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SOLUTION OF THE NONLINEAR PROBLEM $Au = N(u)$ IN A BANACH SPACE

BY

MARTIN SCHECHTER, JACK SHAPIRO AND MORRIS SNOW

ABSTRACT. We solve a nonlinear problem $Au = N(u)$ where A is semi-Fredholm and N is a nonlinear compact operator.

1. Introduction. The operator equation $Au - N(u) = f$ where A is a linear and N is a nonlinear operator from a Banach space X to a Banach space Y has been studied extensively in recent literature. For discussions of the problem when one of the operators is monotone, see for instance [3], [7], [8], [10]. Operators of type S, S^+ and M from X to X^* have also been discussed in relation to the above problem in [3], [7], [10], [11] and others. The Hammerstein integral equation, a special case of the above problem, is considered in [1], [2], [4] (see also the bibliography to these papers). Gustafson and Sather [9] consider the problem when A is normally solvable and N belongs to a special class of large nonlinearities. The theorems proved in [3], [5], [6] apply when A is a Φ_+ operator, N is compact and the operators satisfy certain inequalities. We will be working with A a Φ_- operator and we have attempted to simplify the conditions on A and N and to present a simple proof of our main theorem based on a fixed point theorem. The technique is similar to that of [13].

The following theorem follows from our main result:

Let X and Y be Banach spaces, A a Φ_- operator from X to Y, and N a compact operator from X to Y. Let Z be such that $Y = Z \oplus R(A)$. Z can be renormed to form a Hilbert space. Let S be a bounded linear operator from Y to X such that $Q = AS$ is a bounded projection onto $R(A)$, and let P be a bounded projection of Y onto Z along $R(A)$. Let T be a compact map from Z to $N(A)$.

Put $Gu = SNu$, $Hu = PNu$ and

$$\alpha = \limsup_{\|z\|\to\infty} \frac{\|Tz\|}{\|z\|}, \qquad \beta = \limsup_{\|u\|\to\infty} \frac{\|Gu\|}{\|u\|} \quad \text{and}$$

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$$\gamma = \limsup_{\|u\| \to \infty} \frac{\|Hu\|}{\|u\|}.$$

Suppose that α and γ are finite and suppose further that there are positive constants ρ, δ and K_0 such that $\alpha\rho + \beta < 1$ and

$$(z, H(Tz + Gu)) \geqslant \delta \tag{1.1}$$

for $u \in X$, $z \in Z$, $\|z\| \geqslant K_0$, $\|z\| \geqslant \rho\|u\|$. We can then conclude that there exists $u \in D(A)$ such that $Au = N(u)$.

It should be noted that our inequality (1.1) is less restrictive than that of Fučík [6] as is illustrated by our application.

2. The abstract theory. We state several results in Banach space involving nonlinear perturbations of linear operators. Their proofs will be given in §4, and will be based on an abstract theorem (Theorem 3.1) proved in the next section. Applications will be given in §5.

Let X, Y be Banach spaces, and let A be a closed linear operator from X to Y, and N a compact map from X to Y. Assume

H1. There is an operator $S \in B(Y, X)$ such that $R(S) \subset D(A)$ and $Q = AS$ is a bounded projection onto $R(A)$.

H2. There is a finite dimensional Hilbert space Z and a bounded operator P from Y to Z such that $N(P) \subset R(A)$.

H3. There is a compact map T from Z to $N(A)$.

H4. Put $Gu = SNu$, $Hu = PNu$, and

$$\alpha = \limsup_{\|z\| \to \infty} \frac{\|Tz\|}{\|z\|}, \qquad \beta = \limsup_{\|u\| \to \infty} \frac{\|Gu\|}{\|u\|} \quad \text{and}$$

$$\gamma = \limsup_{\|u\| \to \infty} \frac{\|Hu\|}{\|u\|}.$$

Then α and γ are finite.

H5. There are positive constants ρ, δ and K_0 such that $\alpha\rho + \beta < 1$ and

$$(z, H(Tz + Gu)) \geqslant \delta \tag{2.1}$$

for $u \in X$, $z \in Z$, $\|z\| \geqslant K_0$, $\|z\| \geqslant \rho\|u\|$.

THEOREM 2.1. *Under hypotheses* H1–H5, *there is a* $u \in D(A)$ *such that*

$$Au = N(u). \tag{2.2}$$

REMARKS. (1). Hypotheses H1 and H2 hold iff $R(A)$ is closed and has finite codimension in Y and $N(A)$ is complemented (cf. [12]).

