums may yield different operators off the set of continuous vectors, Ctn.

Let δ be a positive infinitesimal and define

$$\Delta x = \frac{1}{\delta}\left[T_\delta x - x\right].$$

This δ-infinitesimal difference operator does not depend on δ over an S-dense set of continuous vectors, that is, $t \mapsto T_t x$ *is differentiable on a set of vectors* $\mathrm{Dif}(T) = \mathrm{Dif} \subseteq \mathrm{fin}(^*\mathbf{E})$ *that approximates* Ctn. When x is continuous, we know $y = R_\lambda x$ is finite and direct calculation yields

$$\Delta y \approx \left(\frac{e^{\lambda\delta} - 1}{\delta}\right)(R_\lambda x) - x \approx \lambda y - x$$

so Δ takes finite values on the ranges $R_\lambda(\mathrm{Ctn})$ for standard positive λ, independent of the particular infinitesimal δ. Simple estimates show that $nR_n x \to x$ in the standard sense for continuous x as n tends to infinity. (When n is infinite most of $nR_n x$ comes from $\int_0^\varepsilon n e^{-nt} T_t x\, dt$ where $\varepsilon \approx 0$ and there $T_t x \approx x$.) This proves our assertion about the points of differentiability of the semigroup.

The ranges of R_λ consist of continuous vectors because anywhere Δx is finite, $T_{t+\delta}x - T_t x = \delta T_t(\Delta x)$ and since T_t is a finite operator by hypothesis, $T_{t+\delta}x \approx T_t x$. This also shows that Δ maps differentiable points to continuous ones and $\partial(T_t x)/\partial t \approx T_t(\Delta x)$ is S-continuous in t.

Let x be a differentiable point and let $y = \lambda x - \Delta(x)$. We wish to justify the calculation:

$$\begin{aligned}
&\int_0^\infty e^{-\lambda t} T_t(y)\, dt \\
&\quad \approx \int_0^\infty \lambda e^{-\lambda t} T_t(x)\, dt - \frac{1}{\delta} \int_0^\infty e^{-\lambda t} T_{t+\delta}(x)\, dt + \frac{1}{\delta} \int_0^\infty e^{-\lambda t} T_t(x)\, dt \\
&\quad \approx \int_0^\infty \lambda e^{-\lambda t} T_t(x)\, dt - \frac{1}{\delta} \int_\delta^\infty e^{-\lambda(t-\delta)} T_t(x)\, dt + \frac{1}{\delta} \int_0^\infty e^{-\lambda t} T_t(x)\, dt \\
&\quad \approx \lambda \int_0^\infty e^{-\lambda t} T_t(x)\, dt - \frac{1 + \lambda \cdot \delta}{\delta} \int_\delta^\infty e^{-\lambda t} T_t(x)\, dt \\
&\qquad + \frac{1}{\delta} \int_0^\infty e^{-\lambda t} T_t(x)\, dt \\
&\quad \approx \frac{1}{\delta} \int_0^\delta e^{-\lambda t} T_t(x)\, dt \approx x.
\end{aligned}$$

When T is a standard continuous semigroup this shows that $R_\lambda(\lambda x - \Delta(x)) \approx x$ or that $\hat{R}_\lambda$ equals $[\lambda I - \mathrm{st}(\Delta)]^{-1}$. When T is only a finite continuous semigroup, additional justification is required and in particular the meaning of infinitesimal integrals needs to be clarified. Suffice it to say that we can work backwards from such an x, partition $[0, \delta]$ into a number of parts, say 1 to ω, and use Riemann sums from 1 to $\Omega\omega$, an infinite number, in place of the improper integrals. This takes us back to the Riemann sum for the first improper integral and in particular shows that it exists. This calculation shows that an internal sum for that integral, $R_\lambda y$, is an approximate left inverse of $\lambda I - \Delta$ when $\lambda I - \Delta$ is restricted to the differentiable points of the semigroup. We already saw that it is an approximate right inverse in the density argument above.

