ABSTRACT. Klingenberg planes are generalizations of Hjelmslev planes. If $R$ is a local ring, one can construct a projective Klingenberg plane $V(R)$ and a derived affine Klingenberg plane $A(R)$ from $R$. If $V$ is a projective Klingenberg plane, if $R_1$, $R_2$ and $R_3$ are local rings, if $s_1$, $s_2$ and $s_3$ are the sides of a nondegenerate triangle in $V$, and if each of the derived affine Klingenberg planes $A(V, s_i)$ is isomorphic to $A(R_i)$, then the rings $R_1$, $R_2$ and $R_3$ are isomorphic, and $V$ is isomorphic to $V(R_i)$; also, if $g$ is a line of $V$, then the derived affine Klingenberg plane $A(V, g)$ is isomorphic to $A(R)$ whenever $m = b$, or $m$ is a line through $B$ which is not neighbor to $b$.

1. Desarguesian projective Klingenberg planes.

1.1 Definition. Let $S_V = (\mathcal{B}, \gamma, \Pi)$ be an incidence structure, and let $\sim$ be an equivalence relation on the points and lines of $S_V$ such that no point is related to any line. We say that $V = (S_V, \sim)$ is a projective Klingenberg plane (abbreviated PK-plane) and that $\sim$ is the neighbor relation of $V$ whenever there is a map $\varphi: V \rightarrow V'$ satisfying the following condition.

(PK) $\varphi$ is a surjective incidence structure homomorphism from $V$ to a projective plane $V' = (S_{V'}, \sim')$ where $\sim'$ is the equality relation on the points and lines of $S_{V'}$ such that the following four conditions are satisfied for all $P, Q \in \mathcal{B}; g, h \in \mathcal{G}$.

(i) If $\varphi P \neq \varphi Q$, then there is a unique line of $V$ joining $P$ and $Q$. We denote this line by $PQ$ or $P \n V Q$.

(ii) If $\varphi g \neq \varphi h$, then $g$ and $h$ meet in a unique point of $V$. We denote this point by $g \cap h$.

(iii) $P \sim Q \iff \varphi P = \varphi Q$.

(iv) $g \sim h \iff \varphi g = \varphi h$.

Remark. It is easily seen that any projective plane with neighbor elements [12, D0] together with its neighbor relation is a projective Klingenberg plane. Projective planes with neighbor elements are frequently called projective...
Hjelmslev planes. Lenz and Drake [14] give a slightly different definition of (projective) Klingenberg plane.

1.2 Definition. Let $R$ be a local ring with maximal ideal $N$. If $x_1, x_2, x_3 \in R$, let $r(x_1, x_2, x_3)$ denote the set $\{(a_1, a_2, a_3) \mid 3 \in R \setminus N \}$ for $j = 1, 2, 3$. If $y_1, y_2, y_3 \in R$, let $(y_1, y_2, y_3)s$ denote the set $\{(b_1, b_2, b_3) \mid 3 \in R \setminus N \}$ for $j = 1, 2, 3$. Let $\mathfrak{B} = \{r(x_1, x_2, x_3) \mid 3 \in R \setminus N \}$ and let $g = \{(y_1, y_2, y_3)s \mid 3 \in R \setminus N \}$. Define incidence by $r(x_1, x_2, x_3) I (y_1, y_2, y_3)s$ if and only if $x_1y_1 + x_2y_2 + x_3y_3 = 0$. Incidence can be seen to be well defined. We frequently write $r(x_1, x_2, x_3)$ as $r_x$, and $(y_1, y_2, y_3)s$ as $y_s$. Let $\mathcal{S}_\mathfrak{B}(R) = (\mathfrak{B}, g, I)$. If $\theta: R \to R'$ is a local ring homomorphism, let $\mathcal{S}_\mathfrak{B}(\theta): \mathcal{S}_\mathfrak{B}(R) \to \mathcal{S}_\mathfrak{B}(R')$ be defined by $\mathcal{S}_\mathfrak{B}(\theta)(r_x) = r(\theta x_1, \theta x_2, \theta x_3)$, and similarly for lines. Let $\nu: R \to R/N$ be the quotient map. Define a relation $\sim$ by letting $r_x \sim r_a$ whenever $\mathcal{S}_\mathfrak{B}(\nu)(r_x) = \mathcal{S}_\mathfrak{B}(\nu)(r_a)$ and $y_s \sim b_s$ whenever $\mathcal{S}_\mathfrak{B}(\nu)(y_s) = \mathcal{S}_\mathfrak{B}(\nu)(b_s)$. Let $\mathcal{V}(R) = (\mathcal{S}_\mathfrak{B}(R), \sim)$. If $\theta: R \to R'$ is a local ring homomorphism, let $\mathcal{V}(\theta): \mathcal{V}(R) \to \mathcal{V}(R')$ have the same action on points and lines of $\mathcal{V}(R)$ that $\mathcal{S}_\mathfrak{B}(\theta)$ does.

1.3 Proposition [5, Proposition 5.2.3]. Let $R$ be a local ring with maximal ideal $N$, and let $\nu: R \to R/N$ be the quotient map; then $\mathcal{V}(R)$ is a PK-plane and $\mathcal{V}(\nu): \mathcal{V}(R) \to \mathcal{V}(R/N)$ satisfies condition (PK).

1.4 Definition. Let $V, V'$ be PK-planes. We define a PK-plane homomorphism $\omega: V \to V'$ to be an incidence structure homomorphism which preserves the neighbor relation; that is, $P I g$ in $V$ implies $\omega P I \omega g$ in $V'$; $P \sim Q$ in $V$ implies $\omega P \sim \omega Q$ in $V'$, and $g \sim h$ in $V$ implies $\omega g \sim \omega h$ in $V'$.

1.5 Definition. Let $B$ be an invertible $3 \times 3$ matrix over a local ring $R$. We define a map $\sigma_B: \mathcal{V}(R) \to \mathcal{V}(R)$ by $\sigma_B(r_x) = r((x_1, x_2, x_3)B)$ and $\sigma_B(y_s) = ((B^{-1}(y_1, y_2, y_3))^t)s$ where $^t$ is the transpose operator. Observe that $\sigma_B$ is well defined. We call $\sigma_B$ a projective map of $\mathcal{V}(R)$.

