THE PRODUCT OF NONPLANAR COMPLEXES
DOES NOT IMBED IN 4-SPACE

BY

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Abstract. We prove that if $K_1$ and $K_2$ are nonplanar simplicial complexes, then $K_1 \times K_2$ does not imbed in $\mathbb{R}^4$.

In this paper a proof is given of the following theorem:

Theorem P. If $K_1$ and $K_2$ are finite simplicial complexes neither of which is homeomorphic to a subset of the euclidean plane $\mathbb{R}^2$, then their cartesian product $K_1 \times K_2$ is not homeomorphic to any subset of euclidean 4-space $\mathbb{R}^4$.

This result answers a question originally posed by Professor Karl Menger in [5]. I wish to thank Professor Joseph Zaks for showing me this problem.

1. Preliminaries. We say a space $X$ imbeds in euclidean $n$-space $\mathbb{R}^n$ if there is an imbedding (i.e., homeomorphism into) $f: X \to \mathbb{R}^n$. If $X$ imbeds in $\mathbb{R}^2$ we say that $X$ is planar. For a proof of the following see [4]:

Proposition 1.1. If $K$ is a finite nonplanar simplicial complex then $K$ contains a subspace homeomorphic to one of the following spaces:

a. $K_5$, the complete graph on 5 vertices (or, if you prefer, the 1-skeleton of a 4-simplex);

b. $K_{3,3}$, the join of 2 3-point sets;

c. $S^2$, the 2-sphere; or

d. $Q^2 = \{ (x, y, z) \in \mathbb{R}^3 : x_3 = 0$ and $x_1^2 + x_2^2 < 1$ or $x_1 = x_2 = 0$ and $0 < x_3 < 1 \}$.

We will henceforth assume that the complexes $K_1$ and $K_2$ of Theorem P are chosen from the list of Proposition 1.1, and that we have chosen imbeddings $f_1: K_1 \to \mathbb{R}^3$ and $f_2: K_2 \to \mathbb{R}^3$.

To clarify notations we recall some standard definitions. Let $\pi = \{1, \tau\}$ be the multiplicative group of order 2. A $\pi$-space $X$ is a Hausdorff space together with a fixed point free involution $\tau: X \to X$; this involution defines a free...
\(\pi\)-action on \(X\) and we denote the orbit space of this action by \(X/\pi\). The natural projection \(p: X \rightarrow X/\pi\) is then a 2-fold covering. If \(X\) and \(Y\) are \(\pi\)-spaces then a map \(f: X \rightarrow Y\) is \(\pi\)-equivariant if \(f \cdot \tau = \tau \cdot f\). Homotopies are equivariant if they are \(\pi\)-equivariant at each stage. If \(K\) is a Hausdorff space, the deleted product of \(K\) is

\[
D_2K = \{(x, y) \in K \times K : x \neq y\}.
\]

Using the action \(\tau(x, y) = (y, x)\), \(D_2K\) is a \(\pi\)-space and we denote the orbit space by \(\Sigma_2K\). Let \(S^\infty = \text{proj lim } S^n\) under the natural inclusions and \(\tau: S^\infty \rightarrow S^\infty\) be the limit of the antipodal maps; set \(P^\infty = S^\infty/\tau\). If \(K\) is paracompact there is a \(\pi\)-equivariant map \(\hat{k}: D_2K \rightarrow S^\infty\) and any two such maps are \(\pi\)-equivariantly homotopic (cf. [2]). Using the induced map \(k: \Sigma_2K \rightarrow P^\infty\) and singular cohomology with \(\mathbb{Z}_2\) coefficients we define the \(n\)th mod-2 imbedding class of \(K\) by

\[
\Phi^2_n(K) = k^*(w^n) \in H^n(\Sigma_2K; \mathbb{Z}_2)
\]

where \(w^n\) is the nonzero element of \(H^n(P^\infty; \mathbb{Z}_2)\), \(n > 0\). The following is an immediate consequence of the definition.

**Proposition 1.2.** a. If \(K\) and \(L\) are paracompact, \(f: K \rightarrow L\) is an imbedding, and \(F: \Sigma_2K \rightarrow \Sigma_2L\) is the induced map, then \(F^*(\Phi^2_n(L)) = \Phi^2_n(K)\). b. For \(n > 0\), \(D_2R^n\) is \(\pi\)-equivariantly homotopy equivalent to \(S^{n-1}\) (with antipodal action); thus \(\Phi^2_n(R^n) \neq 0\) iff \(0 < i < n - 1\).

