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3-PSEUDOMANIFOLDS WITH PREASSIGNED LINKS

BY

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ABSTRACT. A 3-pseudomanifold is a finite connected simplicial 3-complex $\mathcal{K}$ such that every triangle in $\mathcal{K}$ belongs to precisely two 3-simplices of $\mathcal{K}$, the link of every edge in $\mathcal{K}$ is a circuit, and the link of every vertex in $\mathcal{K}$ is a closed 2-manifold. It is proved that for every finite set Σ of closed 2-manifolds, there exists a 3-pseudomanifold $\mathcal{K}$ such that the link of every vertex in $\mathcal{K}$ is homeomorphic to some $S \in \Sigma$, and every $S \in \Sigma$ is homeomorphic to the link of some vertex in $\mathcal{K}$.

I. Introduction. The term *pseudomanifold* appears in the literature in more than one meaning (see, e.g., [**4**] and [**10**]). We adopt here the meaning given to it by Pachner [**10**], [**11**]. (In fact, Pachner's questions [**11**, Problems 2, 3] motivated the present article.) For the reader's convenience we repeat here the definition. However, since our interest in this paper is restricted to the (simplicial) 3-dimensional case only, we find it convenient to modify Pachner's general definition [**10**, Definition 8], which is inductive in nature, to the simplicial 3-dimensional case. First we introduce some terminology.

Let $\mathcal{K}$ be a simplicial n-complex. The elements of $\mathcal{K}$ are its *faces*; the 0-elements (1-elements, 2-elements) are also called *vertices* (*edges*, *triangles*) and the n-elements, *facets*. For every $A \in \mathcal{K}$ define $\mathrm{st}(A, \mathcal{K})$, the *star* of A in $\mathcal{K}$, to be the set $\{B \in \mathcal{K}: A \subset B\}$; define $\mathrm{ast}(A, \mathcal{K})$, the *antistar* of A in $\mathcal{K}$, to be the complex $\{B \in \mathcal{K}: A \cap B = \varnothing\}$; define $\mathrm{clst}(A, \mathcal{K})$ to be the smallest subcomplex of $\mathcal{K}$ which contains $\mathrm{st}(A, \mathcal{K})$, and define $\mathrm{link}(A, \mathcal{K})$, the *link* of A in $\mathcal{K}$, to be $\mathrm{clst}(A, \mathcal{K}) \cap \mathrm{ast}(A, \mathcal{K})$. All the manifolds mentioned in this paper are connected and without boundary.

DEFINITION 1. *A 3-pseudomanifold* (briefly: 3-pm) is a finite connected simplicial 3-complex $\mathcal{K}$ such that:

(i) every triangle in $\mathcal{K}$ is a face of precisely two facets of $\mathcal{K}$;

(ii) for every edge $A \in \mathcal{K}$, $|\mathrm{link}(A, \mathcal{K})|$ is a (connected) 1-manifold, i.e., a circuit;

(iii) for every vertex $x \in \mathcal{K}$, $|\mathrm{link}(x, \mathcal{K})|$ is a (connected) 2-manifold, i.e.

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an orientable or nonorientable surface, without boundary.

Notice that under this definition, the complexes mentioned in [3, Remarks 4.3, 4.4] are 3-pm's. Also notice that if in condition (ii), instead of being a 2-manifold, $|\mathrm{link}(x, \mathcal{K})|$ is required–for every vertex $x \in \mathcal{K}$–to be a 2-sphere, then $\mathcal{K}$ becomes just a combinatorial 3-manifold. We use here the term "combinatorial 3-manifold" in the sense of [3], which is, essentially, a triangulation of a 3-manifold. Clearly, every combinatorial 3-manifold is a 3-pm.

DEFINITION 2. Let Σ be a finite set of 2-manifolds. Σ is said to be *pm-realizable* if there exists a 3-pm $\mathcal{K}$ such that for every vertex $x \in \mathcal{K}$, $|\mathrm{link}(x, \mathcal{K})|$ is homeomorphic to some $S \in \Sigma$, and, on the other hand, for every $S \in \Sigma$ there is some vertex $x \in \mathcal{K}$ such that $|\mathrm{link}(x, \mathcal{K})|$ is homeomorphic to S. In this case we also say that $\mathcal{K}$ *pm-realizes* Σ.

Our main result can now be stated as follows:

THEOREM 1. *Every finite set Σ of 2-manifolds is pm-realizable.*

In particular, by letting Σ consist of just one 2-manifold, we get

THEOREM 2. *For every 2-manifold S, there is a 3-pm in which the link of every vertex is homeomorphic to S.*

Theorem 2 will be proved first, and will form a major step in the proof of Theorem 1. The proof of Theorem 1 is constructive: we shall describe inductively how to construct the 3-pm pm-realizing Σ. The construction is obtained by assembling some copies of certain three basic 3-pm's.

Where no confusion is likely to arise, we shall often consider a triangulation of a manifold as the manifold itself (see, e.g., the statement of Theorem 2 above). Thus we will often say "$\mathrm{link}(x, \mathcal{K})$ is a torus" where, strictly speaking, $\mathrm{link}(x, \mathcal{K})$ is a triangulation of a torus, and $|\mathrm{link}(x, \mathcal{K})|$ is the torus. Recall that all our manifolds are connected and without boundary.

As is well known, every abstract finite simplicial n-complex can be rectilinearly embedded in the $(2n + 1)$-dimensional Euclidean space. Thus we may–and do–deal with our complexes–which are all simplicial–as abstract complexes. An n-simplex whose vertices are $a_0, a_1, \ldots, a_n$ is denoted by $a_0 a_1 \cdots a_n$.

In §2 we introduce some preparatory lemmas, as well as the basic process of *assembling* certain 3-pm's to yield another 3-pm. In §3 we first prove Theorem 2 by assembling some copies of two specific 3-pm's $\mathcal{T}$ and $\mathcal{P}$, which form the "heart" of the entire work, and next we prove the main Theorem 1 by using an additional specific 3-pm $\mathcal{C}$. In §4 we study in detail the structure of the 3-pm $\mathcal{T}$, and the 3-pm $\mathcal{P}$ is similarly investigated in §5. The 3-pm $\mathcal{C}$ is

a 3-sphere which is well known in the literature, and needs no special investigation here.

In the proof of Theorem 1, no effort was made to be economical, i.e., the 3-pm $\mathfrak{M}$ constructed to realize the given set Σ of 2-manifolds usually possesses many redundant vertices. Therefore we show, in §6, how the number of vertices of $\mathfrak{M}$ can often be reduced. Finally, in §7, we investigate the possibility of embedding 3-pm's in the boundary complexes of convex polytopes.

A *convex d-polytope* (briefly: d-polytope) is a d-dimensional convex body in a Euclidean space, which is the convex hull of a finite set of points. Thus a d-polytope can be considered a subset of $\mathbf{R}^d$. The boundary of a d-polytope is well known to be homeomorphic to a $(d-1)$-sphere. A particularly interesting case of a d-polytope is the *cyclic* d-polytope, which is defined as follows. In $\mathbf{R}^d$ consider the *moment curve* M_d defined parametrically by $x(t) = (t, t^2, \ldots, t^d)$. A cyclic d-polytope $C(v, d)$ is the convex hull of $v \geqslant d+1$ distinct points $x(t_i)$ $(1 \leqslant i \leqslant v)$ on M_d. Those v points $x(t_i)$ are the vertices of $C(v, d)$, and $C(v, d)$ has the remarkable property of being $[d/2]$-neighborly, i.e., for every positive integer k such that $2k \leqslant d$, every k vertices of $C(v, d)$ determine a $(k-1)$-dimensional face of $C(v, d)$.

Our terminology and notation, in particular, that related to convex polytopes, follows Grünbaum's Convex polytopes [**6**]. To denote the end of a proof we use the sign. $\square$

2. Preliminary results and basic constructions. For every simplicial complex $\mathfrak{M}$, we denote by $f_i(\mathfrak{M})$ the number of i-faces of $\mathfrak{M}$, and by $f_i(v, \mathfrak{M})$–v being a vertex in $\mathfrak{M}$–the number of i-faces of $\mathfrak{M}$ incident to v. $\mathrm{Skel}_k\mathfrak{M}$, the k-skeleton of $\mathfrak{M}$, is the subcomplex of $\mathfrak{M}$ composed of all the i-faces of $\mathfrak{M}$, where $i \leqslant k$. $\mathrm{skel}_0\mathfrak{M}$, the set of all the vertices of $\mathfrak{M}$, is also denoted by $\mathrm{vert}\,\mathfrak{M}$. $\mathfrak{M}$ is *k-neighborly* iff every k vertices of $\mathfrak{M}$ form the vertices of some $(k-1)$-face of $\mathfrak{M}$. The *boundary complex* $\mathrm{bd}\,\Delta$ of a convex polytope Δ is the complex of all the i-faces of Δ, where $i < \dim \Delta$. A simplex Δ is a *missing face* of $\mathfrak{M}$ if all the vertices of Δ belong to $\mathfrak{M}$, but Δ itself does not belong to $\mathfrak{M}$. The *f-vector* $f(\mathfrak{M})$ of $\mathfrak{M}$ is the vector $(f_0(\mathfrak{M}), f_1(\mathfrak{M}), \ldots, f_n(\mathfrak{M}))$, where n is the dimension of $\mathfrak{M}$.

For a 2-manifold S, we denote by $q(S)$ the connectivity of S, and by $\chi(S)$ the Euler characteristic of S [**2**, §5.1]. Thus Euler's equation for a triangulated 2-manifold S states that

$$\chi(S) = f_0(S) - f_1(S) + f_2(S) = 2 - q(S).$$

If, in addition, S is orientable, then $q(S) = 2g(S)$, where $g(S)$ denotes the genus of S.

For a combinatorial 3-manifold $\mathfrak{M}$, the well-known Euler-Poincaré relation states that $\Sigma_{i=0}^{3}(-1)^i f_i(M) = 0$. The analogous relation for 3-pm's is given by the following lemma. Since a combinatorial 3-manifold is a particular case of a 3-pm, it can be considered a generalization of the above Euler-Poincaré relation.

LEMMA 3. *For every 3-pm $\mathfrak{M}$, the equalities*
(a) $f_2(\mathfrak{M}) = 2f_3(\mathfrak{M})$,
(b) $\Sigma_{i=0}^{3}(-1)^i f_i(\mathfrak{M}) = \frac{1}{2}\Sigma_v q(\mathrm{link}(v, \mathfrak{M}))$
hold, where the sum on the right side of (b) *ranges over all the vertices v of $\mathfrak{M}$.*

PROOF. Every triangle in $\mathfrak{M}$ belongs to two facets of $\mathfrak{M}$, and every facet of $\mathfrak{M}$ contains four triangles. Hence a double counting of the triangles in $\mathfrak{M}$ yields $2f_2(\mathfrak{M}) = 4f_3(\mathfrak{M})$, and (a) follows.

