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PROCESSES WITH INDEPENDENT INCREMENTS ON A LIE GROUP(1)

BY

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ABSTRACT. The Lévy-Khinchin representation for processes with independent increments is extended to processes taking values in a Lie group.

The basis of the proof is to approximate continuous time processes by Markov chains. The processes involved are handled by the technique, developed by Stroock and Varadhan, of characterizing Markov processes by associated martingales.

1. **Introduction.** A stochastically continuous real-valued process $x(t)$ with independent increments satisfies

$$\langle e^{i\xi x(t)}\rangle = \exp\left[i\xi m(t) - \tfrac{1}{2}\xi^2 A(t) + \int_{\mathbf{R}-\{0\}} \left(e^{i\xi u} - 1 - \frac{i\xi u}{1+u^2} \right) M(t, du) \right],$$

where $\langle\ \rangle$ denotes expected value.

Here $m(t)$ is a continuous function, $A(t)$ is a nonnegative, increasing, continuous function and $M(t, du)$ is a weakly continuous, increasing family of measures such that the integral of any bounded smooth function f with $f(0) = f'(0) = 0$ is finite.

So the finite-dimensional distributions of $x(t)$ are determined by these three functions, parameters of the process, (m, A, M).

The question is how does this generalize to processes taking values in a topological group? For locally compact abelian groups Fourier analysis can be applied; this theory is done in [7]. See also [1]. For a discussion of this theory and for a general survey of results up to 1966 concerning probability distributions on topological groups, refer to the article [8].

The method adopted here is basically that developed by Stroock and Varadhan [10] to deal with the Gaussian part. The method depends on the

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characterization of Markov processes by associated martingales; the fundamental theory for Markov processes in general was developed in [9].

To see the idea, suppose G is a Lie group. A stochastically continuous process with independent increments, $x(t)$, means, by definition:

1. $x(0) = \varepsilon$ with probability one.

2. Given any partition of $[0, 1]$, $0 \leqslant t_1 \leqslant t_2 \leqslant \cdots \leqslant t_n \leqslant 1$, the increments

$$x(t_1), x(t_1)^{-1}x(t_2), \ldots, x(t_{j-1})^{-1}x(t_j), \ldots$$

are independent random elements.

3. For every U, a neighborhood of ε, $P(x(s)^{-1}x(t) \not\in U)$ converges to zero as $|s - t|$ goes to zero, uniformly for s, t in $[0, 1]$.

Suppose further that the process is time-homogeneous; that is, the distribution of an increment $x(s)^{-1}x(t)$ depends only on $t - s$. Then $x(t)$ is certainly a time-homogeneous Markov process so that the family of operators $T(t)$ defined by $T(t)f(x) = \langle f(x(t))\rangle_{x(0)=x}$ form a semigroup with an infinitesimal generator L, having a domain that contains $C_k^\infty(G)$-smooth functions with compact support. For f in C_k^∞, the semigroup formula

$$T(t)f(x) = f(x) + \int_0^t LT(s)f(x)\, ds$$

is equivalent to the statement that $f(x(t)) - \int_0^t Lf(x(s))\, ds$ is a martingale. The case of time-homogenous processes is treated by the semigroup method in [5].

In the case of processes that are not time-homogeneous, the martingale formulation readily generalizes. In the time-dependent case it is reasonable to expect that a process may be characterized by the condition that $f(x(t)) - \int_0^t L(s)f(x(s))\, ds$ is a martingale for any f in C_k^∞. In such a case the process can also be characterized by parameters (m, A, M), analogously to the case $G = \mathbf{R}$.

