PARACOMPACTNESS OF BOX PRODUCTS
OF COMPACT SPACES\(^1\)

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Abstract. We consider box products of countably many compact Hausdorff spaces. Under the continuum hypothesis, the product is paracompact if its Lindelöf degree is no more than the continuum; in particular, the product is paracompact if each space has weight continuum or less, or if each space is dispersed. Some partial results are proved under Martin's axiom.

0. Introduction. All our spaces are \(T_3\) (regular Hausdorff). The box topology on \(\prod_n X_n\) has as a basis arbitrary products of open sets. Such a product of compact spaces is compact only in trivial cases, but it may be paracompact. By van Douwen \([vD\ 2]\) there are compact \(X_n\) of weight \(\omega_2\) such that \(\prod_n X_n\) is not even normal. However, Rudin has shown that the continuum hypothesis (CH) implies that \(\prod_n X_n\) is paracompact if each \(X_n\) is compact metric \([R\ 1]\) or if each \(X_n\) is a compact ordinal \([R\ 2]\).

In this paper, we obtain easier proofs of some stronger positive results. Under CH, when each \(X_n\) is compact, \(\prod_n X_n\) is paracompact iff it is \(\omega_1\)-Lindelöf (see 3.2). In particular if each \(X_n\) has weight \(<\omega_1\), \(\prod_n X_n\) is paracompact under CH (4.1); under Martin's axiom, we can modify our arguments to establish this if we assume also that the \(X_n\) are first countable (4.4). In §5, we show that if each \(X_n\) is a compact dispersed space, \(\prod_n X_n\) is \(c\)-Lindelöf and hence, under CH, paracompact. §1 reviews some general results on paracompact spaces, and §2 applies them to box products. §6 extends the results of §4 and §5 to products of locally compact paracompact spaces.

1. Paracompactness. We collect here some known results and easy consequences thereof.

1.1. Theorem (Michael [M\ 1]). The following are equivalent:
(a) \(X\) is paracompact.
(b) Every open cover of $X$ has a $\sigma$-locally finite open refinement.
(c) Every open cover of $X$ has a $\sigma$-discrete open refinement.

We remind the reader that all our spaces are $T_3$, an assumption needed in the theorem. A corollary of the fact that (b) implies (a) is that Lindelöf spaces are paracompact. We shall need a generalization of this, which we state after recalling some definitions.

1.2. Definition. Let $\kappa$ be an infinite cardinal.
(1) $X$ is $\kappa$-Lindelöf iff every open cover of $X$ has a subcover of cardinality $< \kappa$.
(2) $X$ is $\kappa$-open iff the intersection of less than $\kappa$ open sets in $X$ is open.

Thus, $\omega$-Lindelöf = Lindelöf, and all spaces are $\omega$-open. The fact that Lindelöf spaces are paracompact is the following lemma for $\kappa = \omega$.

1.3. Lemma. If $X$ is $\kappa$-open and $\kappa$-Lindelöf, then $X$ is paracompact.

Proof. If $\kappa > \omega$, then $X$ is zero dimensional, since it is regular and $\omega_1$-open. Let $\mathcal{U}$ be an open cover of $X$. Then $\mathcal{U}$ has a refinement $\{V_\alpha: \alpha < \kappa\}$, where each $V_\alpha$ is clopen. Let

$$W_\alpha = V_\alpha - \bigcup \{V_\beta: \beta < \alpha\}.$$  

Then $\{W_\alpha: \alpha < \kappa\}$ is a disjoint clopen refinement of $\mathcal{U}$.

We shall also need the following result on preservation of paracompactness under closed maps.

1.4. Lemma. Let $F$ be a closed continuous map from $X$ onto $Y$. Then
(1) If $X$ is paracompact, so is $Y$.
(2) If $Y$ is paracompact and $F^{-1}(y)$ is Lindelöf for each $y \in Y$, then $X$ is paracompact.

Proof. For (1), see Michael [M 2]. For (2), let $\mathcal{U}$ be an open cover of $X$. For each $y \in Y$, choose a countable subfamily of $\mathcal{U}$, $\{U'_n: n \in \omega\}$, which covers $F^{-1}(y)$. Let

$$V^y = \left\{ z \in Y: F^{-1}\{z\} \subset \bigcup_{n \in \omega} U'_n \right\}.$$  

Then $y \in V^y$ and $V^y$ is open since $F$ is closed. Let $\{W^y: y \in Y\}$ be a locally finite open refinement of $\{V^y: y \in Y\}$ which is precise (i.e., $W^y \subseteq V^y$). Let $\mathcal{R}_n$ be the family $\{U'_n \cap F^{-1}(W^y): y \in Y\}$. Then $\{\mathcal{R}_n: n \in \omega\}$ forms a $\sigma$-locally finite open refinement of $\mathcal{U}$. Thus, $X$ is paracompact by Theorem 1.1.