There is an obvious 1-1 correspondence between the i-faces in $\mathrm{link}(v, \mathfrak{M})$ and the $(i+1)$-simplices of $\mathfrak{M}$ which contain the vertex v (i.e., the $(i+1)$-simplices in $\mathrm{st}(v, \mathfrak{M})$). Therefore $f_i(\mathrm{link}(v, \mathfrak{M})) = f_{i+1}(v, \mathfrak{M})$ holds for every vertex $v \in \mathfrak{M}$ and for every i, $0 \leqslant i \leqslant 2$. Thus Euler's equation for 2-manifolds yields

$$f_1(v, \mathfrak{M}) - f_2(v, \mathfrak{M}) + f_3(v, \mathfrak{M}) = 2 - q(\mathrm{link}(v, \mathfrak{M})).$$

Summing over all the $f_0(\mathfrak{M})$ vertices of $\mathfrak{M}$, we get

$$\sum_v f_1(v, \mathfrak{M}) - \sum_v f_2(v, \mathfrak{M}) + \sum_v f_3(v, \mathfrak{M}) = 2f_0(\mathfrak{M}) - \sum_v q(\mathrm{link}(v, \mathfrak{M})). \qquad (*)$$

Since each i-face in $\mathfrak{M}$ contains $i+1$ vertices, each i-face of $\mathfrak{M}$ is counted $i+1$ times in $\Sigma_v f_i(v, \mathfrak{M})$, and it follows that $\Sigma_v f_i(v, \mathfrak{M}) = (i+1)f_i(\mathfrak{M})$ for $0 \leqslant i \leqslant 3$. Thus we obtain from $(*)$ that

$$2f_1(\mathfrak{M}) - 3f_2(\mathfrak{M}) + 4f_3(\mathfrak{M}) = 2f_0(\mathfrak{M}) - \sum_v q(\mathrm{link}(v, \mathfrak{M})).$$

Now part (b) follows easily from the last equation and from (a). □

COROLLARY 4. *In every 3-pm $\mathfrak{M}$, the number of vertices $v \in \mathfrak{M}$ for which $q(\mathrm{link}(v, \mathfrak{M}))$ is odd, is even. In particular, there is no 3-pm with seven vertices in which the link of every vertex is a projective plane.*

COROLLARY 5. *If $\mathfrak{M}$ is a 3-pm in which the link of every vertex is either a torus or Klein's bottle, then $f_1(\mathfrak{M}) = f_3(\mathfrak{M})$.*

PROOF. For every vertex $v \in \mathfrak{M}$ we have $q(\mathrm{link}(v, \mathfrak{M})) = 2$, hence $\Sigma_v q(\mathrm{link}(v, \mathfrak{M})) = 2f_0(\mathfrak{M})$, and our corollary follows from Lemma 3. □

For every 2-manifold S, let $\mu(S)$ denote the minimal number of vertices

with which S can be triangulated. Thus, as is well known, μ(2-sphere) = 4, μ(projective plane) = 6, μ(torus) = 7 and μ(Klein's bottle) = 8 (see [5]). The following lemma is obvious:

LEMMA 6. *If Σ is a finite set of 2-manifolds and $\mathfrak{M}$ is a 3-pm pm-realizing Σ, then $f_0(\mathfrak{M}) \geqslant 1 + \max\{\mu(S): S \in \Sigma\}$.*

The next definition and lemma establish the process of assembling certain 3-pm's to yield another 3-pm. This process plays a central role in the entire work.

DEFINITION 3. Let $\mathfrak{M}$ and $\mathfrak{N}$ be 3-complexes which share some common 3-simplices $\Delta_1, \ldots, \Delta_n$. The complex $\mathfrak{K} = \mathfrak{M} \cup \mathfrak{N} \setminus \{\Delta_1, \ldots, \Delta_n\}$ is said to be obtained by *assembling* $\mathfrak{M}$ and $\mathfrak{N}$ at $\Delta_1, \ldots, \Delta_n$.

Note that in the last definition, only the 3-simplices $\Delta_1, \ldots, \Delta_n$, and not any of their proper faces, are removed from $\mathfrak{M} \cup \mathfrak{N}$ to yield the complex $\mathfrak{K}$.

LEMMA 7. *Let $\mathfrak{M}$ and $\mathfrak{N}$ be 3-pm's whose intersection is a (simplicial) 3-complex $\mathfrak{L}$ composed of certain 3-simplices $\Delta_1, \ldots, \Delta_n$, which are pairwise disjoint (i.e., no two of which share a common vertex), and their faces. Let $\mathfrak{K}$ be the complex obtained from $\mathfrak{M}$ and $\mathfrak{N}$ by assembling them at $\Delta_1, \ldots, \Delta_n$. Then:*

(i) *$\mathfrak{K}$ is a 3-pm;*

(ii) *for every vertex $x \in \mathfrak{M} \setminus \mathfrak{L}$ we have* $\operatorname{link}(x, \mathfrak{K}) = \operatorname{link}(x, \mathfrak{M})$, *and similarly for every vertex $x \in \mathfrak{N} \setminus \mathfrak{L}$;*

(iii) *for every vertex $x \in \mathfrak{L}$ we have*

$$q(\operatorname{link}(x, \mathfrak{K})) = q(\operatorname{link}(x, \mathfrak{M})) + q(\operatorname{link}(x, \mathfrak{N})),$$

and $\operatorname{link}(x, \mathfrak{K})$ *is orientable iff both* $\operatorname{link}(x, \mathfrak{M})$ *and* $\operatorname{link}(x, \mathfrak{N})$ *are orientable.*

PROOF. The complex $\mathfrak{K}$ is 3-dimensional because of the pairwise disjointness of $\Delta_1, \ldots, \Delta_n$. It is also connected, since $\mathfrak{M}$ and $\mathfrak{N}$ are, and

$$\mathfrak{M} \cap \mathfrak{N} \setminus \{\Delta_1, \ldots, \Delta_n\} = \mathfrak{L} \setminus \{\Delta_1, \ldots, \Delta_n\} = \operatorname{skel}_2 \mathfrak{L} \neq \varnothing.$$

Let A be a 2-face of $\mathfrak{K}$. If $A \notin \operatorname{skel}_2\mathfrak{L}$, then A belongs either to $\mathfrak{M}$ or to $\mathfrak{N}$, but not to both. Say $A \in \mathfrak{M}$. In $\mathfrak{M}$ A belongs to two facets neither of which is in $\mathfrak{L}$, hence both are in $\mathfrak{K}$. If $A \in \operatorname{skel}_2\mathfrak{L}$, then A is a face of just one 3-simplex in $\mathfrak{L}$ (since the 3-simplices of $\mathfrak{L}$ are pairwise disjoint), say $A \in \operatorname{bd} \Delta_1$. Now in $\mathfrak{M}$, A belongs to two facets Δ_1, Δ', and in $\mathfrak{N}$, A belongs to two facets Δ_1, Δ''. Thus in $\mathfrak{K}$ A belongs precisely to the two facets Δ', Δ''.

Let xy be an edge of $\mathfrak{K}$ ($x, y \in \operatorname{vert} \mathfrak{K}$). If $xy \notin \mathfrak{L}$, then the edge xy is either in $\mathfrak{M}$ or in $\mathfrak{N}$ but not in both. Say $xy \in \mathfrak{M}$. Then clearly $\operatorname{link}(xy, \mathfrak{K}) = \operatorname{link}(xy, \mathfrak{M})$, and $|\operatorname{link}(xy, \mathfrak{K})|$ is therefore a circuit. If $xy \in \mathfrak{L}$, then both vertices x and y belong to the same 3-simplex Δ in $\mathfrak{L}$. Let a, b be the other

two vertices of Δ. Then the two circuits $\mathrm{link}(xy, \mathfrak{M})$, $\mathrm{link}(xy, \mathfrak{N})$ intersect at the common edge ab (and its vertices a, b) only. Thus $\mathrm{link}(xy, \mathfrak{K}) = \mathrm{link}(xy, \mathfrak{M}) \cup \mathrm{link}(xy, \mathfrak{N}) \setminus \{ab\}$, and this is a circuit.

Finally, let x be a vertex in $\mathfrak{K}$. If $x \notin \mathfrak{L}$, then x belongs to $\mathfrak{M}$ or to $\mathfrak{N}$ but not to both. If $x \in \mathfrak{M}$, then clearly $\mathrm{link}(x, \mathfrak{K}) = \mathrm{link}(x, \mathfrak{M})$ and therefore $\mathrm{link}(x, \mathfrak{K})$ is a 2-manifold. Similarly for $x \in \mathfrak{N}$. If $x \in \mathfrak{L}$, let a, b, c be the other three vertices in the unique 3-simplex in $\mathfrak{L}$ which contains x as a vertex. Then the two 2-manifolds $\mathrm{link}(x, \mathfrak{M})$, $\mathrm{link}(x, \mathfrak{N})$ intersect in the triangle abc (and its faces) only. Hence $\mathrm{link}(x, \mathfrak{K}) = \mathrm{link}(x, \mathfrak{M}) \cup \mathrm{link}(x, \mathfrak{N}) \setminus \{abc\}$, and $|\mathrm{link}(x, \mathfrak{K})|$ is clearly a 2-manifold, which is orientable iff $|\mathrm{link}(x, \mathfrak{M})|$ and $|\mathrm{link}(x, \mathfrak{N})|$ are. If we denote by f_i (f_i', f_i'') the number of i-faces $(0 \leqslant i \leqslant 2)$ in $\mathrm{link}(x, \mathfrak{K})$ ($\mathrm{link}(x, \mathfrak{M})$, $\mathrm{link}(x, \mathfrak{N})$ resp.), then clearly

$$f_0 = f_0' + f_0'' - 3, \quad f_1 = f_1' + f_1'' - 3, \quad f_2 = f_2' + f_2'' - 2;$$

hence

$$\begin{aligned} q(\mathrm{link}(x, \mathfrak{K})) &= 2 - f_0 + f_1 - f_2 \\ &= (2 - f_0' + f_1' - f_2') + (2 - f_0'' + f_1'' - f_2'') \\ &= q(\mathrm{link}(x, \mathfrak{M})) + q(\mathrm{link}(x, \mathfrak{N})). \quad \square \end{aligned}$$

The next lemma is a useful variation of Lemma 7.

LEMMA 8. *Let $\mathfrak{M}$, $\mathfrak{N}$ and $\mathfrak{J}$ be 3-pm's such that $\mathfrak{M}$ intersects $\mathfrak{J}$ at a common facet X (and its faces, i.e., $\mathfrak{M} \cap \mathfrak{J}$ is a 3-simplex X and its faces), $\mathfrak{N}$ intersects $\mathfrak{J}$ at a common facet Y (and its faces), and $\mathfrak{M} \cap \mathfrak{N} = \varnothing$. Then the complex $\mathfrak{K} = (\mathfrak{M} \cup \mathfrak{N}) \cup \mathfrak{J} \setminus \{X, Y\}$ obtained by assembling $\mathfrak{M} \cup \mathfrak{N}$ and $\mathfrak{J}$ at X and Y is a 3-pm, and for every vertex $x \in \mathfrak{K}$ we have:*

If x is not a vertex of either X or Y, then the link of x is not affected by the assembly, i.e., if $x \in \mathrm{vert}\,\mathfrak{M} \setminus \mathrm{vert}\, X$ then $\mathrm{link}(x, \mathfrak{K}) = \mathrm{link}(x, \mathfrak{M})$, and similarly for $x \in \mathrm{vert}\,\mathfrak{N} \setminus \mathrm{vert}\, X$ and for $x \in \mathrm{vert}\,\mathfrak{J} \setminus \mathrm{vert}\, X \cup Y$.