(2) Since Z is finite dimensional, T will be compact if it is continuous and takes bounded sets into bounded sets.

(3) N need not be compact from X to Y but only from $D(A)$ to Y. In this case we replace X by $D(A)$ and use the graph norm of A in place of the norm

of X. This alters hypotheses H4 and H5 slightly. In particular, β and γ are replaced by

$$\tilde{\beta} = \limsup_{\|u\|_A \to \infty} \frac{\|Gu\| + \|QNu\|}{\|u\|_A} \quad \text{and} \quad \tilde{\gamma} = \limsup_{\|u\|_A \to \infty} \frac{\|Hu\|}{\|u\|_A},$$

where $\|u\|_A = \|u\| + \|Au\|$. This change may weaken or strengthen hypothesis H4 or H5.

(4) Hypothesis H5 is not needed when $R(N) \subset R(A)$. For then we can take $Y = R(A)$ and $Z = \{0\}$. If we take $K_0 = 1$, hypothesis H5 will be vacuously satisfied. See the remark following the proof of Theorem 3.1.

(5) The requirement $\alpha\rho + \beta < 1$ is essential. To see this consider the case when A is selfadjoint and the closest other point of $\sigma(A)$ to 0 is a positive eigenvalue λ.

Set $Z = N(A)$ and let P be the orthogonal projection of X onto Z. Since $X = R(A) \oplus N(A)$, A has an inverse S_1 on $R(A)$ with $|\lambda|\,\|S_1\| = 1$. Set $S = S_1(I - P)$. Put $Nu = \lambda P_1 u + f$, where f is arbitrary and P_1 is the orthogonal projection onto $N(A) \oplus N(A - \lambda)$. $\alpha = 1$ because $T = I$. $u = Pu + (P_1 - P)u + (I - P_1)u$. Therefore,

$$SNu = S\big(\lambda(Pu + (P_1 - P)u + f)\big) = \lambda S(P_1 - P)u + Sf.$$

Since $(P_1 - P)u \in N(A - \lambda)$, we have $\lambda S(P_1 - P)u + Sf = (P_1 - P)u + Sf$. Therefore,

$$\beta = \limsup_{\|u\| \to \infty} \frac{\|SNu\|}{\|u\|} = \limsup_{|u| \to \infty} \frac{\|(P_1 - P)u + Sf\|}{\|u\|} = 1.$$

If $z \in N(A)$ and $u \in X$ we have

$$\begin{aligned}(z, H(Tz + Gu)) &= (z, N(z + SNu)) = \big(z, N\big(z + (P_1 - P)u + Sf\big)\big) \\ &= \big(z, \lambda(z + (P_1 - P)u + (P_1 - P)Sf) + f\big) = \lambda\|z\|^2 + (z, f).\end{aligned}$$

Thus all the hypotheses of Theorem 2.1 except $\beta < 1$ are satisfied for any f. But $Au = \lambda Pu + f$ cannot be solved for every f.

(6) Inequality (2.1) in hypothesis H5 can be replaced by

$$(z, H(Tz + Gu)) \leqslant -\delta. \tag{2.4}$$

To see this just replace P by $-P$ in hypothesis H2. None of the other hypotheses is affected.

(7) If the dimension of Z is > 1, we can replace (2.1) by

$$|(z, H(Tz + Gu))| \geqslant \delta. \tag{2.5}$$

This follows from the fact that for fixed u, the left-hand side of (2.1) is a continuous function of z. Since the set $\|z\| > K$ is connected in dimensions higher than one, (2.5) implies that the left-hand side of (2.1) cannot change

sign on the set considered. Thus inequality (2.5) implies either (2.1) or (2.4).

(8) It will be clear from our proof that we can replace (2.1) by

$$(z, H(Tz + Gu)) \geqslant \delta \min\{1, \|H(Tz + Gu)\|^2\}. \tag{2.6}$$

(9) If $\dim Z = 1$, then (2.1) can be replaced by

$$(z, H(Tz + Gu)) \geqslant 0. \tag{2.7}$$

To see this, suppose $z_0 \in Z$ has norm 1. Then

$$\|H(Tz + Gu)\|^2 = (z_0, H(Tz + Gu))^2.$$

Hence (2.6) is implied by

$$\|z\|^2(z, H(Tz + Gu)) \geqslant \delta(z, H(Tz + Gu))^2.$$

This in turn is implied by (2.7) and the fact that α, β, and γ are finite.