1.6 Proposition [5, Proposition 5.2.5]. Let $R$ be a local ring with maximal ideal $N$. A projective map $\sigma_B$ of $\mathcal{V}(R)$ is a PK-plane automorphism.

1.7 Definition. Let $V$ be a PK-plane, and let $g$ be a line of $V$. We say a point $P$ is near the line $g$, and write $P \simeq g$, if there is a point $Q$ on $g$ such that $Q$ is neighbor to $P$. We denote the negations of $\simeq$ and $\sim$ by $\not\simeq$.

The following proposition is easily shown.

1.8 Proposition. Let $V$ be a PK-plane, and let $\varphi: V \to V'$ be a map satisfying condition (PK) of the definition of PK-plane. Then $P \simeq g$ in $V$ if and only if $\varphi P I \varphi g$ in $V'$.

1.9 Definition. Let $V$ be a PK-plane. We say that a sequence $(P_1, P_2, P_3)$ of three pairwise nonneighbor points of $V$ is a nondegenerate triangle in $V$. License or copyright restrictions may apply to redistribution; see https://www.ams.org/journal-terms-of-use
and that \( P_1P_2, P_2P_3 \) and \( P_3P_1 \) are the sides of \((P_1, P_2, P_3)\) if there is no line \( g \) of \( V \) such that all three points are near \( g \).

1.10 Definition. Let \( V \) be a PK-plane. We say a sequence of four distinct points of \( V \) is a nondegenerate quadrangle if any subsequence of three distinct points is a nondegenerate triangle.

1.11 Definition. Let \( R \) be a local ring. In \( V(R) \) we denote the point \( r(1, 0, 0) \) by \( O' \); \( r(0, 1, 0) \) by \( O'' \); \( r(0, 0, 1) \) by \( O''' \), and \( r(1, 1, 1) \) by \( E \).

1.12 Lemma [5, Lemma 5.2.9]. Let \( R \) be a local ring, let \( N \) be the maximal ideal of \( R \), and let \( D = R/N \). Then a matrix \( A \) in \( R_{n \times n} \) is nonsingular (that is, invertible) if and only if its image in \( D_{n \times n} \) under the quotient map \( r: R_{n \times n} \rightarrow D_{n \times n} \) is nonsingular.

1.13 Proposition. Let \( R \) be a local ring. The projective maps of \( V(R) \) are transitive on the nondegenerate quadrangles of \( V(R) \).

Proof. Observe that \((O', O'', O''', E)\) is a nondegenerate quadrangle of \( V(R) \). Since the projective maps form a group under composition, it will suffice to show that for any given nondegenerate quadrangle \((P_1, P_2, P_3, P_4)\) there is a projective map which takes \((P_1, P_2, P_3, P_4)\) to \((O', O'', O''', E)\). Form a matrix \( Z \) by letting the \( i \)th row of \( Z \) be an element of \( P_i \) for \( i = 1, 2, 3 \). By Lemma 1.12, a \( 3 \times 3 \) matrix over a local ring \( R \) is invertible if and only if its image in \( R/N \) is invertible. Hence, \( Z \) is invertible. Let \((x_1, x_2, x_3)\) be an element of \( O''-(P_4) \); then, since \( O''-1 \) is an automorphism, \( O''-1(P_4) \) is not near \( O'O'' \), \( O'O''' \) and \( O''O''' \); hence \( x_j \in R \setminus N \) for \( i = 1, 2, 3 \). Let \( B \) be the diagonal matrix with \( B_{jj} = x_j^{-1} \) for \( j = 1, 2, 3 \). Then \( O''-1B \) is the desired projective map.

1.14 Definition. Let \( V \) be a PK-plane. We say \( V \) is desarguesian if there is a local ring \( R \) and a PK-plane isomorphism \( \mu: V \rightarrow V(R) \). We call such an isomorphism a (desarguesian) representation of \( V \).

2. Desarguesian derived affine Klingenberg planes.

2.1 Definition. Let \( V \) be a PK-plane, and let \( g \) be a line of \( V \). We obtain an incidence structure \( S_{\mathcal{A}(V, g)} \) by removing all the points near \( g \) and lines neighbor to \( g \) from \( V \) and restricting the incidence relation appropriately. We define an equivalence relation \( \parallel \) on the lines of \( S_{\mathcal{A}(V, g)} \) by letting \( h\parallel k \) whenever \( h, k \) and \( g \) have a point in common in \( V \). We restrict the neighbor relation of \( V \) to the points and lines of \( S_{\mathcal{A}(V, g)} \) and we denote this new relation by \( \sim \). We let \( \mathcal{B}(V, g) = (S_{\mathcal{A}(V, g)}, \parallel, \sim) \), and we call \( \mathcal{B}(V, g) \) a derived affine Klingenberg plane (abbreviation derived AK-plane). We call \( \parallel \) the parallel relation, and \( \sim \) the neighbor relation of \( \mathcal{B}(V, g) \).

2.2 Definition. Let \( A, A' \) be derived AK-planes. A map \( \omega: A \rightarrow A' \) is said to be a derived AK-plane homomorphism if \( \omega \) is an incidence structure
homomorphism which preserves both the neighbor and parallel relations.

2.3 Definition. Let \( A = \mathfrak{C}(V, g) \) be a derived AK-plane. We say \( A \) is desarguesian if there is a local ring \( R \) and a derived AK-plane isomorphism \( \alpha: A \to \mathfrak{C}(V(R), (1, 0, 0)s) \), and we call such an isomorphism a (desarguesian) representation of \( A \). For any local ring \( R \), let \( A(R) = \mathfrak{C}(V(R), (1, 0, 0)s) \).

2.4 Definition. Let \( R \) be a local ring. Observe that those points and lines of \( V(R) \) which are in \( A(R) \) are \( r(1, x, y) \) for \( x, y \in R \); \( (d, m, -1)s \) for \( d, m \in R \), and \( (v, -1, u)s \) for \( v, u \in R \). For convenience, we denote \( r(1, x, y) \) by \( (x, y) \); \( (d, m, -1)s \) by \( [m, d] \), and \( (v, -1, u)s \) by \( [u, v]' \) for all \( x, y, w, d, u, v \in R \). We denote the set of lines of \( A(R) \) which meet at the point \( r(0, 1, m) \) on \( (1, 0, 0)s \) by \( (m) \) and the set of lines of \( A(R) \) which meet at the point \( r(0, u, 1) \) on \( (1, 0, 0)s \) by \( (u) \). We denote the line \([0, 0]'\) by \( g_x \), the line \([0, 0]\) by \( g_y \) and the point \((1, 1)\) by \( E \).