Thus \(\Phi^2_n(K) = 0\) is a necessary condition for a paracompact space to imbed in \(R^n\). In §3 we prove Theorem P by showing that \(\Phi^2_n(K_1 \times K_2) \neq 0\). The information we need about the deleted products of \(K_1\) and \(K_2\) is summarized in the following:

**Proposition 1.3.** If \(K\) is one of the four complexes \(K_5, K_3^3, S^2\) or \(Q^2\) of Proposition 1.1, then

a. \(\Phi^2_n(K) \neq 0\);

b. \(D_2K\) is \(\pi\)-equivariantly homotopy equivalent to a closed 2-manifold of genus \(g\), where \(g = 6\) if \(K = K_5\), \(g = 4\) if \(K = K_3^3\) and \(g = 0\) if \(K = S^2\) or \(Q^2\);

c. if \(f: K \rightarrow R^3\) is an imbedding and \(\hat{F}: D_2K \rightarrow D_2R^3\) is the induced map, then \(\hat{F}^*: H^2(D_2R^3) \rightarrow H^2(D_2K)\) is an isomorphism.

**Proof.** For a and b see [7] and [8]. For c we have the following commutative diagram whose rows are exact Gysin sequences (where we interpret a 2-fold covering as a 0-sphere bundle cf. [6]):

\[
\begin{array}{ccccccc}
\rightarrow & H^2(\Sigma_2R^3) & \overset{p^*}{\rightarrow} & H^2(D_2R^3) & \overset{p^*}{\rightarrow} & H^2(\Sigma_2R^3) & \rightarrow 0 \\
\downarrow F^* & & \downarrow F^* & & \downarrow F^* & \\
\rightarrow & H^2(\Sigma_2K) & \overset{p^*}{\rightarrow} & H^2(D_2K) & \overset{p^*}{\rightarrow} & H^2(\Sigma_2K) & \rightarrow 0 \\
\end{array}
\]
where $F: \Sigma_2 K \to \Sigma_2 \mathbb{R}^3$ is the map induced by $\hat{F}$. All six groups in the diagram are isomorphic to $\mathbb{Z}_2$ and so $\rho$ and $\rho'$ are isomorphisms. Thus $\hat{F}^*$ is an isomorphism.

Using $[g_1, \ldots, g_n]$ to denote the $\mathbb{Z}_2$-module with basis $\{g_1, \ldots, g_n\}$ or the zero module if $n = 0$, we can write

$$
\begin{align*}
H^0(K_1) &= \langle \omega^0 \rangle, & H^0(K_2) &= \langle \mu^0 \rangle, \\
H^1(K_1) &= \langle \omega^1, \ldots, \omega^n \rangle, & H^1(K_2) &= \langle \mu^1, \ldots, \mu^n \rangle,
\end{align*}
$$

where $\eta$ (or $\sigma$) is 0, 0, 4, or 6 depending upon whether $K_1$ (or $K_2$) is $S^2$, $Q^2$, $K_{1,3}$, or $K_{2,1}$. Here the superscripts denote dimension rather than exponents. If $K_1$ (or $K_2$) is $S^2$ or $Q^2$ we denote $H^2(K_1) = \langle \omega^2 \rangle$ ($H^2(K_2) = \langle \mu^2 \rangle$); otherwise $H^2(K_1) = 0$ ($H^2(K_2) = 0$). We also need to assume that if $\eta \neq 0$ ($\sigma \neq 0$) then the above basis for $H^1(K_1)$ ($H^1(K_2)$) is dual to a basis which satisfies the following:

**Lemma 1.4.** If $K$ is a finite 1-dimensional simplicial complex and $i: D_2 K \to K \times K$ is the inclusion map, then there is a basis $\{\beta_1, \ldots, \beta_m\}$ for $H_1(K)$ such that if $\beta \in H_2(D_2 K)$ then $i_*(\beta) = \sum c_{ij} (\beta_i \times \beta_j)$ where $c_{ij} \in \mathbb{Z}_2$, $c_{ii} = 0$ for $i = 1, \ldots, m$ and "\times" denotes cross product.