3. The basic theorem. In this section we prove a basic result which implies Theorem 2.1.

THEOREM 3.1. *Let X be a Banach space and Z a finite dimensional Hilbert space. Let T, G, H be compact mappings from Z to X, X to X and X to Z, respectively. Assume that*

$$\alpha = \limsup_{\|z\|\to\infty} \|Tz\|/\|z\|,$$

$$\beta = \limsup_{\|u\|\to\infty} \|Gu\|/\|u\|,$$

$$\gamma = \limsup_{\|u\|\to\infty} \|Hu\|/\|u\|$$

are finite, and that there are positive constants ρ, δ and K_0 such that

$$\alpha\rho + \beta < 1 \tag{3.1}$$

and

$$(z, H(Tz + Gu)) \geqslant \delta \tag{3.2}$$

whenever $z \in Z$, $u \in X$, $\|z\| \geqslant K_0$ and $\|z\| \geqslant \rho\|u\|$. Then there are elements $z \in Z$ and $u \in X$ such that

$$u = Tz + Gu, \qquad Hu = 0. \tag{3.3}$$

PROOF. By hypothesis there are constants α_1, β_1, γ_1 and R such that $\alpha_1\rho + \beta_1 < 1$ and

$$\|Tz\| \leqslant \alpha_1\|z\|, \quad \|z\| > R, \tag{3.4}$$

$$\|Gu\| \leqslant \beta_1\|u\|, \quad \|Hu\| \leqslant \gamma_1\|u\|, \quad \|u\| > R. \tag{3.5}$$

Also there are constants K_1, K_2, K_3 such that

$$\|Tz\| \leqslant K_1, \quad \|z\| \leqslant R, \tag{3.6}$$

$$\|Gu\| \leqslant K_2, \quad \|Hu\| \leqslant K_3, \quad \|u\| \leqslant R. \tag{3.7}$$

Put $\theta = (1 - \beta_1)/\alpha_1$ and

$$M = \max\left\{ \frac{K_0}{\rho}, \frac{K_1}{\alpha_1\theta}, \frac{K_2}{\beta_1}, \frac{K_3}{\gamma_1} \right\}. \tag{3.8}$$

Assume

$$\|u\| \leqslant M, \qquad \|z\| \leqslant \theta M \tag{3.9}$$

and consider the mapping $\{u, z\} \to \{u^*, z^*\}$ given by

$$u^* = Tz + Gu, \quad z^* = z - \varepsilon Hu^*, \tag{3.10}$$

where ε is a positive constant to be determined alter. This mapping is clearly compact. Our aim is to show that it has a fixed point. By the Schauder fixed point theorem, it suffices to show that we can find an $\varepsilon > 0$ such that the mapping (3.10) takes the set (3.9) into itself. This we proceed to do. First we note that if u, z satisfy (3.9), then

$$\|Tz\| \leqslant \alpha_1\theta M, \qquad \|Gu\| \leqslant \beta_1 M, \quad \|Hu\| \leqslant \gamma_1 M. \tag{3.11}$$

For if $\|z\| \leqslant R$, then $\|Tz\| \leqslant K_1 < \alpha_1\theta M$ by (3.6) and (3.8). If $\|z\| > R$, then $\|Tz\| \leqslant \alpha_1\|z\| \leqslant \alpha_1\theta M$ by (3.4) and (3.9). Thus the first inequality of (3.11) holds. Similarly, if $\|u\| \leqslant R$, then $\|Gu\| \leqslant K_2 \leqslant \beta_1 M$ and $\|Hu\| \leqslant K_3 \leqslant \gamma_1 M$ by (3.7) and (3.8). If $\|u\| > R$, then $\|Gu\| \leqslant \beta_1\|u\| \leqslant \beta_1 M$ and $\|Hu\| \leqslant \gamma_1\|u\| \leqslant \gamma_1 M$ by (3.5) and (3.9). Now

$$\|u^*\| \leqslant \|Tz\| + \|Gu\| \leqslant (\alpha_1\theta + \beta_1)M = M$$

by (3.11). Hence u^* satisfies the first inequality of (3.9), and consequently

$$\|Hu^*\| \leqslant \gamma_1 M$$

by (3.11). Now if $\|z\| \leqslant \rho M$, then

$$\|z^*\| \leqslant \|z\| + \varepsilon\|Hu^*\| \leqslant (\rho + \varepsilon\gamma_1)M. \tag{3.12}$$

Since $\rho < \theta$, this can be made less than θM by taking ε sufficiently small. If $\|z\| > \rho M \geqslant \rho\|u\|$, we can apply (3.2). Thus

$$\begin{aligned} \|z^*\|^2 &= \|z\|^2 - 2\varepsilon(z, Hu^*) + \varepsilon^2\|Hu^*\|^2 \\ &\leqslant \|z\|^2 - 2\varepsilon\delta + \varepsilon^2\gamma_1^2 M^2. \end{aligned}$$

This can be made $\leqslant \|z\|^2$ by taking ε sufficiently small. Thus we may arrange it so that u^*, z^* satisfy (3.9).