3. The Triangle Theorem.

3.1 Lemma. Let \( V \) be a PK-plane. Let \( (P', P'', P''', F) \) be a nondegenerate quadrangle in \( V \); let \( k = P'' \cap P''' \), and let \( \alpha: \mathfrak{C}(V, k) \to A(R) \) be a representation of \( A(V, k) \). Then there is a representation \( \alpha': \mathfrak{C}(V, k) \to A(R) \) such that \( \alpha'(P') = (0, 0) \); \( \alpha'(F) = (1, 1) \); \( \alpha'(P' \cup P'') \equiv [0, 0] \), and \( \alpha'(P' \cup P''') = [0, 0]' \).

Proof. In \( V(R) \),

\[
Z = \{\alpha(P'), \alpha(P' \cup P'') \cap (1, 0, 0)s, \alpha(P' \cup P''') \cap (1, 0, 0)s, \alpha(F)\}
\]

is a nondegenerate quadrangle; hence by Proposition 1.13, there is a projective map \( \sigma_B \) of \( V(R) \) which takes \( Z \) to \( (O', O'', O''', E) \). Since \( \sigma_B \) is an automorphism of \( V(R) \) which fixes the line \((1, 0, 0)s\), \( \sigma_B \) induces an automorphism \( \alpha_B \) of \( A(R) \). Let \( \alpha' = \alpha_B \alpha \). Then \( \alpha' \) has the desired properties.

3.2 Definition. Assume we have desarguesian representations of AK-planes which are distinguished by superscripts; for example, assume we have \( \alpha': A' \to A(S') \) and \( \alpha'': A'' \to A(S'') \). Then we denote \( (\alpha')^{-1}(x, y) \) by \( '(x, y) \) and use other similar notation. For example, \( g_x = (\alpha'')^{-1}g_x \) where \( g_x = (0, 0, 1)s \). If \( A' = \mathfrak{C}(V, k') \), we denote \( (\alpha')^{-1}[m, 0] \cap k' \) by \( '(m) \) and so on.

3.3 Lemma. Let \( V \) be a PK-plane, let \( k' \) and \( k'' \) be nonneighbor lines of \( V \), and let \( A' = \mathfrak{C}(V, k') \) and \( A'' = \mathfrak{C}(V, k'') \). Assume that there are local rings \( H' \) and \( H'' \) such that \( A' \) is isomorphic to \( A(H') \), and \( A'' \) is isomorphic to \( A(H'') \). Then the following conditions hold.

(i) \( H'' \) is isomorphic to \( H' \); hence \( A'' \) is isomorphic to \( A(H'') \) and \( A' \).

(ii) If \( \beta': A' \to A(S) \) is a desarguesian representation of \( A' \), and if \( 'g_y = k'' \), then there is a desarguesian representation \( \beta'': A'' \to A(S) \) of \( A'' \) such that \( g_x = k' \); \( g_y = 'g_x \) and \( '(1, y) = "(y, 1) \) for every \( y \in S \).

(iii) If \( \alpha': A' \to A(R) \) and \( \alpha'': A'' \to A(R) \) are desarguesian representations of
A' and A'' such that \( g_y = k'' \); \( g_x = k' \); \( g_y = g_x \) and \( (1, y) = (y, 1) \) for every \( y \in R \), then \( [m, d] = [d, m] \) for all \( d, m \in R \), and if \( x \in R \setminus N \) where \( N \) is the maximal ideal of \( R \), then \( (x, y) = (x^{-1}y, x^{-1}) \).

**Proof.** (i) Let \( \varphi: V \to V' \) be a map satisfying condition (PK) of the definition of PK-plane. Since \( \varphi \) is surjective, there is, by Proposition 1.8, a point \( K'' \) on \( k'' \) which is not near \( k' \), and a point \( K' \) on \( k' \) which is not near \( k'' \). Let \( k = K' \cap K'' \), and let \( O = k'' \cap k'' \). Let \( Z \) be a point not near \( k', k'', k \); such a point exists as can be seen by looking at the images of \( k', k'', k \) in \( V' \). Then \((K'', K', O, Z)\) is a nondegenerate quadrangle in \( V \) as can be seen by looking at its image in \( V' \). Let \( \xi: A' \to A(M') \) be a derived AK-plane isomorphism. By Lemma 3.1, there is a representation \( \xi: A' \to A(M') \) such that \( \xi(k'') = \xi(K'' \cap O) = g_y \).

(ii) and a continuation of (i). Assume \( A' \) has a desarguesian representation \( \beta': A' \to A(M') \) such that \( g_y = k'' \). The construction of the preceding paragraph satisfies this requirement. Let \( G'' = g_x \cap k' \), and \( G''' = g_y \cap k' \). Then \((G'', G''', O', E')\) is a nondegenerate quadrangle in \( V \). Let \( \beta: A'' \to A(M'') \) be a representation of \( A'' \). By Lemma 3.1, there is a representation \( \beta'': A'' \to A(M'') \) such that \( O' = G'' \); \( E' = E'' \); \( g_x = k' \), and \( g_y = g_x \). The line \( \{0, 1\} \) is equal to the line \( \{(0, 1) \} \). Observe that \( (1, n) \sim (1, 0) \) if and only if \( n \) is in the maximal ideal of \( M' \), and that \( (m, 1) \sim (0, 1) \) if and only if \( m \) is in the maximal ideal of \( M'' \). Thus we can change the symbols of \( M'' \) to match those of \( M' \) in such a way that the maximal ideals are equal and \( (1, y) = (y, 1) \) for all \( y \in M' \).