We remark that (2) is well known in the case that each $F^{-1}(y)$ is compact, and the proof above is an easy modification of the standard proof for that case.
Lindelöf degree has the same preservation property, with a similar, but easier proof.

1.5. Lemma. Let $F$ be a closed continuous map from $X$ onto $Y$, such that $F^{-1}(y)$ is Lindelöf for each $y \in Y$. Then for any infinite $\kappa$, $Y$ is $\kappa$-Lindelöf iff $X$ is.

2. The reduced box product. We shall define a certain reduced product, $\Pi_n X_n/\sim$. This will be easier to deal with than $\Pi_n X_n$, as it is $\omega_1$-open. However, for compact $X_n$, $\Pi_n X_n$ and $\Pi_n X_n/\sim$ will have similar paracompactness and Lindelöf properties. The reduced product was first defined by Knight [Kn], and was applied by Rudin [R 2] to obtain information about box products.

$\Pi_n X_n/\sim$ is defined as follows. If $p, q \in \Pi_n X_n$, we say $p$ is equivalent to $q$ ($p \sim q$) iff $p(n) = q(n)$ for all but finitely many $n$. Let $E(p) = \{q: q \sim p\}$, and, for $A \subseteq \Pi_n X_n$, $E(A) = \bigcup \{E(p): p \in A\}$. We give the quotient space $\Pi_n X_n/\sim = \{E(p): p \in \Pi_n X_n\}$ the usual quotient topology and let $\sigma: \Pi_n X_n \to \Pi_n X_n/\sim$ be the usual projection. So $\sigma(p) = E(p)$; we use $\sigma(p)$ when we are thinking of a point in $\Pi_n X_n/\sim$ and $E(p)$ when we are thinking of a subset of $\Pi_n X_n$.

It is easy to see that each $E(p)$ is closed. Also, the map $\sigma$ is open, so that the $\sigma(U)$ for $U$ basic in $\Pi_n X_n$ form a base for $\Pi_n X_n/\sim$. We shall rely heavily on the following

2.1. Theorem (Rudin). $\Pi_n X_n/\sim$ is $\omega_1$-open.

2.2. Theorem. If each $X_n$ is compact, the map $\sigma$ is closed.

Before proving these, we remark that the interest of 2.2 is in its corollary,

2.3. Corollary. If each $X_n$ is compact, then
(a) $\Pi_n X_n$ is paracompact iff $\Pi_n X_n/\sim$ is.
(b) For any infinite $\kappa$, $\Pi_n X_n$ is $\kappa$-Lindelöf iff $\Pi_n X_n/\sim$ is.

Proof. Point inverses under $\sigma$ are $\sigma$-compact, and hence Lindelöf, so 1.4 and 1.5 apply.

We turn now to the proofs of Theorems 2.1 and 2.2. 2.1 is essentially proved in [R 2], but we include a proof to show that 2.1 and 2.2 actually both follow from the same Lemma 2.4. For each $k$, let $\tau_k$ be the canonical projection from $\Pi_n X_n = \Pi_{n<k} X_n \times \Pi_{n\geq k} X_n$ onto $\Pi_{n\geq k} X_n$.

2.4. Lemma. If, for each $k$, $U^k$ is an open subset of $\Pi_{n\geq k} X_n$, then $\bigcap \{\tau_k^{-1}(U^k): k < \omega\}$ is open in $\Pi_n X_n$.

Proof. Fix $p \in \bigcap \{\tau_k^{-1}(U^k): k < \omega\}$. Then for each $k$, there is a neighborhood $V^k$ of $p$ contained in $\tau_k^{-1}(U^k)$, which may be taken to be of the form
\[ V^k = X_0 \times X_1 \times \cdots \times X_{k-1} \times V^k_k \times V^k_{k+1} \times V^k_{k+2} \times \ldots, \]
each \( V^k_n (n > k) \) being open in \( X_n \). Let \( W_n = \cap \{ V^k_k : k < n \} \), and \( W = \cap \{ \Pi_n W_n \} \). Then \( W = \cap \{ V^k : k < \omega \} \) is a neighborhood of \( p \) contained in each \( \tau^{-1}(U^k) \).