If $x \in \mathrm{vert}\, X$ then $q(\mathrm{link}(x, \mathfrak{K})) = q(\mathrm{link}(x, \mathfrak{M})) + q(\mathrm{link}(x, \mathfrak{J}))$, and $\mathrm{link}(x, \mathfrak{K})$ is orientable iff both $\mathrm{link}(x, \mathfrak{M})$ and $\mathrm{link}(x, \mathfrak{J})$ are orientable.

If $x \in \mathrm{vert}\, Y$ then $q(\mathrm{link}(x, \mathfrak{K})) = q(\mathrm{link}(x, \mathfrak{N})) + q(\mathrm{link}(x, \mathfrak{J}))$, and $\mathrm{link}(x, \mathfrak{K})$ is orientable iff both $\mathrm{link}(x, \mathfrak{N})$ and $\mathrm{link}(x, \mathfrak{J})$ are orientable.

PROOF. Apply Lemma 7 first to $\mathfrak{M}$ and $\mathfrak{J}$ at X, and next to $\mathfrak{M} \cup \mathfrak{J} \setminus \{X\}$ and $\mathfrak{N}$ at Y. $\square$

Notice that in Lemma 8, the 3-pm $\mathfrak{J}$ serves as a joint by which $\mathfrak{M}$ and $\mathfrak{N}$ are fastened together (while in Lemma 7 $\mathfrak{M}$ and $\mathfrak{N}$ are fastened together without such a joint). Since the 3-simplices X, Y are disjoint, the joint $\mathfrak{J}$ should contain at least eight vertices. Later we will use three distinct "basic" 3-pm's $\mathfrak{T}$, $\mathfrak{P}$ and $\mathfrak{C}$ to serve as such joints, and each of them will have

precisely eight vertices. If the joint $\mathcal{J}$ has precisely eight vertices, then each of the 3-simplices X, Y is uniquely determined by the other, and forms the *complement* of the other. Together they form a *complementary* pair.

Also note that the joint $\mathcal{J}$ occupies four vertices of $\mathcal{M}$ and four vertices of $\mathcal{N}$. The usual situation in later applications of Lemma 8 will be that $\mathcal{N}$ is the image of $\mathcal{M}$ under some combinatorial equivalence φ. We would like to fasten $\mathcal{M}$ and $\mathcal{N} = \varphi(\mathcal{M})$ together by more than one joint, such that all the joints are disjoint to each other, each joint joins a facet $X \in \mathcal{M}$ to its counterpart $Y = \varphi(X)$ in $\mathcal{N}$, and together all the joints occupy all the vertices of $\mathcal{M}$ (and hence of $\mathcal{N}$ too). For this to be possible, $\mathcal{M}$ must possess a certain structure, a *cover*, in terms of the next definition.

DEFINITION 4. Let $\mathcal{M}$ be a 3-pm, and let F be a subset of the set of facets of $\mathcal{M}$. F is called a *cover* of $\mathcal{M}$ if every vertex of $\mathcal{M}$ belongs to some $X \in F$, and no two elements of F share a common vertex. Clearly, if $\mathcal{M}$ has a cover of cardinality m, then $\mathcal{M}$ has precisely $4m$ vertices. If $F = \{\Delta_1, \Delta_2\}$ is a cover of $\mathcal{M}$ (and $\mathcal{M}$ thus has eight vertices), then the pair of facets Δ_1, Δ_2 is also called a *complementary pair* in $\mathcal{M}$, and each of Δ_1, Δ_2 is the complement of the other (with respect to $\mathcal{M}$).

Note that two distinct covers of the same 3-pm $\mathcal{M}$ are not necessarily disjoint, unless $\mathcal{M}$ has precisely eight vertices and therefore each cover of $\mathcal{M}$ is of cardinality two (a complementary pair), in which case each facet in the cover uniquely determines its complement.

DEFINITION 5. Let $\mathcal{M}$ be a 3-pm with cover $F = \{X_1, \ldots, X_m\}$, let $\mathcal{N}$ be a 3-pm combinatorially equivalent to $\mathcal{M}$ such that $\mathcal{M} \cap \mathcal{N} = \varnothing$, and let φ be a combinatorial equivalence such that $\varphi(\mathcal{M}) = \mathcal{N}$. (Then clearly $f_0(\mathcal{M}) = f_0(\mathcal{N}) = 4m$, and $\varphi(F) = \{\varphi(X_1), \ldots, \varphi(X_m)\}$ is a cover of $\mathcal{N}$.) Let $\mathcal{J}_1, \ldots, \mathcal{J}_m$ be pairwise disjoint 3-pm's such that X_i, $\varphi(X_i) \in \mathcal{J}_i$, $\operatorname{vert} \mathcal{M} \cap \operatorname{vert} \mathcal{J}_i = \operatorname{vert} X_i$ and $\operatorname{vert} \mathcal{N} \cap \operatorname{vert} \mathcal{J}_i = \operatorname{vert} \varphi(X_i)$ for every $1 \leqslant i \leqslant m$. The complex

$$\mathcal{K} = \mathcal{M} \cup \mathcal{N} \cup \mathcal{J}_1 \cup \cdots \cup \mathcal{J}_m \setminus \bigcup_{i=1}^{m} \{X_i, \varphi(X_i)\}$$

is said to be obtained by *totally assembling* $\mathcal{M}$ *and its image* $\varphi(\mathcal{M})$ *at the cover* F, *with joints* $\mathcal{J}_1, \ldots, \mathcal{J}_m$, or, if φ and F play no role, we say that $\mathcal{K}$ is obtained by *totally assembling* $\mathcal{M}$ *and its image, with joints* $\mathcal{J}_1, \ldots, \mathcal{J}_m$.

LEMMA 9. *In the notation of Definition* 5, *the complex* $\mathcal{K}$ *is a* 3-*pm, and for every vertex* $x \in \mathcal{K}$ *we have*:

(a) *If* $x \in \operatorname{vert} X_i$ *for some* $1 \leqslant i \leqslant m$, *then* $q(\operatorname{link}(x, \mathcal{K})) = q(\operatorname{link}(x, \mathcal{M})) + q(\operatorname{link}(x, \mathcal{J}_i))$, *and* $\operatorname{link}(x, \mathcal{K})$ *is orientable iff both* $\operatorname{link}(x, \mathcal{M})$ *and* $\operatorname{link}(x, \mathcal{J}_i)$ *are orientable.*

(b) *If* $x \in \operatorname{vert} \varphi(X_i)$ *for some* $1 \leqslant i \leqslant m$, *then* $q(\operatorname{link}(x, \mathcal{K})) =$

$q(\mathrm{link}(x, \varphi(\mathfrak{M}))) + q(\mathrm{link}(x, \mathcal{J}_i))$, *and* $\mathrm{link}(x, \mathcal{K})$ *is orientable iff both* $\mathrm{link}(x, \varphi(\mathfrak{M}))$ *and* $\mathrm{link}(x, \mathcal{J}_i)$ *are orientable.*

(c) *If x is not a vertex of any X_i or $\varphi(X_i)$, $1 \leqslant i \leqslant m$, then the link of x is not affected by the assembly.*

PROOF. Define inductively:

$\mathfrak{N}^1 = \mathfrak{M} \cup \varphi(\mathfrak{M}) \cup \mathcal{J}_1 \setminus \{X_1, \varphi(X_1)\}$,

$\mathfrak{N}^k = \mathfrak{N}^{k-1} \cup \mathcal{J}_k \setminus \{X_k, \varphi(X_k)\}$ for $2 \leqslant k \leqslant m$.

We claim that for every $1 \leqslant k \leqslant m$, $\mathfrak{N}^k$ is a 3-pm, and that the assertions (a), (b), (c), modified by replacing $\mathcal{K}$ by $\mathfrak{N}^k$ and m by k, hold. Indeed, this is true for $k = 1$ by Lemma 8, and for every $1 < k \leqslant m$ by inductively using Lemma 7.

Finally notice that $\mathcal{K} = \mathfrak{N}^m$, and thus the proof of our lemma is complete. □

REMARK. If, in the notation of Definition 5, $f_0(\mathcal{J}_i) = 8$ for every $1 \leqslant i \leqslant 8$–and this is the usual situation in the present article–then no vertex of $\mathcal{K}$ belongs to the category (c) of Lemma 9.

3. Proof of the main theorem. The proof of Theorem 1 is done in three stages. First we prove it in the particular case (Theorem 10) where $\Sigma = \{S\}$ and S is orientable. Here we start with certain basic 3-pm $\mathcal{T}$, and using Lemma 9 and copies of $\mathcal{T}$, we inductively construct a pm-realization for $\{S\}$. Next we prove another particular case of Theorem 1, namely the case where $\Sigma = \{S\}$ and S is not orientable (Theorem 11). The proof follows exactly the same lines of the proof of Theorem 10, only that the basic 3-pm $\mathcal{T}$ is replaced by another basic 3-pm, denoted by $\mathcal{P}$. Thus Theorems 10, 11 together yield Theorem 2. Finally, for the proof of Theorem 1, we first use Theorem 2, for pm-realizing each element of Σ separately, and next we assemble those pieces, using Lemma 8 and some copies of a third basic 3-pm, denoted by $\mathcal{C}$.

Each of the basic 3-pm's $\mathcal{T}$, $\mathcal{P}$ and $\mathcal{C}$ should have exactly eight vertices and the following conditions should be satisfied:

(1) The link of every vertex in $\mathcal{T}$ is a torus; in $\mathcal{P}$, a projective plane; and in $\mathcal{C}$, a 2-sphere (thus $\mathcal{C}$ is simply a 3-manifold).

(2) Each of $\mathcal{T}$ and $\mathcal{P}$ has at least two complementary pairs of facets (i.e., covers of cardinality 2), and $\mathcal{C}$ has at least one complementary pair of facets.

As the 3-pm $\mathcal{C}$, we may take the boundary complex of $C(8, 4)$, the cyclic 4-polytope with eight vertices (see [**6**, §4.7]). It is a 3-sphere, and is easily seen to have the required properties.

The 3-pm's $\mathcal{T}$ and $\mathcal{P}$ are described in Tables 1 and 2, respectively. It is not difficult to check directly that each of them satisfies condition (1) above, and we also do it explicitly and economically in the next two sections. As for condition (2), it is readily seen that *each* facet both in $\mathcal{T}$ and in $\mathcal{P}$ has a

complement. Thus $\mathcal{T}$ with its 28 facets has 14 complementary pairs of facets, and $\mathcal{P}$ with its 24 facets has 12 complementary pairs of facets. In a sense, the 3-pm's $\mathcal{T}$ and $\mathcal{P}$ form the heart of the present work. Thus the next two sections are devoted to a detailed investigation of those two 3-pm's.