(iii) and a continuation of (i) and (ii). Assume we have representations \( \alpha': A' \to A(R') \) and \( \alpha'': A'' \to A(R'') \) such that \( g_y = k'' \); \( g_x = k' \); \( g_y = g_x \); there is a set \( R \) which is the symbol set of both \( R' \) and \( R'' \); the maximal ideals of \( R' \) and \( R'' \) are equal, and \( (1, y) = (y, 1) \) for every \( y \in R \). The construction of the preceding paragraph satisfies these requirements. Let \( R' = (R, +, \#) \) and \( R'' = (R, \ast, \cdot) \), and write \( r \# s \) as \( rs \) for all \( r, s \in R \). Let \( E = E'' \). Let \( N \) be the maximal ideal of \( R' \). For each \( x \in R \setminus N \), there is a multiplicative inverse \( x' \) in \( R' \) and a multiplicative inverse \( x'' \) in \( R'' \). We wish to show that \( x' = x'' \).

Let \( (m) = [m, 0] \cap k' \); let \( (u) = [u, 0] \cap k' \); let \( (m) = [m, 0] \cap k' \), and let \( (u) = [u, 0] \cap k' \) for all \( m, u \in R \). The line \( (m, 0) \) joins \( (0, 0) \) and \( (1, m) \). Also, \( (m) = [m, 0] \cap k' = \{(m, 1) \cap (0)\} \cap [0, 0] = (m, 0) \).

Once we have shown a relationship between the coordinates of \( A' \) and \( A'' \), we have, by the symmetry of the situation, that a similar relationship holds with the roles of \( A' \) and \( A'' \); \( R' \) and \( R'' \); \( k' \) and \( k'' \), and the second and third positions of the triples interchanged. Since \( [m, 0] \cap k' = (m) = (m, 0) \), we
have that \( (0, m) = "(m) \) for all \( m \in \mathbb{R} \). Also,
\[
[m, b] = '(m) \lor '(0, b) = "'(m, 0) \lor "'(b) = "[b, m]'
\]
for all \( m, b \in \mathbb{R} \). Since \( [m, b] = "'[b, m]" \), and since \( '(1, y) = "'(y, 1) \) for all \( m, b, y \in \mathbb{R} \), and since \( \mathcal{O} \mathcal{E} \) where \( \mathcal{O} = k' \cap k'' \) meets \( [m, b] \) at \( '(1, m + b) \) and \( "[b, m]" \) at \( "'(m \ast b, 1) \), we have \( m + b = m \ast b \) for all \( m, b \in \mathbb{R} \).

Let \( x \in \mathbb{R} \setminus \mathbb{N} \). Then since \( x' \) is the inverse of \( x \) in \( \mathbb{R}' \); \( x'' \), in \( \mathbb{R}'' \), we have
\[
'(x, 1) = '[x', 0] \cap '[0, 1] = "'[0, x']' \cap "'[1, 0]'' = "'(x', x')".
\]
Using this, we have
\[
'[0, x'] = '(x, 1) \lor \mathcal{O} = "'(x', x') \lor \mathcal{O} = "'[0, x']';
\]
hence
\[
'(x, 0) = '[0, x'] \cap '[0, 0] = "'[0, x'] \cap "'[0, 0]'' = "'(0, x')".
\]
Using the interchange symmetry, we have that \( "'(0, y) = '(y', 0) \) for all \( y \in \mathbb{R} \setminus \mathbb{N} \). Hence, \( '(x, 0) = "'(0, x') = '((x')', 0) \). Thus, \( x' = x'' \) for all \( x \in \mathbb{R} \setminus \mathbb{N} \). Let \( x^{-1} = x' = x'' \) for all \( x \in \mathbb{R} \setminus \mathbb{N} \).

Let \( x \in \mathbb{R} \setminus \mathbb{N} \) and \( y \in \mathbb{R} \). Then,
\[
'(x, y) = '[0, x'] \cap '[0, y] = "'[0, x^{-1}]' \cap "'[y, 0]'' = "'(x^{-1} \cdot y, x^{-1})".
\]
By the interchange symmetry, \( "'(x, y) = '(y^{-1}, y^{-1} \cdot x)" \).

We wish to show that \( a \cdot b = ab \) for all \( a, b \in \mathbb{R} \). Let \( a \in \mathbb{R} \setminus \mathbb{N} ; b \in \mathbb{R} \). From our relations above we have with \( x = a^{-1} ; y = b \), that \( '(a^{-1}, b) = "'(a \cdot b, a) \), and with \( x = a \cdot b ; y = a \), that \( "'(a \cdot b, a) = '(a^{-1}, a^{-1} \cdot (a \cdot b)) \). Hence \( b = a^{-1} (a \cdot b) \), and \( ab = a \cdot b \) when \( a \in \mathbb{R} \setminus \mathbb{N} ; b \in \mathbb{R} \). Let \( a \in \mathbb{N} ; b \in \mathbb{R} \). Then since \( 1 \notin \mathbb{N} \), and since \( (\mathbb{N}, +) \) is a subgroup of \((\mathbb{R}, +) \), we have \( a - 1 \notin \mathbb{N} \); hence
\[
 a \cdot b = ((a - 1) + 1) \cdot b = (a - 1) \cdot b + b = (a - 1)b + b = ab.
\]
Hence, \( a \cdot b = ab \) for all \( a, b \in \mathbb{R} \). Thus, \( \mathbb{R}' = (\mathbb{R}, +, \#) = (\mathbb{R}, *, \cdot) = \mathbb{R}'' \), and we are done.

3.4 THE TRIANGLE THEOREM. Let \( V \) be a projective Klingenberg plane which has a nondegenerate triangle with sides \( k', k'' \) and \( k''' \) such that each of the derived affine Klingenberg planes \( A' = \mathcal{E}(V, k') \), \( A'' = \mathcal{E}(V, k'') \) and \( A''' = \mathcal{E}(V, k''') \) is desarguesian. Then \( V \) is desarguesian, and if \( R', R'' \) and \( R''' \) are local rings such that \( A' \) is isomorphic to \( A(R') \); \( A'' \) to \( A(R'') \), and \( A''' \) to \( A(R''') \), then \( R', R'' \) and \( R''' \) are isomorphic, and \( V \) is isomorphic to \( V(R') \).

PROOF. If \( R', R'' \) and \( R''' \) are local rings such that \( A' \) is isomorphic to \( A(R') \); \( A'' \) to \( A(R'') \), and \( A''' \) to \( A(R''') \), then by Lemma 3.3, \( R', R'' \) and \( R''' \) are isomorphic.