REMARK. If Hu vanishes identically, then hypothesis (3.2) of Theorem 3.1 is unnecessary. For then $z^* = z$, and the inequality (3.2) is not needed to verify that z^* satisfies (3.9). Inequality (3.2) can be replaced by (2.4) merely by replacing $-\varepsilon$ by ε in (3.10). Moreover, if $\dim[Z] > 1$, (3.2) can be replaced by (2.5) (see Remark 8 of §2).

4. The proof. Now we give the proof of the theorem of §2.

PROOF OF THEOREM 2.1. The operators T, G, H are compact from Z to X, X to X and X to Z respectively. α, β, γ, δ and ρ are the same as in Theorem 3.1, and the hypotheses of the latter theorem are satisfied. Consequently, we know that there exist u, z such that (3.3) holds. Now the ranges of T and G are both in $D(A)$. Hence $u \in D(A)$ and $Au = AGu = QNu$. Since $PNu = Hu = 0$, we know by hypothesis H2 that $Nu \in R(A)$. In view of hypothesis H1, $QNu = Nu$, and consequently u is a solution of (2.2).

5. Application. We wish to solve

$$u'' + u = g(u) + h, \qquad u(0) = u(\pi) = 0 \tag{5.1}$$

where $u(x) \in C''[0, \pi]$, $h \in L^2[0, \pi]$, and

(i) g is continuous, odd, and monotone on R,

(ii) $\lim_{\xi\to\infty} g(\xi) = \infty$,

(iii) $|g(\varepsilon)| \leqslant C_1 + C_2|\xi|$.

Let A: $D(A) \subset L^2[0, \pi] \to L^2[0, \pi]$ be defined by $A(u) = u'' + u$, $u(0) = u(\pi) = 0$, and N: $L^2[0, \pi] \to L^2[0, \pi]$ be defined by $N(u) = g(u) + h$. It is clear that $N(A)$, the null space of A, is equal to $\{\sin x\}$. Let $X = Y = L^2[0, \pi]$ and $Z = \{\sin x\}$. Z is orthogonal to $R(A)$ and we choose P to be the orthogonal projection of X onto Z. Since $Z = N(A)$, we take T to be the identity map. If $u'' + u = f$ then $u = S_1(f)$ where

$$\begin{aligned} S_1(f) &= C \sin x + \int_0^x f(y)\sin(x - y)\, dy \\ &= C'\sin x - \int_x^\pi f(y)\sin(x - y)\, dy. \end{aligned}$$

We set $S = S_1(I - P)$. C and C' will be chosen so that $\int_0^\pi S(f(x)) \sin x\, dx = 0$, i.e. $R(S) \perp N(A)$. This will assure that $\|S\|$ is as small as possible. Then,

$$(z, H(Tz + Gu)) = \int_0^\pi t \sin x\Big[g\big(t \sin x + S\big(g(u(x)) + h(x)\big)\big) + h(x)\Big]\, dx.$$

We can get $(z, H(Tz + Gu)) \geqslant \delta > 0$ by establishing conditions which make $t \cdot \sin x + S(g(u(x)) + h(x))$ sufficiently large for x bounded away from 0.

We wish to find an upper bound for $S(g(u(x)) + h(x))$ when $0 \leqslant x \leqslant \pi/2$. By definition,

$$S\big(g(u(x)) + h(x)\big) = C \sin x + \int_0^x \big[g(u(y)) + h(y)\big] \sin(x - y)\, dy.$$