**Proof.** Let $D_2^\circ(K) = \{(x_1, x_2) \in K \times K: c(x_1) \cap c(x_2) = \emptyset\}$ where $c(x_j)$ is the smallest closed simplex of $K$ containing $x_j$. Then, by [9], $D_2^\circ(K)$ is a strong $\pi$-equivariant deformation retract of $D_2 K$, and so we can use the inclusion $j: D_2^\circ K \to K \times K$ instead of $i$. Let $\{\sigma_0, \ldots, \sigma_{m+n}\}$ be the 1-simplices of $K$ numbered so that $\{\sigma_{m+1}, \ldots, \sigma_{m+n}\}$ form a maximal tree $T$ of $K$. We also use $\sigma_i$ to denote the linear singular 1-simplex whose image is $\sigma_i$; there is no orientation problem since we are using $\mathbb{Z}_2$ coefficients. For $i = 1, \ldots, m$ set $\beta_i = [\sigma_i + \lambda_i] \in H_1(K_1)$ where $\lambda_i$ is a sum of simplices of $T$. For $i > m$ we set $\lambda_i = \sigma_i$. Suppose $\beta \in H_2(D_2^\circ K)$. Then $\beta = [\Sigma k_{ij} (\sigma_i \times \sigma_j)]$ where $k_{ij} \in \mathbb{Z}_2$ and $k_{ii} = 0$ for all $i$. We have

$$
\sigma_i \times \sigma_j = (\sigma_i + \lambda_i - \lambda_j) \times (\sigma_j + \lambda_j - \lambda_i)
$$

$$
= (\sigma_i \times \lambda_i) \times (\sigma_j \times \lambda_j) - \lambda_i \times \sigma_j - \sigma_i \times \lambda_j + \lambda_i \times \lambda_j.
$$

So if $i \neq j$, $\sigma_i \times \sigma_j = (\sigma_i + \lambda_i) + \gamma_{ij}$ where $\gamma_{ij}$ is a 2-chain of $X \times \Gamma \cup \Gamma \times X$. So $\beta = [\Sigma k_{ij} (\sigma_i + \lambda_i) \times (\sigma_j + \lambda_j) + \gamma]$ where $\gamma$ is a 2-chain of $\Gamma \times X \cup X \times \Gamma$. In fact $\gamma$ is a 2-cycle of $\Gamma \times X \cup X \times \Gamma$ and since $H_2(\Gamma \times X \cup X \times \Gamma) = 0$, $j_*(\gamma) = 0$. Thus $j^*(\beta) = [\Sigma k_{ij} (\beta_i \times \beta_j)]$. Thus $\{\beta_1, \ldots, \beta_n\}$ is the desired basis. This proves Lemma 1.4.

**Lemma 1.5.** If $K = K_{3,1}^1$, $K_{3,3}^1$, $S^2$ or $Q^2$, then the inclusion map $j: D_2 K \to K \times K$ induces an isomorphism $j^*: H^1(K \times K) \to H^1(D_2 K)$.

**Proof.** For $K = S^2$ or $Q^2$ both groups are trivial. If $K = K_{3,1}^1$ or $K_{3,3}^1$ then
by [8] $D_2(CK)$ is $\pi$-equivariantly homotopy equivalent to $S^2$. So in the exact sequence (cf. [1])
\[ H^n(D_2CK) \rightarrow H^n(K) \oplus H^n(K) \xrightarrow{\alpha} H^n(D_2K) \rightarrow H^{n+1}(D_2CK) \rightarrow \]
where $\alpha(u, v) = \varphi_1^*(u) + \varphi_2^*(v)$, $\varphi_i: D_2K \rightarrow K$ given by $\varphi_i(x_1, x_2) = x_i$, we have $\alpha$ is an isomorphism if $n = 1$. Using this and the Künneth Theorem proves Lemma 1.5.

Using Lemma 1.5, Proposition 1.5 and the above bases we have:

\[ H^0(D_2K_1) = [\omega^0 \times \omega^0], \quad H^1(D_2K_1) = [\omega^0 \times \omega^1, \omega^1 \times \omega^0; i = 1, \ldots, \eta], \]
\[ H^2(D_2K_1) = [\Omega^2], \]
and

\[ H^0(D_2K_2) = [\mu^0 \times \mu^0], \quad H^1(D_2K_2) = [\mu^0 \times \mu^1, \mu^1 \times \mu^0; i = 1, \ldots, \sigma], \]
\[ H^2(D_2K_2) = [\Lambda^2] \]
where "$\times$" denotes cross product followed by restriction.