Let \( \alpha : A' \rightarrow A(R') \) be a desarguesian representation of \( A' \), and let \( R = R' \).
Let \( \varphi : V \to V' \) be a map satisfying condition (PK). Let \( F \) be a point such that \( \varphi F \) is not on \( \varphi k', \varphi k'', \varphi k''' \). It is easily seen that \( (k' \cap k''', k'' \cap k'', k'' \cap k', F) \) is a nondegenerate quadrangle of \( V \). By Lemma 3.1, there is a representation \( \alpha' : A' \to A(R) \) of \( A' \) with \( 'g_y = k'' \); \( 'g_x = k''' \); and \( 'E = F \).

Let \( N \) be the maximal ideal of \( R \).

By Lemma 3.3, there is a representation \( \alpha'' : A'' \to A(R) \) of \( A'' \) such that \( ''g_y = 'g_x = k''' \); \( ''g_x = k'' \); \( ''E = E = F \), and \( '(1, y) = ''(y, 1) \) for all \( y \in R \). Thus, if \( P \not\in k''', k'' \) and if \( P = ''(x, y) \), then \( x \not\in N \), and by Lemma 3.3, \( '(x, y) = ''(x \cdot y, x^{-1}) \). Similarly, by Lemma 3.3, there is a representation \( \alpha''' : A''' \to A(R) \) of \( A''' \) such that \( ''''g_y = ''''g_x = k'' \); \( ''''g_x = k''' \); \( ''''E = F \), and if \( Q \not\in k'', k''' \), and \( Q = ''''(x, y) \), then \( x \not\in N \), and \( ''''(x, y) = ''''(x \cdot y, x^{-1}) \).

We wish to show that if \( P \not\in k''', k'' \) and \( P = ''''(x, y) \), then \( x \not\in N \), and \( ''''(x, y) = ''''(x \cdot y, x^{-1}) \). Let \( Q \) be a point of \( V \) which is not near \( k', k'', k''' \). Then \( Q = ''''(x, y) \) where \( x, y \not\in N \). Using the relations above, \( Q = ''''(y^{-1}, x^{-1}) \). If \( x, y \not\in N \), then \( ''''(x, y) \not\in k', k'', k''' \). Then \( n \in N \), \( d \not\in N \). The line \( ''''[0, d] \) contain the points \( 'O' \) and \( ''''(d, d) = ''''(1, d^{-1}) \); hence \( ''''[0, d] = ''''[d, 0] \), and \( ''''(0, d) = ''''(d') \). Since \( n \in N \) and \( d \not\in N \), we have \( n + d \not\in N \); so that \( ''''(1, n + d) = ''''(n + d, 1) \), and since \( ''''(0, d) = ''''(d') \), we have \( ''''[n, d] = ''''[d, n] \); so that \( ''''(n) = ''''(n, 0) \), and hence \( ''''[n, 0] = ''''[0, n] \). Thus,

\[
''''(1, n) = ''''[0, n] \cap ''''[0, 1] = ''''[0, n] \cap ''''[0, 1] = ''''(n, 1).
\]

If \( G \not\in k''', k'' \), and if \( G = ''''(x, y) \), then \( x \not\in N \), and by Lemma 3.3, \( ''''(x, y) = ''''(x \cdot y, x^{-1}) \). Thus the relationships between the three representations \( \alpha' \), \( \alpha'' \) and \( \alpha''' \) can now be treated in a cyclically symmetric fashion.

For each \( i \), define \( \lambda^i : (S^i, \sim^i) \to V(R) \) where \( S^i \) is the incidence structure of \( A^i ; \sim^i \), the neighbor relation of \( A^i \), by \( \lambda^i((x, y)) = r(1, x, y) ; \lambda^i''((x, y)) = r(y, 1, x) \) or \( \lambda^i''''((x, y)) = r(x, y, 1) \), and correspondingly for lines. Observe that since each of the \( \alpha^i \) is an automorphism, each of the \( \lambda^i \) is injective and preserves and reflects both the incidence and neighbor relations.

Let \( \lambda : V \to V(R) \) be defined by \( \lambda P = \lambda^i(P) \) and \( \lambda g = \lambda^i(g) \) whenever \( P, g \) are in \( S^i \). We wish to show that \( \lambda \) is well defined. Since \( k', k'', k''' \) are the sides of a nondegenerate triangle, the map \( \lambda \) is defined for all points and lines of \( V \). Let \( P \) be a point in \( A' \) and in \( A'' \). Assume \( P = ''''(x, y) \). Then \( P = ''''(x \cdot y, x^{-1}) \) and \( \lambda(P) = \lambda''''(P) \). By symmetry, \( \lambda \) is well defined on points. Let \( g \) be a line in \( A' \) and in \( A'' \). Since \( g = ''''[m, d] \) if and only if \( g = ''''[d, m] \), there are two cases. Assume first that \( g = ''''[m, d] \). Then \( \lambda(g) = (d, m, -1)s = \lambda''''(g) \). If \( g \) cannot be written as \( ''''[m, d] \) for any \( m, d \in R \), then \( g = ''''[u, v] \) for some \( u \in N, v \in R \); similarly, if \( g \) cannot be written as \( ''''[d, m] \) for any \( d, m \in R \), then \( g = ''''[w, z] \) for some \( w \in N, z \in R \). Assume now that \( g = ''''[u, v], u \in N, v \in R, and g = ''''[w, z], w \in N. Let P = g \cap''''g_x = g \cap''''g_y,
Then \( P = '((v, 0) = "(0, z); \) hence \( z = v^{-1} \). Let \( Q = g \cap [1, 0] = g \cap \ "[0, 1]\). Then
\[
Q = '(v(1 - w)^{-1}, v(1 - w)^{-1}) = "(1, w + v^{-1}) .
\]
Thus, \( Q = "(1, (1 - u)v^{-1}); \) hence \( w = -(uv)^{-1} \). Observe that
\[
\lambda(g) = (v, -1, u)s = (-1, v^{-1}, -uv^{-1})s = \lambda''(g).
\]
By symmetry, \( \lambda \) is well defined on lines. Thus, \( \lambda \) is well defined.