However, not all processes can be characterized directly by martingales. For example, if $z(t)$ is determined by martingales of the form suggested above and $m(t)$ is a continuous function, but not of bounded variation, then $x(t) = z(t)m(t)$ has no martingales due to the nondifferentiability of $m(t)$. Also a change in the time scale may be required. The thrust of the following is that these are the only difficulties that arise. That is, modulo a nonrandom change in the time scale and/or multiplication by a continuous function, all stochastically continuous processes with independent increments are determined by martingales of the type suggested. The basis of the proof is to approximate the continuous time processes by Markov chains (random walks) where the possible difficulties, such as controlling $m(t)$, can be handled directly.

The paper is organized as follows. §2 consists of a detailed review of special topics concerning Lie groups, weak convergence of measures, and martingales. §3 consists of an exposition of the results. Their proofs comprise §§4–6.

Note. Most notations are standard or explained in passing. The following, however, should be observed:

For $(\Omega, \mathcal{F}, P)$ a probability space and R in $L^1(dP)$, $\langle R \rangle = \int R\, dP$.

Similarly, $\langle R \rangle_n = \int R\, dP_n$.

For $\mathcal{F}_n$ a sub σ-field of $\mathcal{F}$, $(R \| \mathcal{F}_n)$ = expected value of R given $\mathcal{F}_n$.

For a Lie group G, C_k^∞ denotes the smooth functions with compact support.

$\tilde{E}$ denotes the complement of a set E.

A^* denotes the transpose of the (real) matrix A.

2. Review of special facts.

a. *Remarks on Lie groups* [4], [13]. In the following, G is a Lie group of dimension d with Lie algebra Γ, and G is assumed to be second countable as a topological space. The elements ξ of Γ are considered as left-invariant vector fields (i.e., 'right' derivatives). For f in $C^\infty(G)$,

$$\xi f(x) = \frac{d}{dt}\bigg|_0 f(xe^{t\xi}).$$

Choose $\{\xi_j\}_{1 \leq j \leq d}$ a fixed basis for Γ. Having chosen $\{\xi_j\}$, take a coordinate system (ϕ, W) around ε, the identity element of G, such that:

1. W has compact closure.
2. $\phi = (\phi_1, \ldots, \phi_d)$: $G \to \mathbf{R}^d$, $\phi(\varepsilon) = 0$.
3. $\xi_i \phi_j(\varepsilon) = \delta_{ij} = 1$ if $i = j$, 0 if $i \neq j$.

Define the numbers ρ_k^{ij} to equal $\xi_i \xi_j \phi_k(\varepsilon)$.

(1) *Adjoint representation*. An element m in G determines a C^∞ mapping, the inner autmorphism, aut_m: $G \to G$ defined by $\mathrm{aut}_m(x) = mxm^{-1}$. Since $\mathrm{aut}_m(\varepsilon) = \varepsilon$, a linear mapping Ad_m: $\Gamma \to \Gamma$ is induced. With respect to the basis $\{\xi_j\}$, Ad_m can be expressed as a matrix satisfying

$$\mathrm{Ad}_m(\xi_j) = \sum_{k=1}^{d} \mathrm{Ad}_m^{kj} \xi_k.$$

Recall that Ad_m has the following essential properties:

1. Ad_m is a homomorphism from G to $\mathrm{End}(\Gamma)$. That is, $\mathrm{Ad}_m \mathrm{Ad}_n = \mathrm{Ad}_{mn}$.
2. Ad_m is a smooth function of m.
3. $\mathrm{aut}_m(e^{t\xi}) = me^{t\xi}m^{-1} = e^{t\,\mathrm{Ad}_m \xi}$.