To deduce 2.1 from 2.4, let \( \sigma_k \) be the map from \( \Pi_{n>k}^\times X_n \) onto \( \Pi_{n} X_n / \sim \) such that \( \sigma = \sigma_k \circ \tau_k \). If \( V^k \) is open in \( \Pi_{n>k}^\times X_n / \sim \) for each \( k < \omega \), then \( \sigma_k^{-1}(V^k) \) is open in \( \Pi_{n>k}^\times X_n \), so \( \sigma^{-1}(\cap \{ V^k_k : k < \omega \}) \), which is equal to \( \cap \{ \tau_{k+1}^{-1}(\tau_k^{-1}(V^k)) : k < \omega \} \), is open in \( \Pi_{n} X_n / \sim \), so \( \cap \{ V^k : k < \omega \} \) is open in \( \Pi_{n} X_n / \sim \).

To deduce 2.2, note that showing \( \sigma \) is closed is equivalent to showing that whenever \( K \) is closed in \( \Pi_{n} X_n \), so is \( \sigma^{-1}K \). But this set is just \( \cup \{ \tau_{k}^{-1}(\tau_{k}K) : k < \omega \} \). Hence, if each \( \tau_{k}K \) is closed in \( \Pi_{n>k}^\times X_n \), then by 2.4, \( \sigma^{-1}K \) is closed. But each \( \tau_{k}K \) is closed, since projection from a compact factor is a closed map.

3. On paracompactness and the Lindelöf degree. Under CH, our main tool for proving paracompactness is

3.1. Theorem (CH). If each \( X_n \) is compact, then \( \Pi_{n} X_n \) is paracompact iff it is \( \omega_1 \)-Lindelöf.

Proof. If \( \Pi_{n} X_n \) is \( \omega_1 \)-Lindelöf, then so is \( \Pi_{n} X_n / \sim \) by 2.3. Since \( \Pi_{n} X_n / \sim \) is also \( \omega \)-open by 2.1, it is paracompact by 1.3, so \( \Pi_{n} X_n \) is paracompact by 2.3.

The other direction is more difficult and is not needed for the rest of this paper. We shall in fact establish the following, without CH.

3.2. Theorem. If each \( X_n \) is compact and \( \Pi_{n} X_n \) is paracompact, then it is \( c \)-Lindelöf.

We remark that we know of no example, under any set-theoretic axioms, of a box product of compact spaces which is normal but not paracompact.

To prove 3.2, we need the following unpublished result of Arhangel’skii:

3.3. Theorem. If \( Y \) is compact and \( \mathcal{B} \) is a cover of \( Y \) by closed \( G_n \) sets and \( \mathcal{B} \) satisfies

\[ \forall H \in \mathcal{B} \{ \{ K \in \mathcal{B} : H \cap K \neq 0 \} \} < c, \]
\( (*) \)

then \( |\mathcal{B}| < c \).

Of course, if the elements of \( \mathcal{B} \) are points, 3.3 is just the well-known result of [A], a simple proof of which is due to Pol [P]. Pol’s proof can easily be modified to yield 3.3.

We wish to apply 3.3 with \( Y \) the product of the spaces \( X_n \) under the usual (Tychonov) topology. To do this we need

3.4. Lemma. Let \( Z \) be paracompact and \( \mathcal{B} \) any base for \( Z \). Then every open
cover $\mathcal{U}$ for $Z$ has a refinement to a cover $\mathcal{F}$ such that

1. Elements of $\mathcal{F}$ are of the form $\cap_n B_n$, where each $B_n \in \mathcal{B}$, and $B_0 \supseteq cl(B_1) \supseteq B_1 \supseteq cl(B_2) \supseteq \ldots$.

2. $\mathcal{F}$ satisfies $(\ast)$.

We apply 3.4 with $Z$ the box product $\prod_n X_n$ and $\mathcal{B}$ the usual base. Then elements of $\mathcal{F}$ will be closed $G_\delta$ sets in $Y$, so $|\mathcal{F}| < c$ by 3.3. Thus, if $Z$ is paracompact, it is $c$-Lindelöf.