For Hausdorff spaces $K$ and $L$ we define

\[ J_0(K, L) = D_2K \times D_2L, \quad J_1(K, L) = D_2K \times (L \times L), \]
\[ J_2(K, L) = (K \times K) \times D_2L. \]

Using $\tau(x_1, x_2, y_1, y_2) = (x_2, x_1, y_2, y_1)$, $J_k(K, L)$ becomes a $\pi$-space and we denote the quotient spaces by $J_k(K, L)$ for $k = 0, 1, 2$.

**Lemma 1.6.** If $K$ and $L$ are Hausdorff spaces then $D_2(K \times L)$ is $\pi$-equivariantly homeomorphic to $J_1(K, L) \cup J_2(K, L)$. Moreover, $\{J_1(K, L), J_2(K, L)\}$ is an excisive couple and $J_1(K, L) \cap J_2(K, L) = J_0(K, L)$.

**Proof.** Clearly $\phi: D_2(K \times L) \rightarrow J_1(K, L) \cup J_2(K, L)$ defined by

\[ \phi(x_1, y_1, x_2, y_2) = (x_1, x_2, y_1, y_2) \]

is a $\pi$-equivariant homeomorphism. Since $J_1(K, L)$ and $J_2(K, L)$ are open in their union, the couple is excisive.

For $k = 0, 1, 2$ let

\[ J_k = J_k(K_1, K_2), \quad J'_k = J_k(R^3; R^3), \]
\[ J_k = J_k(K_1, K_2), \quad J'_k = J_k(R^3; R^3), \]
and $\hat{J}_k: J_k \rightarrow J_k$, $F_k: J_k \rightarrow J'_k$ denote the maps induced by the imbeddings $f_j: K_j \rightarrow R^3, j = 1, 2$. We also have maps

\[ \hat{F}: D_2(K_1 \times K_2) \rightarrow D_2(R^3 \times R^3) \]

and

\[ F: \Sigma_2(K_1 \times K_2) \rightarrow \Sigma_2(R^3 \times R^3). \]
Finally let \( \tilde{i}_k: \tilde{J}_0 \to \tilde{J}_k \) and \( i_k: J_0 \to J_k \) be the inclusions for \( k = 1, 2 \) and \( p_k: \tilde{J}_k \to J_k, p'_k: \tilde{J}'_k \to J'_k \) be the natural projections for \( j = 0, 1, 2 \).

**Lemma 1.7.** \( F^*: H^4(J_0) \to H^4(J_0) \) is an isomorphism.

**Proof.** \( \tilde{J}'_0 \) and \( \tilde{J}_0 \) are \( \pi \)-equivariantly homotopy equivalent to closed 4-manifolds; hence \( J_0 \) and \( J'_0 \) are homotopy equivalent to closed 4-manifolds. Thus in the commutative diagram

\[
\begin{array}{ccc}
H^4(\tilde{J}'_0) & \xrightarrow{\rho'} & H^4(J_0) \\
\downarrow F_0^* & & \downarrow F_0^* \\
H^4(\tilde{J}_0) & \xrightarrow{\rho} & H^4(J_0)
\end{array}
\]

where \( \rho' \) and \( \rho \) are from the appropriate Gysin sequences, and Proposition 1.3.c, \( F_0^* \) is an isomorphism. This proves Lemma 1.7.

2. The spectral sequence of a double covering. The proof of Theorem P requires the following in which we use the notation of §1.