(2) *Taylor's expansion*. Consider U a neighborhood of ε, $U \subset W$. Let f be in C_k^∞. Since ϕ: $G \to \mathbf{R}^d$, the function $g = f\phi^{-1}$ is in $C^\infty(\phi(W))$. Suppose a, b are in U. Then Taylor's theorem for functions on $\mathbf{R}^d$ yields

$$f(b) - f(a) = \sum_i (\phi_i(b) - \phi_i(a)) \frac{\partial g}{\partial \phi_i}(\phi(a))$$
$$+ \tfrac{1}{2} \sum_i \sum_j (\phi_i(b) - \phi_i(a))(\phi_j(b) - \phi_j(a)) \frac{\partial^2 g}{\partial \phi_i \partial \phi_j}(\phi(a))$$
$$+ R(a, b) \cdot \sum_i (\phi_i(b) - \phi_i(a))^2,$$

where $R(a, b)$ is smooth in a and b and converges to zero uniformly for a, b in U as $|\phi(b) - \phi(a)|$ goes to zero.

Denote $\partial g(\phi(a))/\partial \phi_i$ by $\partial f(a)/\partial \phi_i$ and $\partial^2 g(\phi(a))/\partial \phi_i \partial \phi_j$ by $\partial^2 f(a)/\partial \phi_i \partial \phi_j$.

How can these derivatives be expressed using the Lie algebra, when $a = \varepsilon$? for the first order terms, set $L_k f = \partial f(\varepsilon)/\partial \phi_k$. And for $\xi_j = \Sigma a_{jk} L_k$, by choice of ϕ,

$$\xi_j \phi_p(\varepsilon) = \delta_{jp} = \sum a_{jk} \delta_{kp} = a_{jp}.$$

Thus, $L_j f = \xi_j f(\varepsilon)$. For the second order terms, set $L_{jk} f = \partial^2 f(\varepsilon)/\partial \phi_j \partial \phi_k$. The operator $D_{jk} f = L_{jk} f - \xi_j \xi_k f(\varepsilon)$ is actually a first order operator. Defining β_r^{jk} by the relation $D_{jk} = \Sigma_r \beta_r^{jk} \xi_r$ yields

$$\beta_r^{jk} = D_{jk} \phi_r = L_{jk} \phi_r - \xi_j \xi_k \phi_r(t) = -\rho_r^{jk}.$$

Thus

$$L_{jk} f = \left(\xi_j \xi_k - \sum_r \rho_r^{jk} \xi_r \right) f(\varepsilon).$$

Using these expressions, Taylor's formula can be written:

$$f(b) - f(\varepsilon) = \sum_i \phi_i(b) \xi_i f(\varepsilon) + \frac{1}{2} \sum_i \sum_j \phi_i(b) \phi_j(b) \left(\xi_i \xi_j - \sum_k \rho_k^{ij} \xi_k \right) f(\varepsilon)$$
$$+ \text{3rd order remainder.}$$

(3) *Functions of bounded variation.* Suppose $b(t)$ is a continuous function from $[0, 1]$ to G. The path $\{b(t)\colon 0 \leqslant t \leqslant 1\}$ is compact and so can be covered by a finite number of coordinate patches (ϕ^j, N^j). $b(t)$ is of bounded variation by definition when each $\phi^j(b(t))$ is of bounded variation, equivalently, when each $\phi_k^j(b(t))$ is of bounded variation as a function from $\mathbf{R}$ to $\mathbf{R}$.

Suppose $b(s)$ and $b(t)$ are in the coordinate patch (ψ, N). For f in C_k^∞, $f(b(u)) = f\psi^{-1}\psi(b(u))$ for $s \leqslant u \leqslant t$. Since the composed functions $f\psi^{-1}$: $\mathbf{R}^d \to \mathbf{R}$ and ψb: $\mathbf{R} \to \mathbf{R}^d$, the chain rule applies to yield

$$f(b(t)) - f(b(s)) = \int_s^t \sum_{j=1}^d \frac{\partial f}{\partial \psi_j}(b(u)) d\psi_j(b(u))$$
$$= \int_s^t \sum_{k=1}^d \xi_k f(b(u)) db_k(u),$$

where, defining $a_{jk}(u)$ by the equation

$$\frac{\partial f}{\partial \psi_j}(b(u)) = \sum_k a_{jk}(u)\xi_k f(b(u)),$$

then

$$db_k(u) = \sum_j a_{jk}(u)d\psi_j(b(u)).$$

This can also be written invariantly as an equation of measures on [0, 1]:

$$df(b(t)) = \sum_{k=1}^{d} \xi_k f(b(t))db_k(t).$$

b. *Facts concerning weak convergence of measures* [3], [11], [12]. The processes considered will correspond to measures on the space $D^G[0, 1] = \{\omega: [0, 1] \to G$ such that $\omega(t + 0)$ and $\omega(t - 0)$ exist for each t, $\omega(t) = \omega(t + 0)$ for $t < 1$, $\omega(1) = \omega(1 - 0)\}$. Properties of this space regarding weak convergence of measures are exactly analogous to those in the familiar case $G = \mathbf{R}$. In particular, the following are important.

(1) *Condition for compactness–uniform stochastic continuity*. Let P_k be measures on $D^G[0, 1]$ corresponding to processes $x_k(t)$. For any stopping time τ, let $\mathcal{F}_\tau$ denote the σ-field of events up to time τ.

Define for any set $E \subset G$ and T in [0, 1],

$$\tau_E^T = \begin{cases} \text{amount of time to exit from } E \text{ starting from time } T, \text{ except} \\ \quad = \infty \text{ when the process does not exit from } E \text{ by time 1.} \end{cases}$$

COMPACTNESS CRITERION. *Suppose that for each neighborhood of* ε, U, *and compact sets* C:

1. $P_k(\tau^\sigma_{x(\sigma)U} < \delta \| \mathcal{F}_\sigma) \leqslant \psi_U(k, \delta)$ *a.s. for every stopping time* $\sigma \leqslant 1$, *and any* $0 < \delta \leqslant 1$ *where* $\operatorname{lt}\sup_{k\to\infty}\psi_U(k, \delta) = \psi_U(\delta)$ *such that* $\operatorname{lt}_{\delta\downarrow 0}\psi_U(\delta) = 0$.

2.

$$\operatorname*{lt}_{C\uparrow G} \operatorname*{lt\,sup}_{k\to\infty} \left\langle \sup_{0\leqslant t\leqslant 1} \chi_{\tilde{C}}(x(t)) \right\rangle_k = 0.$$

Then the measures P_k *are weakly compact.*

COMPACTNESS LEMMA. *In the above criterion, condition* 2 *may be replaced by* 2a.

$$P_k\left(x(\sigma)^{-1}x(\tau^\sigma_{x(\sigma)U}) \in \tilde{C};\ (\tau^\sigma_{x(\sigma)U} \leqslant 1) \| \mathcal{F}_\sigma\right) \leqslant \varepsilon_k(C)$$

for every U *and* σ *as above, where* $\operatorname{lt}\sup_{k\to\infty}\varepsilon_k(C) = \varepsilon(C)$ *and* $\operatorname{lt}_{C\uparrow G}\varepsilon(C) = 0$.

PROOF. Define inductively:

$$\tau_1 = \tau_U^0,\ \tau_2 = \tau^{\tau_1}_{x(\tau_1)U}, \ldots, \tau_{n+1} = \tau^{\tau_n}_{x(\tau_n)U}, \ldots.$$

Take U to be a neighborhood of ε having compact closure. Choose K to be any compact set containing U. And take C to be any compact set containing $\bar{U}^n K^n$, for $n > 0$. Then, to exit from C, the process either makes n escapes of size U or else takes a jump at least of size K so that the first exit from $x(\tau_{j-1})U$ is such that $x(\tau_{j-1})^{-1}x(\tau_j)$ is in $\tilde{K}$ and

$$\left\langle \sup_{0<t<1} \chi_{\tilde{C}}(x(t)) \right\rangle_k \leqslant P_k\left(\sum_{j=1}^{n} \tau_j \leqslant 1\right) + n\varepsilon_k(K)$$

where $C \uparrow G$ as $K \uparrow G$.