The R_λ operators nearly satisfy the resolvent equation,

$$[R_{(\lambda+\varepsilon)} - R_\lambda]x \approx -\varepsilon R_{(\lambda+\varepsilon)} R_\lambda x$$

for x in Ctn and finite positive λ

$$\begin{aligned}
R_{\lambda+\varepsilon} x &\approx R_{\lambda+\varepsilon}(\lambda I - \Delta) R_\lambda x \\
&= R_{\lambda+\varepsilon}\big(-\varepsilon I + ((\lambda + \varepsilon) I - \Delta)\big) R_\lambda x \\
&\approx -\varepsilon R_{\lambda+\varepsilon} R_\lambda x + R_\lambda x.
\end{aligned}$$

Our remarks so far are leading up to the following result.

(4.9). THEOREM. *The hull of a finite semigroup T_t restricted to the hull of its*

continuous vectors, $\hat{T}$: $[0, \infty) \to \mathrm{Lin}(\hat{\mathrm{C}}\mathrm{tn}, \hat{\mathrm{C}}\mathrm{tn})$, *is a standard continuous semigroup. The pointwise hull of* Δ *defined by* $\hat{\Delta}(\hat{x}) = (\Delta(x))\hat{}$ *for* $x \in \mathrm{Dif}(T)$ *is the standard generator of* $\hat{T}$. (Δ *is defined above.*) *The hulls of the* $\hat{R}_\lambda$ (*defined above*) *form the resolvent of* $\hat{\Delta}$ *for standard* λ.

PROOF. Since T_t is a finite operator and Ctn is invariant, the hulls and restriction are well defined. The internal operators R_λ are also finite and leave Ctn invariant. Since we showed above that Δ sends Dif into Ctn, in order to show that $\hat{\Delta}$ is well defined we only need to show that if $x, y \in \mathrm{Dif}$ and $x \approx y$, then $\Delta(x) \approx \Delta(y)$. By the linearity it is sufficient to show that if $z \approx 0$ and $z \in \mathrm{Dif}$, then $\Delta(z) \approx 0$. We do this now.

Let $w = \Delta(z)$ and $\lambda \in {}^\sigma\mathbf{R}^+$. We know $\lambda R_\lambda(\lambda I - \Delta)z \approx \lambda z \approx 0$, from remarks preceding the theorem. Also, $\lambda R_\lambda z \approx 0$, since R_λ is finite and $z \approx 0$. Now $\lambda \hat{\int}_0^\infty e^{-\lambda t} T_t w \, dt \approx 0$ for all standard λ and $\lambda R_\lambda w \to w$ as $\lambda \to \infty$, hence $\Delta(z) = w \approx 0$. Thus we have shown that $\hat{\Delta}$ is pointwise well defined.

REMARKS. The hull of the translation semigroup on $C[0, \infty]$ is itself. A continuous vector is a uniformly S-continuous finite valued internal function. The hull of such a function is a standard continuous function in $C[0, \infty]$ (that is, in $\{{}^\sigma C[0, \infty]\}\hat{}$).

The hull of the semigroup on $C[-\infty, \infty]$ given by

$$(T_t x)(s) = e^{-\lambda t} \sum_{k=0}^{\infty} \frac{(\lambda t)^k}{k!} x(s - k\mu)$$

is defined on the whole infinitesimal hull $\hat{C}[-\infty, \infty]$, since this semigroup is S-continuous.

We could prove an internal version of the representation theorem for semigroups in terms of their infinitesimal generators via an exponential series or alternately apply the standard result to $\hat{T}_t$ and $\hat{\Delta}$. We omit the details.

ACKNOWLEDGEMENT. The author is grateful to Professor L. C. Moore, Jr. for a number of valuable suggestions.

The author was supported by NSF grant MCS 76-06634 during part of the preparation of the manuscript.

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