To prove 3.4, we apply paracompactness $\omega$ times to produce a sequence of covers $\mathcal{V}_n (n \in \omega)$ of $Z$ with $\mathcal{V}_0 = \mathcal{U}$ and

(a) Each $\mathcal{V}_{n+1}$ is a locally finite open refinement of $\mathcal{V}_n$.

(b) If $V_{n+1} \in \mathcal{V}_{n+1}$, $V_{n+1}$ intersects only finitely many members of $\mathcal{V}_n$.

(c) If $V_{n+1} \in \mathcal{V}_{n+1}$, there is a $B \in \mathcal{B}$ and $V_n \in \mathcal{V}_n$ with $V_{n+1} \subseteq B \subseteq cl(B) \subseteq V_n$.

Let $\mathcal{F}$ be the set of all sets of the form $\cap_n V_n = \cap_n B_n$, where each $V_n \in \mathcal{V}_n$, $B_n \in \mathcal{B}$, and $V_0 \supseteq cl(B_0) \supseteq B_0 \supseteq V_1 \supseteq \ldots$. $\mathcal{F}$ satisfies $(\ast)$ by

4. Trivial applications of §§1–3. Theorem 3.1 yields immediately:

4.1. Theorem (CH). If each $X_n$ is compact and of weight $< \omega_1$, then $\prod_n X_n$ is paracompact.

We would call an application of §§1–3 nontrivial if the Lindelöf degree of $\prod_n X_n/\sim$ is established by some bound other than its weight. For example, in §5 we show that if each $X_n$ is a compact dispersed space, $\prod_n X_n/\sim$ is $c$-Lindelöf.

We look now at the situation under Martin’s axiom (MA). See Jech [J] for a complete discussion of MA, including a proof of its consistency with $\neg$ CH. We need here only the well-known

4.2. Lemma. Assume MA. If $\kappa < c$ and $f_\alpha (\alpha < \kappa)$ are functions from $\omega$ into $\omega$, then there is a $g: \omega \rightarrow \omega$ such that for each $\alpha$, $g(n) > f_\alpha (n)$ for all but finitely many $n$.

This gives us, in analogy with Theorem 2.1,

4.3. Theorem. If MA and each $X_n$ is first countable, then $\prod_n X_n/\sim$ is $c$-open.

Proof. Fix $\sigma(p) \in \prod_n X_n/\sim$ and let $U_\alpha (\alpha < \kappa)$ be neighborhoods of $\sigma(p)$, where $\kappa < c$. We show that $\cap \{ U_\alpha : \alpha < \kappa \}$ is a neighborhood of $\sigma(p)$. For
each $n$, let $\{V^n_k : k < \omega\}$ be a base at $p(n)$ in $X_n$. For each $\alpha$, let $f_\alpha : \omega \to \omega$ be such that

$$\sigma \left( \prod_n V^n_{f_\alpha(n)} \right) \subseteq U_\alpha.$$ 

Then if $g$ is as in 4.2, $\sigma(\prod_n V^n_{g(n)})$ is an open neighborhood of $\sigma(p)$ contained in $\bigcap \{U_\alpha : \alpha < \kappa\}$.

Then, analogously to 4.1, we have the following, whose proof is obtained essentially by replacing $\omega_1$ by $c$ in the proof of the easy direction of 3.1.

4.4. Theorem (MA). If each $X_n$ is compact and first countable, then $\prod_n X_n$ is paracompact.

Proof. By Arhangel'skiï [A], each $X_n$ has cardinality $< c$, and thus weight $< c$, so $\prod_n X_n/\sim$ has weight $< c$, so by 4.3, $\prod_n X_n/\sim$, and hence also $\prod_n X_n$ is paracompact.

We do not know whether CH is required for Theorem 4.1, or MA for 4.4. van Douwen [vD 3] has shown that in 4.4, if each $X_n$ is compact metric, then MA may be weakened to the existence of a $\kappa$-scale in $\omega^\omega$ for some $\kappa$. The proof shows that in this case, $\prod_n X_n/\sim$ is $\kappa$-metrizable.

Under $\neg$CH, there is no criterion like Theorem 3.1 for paracompactness in terms of Lindelöf degree. By van Douwen [vD 2], there are always compact $X_n$ of weight $\omega_1$, such that $\prod_n X_n$ is not even normal; under $\neg$CH, this box product would have weight $c$ and hence Lindelöf degree $c$ (since no nontrivial box product has Lindelöf degree $< c$). But the box product of 2-point spaces has the same Lindelöf degree $c$ and is paracompact (in fact discrete).