1. 1348	8. 4578	15. 2567	22. 1236
2. 1248	9. 1578	16. 3567	23. 2346
3. 2458	10. 1568	17. 1367	24. 2347
4. 2358	11. 1268	18. 1467	25. 3457
5. 3568	12. 2678	19. 1247	26. 1345
6. 3468	13. 2378	20. 1257	27. 1456
7. 4678	14. 1378	21. 1235	28. 2456

TABLE 1. The facets of $\mathcal{T}$

1. 1236	7. 1678	13. 4578	19. 2345
2. 1267	8. 1478	14. 3458	20. 2356
3. 1247	9. 1456	15. 3568	21. 2378
4. 1245	10. 1568	16. 3678	22. 2347
5. 1258	11. 1348	17. 3467	23. 2567
6. 1346	12. 1238	18. 2578	24. 4567

TABLE 2. The facets of $\mathcal{P}$

THEOREM 10. *For every integer $g \geqslant 0$, there exists a 3-pm in which the link of every vertex is an orientable 2-manifold of genus g (i.e., of connectivity $2g$).*

PROOF. For $g = 0$ the assertion is trivial, since every 3-manifold will do. We prove, by induction on g, that for every $g \geqslant 1$ there exists a 3-pm $\mathfrak{M}_g$ which satisfies the requirements of the theorem and, in addition, has a cover of cardinality 2^g.

For $g = 1$, we take the basic 3-pm $\mathcal{T}$ as $\mathfrak{M}_1$.

Assuming that $\mathfrak{M}_{g-1}$ has already been constructed, construct $\mathfrak{M}_g$ as follows. Denote $m = 2^{g-1}$, let $F = \{X_1, \ldots, X_m\}$ be a cover of $\mathfrak{M}_{g-1}$, let $\mathfrak{M}'_{g-1}$ be a copy of $\mathfrak{M}_{g-1}$, disjoint to $\mathfrak{M}_{g-1}$, and let φ be a combinatorial equivalence between $\mathfrak{M}_{g-1}$ and $\mathfrak{M}'_{g-1}$, $\varphi(\mathfrak{M}_{g-1}) = \mathfrak{M}'_{g-1}$. Denote $\varphi(X_i) = Y_i$, $1 \leqslant i \leqslant m$. For every $1 \leqslant i \leqslant m$, let $\mathcal{T}_i$ be a copy of the 3-pm $\mathcal{T}$, such that vert $\mathcal{T}_i$ = vert $X_i \cup$ vert Y_i and such that X_i, and therefore also Y_i, is a facet in $\mathcal{T}_i$. Thus X_i, Y_i form a complementary pair of facets in $\mathcal{T}_i$. Now define $\mathfrak{M}_g$ to be the complex obtained (see Definition 5) by totally assembling $\mathfrak{M}_{g-1}$ and its image $\varphi(\mathfrak{M}_{g-1})$ at the cover F, with joints $\mathcal{T}_1, \ldots, \mathcal{T}_m$.

By Lemma 9, $\mathfrak{M}_g$ is a 3-pm. Notice that

$$\bigcup_{i=1}^{m} (\text{vert } X_i \cup \text{vert } Y_i) = \text{vert } \mathfrak{M}_{g-1} \cup \text{vert } \mathfrak{M}'_{g-1} = \text{vert } \mathfrak{M}_g. \qquad (*)$$

Since the link of every vertex in $\mathfrak{M}_{g-1}$ is an orientable 2-manifold of connectivity $2(g-1)$ and the link of every vertex in $\mathfrak{T}$ is an orientable 2-manifold of connectivity 2 (a torus), it follows from Lemma 9 that the link of every vertex in $\mathfrak{M}_g$ is an orientable 2-manifold of connectivity $2(g-1)+2 = 2g$, i.e., of genus g.

Finally, we have to show that $\mathfrak{M}_g$ has a cover of cardinality 2^g. Since $\mathfrak{M}_{g-1}$ has a cover of cardinality 2^{g-1}, it has 2^{g+1} vertices. Thus $\mathfrak{M}_g$, by $(*)$, has 2^{g+2} vertices. Each $\mathfrak{T}_i$, being a copy of $\mathfrak{T}$, has, besides the cover $\{X_i, Y_i\}$, another cover $\{Z_i, W_i\}$. It is clear that the set $\bigcup_{i=1}^{m}\{Z_i, W_i\}$ forms the desired cover for $\mathfrak{M}_g$. □

THEOREM 11. *For every integer $q > 0$, there exists a 3-pm in which the link of every vertex is a nonorientable 2-manifold of connectivity q.*

PROOF. The proof follows the lines of the proof of Theorem 10. For every $q \geqslant 1$ we construct inductively a 3-pm $\mathfrak{M}^q$ which satisfies the requirements of the theorem and, in addition, has a cover of cardinality 2^q.

For $q = 1$, we take the basic 3-pm $\mathfrak{P}$ as $\mathfrak{M}^1$. Assuming that $\mathfrak{M}^{q-1}$ has already been constructed, we can proceed to construct $\mathfrak{M}^q$ in precisely the same manner as $\mathfrak{M}_g$ has been constructed from $\mathfrak{M}_{g-1}$ in the proof of Theorem 10, with the exception that copies $\mathfrak{P}_i$ of $\mathfrak{P}$ replace the copies $\mathfrak{T}_i$ of $\mathfrak{T}$. In other words, $\mathfrak{M}^q$ is obtained by totally assembling $\mathfrak{M}^{q-1}$ and its image at any cover of $\mathfrak{M}^{q-1}$ (which exists by the induction hypothesis), with joints $\mathfrak{P}_1, \ldots, \mathfrak{P}_{2^{q-1}}$, which are disjoint copies of $\mathfrak{P}$.

Lemma 9 and the fact that $\mathfrak{P}$ has at least two distinct complementary pairs of facets can be used here, as is done in the proof of Theorem 10, to show that $\mathfrak{M}^q$ satisfies the requirements. □

By this, Theorem 2 is proved also, since it is just a combination of Theorems 10, 11.

PROOF OF THEOREM 1. The proof is by induction on the cardinality $|\Sigma|$ of the set Σ if 2-manifolds, which is to be pm-realized. If $|\Sigma| = 1$, then Σ is pm-realized by Theorem 2. Assume the theorem has been proved for every suitable set Σ' of cardinality $< k$, and let Σ be a set of 2-manifolds, with $|\Sigma| = k$. Choose some $S \in \Sigma$, and let $\Sigma' = \Sigma \setminus \{S\}$. By the induction hypothesis, Σ' can be pm-realized. Let $\mathfrak{M}'$ be a 3-pm which pm-realizes Σ'. $\{S\}$ can be pm-realized by Theorem 2. Let $\mathfrak{M}''$ be a 3-pm pm-realizing $\{S\}$, such that $\mathfrak{M}' \cap \mathfrak{M}'' = \varnothing$.

We will now use the basic 3-pm $\mathcal{C}$, or rather a copy of it, as a joint by which $\mathfrak{M}'$ and $\mathfrak{M}''$ will be fastened together to yield the desired 3-pm $\mathfrak{M}$ which pm-realizes Σ.

Choose some facet $X \in \mathfrak{M}'$ and some facet $Y \in \mathfrak{M}''$. Take a copy of $\mathcal{C}$, which we denote again by $\mathcal{C}$, such that vert $\mathcal{C}$ = vert $X \cup$ vert Y and such that both X and Y are facets of C. This is possible, since the original $\mathcal{C}$ has eight vertices and contains a complementary pair of facets. Finally define $\mathfrak{M} = \mathfrak{M}' \cup \mathfrak{M}'' \cup \mathcal{C} \setminus \{X, Y\}$.

By Lemma 8, $\mathfrak{M}$ is a 3-pm, and clearly vert $\mathfrak{M}$ = vert $\mathfrak{M}' \cup$ vert $\mathfrak{M}''$. Since the link of every vertex in $\mathcal{C}$ is a 2-sphere, i.e., orientable and of connectivity 0, it also follows from Lemma 8 that for every vertex $x \in \mathfrak{M}$ we have that $|\text{link}(x, \mathfrak{M})|$ is homeomorphic to $|\text{link}(x, \mathfrak{M}')|$ if $x \in \mathfrak{M}'$, and to $|\text{link}(x, \mathfrak{M}'')|$, i.e., to S, if $x \in \mathfrak{M}''$. Thus $\mathfrak{M}$ pm-realizes Σ. $\square$

4. The 3-pseudomanifold $\mathfrak{T}$. We first constructed $\mathfrak{T}$ by using a simple and natural modification of the algorithm given in [3] for constructing 3-manifolds with eight vertices. (This method is applied here in detail, for constructing a certain 3-pm $\mathfrak{B}$, at the end of §6.) However, having $\mathfrak{T}$ at hand, we find that it can be much more easily constructed and described as follows.

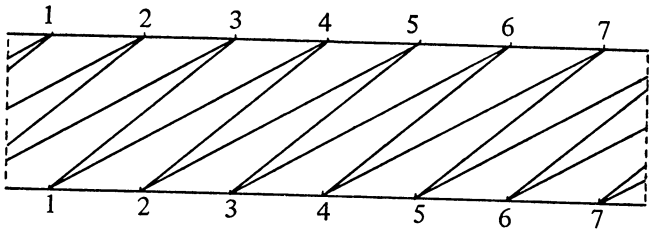


FIGURE 1. link(8, $\mathfrak{T}$)

Consider the minimal triangulation of the torus, shown in Figure 1. It has seven vertices, denoted $1, 2, \ldots, 7$, and 14 triangles. To each of the 14 triangles we "add" a fourth vertex 8, thus getting the 14 3-simplices denoted by numbers 1–14 in Table 1. For each of those 14 3-simplices we construct its "complement" with respect to the set $\{1, 2, \ldots, 8\}$, thus getting the 28 3-simplices listed in Table 1, which are precisely the facets of our 3-pm $\mathfrak{T}$. Here, for each $1 \leqslant i \leqslant 14$, the facets numbered i and numbered i + 14 form a complementary pair. Briefly we can say that the facets of $\mathfrak{T}$ are obtained by coning the 2-complex shown in Figure 1 from an eighth vertex 8, and taking complements.

So far we know only that $\mathfrak{T}$ is a simplicial 3-complex with 8 vertices and 28 facets, and that its facets split into 14 complementary pairs. It is not difficult to check directly that $\mathfrak{T}$ is a 3-neighborly 3-pm and that the link of every vertex of $\mathfrak{T}$ is a torus which contains all the other seven vertices of $\mathfrak{T}$. This is best done as follows.