Clearly,

$$\left|\int_0^x [g(u(y)) + h(y)] \sin(x-y)\, dy\right| \leqslant \sin x \cdot \left|\int_0^x [g(u(y)) + h(y)]\, dy\right|$$

$$\leqslant \sin x\left(\int_0^x (C_1 + C_2|u(y)| + h(y))\, dy\right)$$

$$\leqslant \sin x \cdot \left(C_1 x + C_2\|u\|\sqrt{x} + \int_0^{\pi/2} |h(y)|\, dy\right)$$

$$\leqslant \left(C_3 + C_2\sqrt{\pi/2} \cdot \|u\|\right)\sin x. \tag{1}$$

We now choose C so that $S(g(u(x)) + h(x))$ is orthogonal to $\sin x$. Hence,

$$0 = C\int_0^\pi \sin^2 x\, dx + \int_0^\pi \int_0^x [g(u(y)) + h(y)]\sin(x-y) \sin x\, dy\, dx$$

and

$$C = -2(\pi)^{-1}\int_0^\pi \int_0^x [g(u(y)) + h(y)]\sin(x-y) \sin x\, dy\, dx$$

$$= -2(\pi)^{-1}\int_0^\pi \int_y^\pi [g(u(y)) + h(y)] \sin(x-y) \sin x\, dx\, dy$$

$$= -2(\pi)^{-1}\int_0^\pi [g(u(y)) + h(y)]\int_y^\pi (2)^{-1}(\cos y - \cos(2x-y))\, dx\, dy$$

$$= -2(\pi)^{-1}\int_0^\pi [g(u(y)) + h(y)]\left[(\pi/2 - y/2)\cos y + (2)^{-1}\sin y\right] dy.$$

Since $|g(u(y))| \leqslant C_1 + C_2|u(y)|$, we have

$$|C| \leqslant K_1 + (\pi)^{-1}C_2\int_0^\pi |u(y)| \cdot |(\pi - y)\cos y + \sin y|\, dy$$

$$\leqslant K_1 + (\pi)^{-1}C_2\|u\|\left(\int_0^\pi [(\pi - y)\cos y + \sin y]^2\, dy\right)^{1/2}$$

$$\leqslant K_1 + (\pi)^{-1}C_2\|u\|(\pi^3/6 + 5\pi/4)^{1/2}$$

and hence

$$|C| \leqslant K_1 + C_2\|u\|(\pi/6 + 5/4\pi)^{1/2}. \tag{2}$$

Therefore, from (1) and (2), we conclude the following: If

$$t > K_2 + \left[\left(\pi(6)^{-1} + 5(4\pi)^{-1}\right)^{1/2} + \sqrt{\pi/2}\,\right]C_2\|u\|$$

for K_2 sufficiently large, then $t \sin x + S(g(u(x)) + h(x)) > 0$ for $x \in (0, \pi/2]$ and $t \sin x + S(g(u(x)) + h(x)) \geqslant \delta_1$ for x bounded away from 0. Since $z = t \sin x$ and $\|\sin x\| = \sqrt{\pi/2}$, we can conclude the above if $\|z\| > K_3 + [(\pi^2/12 + 5/8)^{1/2} + \pi/2]C_2\|u\|$.

We now find an upper bound for $S(g(u(x)) + h(x))$ when $\pi/2 \leqslant x \leqslant \pi$. We

have

$$S\big(g(u(x)) + h(x)\big) = C' \sin x - \int_x^\pi \big[\, g(u(y)) + h(y)\big] \sin(x - y)\, dy.$$

Proceeding as above, it can easily be shown that to make $S(g(u(x)) + h(y))$ orthogonal to $\sin x$ we must take $C' = (\pi)^{-1}\int_0^\pi [g(u(y)) + h(y)]\cdot y \cos y\, dy$. Clearly,

$$\begin{aligned}\left|\int_x^\pi \big[\, g(u(y)) + h(y)\big] \sin(x - y)\, dy\right| &\leqslant \sin x \int_x^\pi |g(u(y)) + h(y)|\, dy \\ &\leqslant \sin x \int_x^\pi \big(C_1 + C_2|u(y)| + |h(y)|\big)\, dy \\ &\leqslant \sin x\big[\, C_4 + C_2\|u\|\sqrt{\pi - x}\,\big] \\ &\leqslant \big(C_4 + C_2\sqrt{\pi/2}\,\|u\|\big) \sin x.\end{aligned}$$

$$\begin{aligned}\left|\int_0^\pi \big[\, g(u(y)) + h(y)\big]\cdot y \cos y\, dy\right| &\leqslant \int_0^\pi \big(C_1 + C_2|u(y)| + |h(y)|\big) y|\cos y|\, dy \\ &\leqslant K_4 + C_2\|u\|\left(\int_0^\pi y^2 \cos^2 y\, dy\right)^{1/2}\end{aligned}$$

and

$$C' \sin x \leqslant K_5 + C_2\|u\|(\pi/6 + 1/4\pi)^{1/2} \sin x.$$

Therefore,

$$\big|S\big(g(u(x)) + h(x)\big)\big| \leqslant K_6 \sin x + \big(\sqrt{\pi/2} + [\pi/6 + 1/(4\pi)]\big)C_2\|u\| \sin x.$$