Let \( h' = (1, 0, 0)s; h'' = (0, 1, 0)s \) and \( h''' = (0, 0, 1)s \). It is easily seen that \( \lambda \) takes the points and lines of \( \mathcal{A}^i \) onto the points and lines of \( \mathcal{O}(V(R), h^i) \). Assume \( P \) is \( g \) in \( V \). Since \( P \) is not near one of \( k', k'', k''', \) say \( k' \), both \( P \) and \( g \) are in \( A^j \). Since \( \lambda^j \) preserves incidence, \( \lambda P \) is \( g \) in \( V(R) \); hence \( \lambda \) preserves incidence. Assume \( \lambda Q \) is \( f \) in \( V(R) \). Since \( \lambda Q \) is not near one of the \( h^i \), say \( h^j \), both \( \lambda Q \) and \( \lambda f \) are in \( \mathcal{O}(V(R), h^j) \). Since \( \lambda^j \) reflects incidence, \( Q \) is \( f \) in \( V \).

Hence \( \lambda \) reflects incidence. Similar arguments show that \( \lambda \) preserves and reflects the neighbor relation.

Observe that any point of \( V(R) \) can be written as \( rx_i \), with one of the \( x_i \) equal to 1, and that any line of \( V(R) \) can be written as \( y_j s \) with one of the \( y_j \) equal to \(-1\); hence \( \lambda \) is surjective. Assume that \( \lambda P = \lambda Q; \) \( P \neq Q \). Then \( P \) and \( Q \) are not in the same \( A^i \). Assume that \( P \) is in \( A^j \) and that \( Q \) is in \( A^\prime \). Then \( P = '(x, y) \) where \( x \in N \). Thus, \( \lambda P = rx_i, \) then \( x_2 \in N \), a contradiction to \( Q \) being in \( A^\prime \). The other cases are similar by symmetry; hence \( \lambda \) is injective on points. For each \( i \), each line of \( S^i \) has at least two pairwise nonneighbor points on it by Proposition 1.8; hence \( \lambda \) is injective on lines. Hence \( \lambda: V \to V(R) \) is an isomorphism and \( V \) is desarguesian.

3.5 Proposition. Let \( V \) be a desarguesian \( PK \)-plane with a representation \( \mu: \)
\( V \to V(R), \) and let \( g \) be a line of \( V \). Then \( \mathcal{O}(V, g) \) is desarguesian, and there is
a representation \( \alpha: \mathcal{O}(V, g) \to \mathcal{A}(R). \)

Proof. One can construct a nondegenerate quadrangle \( Z = (Q', Q'', Q''', F) \) of \( V \)
such that \( g = Q'' \lor Q''' \). By Proposition 1.13, there is a projective map \( \sigma_B \) of \( V(R) \) which takes \( Z \) to \( (O', O'', O''', E) \). Then \( \sigma_B \mu \)
induces a derived AK-plane isomorphism \( \alpha: \mathcal{O}(V, g) \to \mathcal{A}(R). \)

3.6. Proposition. Let \( R \) and \( S \) be local rings. Then, \( \mathcal{A}(R) \) is isomorphic to
\( \mathcal{A}(S) \) if and only if \( R \) is isomorphic to \( S. \)

Proof. If \( R \) is isomorphic to \( S, \) then \( \mathcal{A}(R) \) is isomorphic to \( \mathcal{A}(S). \)
Assume \( \xi: \mathcal{A}(R) \to \mathcal{A}(S) \) is an isomorphism. Observe that \( Z = (\xi O', \xi E, \xi g_x \)
\( \cap (1, 0, 0)s, \xi g_y \cap (1, 0, 0)s) \) is a nondegenerate quadrangle in \( V(S). \) By
Proposition 1.13, there is a projective map \( \sigma_B \) which takes \( Z \) to \( (O', E, O'', O''') \). The map \( \sigma_B \) induces an automorphism \( \alpha_B \) of \( \mathcal{A}(S). \) Then
\( \eta = \alpha_B \xi \) takes \( (0, 0), (1, 1), g_x \) and \( g_y \) of \( \mathcal{A}(R) \) to \( (0, 0), (1, 1), g_x \) and \( g_y \) of
A(S). Also, \( \eta \) takes the points of \([1, 0] \) to the points of \([0, 1] \).

Define \( \varphi : R \rightarrow S \) by letting \( \varphi(a) = b \) whenever \( \varphi(a, a) = (b, b) \).

Then using the lines through \( O'' \) and \( O''' \), one can see that \( \eta(x, y) = (\varphi x, \varphi y) \)

for all \( x, y \in R \). The line \([m, 0] \) joins \( O' \) and \((1, m) \) in \( A(R) \), and \([\varphi m, 0] \) joins \( O' \) and \((1, \varphi m) \) in \( A(S) \); then \( \eta[m, 0] = [\varphi m, 0] \).

The line \([m, d] \) goes through \((0, d) \) and is parallel to \([m, 0] \) in \( A(R) \), and \([\varphi m, \varphi d] \) goes through \((0, \varphi d) \) and is parallel to \([\varphi m, 0] \) in \( A(S) \); thus \( \eta[m, d] = [\varphi m, \varphi d] \).

Thus, \( (\varphi x, \varphi(xm + d)) = (\varphi x, \varphi x(\varphi m) + \varphi d) \) for all \( x, m, d \in R \);

hence \( \varphi \) is a ring homomorphism. Since \( \varphi \) is a bijection, \( R \) is isomorphic to \( S \).

3.7 PROPOSITION. Let \( R \) and \( S \) be local rings. Then \( V(R) \) is isomorphic to \( V(S) \) if and only if \( R \) is isomorphic to \( S \).

PROOF. If \( R \) is isomorphic to \( S \), then \( V(R) \) is isomorphic to \( V(S) \).

Assume that \( \mu : V(R) \rightarrow V(S) \) is an isomorphism.

Then \( Z = (\mu O', \mu O'', \mu O''', \mu E) \) is a nondegenerate quadrangle in \( V(S) \).

By Proposition 1.13, there is a projective map \( \sigma_B \) which takes \( Z \) to \((O', O'', O''', E) \).

Then \( \sigma_B \mu \) induces an isomorphism from \( A(R) \) to \( A(S) \).

By Proposition 3.6, \( R \) is isomorphic to \( S \).

4. Desarguesian Hjelmslev planes. The term affine Hjelmslev plane (abbreviated AH-plane) is defined in [17, Definition 2.3]; also see [17, Satz 2.6] for an equivalent definition.