**Lemma 2.1.** \( \ker(p^*_1: H^4(J_1) \to H^4(\tilde{J}_1)) \subseteq \text{Im}(i^*_1: H^4(J_1) \to H^4(J_0)). \)

The proof of Lemma 2.1 requires using the cohomology spectral sequence of a covering (cf. [3]) specialized to the case of a double covering which allows explicit identification of the \( E_1 \)-term and the \( E_1 \) differential operators. The properties of this spectral sequence are summarized in the following:

**Proposition 2.2.** If \( X \) is a \( \pi \)-space, there is a natural first quadrant \( E_1 \)-spectral sequence \( \{ E_1^{p,q}(X), d_1^{p,q} \}_{p=1}^{\infty} \) convergent to \( H^*(X/\pi; Z_2) \) with the following properties:

a. \( E_1^{p,q}(X) = H^q(X; Z_2) \),

b. \( d_1^{p,q}: E_1^{p,q}(X) \to E_1^{p+1,q}(X) \) is given by \( d_1^{p,q}(x) = x + \tau x \) where \( \tau : H^1(H; Z_2) = H^1(X, Z_2) \) is the homomorphism induced by the involution \( \tau : X \to X \).

c. For each \( n \) there is a natural decreasing filtration \( \{ F_p H^n(X/\pi) \}_{p=0}^{\infty} \) of \( H^n(X, Z_2) \) such that \( F_0 H^n(X/\pi) = H^n(X/\pi; Z_2) \), \( F_{n+1} H^n(X/\pi) = 0 \) and for each \( p > 0 \) there is a natural short exact sequence

\[
0 \to F_{p+1} H^n(X/\pi) \to F_p H^n(X/\pi) \to E_\infty^{p,n-p}(X) \to 0.
\]

d. The projection induced map \( p^*: H^n(X/\pi; Z_2) \to H^n(X; Z_2) \) is the composition:

\( H^n(X/\pi) = F_0 H^n(X/\pi) \to E_\infty^{0,n}(X) \subseteq E_1^{0,n}(X) = H'(X) \).

For the remainder of this section and the next we assume all coefficients to be \( Z_2 \) and suppress this in the notation.
Lemma 2.3. If \( p > 1 \) then \( \tilde{d}^{2,3}_{K} : E^{p,2}_2(D_2K_i) \to E^{p+3,0}_2(D_2K_i) \) is an isomorphism with \( E^{p,2}_2(D_2K_i) = [\Omega^2] \).

Proof. Using the calculations of §1 and Proposition 2.2.b,

\[
E^{p,0}_2(D_2K_i) = [\omega^0 \times \omega^0] , \quad p \geq 0, \quad E^{p,1}_2(D_2K_i) = 0, \quad p \geq 1, \\
E^{p,2}_2(D_2K_i) = [\Omega^2] , \quad p > 0.
\]

Thus \( E^{p,0}_2(D_2K_i) = E^{p,0}_2(D_2K_i) = [\omega^0 \times \omega^0] \) if \( p > 3 \) and \( E^{p,2}_2(D_2K_i) = E^{p,2}_2(D_2K_i) = [\Omega^2] \) if \( p > 0 \). Since \( H^n(\Sigma_2K_i) = 0 \) if \( n > 2 \),

\[
d^{2,3}_K : E^{p,2}_2(D_2K_i) \to E^{p+3,0}_2(D_2K_i)
\]

must be an isomorphism. This proves Lemma 2.3.

Lemma 2.4. \( E^{p,4-p}_2(J_0) = 0 \) if \( p > 3 \).

Proof. Using the calculations of §1 and the Künneth formula,

\[
\begin{align*}
    H^0(J_0) &= [\omega^0 \times \omega^0] \otimes [\mu^0 \times \mu^0], \\
    H^1(J_0) &= [\omega^0 \times \omega^0] \otimes [\mu^1 \times \mu^0, \mu^0 \times \mu^1; i = 1, \ldots, \sigma] \\
    &\quad \oplus [\omega^1 \times \omega^0, \omega^0 \times \omega^1; i = 1, \ldots, \eta] \otimes [\mu^0 \times \mu^0],
\end{align*}
\]

where \( \eta \) and \( \sigma \) are the ranks of \( H^1(K_1) \) and \( H^1(K_2) \) respectively. Using the diagonal \( \tau \)-action and Proposition 2.2.b

\[
E^{p,0}_2(J_0) = [\omega^0 \times \omega^0] \otimes [\mu^0 \times \mu^0], \quad E^{p,1}_2(J_0) = 0 \quad \text{if} \quad p > 0.
\]