5. Products of compact dispersed spaces. Rudin [R 2] shows that under CH, if each $X_n$ is a compact ordinal (i.e., a successor ordinal with the order topology), then $\prod_n X_n$ is paracompact. An easier proof of this may be obtained by first establishing without any set-theoretic assumptions, that $\prod_n X_n$ is $c$-Lindelöf, and then quoting 3.1. Following a suggestion of Scott Williams, we can also generalize this to dispersed spaces:

5.1. Theorem. If each $X_n$ is a compact dispersed space, then $\prod_n X_n$ is $c$-Lindelöf.

Here, $X$ is dispersed iff each subspace $Y$ contains an isolated (in $Y$) point. Ordinals are dispersed, since the first point in any subspace of an ordinal is isolated.

For any $X$, one can form the Cantor-Bendixon sequence $X^{(\alpha)}$, where $X^{(0)} = X$, $X^{(\alpha+1)} = \{x \in X^{(\alpha)} : x$ is not isolated in $X^{(\alpha)}\}$, and $X^{(\gamma)} = \bigcap \{X^{(\alpha)} : \alpha < \gamma\}$ for $\gamma$ a limit. Then each $X^{(\alpha)}$ is closed in $X$, the $X^{(\alpha)}$ are nonincreasing, and, if $Y \subseteq X$, $Y^{(\alpha)} \subseteq X^{(\alpha)}$ for all $\alpha$. $X$ is dispersed iff $X^{(\alpha)} = 0$ for some $\alpha$. If $X$ is compact dispersed, then the first $\alpha$ such that
$X^{(\alpha)} = 0$ is a successor ordinal, $\beta + 1$, and $X^{(\beta)}$ is finite; we call $\beta$ the rank of $X$.

To prove Theorem 5.1, let $\mathcal{U}$ be an open cover of $\prod_n X_n$. Following [R 2], we use a tree to index our attempts to refine $\mathcal{U}$, but we look for a closed rather than an open refinement.

Let $T = c^\omega = \bigcup \{ c^\xi : \xi < \omega_1 \}$. Elements $s \in T$ are viewed as $c$-ary sequences of countable ordinal length, so $T$ is the complete $c$-ary tree of height $\omega_1$. We write $\text{lh}(s)$ for the length (domain) of $s$, and, if $\eta < \text{lh}(s)$, $s \upharpoonright \eta$ is the sequence of length $\eta$ formed by restricting $s$ to $\eta$. If $s \in T$ and $\mu \in c$, then $s_\mu$ is the sequence of length $\text{lh}(s) + 1$ which begins with $s$ and ends with the element $\mu$. 0 is the empty sequence, of length 0.

$K \preceq \mathcal{U}$ means $\exists U \in \mathcal{U}$ ($K \subseteq U$). We shall define, by induction on $\text{lh}(s)$, $K(s) = \prod_n K_n(s)$, a closed box in $\prod_n X_n$. We shall then check that $\forall p \in \prod_n X_n \exists s \in T (p \in K(s) \preceq \mathcal{U})$, which, since $T$ has cardinality $c$, will imply that $\mathcal{U}$ has a subcover of cardinality $c$.

$K(s)$ will be defined so that $K(0) = \prod_n X_n$, and, for each $s \in T$,
(1) $K(s) = \bigcup \{ K(s_\mu) : \mu < c \}$.
(2) If $\text{lh}(s)$ is a limit, $K(s) = \bigcap \{ K(s \upharpoonright \xi) : \xi < \text{lh}(s) \}$.
(3) For each $\mu < c$, either $K(s_\mu) \preceq \mathcal{U}$ or $\exists n [\text{rank}(K_n(s_\mu)) < \text{rank}(K_n(s))]$.

By (1) and (2), if $p \in \prod_n X_n$, there is an $f \in c^\omega$ with $p \in K(f \upharpoonright \xi)$ for all $\xi < \omega_1$. Then, for each $n$, the ranks of the $K_n(f \upharpoonright \xi)$ are nonincreasing and hence eventually constant. If they are constant past $\xi$ for each $n$, then (3) insures that $K(f \upharpoonright \xi) \preceq \mathcal{U}$. Hence, we shall be done if (1)–(3) can be accomplished.