The term projective plane with neighbor elements is defined in [12, DO]. We will usually call a projective plane with neighbor elements a projective Hjelmslev plane (abbreviated PH-plane).

4.1 DEFINITION. A local ring \( S \) with maximal ideal \( N \) is said to be an AH-ring if every element of \( N \) is both a right and left zero divisor, and whenever \( x, y \in N \), there is an \( m \in S \) such that \( y = xm \), or there is a \( u \in S \) such that \( x = yu \).

An AH-ring \( S' \) with maximal ideal \( N' \) is said to be an H-ring if whenever \( m, n \in N' \), there is an \( x \in S \) such that \( n = xm \), or there is a \( y \in S \) such that \( m = yn \).

4.2 PROPOSITION. Let \( R \) be a local ring. Let \( R' = (R, +, \cdot) \) be the ring anti-isomorphic to \( R \); that is, \( a \cdot b = ba \) for all \( a, b \in R \).

The incidence structure of \( V(R') \) is the dual of that of \( V(R) \) and the neighbor relation of \( V(R') \) is the same as that of \( V(R) \).

4.3 PROPOSITION. Let \( R \) be a local ring with maximal ideal \( N \). Any two neighbor points of \( V(R) \) are joined by at least one line if and only if whenever \( x, y \in N \), there is an \( m \in R \) such that \( y = xm \), or there is a \( u \in R \) such that \( x = yu \).

PROOF. By the transitivity on nondegenerate quadrangles, it suffices to consider the pairs of points \( r(1, 0, 0), r(1, x, y) \) such that \( x, y \in N \).
4.4 Proposition. Let \( R \) be a local ring with maximal ideal \( N \). No two neighbor lines of \( V(R) \) meet in exactly one point if and only if whenever \( n \in N \), there is a \( k \in R \setminus \{0\} \) such that \( kn = 0 \).

Proof. By the transitivity on nondegenerate quadrangles, it suffices to consider the incidences \( r(1, 0, 0), r(1, k, 0) I (0, 0, -1)s, (0, n, -1)s \) such that \( n \in N \).

By Proposition 4.2, the following two propositions follow immediately.

4.5 Proposition. Let \( R \) be a local ring with maximal ideal \( N \). No two neighbor points of \( V(R) \) are joined by exactly one line if and only if whenever \( n \in N \), there is a \( k \in R \setminus \{0\} \) such that \( nk = 0 \).

4.6 Proposition. Let \( R \) be a local ring with maximal ideal \( N \). Any two neighbor lines of \( V(R) \) meet in at least one point if and only if whenever \( m, n \in N \), there is an \( x \in R \) such that \( n = xm \), or there is a \( y \in R \) such that \( m = yn \).

By transitivity on nondegenerate quadrangles, propositions corresponding to Propositions 4.3, 4.4 and 4.5 hold in \( A(R) \); we call these new propositions Propositions 4.3a, 4.4a and 4.5a. Baker [7] shows Propositions 4.3a and 4.4a, and a result closely related to Proposition 4.5a.

Remark. Lorimer [15] proves that if \( R \) is an AH-ring, then \( A(R) \) is an AH-plane. Another proof of this result is given in Bacon [4]; also see Lorimer and Lane [16].

The following three propositions now follow easily from Propositions 4.3-4.6 and 4.3a-4.5a.

Baker did not state the following proposition in [7]; however it is an easy consequence of results she obtains there.

4.7 Proposition. Let \( R \) be a local ring. If \( A(R) \) is an AH-plane, then \( R \) is an AH-ring.

4.8 Proposition. Let \( R \) be a local ring. If \( V(R) \) is a PH-plane, then \( R \) is an H-ring.

4.9 Proposition. If \( R \) is an H-ring, then \( V(R) \) is a PH-plane.

Remark. Klingenberg [12, S 28] claims to have shown the above result. There is an error in his argument for S 28 where he says that the lines \((u_1, 1, u_3)s \) and \((u_1, 1 + \tilde{c}, u_3)s \) are distinct. This fails as follows: let \( K \) be a field, and let \( \tilde{c}, u_1, u_3 \) be in the maximal ideal \( \{kX|k \in K\} \) of the ring \( K[X]/(X^2) \); then, by multiplying the triple \((u_1, 1 + \tilde{c}, u_3)s \) on the right by \((1 - \tilde{c}) \), we see that the two lines are equal.

Remark. There are AH-rings which are not H-rings; see [4, Construction 1.1].
4.10 Proposition. Let $V$ be a projective Hjelmslev plane, and let $s_1, s_2, s_3$ be the sides of a nondegenerate triangle in $V$. If there are local rings $R_1, R_2, R_3$ such that $\mathbb{A}(V, s_i)$ is isomorphic to $\mathbb{A}(R_i)$ for $i = 1, 2, 3$, then $V$ is isomorphic to $V(R_1)$, and $R_2$ and $R_3$ are isomorphic to $R_1$. Thus, $R_1$ is an $H$-ring.

Proof. This is immediate by the Triangle Theorem and Proposition 4.8.

Remark. The preceding proposition is quoted and used in Dugas [10].

4.11 Proposition. Let $V$ be a PH-plane, and let $k'$ and $k''$ be nonneighbor lines of $V$. If $A' = \mathbb{A}(V, k')$ and $A'' = \mathbb{A}(V, k'')$ are desarguesian, and if $\alpha: A' \to A(S)$ is a representation of $A'$, then $S$ is an $H$-ring.

Proof. Obviously $V$ is a PK-plane. One can construct a nondegenerate quadrangle $(Q', Q'', Q''', F)$ such that $k' = Q'' \cup Q'''$ and $k'' = Q' \cup Q'''$. By Lemmas 3.1 and 3.3, if $\alpha: A' \to A(S)$ is a representation, then there are representations $\alpha': A' \to A(S)$ and $\alpha'': A'' \to A(S)$ such that $\alpha' : k' = k''$; $\alpha'' : E = F$, and $[m, d] = [d, m]'$ for all $m, d \in S$.