Let $G(\mathfrak{T})$ denote the group of all the (combinatorial) automorphisms of $\mathfrak{T}$. By computation, one can see that $G(\mathfrak{T})$ contains the (automorphisms induced by the) permutation $\varphi = (1, 3, 4, 8, 6, 7, 2, 5)$, and hence, using powers of φ, $G(\mathfrak{T})$ is transitive on all the vertices of $\mathfrak{T}$. Since, by the construction, link$(8, \mathfrak{T})$ is a torus, we deduce that the link of every vertex in $\mathfrak{T}$ is a torus isomorphic (as a complex) to link$(8, \mathfrak{T})$ (Figure 1). $G(\mathfrak{T})$ also contains the permutation $\psi = (1, 2, 3, 4, 5, 6, 7)$ which preserves the vertex 8, hence, using powers of φ and ψ, $G(\mathfrak{T})$ is easily seen to be transitive on the ordered pairs of vertices of $\mathfrak{T}$. Thus $\mathfrak{T}$ is 2-neighborly and, since link$(12, \mathfrak{T})$ is a circuit of length 6, it follows that the link of every edge in $\mathfrak{T}$ is a circuit of length 6. $G(\mathfrak{T})$ also contains the permutation $\eta = (1, 3, 2, 6, 4, 5)$ which preserves the edge 78, and therefore, using powers of φ, ψ and η, $G(\mathfrak{T})$ is easily seen to be transitive on the ordered triples of vertices of $\mathfrak{T}$. Thus $\mathfrak{T}$ is 3-neighborly, and since the triangle 123 belongs to precisely two facets of $\mathfrak{T}$, the same holds for every triangle in $\mathfrak{T}$. $\mathfrak{T}$ is clearly connected.

Thus $\mathfrak{T}$ is indeed a 3-pm and satisfies all the requirements imposed in the beginning of §3, and enjoys a high degree of symmetry. Indeed, it follows from the above that the group $G(\mathfrak{T})$ is of order $8 \cdot 7 \cdot 6 = 336$ (since $G(\mathfrak{T})$ acts transitively on 2-simplices and only the identity keeps a particular 2-simplex pointwise fixed), and is transitive on the facets of $\mathfrak{T}$ (as unordered quadruples).

From the 3-neighborliness of $\mathfrak{T}$ it follows easily that the fundamental group of $|\mathfrak{T}|$ is trivial. It can also be easily checked that $\mathfrak{T}$ is orientable. Notice that

$$f_0(\mathfrak{T}) = 8, \quad f_1(\mathfrak{T}) = \binom{8}{2} = 28, \quad f_2(\mathfrak{T}) = \binom{8}{3} = 56, \quad f_3(\mathfrak{T}) = 28,$$

which agrees with Lemma 3 and Corollary 5.

Professor M. A. Perles suggested (private communication) that the construction of $\mathfrak{T}$, as well as all its properties mentioned above, can be elegantly obtained as follows.

Let l be a projective line over the finite field $F_7 = Z_7$ with seven elements. The eight points of l are the seven elements $0, 1, \ldots, 6$ of the field and the point ∞, at infinity. If the cross ratio of an ordered quadruple of points $a, b, c, d \in l$ satisfies $(a, b; c, d) = x$, then the 24 permutations of a, b, c, d yield the six cross ratios

$$x, \quad \frac{1}{x}, \quad 1 - x, \quad \frac{1}{1-x}, \quad \frac{x-1}{x}, \quad \frac{x}{x-1}. \tag{*}$$

Choosing $x \in Z_7$ such that

$$x(1 - x) = 1 \tag{**}$$

(i.e., $x = 3$ or $x = 5$), those six cross ratios reduce to only two values: $x = (x - 1)/x = 1/(1 - x)$ and $1/x = 1 - x = x/(x - 1)$. Now define $\mathfrak{T}'$

to be the (abstract) simplicial 3-complex whose facets are all the quadruples $a, b, c, d \in l$ such that either $(a, b; c, d) = x$ or $(a, b; c, d) = 1/x$. It is now easy to see that $\mathfrak{T}'$ is isomorphic to $\mathfrak{T}$.

From this construction, most of the properties of $\mathfrak{T}$ follow easily. For example, for every triple $a, b, c \in l$ there is a unique $d_1 \in l$ and a unique $d_2 \in l$ such that $(a, b; c, d_1) = x$, $(a, b; c, d_2) = 1/x$. Thus $\mathfrak{T}_1$ is 3-neighborly, and every triangle belongs to precisely two facets. We also mention that for every two ordered triples of points of l, there is a unique projective transformation of l which maps the first triple onto the second, and hence the group $G(\mathfrak{T})$ is isomorphic to the group of all the projective transformations of l, which is well known to be of order $8 \cdot 7 \cdot 6 = 336$.

A natural question is whether or not the above construction can be generalized to other finite fields and yield some other interesting 3-pm's. The following consideration, however, shows that F_7 is the only finite field which yields, in the above manner, a 3-pm.

Let F be a finite field and let l be a projective line over F. We assume that F is such that equation (**) has a solution $x \in F$. Then the six numbers in (*) reduce, as above, to only two distinct numbers. It follows that every unordered triple of points in l is contained in precisely two quadruples of points of l, such that the cross ratio of one of the quadruples is x, and the cross ratio of the other quadruple is $1/x$. Thus, the set of all the quadruples of points of l with cross ratio x or $1/x$ yields a simplicial 3-complex $\mathfrak{K}$, which is 3-neighborly.

The group of all the projective transformations of l transforms the set of all the ordered triples of points in l to itself. Therefore this projective group is transitive on all the ordered triples (and hence on the ordered pairs and on single points) in l, and also (since each of our quadruples can be ordered so that its cross ratio is x or $1/x$) on all the unordered quadruples.

We will now show that in $\mathfrak{K}$, the link of every edge is the union of disjoint circuits of length 6, and thus condition (ii) in the definition of a 3-pm (Definition 1) is not satisfied, unless $F = F_7$. (Note that from the 3-neighborliness of $\mathfrak{K}$ it follows that the link of every edge in $\mathfrak{K}$ contains all the other vertices of $\mathfrak{K}$.) Since the projective group acts transitively on edges, it is sufficient to consider one particular edge, and we choose the edge 0∞.

There exist points $c, d \in l$ such that $(0, \infty; c, d) = 1/x$. On the other hand, $(0, \infty; c, d) = c/d$, hence $d = cx$. Thus, c is a vertex in link$(0\infty, \mathfrak{K})$, $d = cx$ is a vertex adjacent to c in link$(0\infty, \mathfrak{K})$, cx^2 is a vertex adjacent to cx in link$(0\infty, \mathfrak{K})$, and so on. The length of the circuit in link$(0\infty, \mathfrak{K})$ which contains the point c is therefore the minimal $n > 0$ such that $x^n = 1$.

Now, from (**) it follows that $x^2 = x - 1$, which implies $x^3 = x(x - 1) = x^2 - x = (x - 1) - x = -1$, hence $x^6 = 1$. If the characteristic of F is not

equal to 2, $1 \neq -1$, and the minimal n for which $x^n = 1$ is either 6 or 2.

If $x^2 = 1$, then $x = 2$ and F has characteristic 3. Conversely, any field of characteristic 3 has a unique root of $x^2 - x + 1 = 0$.

The field of order 4 contains two distinct roots of $x^2 - x + 1 = 0$; they are cube roots of 1. In general, the fields of order 2^{2k+1}, $k \geqslant 0$, will have roots of $x^2 - x + 1$.

Now if $x^2 = 1$ the construction would give a circuit of length 2, which is absurd. If $x^3 = 1$, then all circuits would have length 3. Therefore, in order to achieve a 3-pm the field would have to be the field with 4 elements. Then the resulting 3-pm would have 5 vertices and would be the boundary of the 4-simplex. Otherwise, we have $n = 6$ and the field must be F_7, as asserted.

REMARK 1. (**) is obtained by equating the first and fourth numbers in (*), in order that the six numbers in (*) be reduced to only two. It is easy to see that this is essentially the only way to reduce the six numbers in (*) to only two, as follows.

A priori there are $\binom{6}{2} = 15$ ways to equate two of those six numbers, but because of the particular structure of (*), the 15 possibilities reduce to the following five:

$$\text{(a) } x = \frac{1}{x}, \qquad \text{(b) } x = 1 - x, \qquad \text{(c) } x = \frac{1}{1-x},$$
$$\text{(d) } x = \frac{x-1}{x}, \quad \text{(e) } x = \frac{x}{x-1}.$$

Case (b) yields that $x = \frac{1}{2}$; hence the six numbers in (*) reduce to $\frac{1}{2}$, 2, -1, which are three distinct elements of F if the characteristic of F is other than 3, and reduce to just one element if the characteristic of F is 3. The situation is similar in cases (a) and (e), while each of the cases (c), (d) yields our equation (**).

REMARK 2. Pachner [**11**, Problem 3] raised the question whether there exists an orientable 3-pm which contains a subdivision of the 2-skeleton of the 5-dimensional simplex. Our orientable 3-pm $\mathfrak{T}$ provides a positive answer to this question: since $\mathfrak{T}$ is 3-neighborly, it clearly contains the 2-skeleton of every n-dimensional simplex, $2 \leqslant n \leqslant 7$.

REMARK 3. The complex dual to $\mathfrak{T}$ was independently discovered by B. Grünbaum, and was called by him "the polystroma $\mathcal{H}_{2,1}$" (see [7]).

5. The 3-pseudomanifold $\mathcal{P}$. the eight vertices of $\mathcal{P}$ are denoted, like those of $\mathfrak{T}$, by 1, 2, . . . , 8. Consider the simplicial 2-complex shown in Figure 2. This is a triangulation of the projective plane, with seven vertices 2, 3, . . . , 8. The 3-pm $\mathcal{P}$ is obtained by coning this complex from a new vertex 1, and taking complements. More explicitly: to each of the 12 triangles in the triangulation shown in Figure 2 we "add" a fourth vertex 1, thus getting the twelve 3-simplices numbered 1–12 in Table 2; for each of those 12 3-simplices

we construct its "complement" with respect to the set $\{1, 2, \ldots, 8\}$, and thus get a list of 24 3-simplices (Table 2), which form the facets of our 3-pm $\mathcal{P}$. Here, for each $1 \leqslant i \leqslant 12$, the facets numbered i and $i + 12$ form a complementary pair.

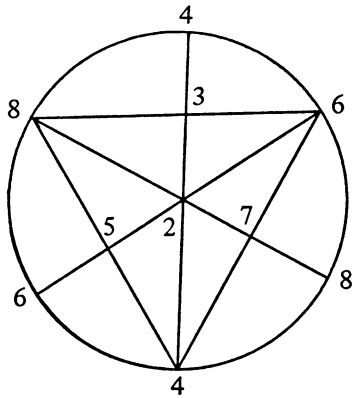


FIGURE 2. link$(1, \mathcal{P})$

It is not difficult to check directly that the complex $\mathcal{P}$ is indeed a 3-pm, and satisfies the requirements imposed on it at the beginning of §3. However, as in the previous section, this is best done by first considering the group $G(\mathcal{P})$ of all the (combinatorial) automorphisms of $\mathcal{P}$.