To define $K(s)$, we proceed by induction, taking intersections at limits, so that (2) holds by definition. For a successor stage, fix $s$. Let $\beta_n = \text{rank}(K_n(s))$ and $Y_n = (K_n(s))^{(\beta_n)}$. Then $Y_n$ is finite. Let $\mathcal{V}$ be a family of open boxes of cardinality $c$ such that $\mathcal{V}$ covers $\prod_n Y_n$ and $\text{cl}(\mathcal{V}) \preceq \mathcal{U}$ for all $\mathcal{V} \in \mathcal{V}$. This is possible since $|\prod_n Y_n| < c$. Let $K(s_\mu)$, for $\mu < c$, enumerate all the $c$ boxes $K = \prod_n K_n$ such that either

(a) $K = \text{cl}(\mathcal{V}) \cap K(s)$ for some $\mathcal{V} \in \mathcal{V}$ or
(b) for some $n$,
   (i) $K_n = K_n(s)$ and $(V^1_n \cup \cdots \cup V^j_n)$ for some $V^1 \ldots V^j \subseteq Y_n \subseteq V^1_n \cup \ldots \cup V^j_n$ and
   (ii) $K_m = K_m(s)$ for all $m \neq n$.

Then (3) holds, since (b)(i) implies that $(K_n)^{\langle \beta_n \rangle} = 0$, whence $\text{rank}(K_n) < \beta_n$. To check (1), fix $p \in K(s)$. If $p$ fails to be in any of the $K(s_\mu)$ of type (b) then, since $Y_n$ is finite, for each $n$ there must be a $q_n \in Y_n$ with $\forall V \in \mathcal{V}$ ($q_n \in V_n \rightarrow p_n \in V_n$). Let $q = \langle q_0 q_1 \ldots \rangle \in \prod_n Y_n$. Then $\forall \mathcal{V} \in \mathcal{V}$ ($q \in \mathcal{V} \rightarrow p \in V$), so, taking $V$ containing $q$, $p$ is in $\text{cl}(\mathcal{V}) \cap K(s)$, which is one of the $K(s_\mu)$ of the type (a).
Thus, the $K(s)$ may indeed be defined to satisfy (1)–(3), which completes
the proof of 5.1.

5.1 can be improved by using a modified Cantor-Bendixon analysis. Let
$w(Z)$ denote the weight of the space $Z$. For any $X$, let $X' = X - \bigcup \{N : N$ is
open and $w(N) < c\}$; let $X^{[\alpha]} = X, X^{[\alpha + 1]} = (X^{[\alpha]})'$, and take intersections
at limits as before. Call $X$ almost dispersed iff $X^{[\alpha]}$ is empty for some $\alpha$
(equivalently, iff every subspace contains a neighborhood of weight $< c$).
The class of almost dispersed spaces contains all dispersed spaces and all
spaces of weight $< c$, and is closed under countable box products.

5.2. Theorem. If each $X_n$ is compact and almost dispersed, then $\Pi_n X_n$ is
c-$\text{Lindelöf}$.

The proof is as for 5.1, with the appropriate modification in the notion of
rank. In the justification for the existence of $\mathcal{V}$, one uses that $\Pi_n Y_n$ has
weight $< c$, rather than cardinality $< c$. In the argument establishing (1), use
the fact that the $Y_n$ are compact, rather than finite.

Then, by 3.1, we have

5.3. Theorem (CH). If each $X_n$ is compact and almost dispersed, then $\Pi_n X_n$ is
paracompact.

6. Products of locally compact spaces. Compactness can be replaced by local
compactness plus paracompactness in the positive results of §4 and §5 by

6.1. Theorem. Assume that each $X_n$ is paracompact and has an open cover
$\mathcal{U}_n$ such that whenever $U_n \in \mathcal{U}_n (n \in \omega)$, $\Pi_n \text{cl}(U_n)$ is paracompact. Then $\Pi_n X_n$
is paracompact.

Thus, e.g., by 4.1, 4.4 and 5.3, 6.2. Corollary. Assume each $X_n$ is a paracompact and locally compact.
(a) (CH) If each $X_n$ is locally of weight $< \omega_1$, then $\Pi_n X_n$ is paracompact.
(b) (MA) If each $X_n$ is first countable, then $\Pi_n X_n$ is paracompact.
(c) (CH) If each $X_n$ is dispersed, then $\Pi_n X_n$ is paracompact.