Since \( E^{2,1}_2(J_0) = 0, E^{s,0}_2(J_0) = E^{s,0}_2(J_0) \) and

\[
\tau_1^* : E^{s,0}_2(D_2K_i) \to E^{s,0}_2(J_0)
\]

is an isomorphism where \( \tau_1 : J_0 \to D_2K_i \) is the projection. From the commutative diagram

\[
\begin{array}{ccc}
    E^{1,2}_3(D_2K_i) & \xrightarrow{\pi^*} & E^{1,2}_3(J_0) \\
    \downarrow d^{1,2}_K & & \downarrow d^{1,2} \\
    E^{4,0}_3(D_2K_i) & \xrightarrow{\pi^*} & E^{4,0}_3(J_0)
\end{array}
\]

and Lemma 2.3, \( d^{1,2}_K : E^{1,2}_3(J_0) \to E^{1,2}_3(J_0) \) is surjective. Thus \( E^{s,0}_2(J_0) = E^{s,0}_2(J_0) = E^{s,0}_2(J_0) = 0. \) From above, \( E^{3,1}_{\infty}(J_0) = 0. \) This proves Lemma 2.4.

Lemma 2.5. \( E^{2,3}_{\infty}(J_1) = 0 \) and \( i^* : E^{2,2}_{\infty}(J_0) \) is the zero homomorphism.

Proof. We first compute \( E^{s,3}_2(J_i) \) in 3-cases. If \( K_2 = K_2^1 \) or \( K_2^{1,3} \), then
\[ H^1(\hat{J}_1) = [\omega^0 \times \omega^0] \otimes [\mu^0 \times \mu^0; i = 1, \ldots, \sigma] \]
\[ \oplus [\omega^0 \times \omega^i, \omega^i \times \omega^0; i = 1, \ldots, \eta] \otimes [\mu^0 \times \mu^0], \]
\[ H^2(\hat{J}_1) = [\omega^0 \times \omega^0] \otimes [\mu^0 \times \mu^0; i = 1, \ldots, \sigma] \]
\[ \oplus [\omega^0 \times \omega^i, \omega^i \times \omega^0; i = 1, \ldots, \eta] \]
\[ \otimes [\mu^0 \times \mu^i, \mu^i \times \mu^0; i = 1, \ldots, \sigma] \]
\[ \oplus [\Omega^2] \otimes [\mu^0 \times \mu^0]. \]

Using the diagonal \( \pi \)-action on \( \hat{J}_1 \) and Proposition 2.2

\[ E_{p,1}^2(\hat{J}_1) = 0 \text{ if } p > 0, \]
\[ E_{2,2}^2(\hat{J}_1) = [\omega^0 \times \omega^0]\otimes[\mu^0 \times \mu^0; i = 1, \ldots, \sigma] \oplus [\Omega^2] \otimes [\mu^0 \times \mu^0], \]
\[ E_{2,1}^3(\hat{J}_1) = 0. \]

If \( K_2 = S^2 \) and we set \( H_2(K_2) = [\mu^2] \) then

\[ H^1(\hat{J}_1) = [\omega^0 \times \omega^0 \times \omega^i, \omega^i \times \omega^0; i = 1, \ldots, \eta] \otimes [\mu^0 \times \mu^0], \]
\[ H^2(\hat{J}_1) = [\omega^0 \times \omega^0 \times \omega^0; i = 1, \ldots, \eta] \]
\[ \otimes [\mu^0 \times \mu^0, \mu^2 \times \mu^0] \oplus [\Omega^2] \otimes [\mu^0 \times \mu^0]. \]

Therefore

\[ E_{p,1}^2(\hat{J}_1) = 0 \text{ if } p > 0, \quad E_{2,2}^2(\hat{J}_1) = [\Omega^2 \times \mu^0 \times \mu^0], \]
\[ E_{2,1}^3(\hat{J}_1) = 0. \]

Finally if \( K_2 = Q_2 \), then

\[ E_{p,1}^2(\hat{J}_1) = 0 \text{ if } p > 0, \quad E_{2,2}^2(\hat{J}_1) = H^2(\hat{J}_1) = [\Omega^2 \times \mu^0 \times \mu^0], \]
\[ E_{2,1}^3(\hat{J}_1) = H^3(\hat{J}_1) = 0. \]