We find it convenient to use the following terminology. A face, all of whose vertices are denoted by even (odd) integers, will be called *an even* (*odd*) *face*. A face which is either even or odd is *homogeneous*. A face which is not homogeneous is *mixed*. A facet *abcd* is said to be a *product* of the edges *ab* and *cd* (or of *ac* and *bd*, etc.). Recall that a simplex $S \notin \mathcal{P}$ all of whose vertices belong to $\mathcal{P}$ is a *missing face* of $\mathcal{P}$. Using this terminology we now make the following observations concerning $\mathcal{P}$.

$\mathcal{P}$ is 2-neighborly but not 3-neighborly. The missing triangles are exactly all the homogeneous triangles. In other words, the missing triangles are exactly all the 2-faces of the two homogeneous missing facets of $\mathcal{P}$, namely 1357 and 2468. Each facet of $\mathcal{P}$ is the product of an odd edge by an even edge. Every product of an odd edge by an even edge yields a facet $\mathcal{P}$, except for the products hinted at in the following diagram:

$$\begin{array}{ccccccccc} 15 & — & 26\ 17 & — & 28\ 13 & — & 24 \\ | & & | \quad | & & | \quad | & & | \\ 48 & — & 37\ 46 & — & 35\ 68 & — & 57 \end{array}$$

From these observations we conclude the following about the structure of the group $G(\mathcal{P})$. $G(\mathcal{P})$ contains the subgroups G_1, G_2, G_3 where, in addition

to the identity, G_1 contains the permutations (1, 3)(5, 7), (1, 7)(3, 5) and (1, 5)(3, 7); G_2 contains (2, 6)(4, 8), (2, 8)(4, 6) and (2, 4)(6, 8); and G_3 contains (1, 2)(3, 4)(5, 6)(7, 8). G_1 and G_2 are Klein groups. It can be easily checked that G_1, G_2 and G_3 yield $4 \cdot 4 \cdot 2 = 32$ elements of $G(\mathcal{P})$, and that those elements yield that $G(\mathcal{P})$ is transitive on the vertices of $\mathcal{P}$. Thus the links of all the vertices in $\mathcal{P}$ are isomorphic to each other and, in particular, they all are isomorphic to link(1, $\mathcal{P}$) which, by construction, is the triangulated projective plane shown in Figure 2.

$G(\mathcal{P})$ is also transitive on the homogeneous edges of $\mathcal{P}$, and on the mixed edges as well. However, no element of $G(\mathcal{P})$ transforms a homogeneous edge to a mixed edge. (If $13 \to 25$ by an element of $G(\mathcal{P})$, then $135 \to 25x$ for some $x \in \{1, 3, 4, 6, 7, 8\}$, but 135 is a missing triangle of $\mathcal{P}$, while $25x$ is a face of $\mathcal{P}$.) It is easy to check that link(12, $\mathcal{P}$) and link(13, $\mathcal{P}$) are circuits of lengths 6 and 4, respectively. Thus it follows that the link of every mixed edge is a circuit of length 6, and that the link of every homogeneous edge in $\mathcal{P}$ is a circuit of length 4.

Thus, in order to prove that $\mathcal{P}$ is indeed a 3-pm, it remains only to show that every triangle in $\mathcal{P}$ belongs to exactly two facets. This is easy to do, and is left to the reader.

The order of $G(\mathcal{P})$ can be found as follows. One can use link(1, $\mathcal{P}$) to show that there are precisely six automorphisms of $\mathcal{P}$ which preserve pointwise the mixed edge 12. (Each such automorphism yields an automorphism of link(1, $\mathcal{P}$) which preserves the vertex 2.) They are: the identity, (6, 8)(5, 7) (3, 5)(4, 6), (3, 7)(4, 8), (3, 5, 7)(4, 6, 8) and (3, 7, 5)(4, 8, 6). Also, there are 32 ordered mixed edges xy in $\mathcal{P}$ to which the edge 12 can be transformed by $G(\mathcal{P})$. (Four possibilities for x even times four possibilities for y odd, and vice versa.) Thus the order of $G(\mathcal{P})$ is $32 \cdot 6 = 192$.

Using Corollary 6.3.9 of **[8]** it is not difficult to check that $|\mathcal{P}|$, like $|\mathcal{T}|$, is simply connected. $\mathcal{P}$ is also easily shown to be nonorientable. Notice that

$$f_0(\mathcal{P}) = 8, \quad f_1(\mathcal{P}) = \binom{8}{2} = 28, \quad f_2(\mathcal{P}) = 48, \quad f_3(\mathcal{P}) = 24,$$

which agrees with Lemma 3, since $q(\text{link}(x, \mathcal{P})) = 1$ for every vertex $x \in \mathcal{P}$.

The method with which we have constructed the 3-pm $\mathcal{P}$ very much resembles the first of the two methods with which the 3-pm $\mathcal{T}$ has been constructed in §4. It is therefore interesting to note that our third basic 3-pm $\mathcal{C}$ (i.e., the boundary complex of the cyclic polytope $C(8, 4)$) cannot be obtained in a similar manner.

6. Improvement of the 3-pseudomanifolds. Theorem 1 states that for every finite set Σ of 2-manifolds, there is a 3-pm $\mathcal{M}$ pm-realizing Σ. This 3-pm is by no means unique, and one may ask for a "best" such $\mathcal{M}$, in a certain sense. One may ask, for example, for a 3-pm for which the right side of the

second equation in Lemma 3 is minimal. Another possibility is to lexicographically minimize the *f-vector* of $\mathfrak{M}$ (i.e., the vector $(f_0(\mathfrak{M}), f_1(\mathfrak{M}), f_2(\mathfrak{M}), f_3(\mathfrak{M})))$. In the present section we make some (incomplete) efforts to minimize $f_0(\mathfrak{M})$, the number of vertices of $\mathfrak{M}$.

More specifically, we try to *improve* the 3-pm $\mathfrak{M}$ which pm-realizes the set $\Sigma = \{S\}$, where S is either an orientable or a nonorientable 3-manifold. We say that $\mathfrak{M}'$ *improves* $\mathfrak{M}$, if $\mathfrak{M}'$ and $\mathfrak{M}$ are 3-pm's pm-realizing the same set Σ, and $f_0(\mathfrak{M}') < f_0(\mathfrak{M})$. A 3-pm $\mathfrak{M}$ is a *best* pm-realization of Σ if $\mathfrak{M}$ pm-realizes Σ and $f_0(\mathfrak{M})$ is minimal.

Our basic 3-pm $\mathfrak{T}$, described in §4, is a best pm-realization of $\{T\}$, where T is a torus. This follows from Lemma 6 and from the fact that a minimal triangulation of the torus contains seven vertices (see, e.g., [5]). Moreover, since the minimal triangulation of the torus (shown in Figure 1) is unique, and since in a 3-pm with 8 vertices which pm-realizes $\{T\}$ the link of every vertex is a torus with seven vertices, our 3-pm $\mathfrak{T}$ is probably *the* best pm-realization of $\{T\}$ (i.e., it is probably unique).

A minimal triangulation of the projective plane P contains six vertices. Thus it follows from Lemma 6 that a 3-pm which pm-realizes $\{P\}$ should contain at least 7 vertices. But seven is impossible because of Corollary 4, and it thus follows that the basic 3-pm $\mathfrak{P}$ with its eight vertices is a best pm-realization of $\{P\}$.

In the following, we suggest some improvements of the 3-pm's $\mathfrak{M}_g$ and $\mathfrak{M}^q$, used in the proofs of Theorems 10 and 11. Any such improvement clearly yields an improvement of the 3-pm constructed in the proof of Theorem 1.

Let S_q denote an orientable 2-manifold of connectivity $q(S_q) = q$, and let S^q denote a nonorientable 2-manifold of connectivity $q(S^q) = q$. Then the 3-pm $\mathfrak{M}_g$ which has been constructed in the proof of Theorem 10 pm-realizes $\{S_{2g}\}$, and the 3-pm $\mathfrak{M}^q$ constructed in the proof of Theorem 11 pm-realizes $\{S^q\}$. Since $\mathfrak{M}_g$ has a cover of cardinality 2^g and M^q has a cover of cardinality 2^q, it follows that

$$f_0(\mathfrak{M}_g) = 2^{g+2}, \quad f_0(\mathfrak{M}^q) = 2^{q+2}$$

Thus there is a large gap between the numbers of vertices in the two 3-pm's which we know so far to pm-realize $\{S^q\}$ and $\{S_{2g}\}$, where q is approximately equal to $2g$. If, e.g., $q = 2g$, we have $f_0(\mathfrak{M}^{2g}) = 2^g \cdot f_0(\mathfrak{M}_g)$. This calls for an improvement of $\mathfrak{M}^q$, which we do as follows:

First improvement. Case (a). For $q = 2g + 1 \geqslant 3$ odd, define $\mathfrak{K}^q$ to be the complex obtained by totally assembling $\mathfrak{M}_g$ and its image (see Definition 5) with joints $\mathfrak{P}_1, \ldots, \mathfrak{P}_{2g}$, which are pairwise disjoint copies of $\mathfrak{P}$. Of course, we have to choose the notation of the vertices of each $\mathfrak{P}_i$ so that each $\mathfrak{P}_i$ shares a facet X_i with $\mathfrak{M}_g$, and shares with the image of $\mathfrak{M}_g$ the facet which

complements X_i with respect to $\mathcal{P}_i$. Recall from the proof of Theorem 10 that $\mathfrak{M}_g$ has a cover of cardinality 2^g, making possible the above construction. In the above notation, $\{X_1, \ldots, X_{2^g}\}$ is a cover of $\mathfrak{M}_g$.

It follows easily from Lemma 9 that $\mathcal{K}^q$ is a 3-pm which pm-realizes $\{S^q\}$, and clearly

$$f_0(\mathcal{K}^q) = 2 \cdot f_0(\mathfrak{M}_g) = 2^{g+3} = 2^{(q-1)/2+3}.$$

Notice that, like $\mathfrak{M}_g$ in the proof of Theorem 10 and $\mathfrak{M}^q$ in the proof of Theorem 11, $\mathcal{K}^q$ has a cover (of cardinality $2^{(q+1)/2}$).

Case (b). For $q \geqslant 4$ even, we first construct $\mathcal{K}^{q-1}$ as in Case (a), and next define $\mathcal{K}^q$ to be the complex obtained by totally assembling $\mathcal{K}^{q-1}$ and its image with $2^{q/2}$ joints which are pairwise disjoint copies of $\mathcal{P}$. (Recall that $2^{q/2}$ is the cardinality of a cover of $\mathcal{K}^{q-1}$. Here too, as in the previous case, the vertices of the joints should be chosen properly.) It follows from Lemma 9 that $\mathcal{K}^q$ is a 3-pm which pm-realizes $\{S^q\}$, and clearly

$$f_0(\mathcal{K}^q) = 2 \cdot f_0(\mathcal{K}^{q-1}) = 2^{q/2+3}.$$

If we define $\mathcal{K}^1 = \mathfrak{M}^1$, $\mathcal{K}^2 = \mathfrak{M}^2$, then it follows from the above that for every positive integer q, $\mathcal{K}^q$ is a 3-pm which pm-realizes $\{S^q\}$, and $f_0(\mathcal{K}^q) = 2^{[q/2]+3}$.