The assumption of paracompactness of the $X_n$ cannot be dropped from
these results, since each $X_n$ is homeomorphic to a closed subspace of $\Pi_n X_n$.
Also, under CH, $\omega_1 \times (\omega + 1)^\omega$ is not even normal (see [Ku]). A special case
of 6.2(b) is that under MA, the box product of locally compact metric spaces
is paracompact. Under CH, this is essentially due to Rudin [R 1], as was
pointed out by van Douwen [vD 1]. [vD 1] also shows that the box product of
non-locally-compact separable metric spaces need not be normal.

To prove 6.1, note first that we can, by criterion (c) of Michael's Theorem
1.1, assume that each $\mathcal{U}_n$ is countable; if not, we pass to $\sigma$-discrete refine-
ments of the $\mathcal{U}_n$ and note that a discrete union of paracompact spaces is
paracompact. Again using paracompactness of the $X_n$, we may assume that each $\mathcal{U}_n$ is locally finite.

Say $\mathcal{U}_n = \{U^k_n: k < \omega\}$. By normality, we can find closed $F^k_n \subseteq U^k_n$ such that $\bigcup_k F^k_n = X_n$. Then, there are open $V^{k,i}_n$ ($i < \omega$) such that

$$U^k_n \supseteq \text{cl} V^{k,0}_n \supseteq V^{k,i}_n \supseteq \text{cl} V^{k,i+1}_n \supseteq \cdots \supseteq F^k_n.$$ 

Whenever $f \in \omega^\omega$, let

$$A_f = \bigcap_i E \left( \prod_n V^{f(n),i}_n \right) = \bigcap_i E \left( \prod_n \text{cl} V^{f(n),i}_n \right).$$

Then the $A_f$ cover $\Pi_n X_n$. Each $A_f$ is closed, as it is the intersection of closed sets, and it is open, as it is the intersection of $\omega$ open sets and $\Pi_n X_n / \sim$ is $\omega_1$-open.

Furthermore, each $A_f$ is paracompact in its relative topology. To see this, note that $A_f$ is covered by the interiors of the $\prod_n \text{cl}(U^g_n)$, where $g$ ranges over the countably many elements of $\omega^\omega$ which are eventually equal to $f$. Also, each $\prod_n \text{cl}(U^g_n)$ is paracompact by assumption. Paracompactness of $A_f$ now follows immediately from the following easy general fact:

6.3. Lemma. If $Y = \bigcup j \text{int}(K_j)$, where each $K_j$ is closed and paracompact, then $Y$ is paracompact.

Proof. If $\mathcal{U}$ is an open cover of $Y$, for each $j$ let $\mathcal{V}_j$ be a family of open (in $K_j$) subsets of $K_j$ which refines $\mathcal{U}_j$, covers $K_j$, and is locally finite in $K_j$. Let $\mathcal{W}_j = \{V \cap \text{int}(K_j): V \in \mathcal{V}_j\}$. $\mathcal{W}_j$ is a locally finite and open (in $Y$) refinement of $\mathcal{U}$ which covers $\text{int}(K_j)$ so the $\mathcal{W}_j$, for $j < \omega$ form a $\sigma$-locally finite open refinement of $\mathcal{U}$ which cover $Y$. Hence by Michael's Theorem 1.1, $Y$ is paracompact.

Finally, the family of $A_f$ for $f \in \omega^\omega$ is closure-preserving; equivalently, since the $A_f$ are closed, each $p$ has a neighborhood $W$ which is disjoint from all the $A_f$ not containing $p$. To see this, let $W = \prod_n W_n$, where

$$W_n = X_n - \bigcup \{ \text{cl} V^{k,i}_n: k, i \in \omega \text{ and } p(n) \notin \text{cl} V^{k,i}_n \}.$$ 

$W_n$ is open since $\mathcal{U}_n$ is locally finite. If $p \notin A_f$, then for some $i$, $p \notin E(\prod_n \text{cl} V^{f(n),i}_n)$. Thus, $W_n \cap \text{cl} V^{f(n),i}_n = 0$ for infinitely many $n$, so $W \cap A_f = 0$.

Thus, if we let $f_\alpha (\alpha < \omega)$ enumerate $\omega^\omega$ and let

$$B_\alpha = A_{f_\alpha} - \bigcup \{ A_{f_\beta}: \beta < \alpha \},$$

then $\Pi_n X_n$ is expressed as $\bigcup_\alpha B_\alpha$, a disjoint union of clopen paracompact subspaces, and is hence paracompact.

References


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