In all cases \( E_{p,0}^2(\hat{J}_1) = [\omega^0 \times \omega^0 \times \mu^0 \times \mu^0] \). Since \( \pi^p_1: H^1(D_2 K_i) \to H^2(\hat{J}_i) \) is an isomorphism, where \( \pi_1: \hat{J}_1 \to D_2 K_1 \) is the projection, and \( E_{p,1}^2(\hat{J}_1) = 0 \) if \( p > 0 \), we have

\[ \pi^p_1: E_{3,0}^p(D_2 K_1) \to E_{3,0}^p(\hat{J}_1) \]

is an isomorphism for \( p > 2 \). Consider the commutative diagram

\[
\begin{array}{ccc}
E_{2,2}^2(D_2 K_1) & \xrightarrow{\pi^p_1} & E_{3,2}^2(\hat{J}_1) \\
\downarrow d_{2,2}^2 & & \downarrow d_{3,2}^2 \\
E_{3,0}^5(D_2 K_1) & \xrightarrow{\pi^p_1} & E_{3,0}^5(\hat{J}_1) \\
\end{array}
\]

Using Lemma 2.3, and \( \pi^p_1(\Omega^2) = [\Omega^2 \times \omega^0 \times \omega^0] \) we have

\[ d_{3,2}^2 ([\Omega^2 \times \omega^0 \times \omega^0]) \neq 0. \]
Therefore
\[ E'^{2,2}(\hat{j}_1) = \begin{cases} 
[\omega^0 \times \omega^0 \times \mu_1 \times \mu_1; i = 1, \ldots, \sigma] & \text{if } K_2 = K_2' \text{ or } K_2', \\
0 & \text{if } K_2 = S^2 \text{ or } Q^2.
\end{cases} \]

To show that \( i^*(\omega^0 \times \omega^0 \times \mu_1 \times \mu_1) = 0 \), let \( \{\alpha_1, \ldots, \alpha_{\sigma}^1\} \) be the basis of \( H_1(K_2) \) dual to \( \{\mu_1, \ldots, \mu_1^1\} \) as in Lemma 1.4 and let \( \alpha^2 \) denote the nonzero element of \( H_2(D_2K_2) \). If \( j: D_2K_2 \to K_2 \times K_2 \) is the inclusion, then by Lemma 1.4
\[
\langle j^*(\mu_1^1 \times \mu_1^1), \alpha^2 \rangle = \langle \mu_1 \times \mu_1, \sum_{j \neq k} c_{jk}(\alpha_j^1 \times \alpha_k^1) \rangle
\]
\[
= \sum_{j \neq k} c_{jk} \langle \mu_1^1, \alpha_j^1 \rangle \langle \mu_1^1, \alpha_k^1 \rangle = \sum_{j \neq k} c_{jk} \delta_{jk} = 0.
\]
Thus \( j^*(\mu_1^1 \times \mu_1^1) = 0 \). So \( i^*(\omega^0 \times \omega^0 \times \mu_1^1 \times \mu_1^1) = 0 \). This proves Lemma 2.5.

**Proof of Lemma 2.1.** By Proposition 2.2.d, if \( p_1: \hat{j}_1 \to J_1 \) is the natural projection then \( p_1^*: H^4(J_1) \to H^4(\hat{j}_1) \) is the composition \( H^4(J_1) = F^0H^4(J_1) \cong E^0_{\infty}(\hat{j}_1) \). So using Proposition 2.2.c we have
\[
\ker[p^*: H^4(J_1) \to H^4(\hat{j}_1)] = \ker[q: F^0H^4(J_1) \to E^0_{\infty}(\hat{j}_1)] = F_1H^4(J_1).
\]
By Lemma 2.5, \( E^2\alpha(\hat{j}_1) = 0 \); thus \( F_2H^4(J_1) = F_1H^4(J_1) \). Thus \( i^*(\ker p^*) = 0 \) if and only if \( i^*(F_2H^4(J_1)) = 0 \). Now consider the following commutative diagram with exact rows:

\[
\begin{array}{cccc}
0 & \to & F_3H^4(J_1) & \to & F_2H^4(J_1) & \to & E^2_{\infty}(\hat{j}_1) & \to & 0 \\
\downarrow i_\alpha & & \downarrow i_\alpha & & \downarrow i_\alpha & & \downarrow i_\alpha & & \\
0 & \to & F_3H^4(J_0) & \to & F_2H^4(J_0) & \to & E^2_{\infty}(\hat{j}_0) & \to & 0
\end{array}
\]

By Lemma 2.4, \( E^4_{\infty}(\hat{j}_0) = 0 \); thus \( F_3H^4(J_0) = 0 \) and so the projection \( F_2H^4(J_0) \to E^2_{\infty}(\hat{j}_0) \) is an isomorphism. But by Lemma 2.5, \( i_\alpha^*: E^2_{\infty}(\hat{j}_0) \to E^2_{\infty}(\hat{j}_0) \) is zero. Thus \( i_\alpha^*: F_2H^4(J_1) \to F_2H^4(J_0) \) is zero. This proves Lemma 2.1.