Second improvement. Step (a). Note that $\mathfrak{M}_2$, which pm-realizes $\{S_4\}$, has 16 vertices. We shall describe now a 3-pm $\mathcal{T}_2$ which pm-realizes $\{S_4\}$, and has only 12 vertices.

Note that the facets 1348 and 2567 form a complementary pair in $\mathcal{T}$ (see Table 1). Let $\mathcal{T}'$ be the 3-pm obtained from $\mathcal{T}$ by replacing the vertices 1, 3, 4, 8 by four new vertices 1′, 3′, 4′, 8′, respectively, and let $\mathcal{T}''$ be the 3-pm obtained from $\mathcal{T}$ by replacing the vertices 2, 5, 6, 7 by the same four new vertices 1′, 3′, 4′, 8′, respectively. (The change in the vertices implies, of course, some natural changes in the faces of higher dimensions.)

Now, we first assemble $\mathcal{T}$ and $\mathcal{T}'$ at 2567 and thus, by Lemma 7, we get a 3-pm $\mathcal{T}^*$ with 12 vertices, in which the link of each of the vertices 2, 5, 6, 7 is (homeomorphic to) S_4, and the link of each of the remaining vertices is a torus. Next, we assemble $\mathcal{T}^*$ and $\mathcal{T}''$ at 1348 and 1′3′4′8 and obtain–again by Lemma 7–a 3-pm $\mathcal{T}_2$ in which the link of every vertex is S_4, and $f_0(\mathcal{T}_2) = 12$. It is easy to see that $\mathcal{T}_2$ is 2-neighborly, and therefore the link of every vertex contains all the other 11 vertices. The f-vector of $\mathcal{T}_2$ is (12, 66, 156, 78).

In order to carry out the next step, it is important to note that $\mathcal{T}_2$ has a cover, e.g., {2458, 4′678′, 11′33′}.

Step (b). Since $\mathcal{T}_2$ has a cover, one can start an inductive process with $\mathcal{T}_2$ along the lines of the proof of Theorem 10, and construct for every integer $g \geqslant 2$ a 3-pm $\mathcal{T}_g$ which pm-realizes $\{S_{2g}\}$, and which has only $12 \cdot 2^{g-2} = 3 \cdot 2^g$ vertices. Thus $\mathcal{T}_g$ improves $\mathfrak{M}_g$ for $g \geqslant 2$. Finally define $\mathcal{T}_1 = \mathcal{T}$.

Step (c). In the beginning of the present section ("First improvement") we used $\mathfrak{M}_g$ to obtain a 3-pm $\mathfrak{K}^q$ which improves $\mathfrak{M}^q$ where $q \geqslant 3$. In exactly the same manner, we can now use $\mathfrak{T}_g$ to obtain, for every integer $q \geqslant 5$, a 3-pm $\mathfrak{P}^q$ which improves $\mathfrak{K}^q$, since $\mathfrak{P}^q$ is a pm-realization of $\{S^q\}$, and $f_0(\mathfrak{P}^q) = 3 \cdot 2^{[q/2]+1}$.

Step (d). Is it possible to improve $\mathfrak{K}^q$ where $q \leqslant 4$? For $q = 1$ this is impossible, since $\mathfrak{K}^1 = \mathfrak{M}^1 = \mathfrak{P}$, and, as already shown in the beginning of the present section, $\mathfrak{P}$ is a best pm-realization of $\{S^1\}$. For $q = 2$, however, $\mathfrak{K}^2 = \mathfrak{M}^2$ can be improved by just repeating the construction of $\mathfrak{T}_2$ in Step (a), with the letter $\mathfrak{P}$ replacing $\mathfrak{T}$ ($\mathfrak{P}'$ replacing $\mathfrak{T}'$, etc.). We do not even have to change the notation of the vertices, since the facets 1348, 2567 appear in $\mathfrak{P}$, and form a complementary pair there. The resulting complex $\mathfrak{P}^2$ is a 3-pm which pm-realizes $\{S^2\}$, and has precisely 12 vertices. $\mathfrak{P}^2$, like $\mathfrak{T}^2$, is 2-neighborly, and therefore the link of every vertex in $\mathfrak{P}^2$ is a triangulation of the Klein bottle with 11 vertices (while the minimal triangulation of the Klein bottle has eight vertices, see [5]). It is easy to see that $f(\mathfrak{P}^2) = (12, 66, 132, 66)$. Again it is important to note that $\mathfrak{P}^2$ has a cover, e.g., $\{1247, 1'568', 33'4'8\}$.

The fact that $\mathfrak{P}^2$ has a cover enables us to totally assemble $\mathfrak{P}^2$ and its image, with three copies of $\mathfrak{T}$ as joints, and thus get a complex $\mathfrak{P}^4$ which, using Lemma 9, is easily seen to pm-realize $\{S^4\}$. Since $f_0(\mathfrak{P}^4) = 24 < f_0(\mathfrak{K}^4) = 32$, we see that $\mathfrak{P}^4$ improves $\mathfrak{K}^4$. In order that the sequence $\{\mathfrak{P}^i\}$ be complete, we define $\mathfrak{P}^1 = \mathfrak{P}$, $\mathfrak{P}^3 = \mathfrak{K}^3$.

All the above information about the 3-pm's $\mathfrak{T}_i$ and $\mathfrak{P}^i$ ($1 \leqslant i$) which pm-realize $\{S_{2i}\}$ and $\{S^i\}$, respectively, is summarized in Theorem 13 below. Theorem 13 also states the f-vectors of the $\mathfrak{T}_i$'s and $\mathfrak{P}^i$'s. Those f-vectors are easily calculated by induction, using the following lemma, whose proof follows easily from the construction in Definition 5 and the f-vectors of $\mathfrak{T}$ and $\mathfrak{P}$, and is therefore left to the reader.

LEMMA 12. *Let $\mathfrak{M}$ be a 3-pm which has a cover of cardinality m, let $\mathfrak{N}_1$ be the 3-pm obtained by totally assembling $\mathfrak{M}$ and its image with m disjoint joints, each of which is a copy of $\mathfrak{T}$, and let $\mathfrak{N}_2$ be the 3-pm obtained by totally assembling $\mathfrak{M}$ and its image with m disjoint joints, each of which is a copy of $\mathfrak{P}$. Then each of $\mathfrak{N}_1$, $\mathfrak{N}_2$ has a cover of cardinality $2m$, and*

$$f(\mathfrak{N}_1) = (2f_0(\mathfrak{M}) = 8m, 2f_1(\mathfrak{M}) + 16m, 2f_2(\mathfrak{M}) + 48m, 2f_3(\mathfrak{M}) + 24m),$$

$$f(\mathfrak{N}_2) = (2f_0(\mathfrak{M}) = 8m, 2f_1(\mathfrak{M}) + 16m, 2f_2(\mathfrak{M}) + 40m, 2f_3(\mathfrak{M}) + 20m).$$

THEOREM 13. (a) *For every integer $g \geqslant 1$ there exists a 3-pm $\mathfrak{T}_g$ in which the link of every vertex is an orientable 2-manifold of genus g, and*

$$f(\mathfrak{T}_1) = (8, 28, 56, 28),$$

$$f(\mathcal{T}_g) = \left(3 \cdot 2^g, 3 \cdot 2^{g-1}(4g+3), 3 \cdot 2^g(6g+1), 3 \cdot 2^{g-1}(6g+1)\right) \quad \textit{for } g \geqslant 2.$$

(b) *For every integer $q \geqslant 1$ there exists a 3-pm $\mathcal{P}^q$ in which the link of every vertex is a nonorientable 2-manifold of connectivity q, and*

$$f(\mathcal{P}^1) = (8, 28, 48, 24),$$

$$f(\mathcal{P}^3) = (16, 88, 192, 96),$$

$$f(\mathcal{P}^q) = \left(3 \cdot 2^{[q/2]+1}, 3 \cdot 2^{[q/2]}(2q + 6 + (-1)^q),\right.$$
$$\left. 3 \cdot 2^{[q/2]+1}(3q + 4 + (-1)^q), 3 \cdot 2^{[q/2]}(3q + 4 + (-1)^q)\right) \quad \textit{for } q \neq 1, 3.$$

Thus, the above $\mathcal{T}_g$'s and $\mathcal{P}^q$'s are the best pm-realizations of $\{S_{2g}\}$ and $\{S^q\}$, respectively, which we know of. However, except for $\mathcal{T}_1$ and $\mathcal{P}^1$, which were proved to be best pm-realizations, we do not wish to commit ourselves to the belief that they are indeed best pm-realizations.

So far, we have discussed improvements of 3-pm's which pm-realize a one-element set Σ. Any such improvement yields, of course, an improvement of the 3-pm constructed in the proof of Theorem 1 to pm-realize a general set Σ. A priori, it might, perhaps, be conjectured that if a 3-pm $\mathcal{N}_i$ is a best pm-realization of $\{R_i\}$ $(1 \leqslant i \leqslant 2)$, where R_1, R_2 are 2-manifolds, and $\mathcal{N}$ is the 3-pm which is assembled from $\mathcal{N}_1$, $\mathcal{N}_2$ according to the proof of Theorem 1, to yield a pm-realization of $\Sigma = \{R_1, R_2\}$, then $\mathcal{N}$ is a best pm-realization of Σ. In the following we construct two examples which show that this is not the case.

1. 1245	9. 1348	16. 2358
2. 1235	10. 1248	17. 3568
3. 1356	11. 1378	18. 3468
4. 1346	12. 1268	19. 4678
5. 1467	13. 1578	20. 4578
6. 1457	14. 1568	21. 2678
7. 1267	15. 2458	22. 2378
8. 1237		

TABLE 3. The facets of $\mathcal{Q}$

The 3-pm $\mathcal{Q}$. Let $\Sigma = \{S_2, S_0\}$, where S_2 is a torus and S_0 is a 2-sphere. The best pm-realization of $\{S_2\}$ is our basic 3-pm $\mathcal{T}$ (which in this section has also been denoted by $\mathcal{T}_1$) with its eight vertices, and the best pm-realization of $\{S_0\}$ is, of course, the boundary complex of a 4-simplex, with its five vertices. The proof of Theorem 1 combines these two 3-pms to yield a 3-pm $\mathcal{N}$ with 13 vertices which pm-realizes Σ. Now let $\mathcal{Q}$ be the simplicial 3-complex whose

22 facets are listed in Table 3. $\mathcal{Q}$ is easily seen to be a 3-pm which pm-realizes Σ: the link of each of the vertices 1, 8 is a torus, and the link of each of the remaining six vertices is a 2-sphere. Since $\mathcal{Q}$ has only eight vertices, $\mathcal{Q}$ improves $\mathcal{N}$. According to Lemma 6, $\mathcal{Q}$ is a best pm-realization of $\Sigma = \{S_2, S_0\}$. Note that $\mathcal{Q}$ is 2-neighborly, and therefore the link of each vertex contains all the other seven vertices. This 3-pm $\mathcal{Q}$ will be discussed again in the next section.