It is well known (see Lüneburg [17, p. 260]) that $A'$ and $A''$ are AH-planes whose neighbor relations are induced from $V$. Since $A'$ is isomorphic to $A(S)$, we have, by Proposition 4.7, that $S$ is an AH-ring. Let $n, m \in S$. The lines $[0, 0]$ and $[m, -n]$ must meet in $A'$ or in $A''$. Let $P$ be a point on both lines. If $P$ is in $A'$, then $P = \{x, 0\}$ for some $x \in S$; hence $xm = n$. If $P$ is in $A''$, then $P = \{0, y\}$ for some $y \in S$; hence $yn = m$. Thus $S$ is an $H$-ring.

Remark. Part of the preceding argument is adapted from Klingenberg’s argument for $S 26$ in [12].

5. Examples.

5.1. Examples. Let $r$ be an integer such that there is a field $F$ with $r$ elements and an affine plane $A$ of order $r$ which is not isomorphic to $A(F)$. Let $k$ be an integer greater than 1, and let $R = F[\{X\}]/(X^{k+1})$. Then $R$ is a commutative $H$-ring and $V(R)$ is a PH-plane. Let $H$ be the set of points which are joined to $O'$ by $r^k$ or more lines. Then the lines of $V(R)$ induce an affine plane $A'$ of order $r$ isomorphic to $A(F)$ on the points of $H$. Let $h'' = (0, 1, 0)s$, and observe that $O'$ is on $h''$. The incidences of $V(R)$ associated with the points of $H$ can be changed in such a way that the resulting incidence structure together with the original neighbor relation is a PH-plane $W$, that the lines of $W$ induce an affine plane on the points of $H$ which has the same parallel classes as $A'$, and which is isomorphic to $A$ (and hence is not isomorphic to $A(F)$), and that $H$ is the set of points which are joined to $O'$ by $r^k$ or more lines. Moreover, this change can be made in such a way that if $g, k$ are lines of $V(R)$, then $P I h''$, $g, k$ in $W$ if and only if $P I h'', g, k$ in $V(R)$. Thus, $\mathbb{A}(W, h'') = \mathbb{A}(V(R), h'')$ is isomorphic to $A(R)$. If $m$ is a line such that $O'$ is near $m$, and $m$ does not meet $H$, then
\( \varrho(W, m) = \varrho(V(R), m) \) is isomorphic to \( A(R) \). It is routine to show that \( W \) is not desarguesian. Let \( B = r(1, 0, X) \). Then \( B \) has \( h'' \), and there are at least \( r^{k+1} + r^k - r^2 + 1 \) lines \( d \) through \( B \) such that \( \varrho(W, d) \) is isomorphic to \( A(R) \), and the less than \( r^2 \) remaining lines are pairwise neighbor.

The above examples were inspired by results in Artmann [1], [2]; also see Bacon [3], [4].

5.2 Examples. There are constructions in Drake [8, Theorem 5.2] and Artmann [2, Satz 2] (see Bacon [4, Proposition 1.3 and Theorem 2.1]) which can be used to construct nondesarguesian PH-planes \( V \) each of which has a line \( g \) such that \( \varrho(V, g) \) is isomorphic to \( A(R) \) for some commutative \( H \)-ring \( R \).

6. Counterexamples and applications. Examples 5.1 and 5.2 are counterexamples to the theorem in Klingenberg [12, p. 110] by S 29 of [12]. Examples 5.1 and 5.2 are counterexamples to Hauptsatz 2 of Klingenberg [13] by the last four lines of Hauptsatz 2 [13, p. 191, lines 1-4] which can be routinely checked in a manner similar to the proof of S 29 of [12]. Examples 5.1 and 5.2 are counterexamples to S 5.17 of Klingenberg [11] by Lemma 6.1 below.

In Bacon [5], various characterizations are given of desarguesian and pappian desarguesian AK- and PK-planes. For example, it is shown that an AK-plane \( V \) is desarguesian if and only if the automorphisms of \( V \) are \((P, g_\infty)\), \((\Gamma, g_\infty)\) and \((\Sigma, g_\infty)\)-transitive for some point \( P \) and some nonneighbor directions \( \Gamma \) and \( \Sigma \). The characterizations of desarguesian and pappian desarguesian PK-planes are applications of the Triangle Theorem. In [5], various characterizations are also given of translation AK-planes, affinely moufang AK-planes, and moufang PK-planes. Some of these characterizations involve biternary rings. Also, PK-planes are classified by elation type in [5].

6.1 Lemma. Let \( R \) be a commutative \( H \)-ring. Then \( A(R) \) satisfies axioms \( \delta \) and \( \Pi \) as defined in [11].

Proof. One can use automorphisms of the type \( \sigma(x, y) = (x + a, y + b) \) to show axiom \( \delta \), and one can use a routine calculation with \( g = [0, 0] \); \( g' = [0, 0]' \); \( P'_1 = (0, 1) \); \( P_2 = (-a, 0) \); \( P_3 = (-b, 0) \); \( p_{21} = [a, -a] \); \( p_{23} = [c, ac] \), and so on, together with Proposition 1.13 above to show axiom \( \Pi \).

Hauptsatz 1 and the last four lines of Hauptsatz 2 of Klingenberg [13] can be used with the Triangle Theorem to obtain the following theorem.

6.2 Theorem. Let \( V \) be a projective Klingenberg plane, and let \( s_1, s_2, s_3 \) be the sides of a nondegenerate triangle of \( V \). Then \( V \) is isomorphic to \( V(R) \) for some local ring \( R \) if and only if each \( \varrho(V, s_i) \) for \( i = 1, 2, 3 \) satisfies axioms \( (d) \) and \( (D) \).
In Bacon [5, Theorem 11.2.4] it is shown that if $W$ is a group of automorphisms of a PK-plane $V$ and if $W$ is $(P, p)$-, $(Q, q)$- and $(R, r)$-transitive where $P \not\sim p$, $Q \not\sim q$, $R \not\sim r$ and $(P, Q, R)$ is a nondegenerate triangle of $V$, then $W$ has a star spine center; if, in addition, $(p, q, r)$ is a nondegenerate triangle in the dual of $V$, then $W$ is moufang, and it is shown, using the Triangle Theorem, that if $W$ is moufang, and $(G, h)$-transitive for some $G, h$ such that $G \not\sim h$, then $V$ is desarguesian, and $W$ is projectively desarguesian [5, Proposition 11.4.4], [6, Correction A.11.1].

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