**3. Proof of Theorem P.** We use the notation of §1 including the assumptions that \( K_1 \) and \( K_2 \) are complexes selected from the list in Proposition 1.1 and that all coefficients are \( Z_2 \). We will prove that \( \Phi_2(K_1 \times K_2) \neq 0 \).

Consider first the following commutative diagram in which the rows are the
exact Mayer Vietoris sequences given by Lemma 1.6:

\[
H^3(J_1) \oplus H^3(J_2) \to H^3(J_0) \xrightarrow{\delta^*} H^4(\Sigma_2(\mathbb{R}^3 \times \mathbb{R}^3)) \to H^4(J_1) \oplus H^4(J_2)
\]

Since \( H^j(J_i) \cong H^j(J_2) \cong H^j(\mathbb{R}P^2) = 0 \) for \( j > 2 \), \( \delta^* \) is an isomorphism. Since \( \Phi_j^2(K_1 \times K_2) \neq 0 \) if \( F^* \neq 0 \), we have \( \Phi_j^2(K_1 \times K_2) \neq 0 \) if \( \text{Im}(F_0^*) \not\subseteq \text{Im}(i_1^* + i_2^*) \).

Now consider the following commutative diagram in which the rows are exact Gysin sequences or sums of Gysin sequences:

\[
\begin{array}{ccc}
H^3(J_1) \oplus H^3(J_2) & \xrightarrow{\Delta^*} & H^4(J_1) \oplus H^4(J_2) \\
\downarrow F_0^* & & \downarrow F_0^* \\
H^3(J_0) & \xrightarrow{\Delta^*} & H^4(J_0) \\
\downarrow i_1^* + i_2^* & & \downarrow i_1^* + i_2^* \\
H^3(J_1) & \xrightarrow{\Delta^* + \Delta_2^*} & H^4(J_1) \oplus H^4(J_2) \\
\downarrow p_1 \oplus p_2 & & \downarrow p_1 \oplus p_2 \\
H^4(J_1) \oplus H^4(J_2) & \xrightarrow{\rho} & H^4(J_0) \\
\end{array}
\]

Since \( \hat{J}_0 \) is \( \pi \)-equivariantly homotopy equivalent to \( S^2 \times S^2 \), \( H^4(\hat{J}_0) \cong H^4(J_0) \cong H^4(J_1) \cong H^4(J_2) \cong \mathbb{Z}_2 \) and \( \Delta^* \) and \( \rho \) are isomorphisms. By Proposition 1.3, \( H^4(J_0) \cong H^4(J_0) \cong \mathbb{Z}_2 \) and by exactness \( \rho \) is an isomorphism. By Lemma 1.7, \( F_0^*: H^4(J_0) \to H^4(J_0) \) is an isomorphism. Let \( \alpha \) be the nonzero element of \( H^3(J_0) \). We need to show there do not exist elements \( \alpha_1 \in H^3(J_1) \) and \( \alpha_2 \in H^3(J_2) \) such that \( i_1^*(\alpha_1) + i_2^*(\alpha_2) = F_0^*(\alpha) \). Since \( F_0^*(\Delta^*(\alpha)) \neq 0 \), it suffices to show that

\[
i_1^*(\Delta_1^*(\alpha_1)) + i_2^*(\Delta_2^*(\alpha_2)) = 0.
\]

By exactness and symmetry this follows if we prove

\[
\ker[p_1^*: H^4(J_1) \to H^4(\hat{J}_1)] \subseteq \ker[i_1^*: H^4(J_1) \to H^4(J_0)].
\]

This is exactly what was proven in Lemma 2.1. Thus the proof of Theorem P is complete.

REFERENCES


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