The 3-pm $\mathcal{B}$. Let $\Sigma = \{S^1, S_0\}$, where S^1 is a projective plane and S_0 is a 2-sphere. The proof of Theorem 1 combines $\mathcal{P} = \mathcal{P}^1$, which best pm-realizes $\{S^1\}$, and the boundary complex of a 4-simplex, to yield a 3-pm with 13 vertices, which pm-realizes Σ. However, the complex $\mathcal{B}$, whose 15 3-simplices are listed in Table 4, is easily seen to be a 3-pm which pm-realizes Σ, and has only seven vertices. Here the links of the vertices 1, 7 are projective planes, and the link of each of the remaining five vertices is a 2-sphere. Since a minimal triangulation of the projective plane has six vertices ($\mu(S^1) = 6$), it follows from Lemma 6 that $\mathcal{B}$ is a best pm-realization of Σ. From the next theorem it follows that $\mathcal{B}$ is *the* best–i.e., the only best–pm-realization of this Σ.

The proof of the next theorem exemplifies how the method described in [3] for constructing 3-manifolds can be modified to construct 3-pseudomanifolds. (See the first paragraph in §4.)

THEOREM 14. *The 3-pm* $\mathcal{B}$ *is the only 3-pm with less than eight vertices which is not a 3-manifold.*

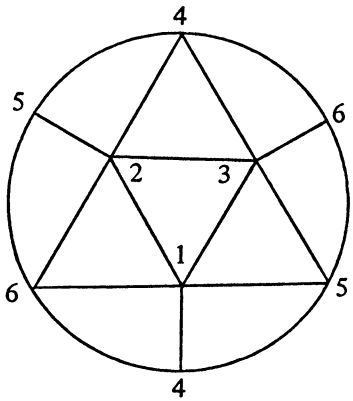


FIGURE 3. link(7, $\mathcal{B}$)

PROOF. The 2-sphere and the projective plane are the only 2-manifolds which can be triangulated with less than seven vertices. Therefore a 3-pm $\mathcal{M}$

with at most seven vertices which is not 3-manifold must contain a vertex x whose link is a projective plane. From Lemma 6 it follows that $\mathfrak{M}$ has exactly seven vertices. Since there exists a unique triangulation of the projective plane with six vertices (it is shown in Figure 3), we get that link(x, $\mathfrak{M}$) is isomorphic to the 2-complex shown in Figure 3. We assume the notation of the vertices of $\mathfrak{M}$ to be such that $x = 7$, and link(7, $\mathfrak{M}$) is precisely the 2-complex of Figure 3.

Thus st(7, $\mathfrak{M}$) contains the first ten 3-simplices of Table 4. We now try to find the remaining facets of $\mathfrak{M}$. So far, the triangle 123 belongs just to one facet (namely, 1237), and therefore there must be another facet of $\mathfrak{M}$ of the form $123x$, where $x \neq 7$, and therefore $x \in \{4, 5, 6\}$. Because of the symmetry in link(7, $\mathfrak{M}$) (Figure 3) we may take $x = 4$, and we thus get the eleventh facet in Table 4. Now the triangle 124 must belong to some additional facet, which must be either 1245 or 1246.

1. 1237	6. 1457	11. 1234
2. 1267	7. 3567	12. 1245
3. 1357	8. 3467	13. 1346
4. 2347	9. 2457	14. 1256
5. 1467	10. 2567	15. 1356

TABLE 4. The facets of $\mathfrak{B}$

If we choose 1245, then the triangle 134 leads us to add the facet 1346 (1345 is impossible since 145 does already belong to two facets), the triangle 125 leads us to add the facet 1256, and the triangle 136 leads to the addition of the facet 1356. Thus we obtain the 15 facets shown in Table 4, and they do indeed form the facets of a 3-pm, the 3-pm $\mathfrak{B}$, which has the required properties.

If, on the other hand, we choose 1246 instead of 1245 as the twelfth facet of $\mathfrak{M}$, then we are led by the triangles 134, 246 and 345, in this order, to add the facets 1345, 2456 and 3456. We thus get another complex $\mathfrak{B}'$ which is easily seen to be a 3-pm. However, this 3-pm $\mathfrak{B}'$ is isomorphic to $\mathfrak{B}$ by the permutation (14)(23) of the vertices, and our theorem is thus proved. □

Notice that a priori, according to Lemma 3 and Corollary 4, a 3-pm $\mathfrak{M}$ with seven vertices, the link of each being either a sphere or a projective plane, might possess two, four or six vertices whose links are projective planes. However, as follows from the last theorem, the last two possibilities do not occur.

7. Embedding 3-pseudomanifolds in polytopes. In his study [**11**] of cuts and coverings of combinatorial spheres, Pachner raised the problem [**11**, Problem

2] of whether or not there exists a 5-dimensional convex polytope K, such that $\mathrm{skel}_3 K$ contains a 3-pm in which the link of no vertex is a sphere. This leads us to the question of whether or not any of our 3-pm's is embeddable in a 5-polytope. If K is a convex polytope and $\mathfrak{M}$ is a complex, then $\mathfrak{M}$ is said to be *embeddable* in bd K (briefly, in K), if $\mathfrak{M}$ is isomorphic to some subcomplex of the boundary complex bd K of K.

The analogous question for 4-polytopes is easily settled in the negative by Theorem 12.10 of [**1**], from which it follows immediately that the only 3-pm contained in a 4-polytope K is the 3-sphere bd K itself. For 5-polytopes we have the following:

THEOREM 15. *If $\mathfrak{M}$ is a 3-pm which contains (a subcomplex isomorphic to) either* $\mathrm{skel}_2\mathfrak{T}$ *or* $\mathrm{skel}_2\mathfrak{P}$, *then $\mathfrak{M}$ is not embeddable in the boundary complex of any 5-dimensional convex polytope.*

PROOF. Assume that $\mathrm{skel}_2\mathfrak{T} \subset \mathfrak{M}$ and $\mathfrak{M}$ is embeddable in bd K' for some 5-polytope K'. Let V be the set of the eight vertices of K' which correspond to the vertices of $\mathrm{skel}_2\mathfrak{T}$, and let $K = \mathrm{conv}\, V$. Since every face of K' spanned by elements of V is a face of K also, it follows that $\mathrm{skel}_2\mathfrak{T}$ is embeddable in K. Thus, since $\mathfrak{T}$ is 3-neighborly, K also is 3-neighborly, but this is impossible by [**6**, Theorem 7.1.4].

Next assume that $\mathrm{skel}_2\mathfrak{P} \subset \mathfrak{M}$ and $\mathfrak{M}$ is embeddable in bd K' for some 5-polytope K'. As before, let V be the set of the eight vertices of K' which correspond to the vertices of $\mathrm{skel}_2\mathfrak{P}$, and let $K = \mathrm{conv}\, V$. It follows, as before, that $\mathrm{skel}_2\mathfrak{P}$ is embeddable in K and, since $\mathfrak{P}$ is 2-neighborly, K is 2-neighborly as well.

Define the *nerve-graph* of a set of 2-simplices to be the graph whose vertices are the 2-simplices in the set, and two vertices are joined by an edge (in the graph) iff the corresponding 2-simplices share a common edge. Thus, since $\mathrm{skel}_2\mathfrak{P}$ is embeddable in K, the nerve-graph of the set of missing 2-faces of K (which, because of the 2-neighborliness of K, are all 2-simplices) is a nerve-graph is, by §5, the graph $K_4 \cup K_4$, i.e., two disjoint copies of K_4, the complete graph on 4 vertices.

According to [**9**], there exist precisely nine 2-neighborly 5-polytopes with eight vertices, and K must therefore be one of them. Let the vertices of each of those nine polytopes be denoted by $1, 2, \ldots, 8$. In Table 5 the distended Gale-diagrams (see [**6**, §6.3]) of those nine polytopes are given, each in the form of a cyclic sequence of the consecutive endpoints of the diameters in the distended Gale-diagram. Here the digit 0 stands for a free endpoint.

In the same Table 5 we also list the set of missing triangles of each of those polytopes–which is readily read from the distended Gale-diagram–and the nerve-graph of this set is marked. Now it is easily seen that none of those nine

nerve-graphs is a subgraph of $K_4 \cup K_4$, and we have the desired contradiction. □

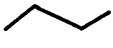
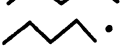
No.	Distended Gale-Diagram	The set D of missing triangles	The nerve-graph of D
1	1,2,3,4,5,6,7,8	{123,234,345,456,567,678,178,128}	
2	1,2,0,3,0,4,0,5,0,0,6,0,7,0,8,0	{345,456,567,678}	
3	1,2,0,3,0,0,4,5,0,0,6,0,7,8,0,0	{123,345,456,678}	
4	1,2,0,3,0,4,0,5,0,6,0,7,0,8	{234,345,456,567,678}	
5	1,2,0,3,0,4,5,0,6,7,8,0	{123,234,345,456,678,128}	
6	1,2,3,0,4,0,5,6,0,7,0,8	{234,345,456,567,678,178}	
7	1,2,3,4,0,5,6,7,0,8	{234,345,456,567,678,178,128}	
8	1,2,0,0,3,0,4,5,0,6,7,0,8,0	{123,234,345,678,128}	
9	1,0,2,0,3,0,4,0,5,0,6,0,7,0 8 at the center	{123,234,345,456,567,167,127}	

TABLE 5

The distended Gale-diagrams of the nine neighborly 5-polytopes with eight vertices, their sets of missing triangles and the nerve-graphs of those sets. Number 1 is the cyclic polytope C(8.5); number 9 is the pyramid over C(7,4).

Notice that with the exception of the 3-pm's $\mathcal{A}$ and $\mathcal{B}$ of §6, all the 3-pm's which have been constructed in the present article ($\mathfrak{M}_g$, $\mathfrak{M}^q$, $\mathfrak{K}^q$, $\mathfrak{T}_g$, $\mathfrak{P}^q$ and those compounded from these to yield the proof of Theorem 1) contain either $\mathrm{skel}_2\mathfrak{T}$ or $\mathrm{skel}_2\mathfrak{P}$. Thus it follows from the last theorem that none of them is embeddable in a 5-polytope. The 3-pm $\mathcal{A}$, however, is easily seen to be contained in the boundary complex of $C(8, 5)$, the cyclic 5-polytope with eight vertices, where the vertices $1, \ldots, 8$ of $C(8, 5)$ are taken in this order on the moment curve used for the definition of $C(8, 5)$ (see [6, §4.7]). But, since $\mathcal{A}$ has some vertices, the link of which are spheres, this does not settle Pachner's problem mentioned in the beginning of the present section.

As for polytopes of dimension 6, one can readily see that the basic 3-pm $\mathfrak{T}$ is contained in the boundary complex of $C(8, 6)$, the cyclic 6-polytope with 8 vertices.

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