Hence,

$$t > K_6 + \left[\sqrt{\pi/2} + (\pi/6 + 1/4\pi)^{1/2}\right]C_2\|u\|,$$

where K_6 is sufficiently large, implies $t \sin x + S(g(u(x)) + h(x)) > 0$ for $x \in (\pi/2, \pi]$ and $t \sin x + S(g(u(x)) + h(x)) \geqslant \delta_1$ for x bounded away from π. Since $\|z\| = |t|\sqrt{\pi/2}$, inequality (3) may be replaced by

$$\|z\| > K_7 + \left[\pi/2 + (\pi^2/12 + 1/8)^{1/2}\right]C_2\|u\|.$$

Since $B_0 = \pi/2 + (\pi^2/12 + 5/8)^{1/2} > \pi/2 + (\pi^2/12 + 1/8)^{1/2}$, $\|z\| \geqslant K_0' + B_0C_2\|u\|$ implies $(z, H(Tz + Gu)) \geqslant \delta > 0$. Since T is the identity map, $\alpha = 1$. $\|G(u)\| = \|SNu\| \leqslant 4/3\|Nu\|$ because $\|S\| = 4/3$. We also have that

$$\begin{aligned}\|N(u)\| &\leqslant \left(\int_0^\pi |g(u(x))|^2\, dx\right)^{1/2} + \|h\| \\ &\leqslant \left(\int_0^\pi (C_1 + C_2|u(x)|)^2\, dx\right)^{1/2} + \|h\| \leqslant C_1\sqrt{\pi} + C_2|u| + \|h\|.\end{aligned}$$

Therefore,

$$\beta = \limsup_{|u|\to\infty} \frac{\|G(u)\|}{\|u\|} \leqslant (4/3)C_2$$

and hence $C_2 < 3/4$ implies $\beta < 1$.

Let $\varepsilon > 0$. If $\|u\| \leqslant K_0'/\varepsilon$, then $\|z\| \geqslant K_0' + B_0C_2K_0'/\varepsilon \geqslant K_0' + B_0C_2\|u\|$. If $|u| > K_0'/\varepsilon$, then $\varepsilon\|u\| > K_0'$ and $\|z\| \geqslant (B_0C_2 + \varepsilon)\|u\| \geqslant B_0C_2|u| + K_0'$. Therefore, the two conditions, $\|z\| \geqslant K_0' + B_0C_2K_0'/\varepsilon$ and $\|z\| \geqslant (B_0C_2 + \varepsilon)\|u\|$ together imply $\|z\| \geqslant K_0' + B_0C_2\|u\|$ which implies $(z, H(Tz + Gu)) \geqslant \delta > 0$. Set $K_0 = K_0' + B_0C_2K_0'/\varepsilon$ and $\rho = B_0C_2 + \varepsilon$. By Theorem 2.1, we need $\alpha\rho + \beta < 1$ or, since $\alpha = 1$, we need $\rho + \beta < 1$. This is implied by $B_0C_2 + \varepsilon + (4/3)C_2 < 1$ or, equivalently, $C_2 < (1 - \varepsilon)(B_0 + 4/3)^{-1}$. Since ε may be arbitrarily small, it suffices to have $C_2 < (B_0 + 4/3)^{-1} \sim .24337$. Thus we have

THEOREM 5.1. *If hypotheses* (i)–(iii) *hold with* $C_2 < .24347$ *then there exists a solution of* (5.1).

In [6], C_2 has to be taken less than $3/4 \cdot \sqrt{\pi}\,(\sqrt{2}\,(\pi^{1/2} + 8))^{-1} \cong .0962$.

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BELFER GRADUATE SCHOOL OF SCIENCE, YESHIVA UNIVERSITY, NEW YORK, NEW YORK 10033

DEPARTMENT OF MATHEMATICS, BARUCH COLLEGE, (CUNY), NEW YORK, NEW YORK 10010

DEPARTMENT OF MATHEMATICS, QUEENS COLLEGE, (CUNY), FLUSHING, NEW YORK 11367