The first probability is estimated as follows:

$$\begin{aligned} P_k\left(\sum_{j=1}^{n} \tau_j \leqslant 1\right) &\leqslant \left\langle \exp\left(1 - \sum_{1}^{n} \tau_j\right)\right\rangle_k = e\left\langle \exp\left(-\sum_{1}^{n} \tau_j\right)\right\rangle_k \\ &= e\left\langle (e^{-\tau_n} \| \tau_1, \ldots, \tau_{n-1}) \cdot \exp\left(-\sum_{1}^{n-1} \tau_j\right)\right\rangle_k \\ &\leqslant e \cdot \rho_k \cdot \left\langle \exp\left(\sum_{1}^{n-1} \tau_j\right)\right\rangle \leqslant e\rho_k^n \end{aligned}$$

by induction, where ρ_k bounds $(e^{-\tau_j} \| \mathfrak{F}_{\tau_{j-1}})_k$. So $\rho_k < 1$ will allow control of the first term. From condition 1 there is a $\psi_U(k, \delta)$ such that, on $(\sum_1^n \tau_j \leqslant 1)$,

$$\begin{aligned} \sup_{1<j<n} \left(e^{-\tau_j} \| \mathfrak{F}_{\tau_{j-1}}\right)_k &\leqslant \sup_j \left[e^{-\delta} P_k\left(\tau_j \geqslant \delta \| \mathfrak{F}_{\tau_{j-1}}\right) + P_k\left(\tau_j < \delta \| \mathfrak{F}_{\tau_{j-1}}\right)\right] \\ &= \sup_j \left[e^{-\delta} + (1 - e^{-\delta}) P_k\left(\tau_j < \delta \| \mathfrak{F}_{\tau_{j-1}}\right)\right] \\ &\leqslant e^{-\delta} + (1 - e^{-\delta})\psi_U(k, \delta). \end{aligned}$$

And

$$\rho = \operatorname{lt\,sup}_{n\to\infty} \rho_k \leqslant e^{-\delta} + (1 - e^{-\delta})\tfrac{1}{2} = \tfrac{1}{2}(1 + e^{-\delta}) < 1$$

by choosing $\delta > 0$ so that $\psi_U(\delta) < \frac{1}{2}$. Thus,

$$\operatorname{lt\,sup}_{k\to\infty} \left\langle \sup_t \chi_{\tilde{C}}(x(t)) \right\rangle_k \leqslant e\rho^n + n\varepsilon(K),$$

where $\rho < 1$.

Choose n first, then K large to make the right-hand side arbitrarily small as $C \uparrow G$.

(2) *Bounded functionals and weak convergence.* Let $F_n(\omega)$ be uniformly bounded functionals on $D^G[0, 1]$ converging to a bounded continuous F uniformly on compact sets. Let P_n be any weakly convergent family of measures with limit P. Let ϕ be a bounded continuous functional.

Then $\langle \phi F_n \rangle_n$ converges to $\langle \phi F \rangle$.

For, choose K compact such that $\sup_n P_n(\tilde{K})$ is less than $\delta/(C_1 + C_2)$, where

$$\sup_\omega |\phi(\omega)| \sup_n \sup_\omega |F_n(\omega)| \leqslant C_1, \quad \sup_\omega |\phi(\omega)| \cdot |F(\omega)| \leqslant C_2.$$

Then

$$\begin{aligned}\left|\int \phi F_n \, dP_n - \int \phi F \, dP\right| &\leqslant \int_K |\phi F_n - \phi F| \, dP_n \\ &\quad + (C_1 + C_2) P_n(\tilde{K}) + \left|\int \phi F (dP_n - dP)\right| \\ &\leqslant \sup_K |\phi F_n - \phi F| + \left|\int \phi F (dP_n - dP)\right| + \delta\end{aligned}$$

for large compact K, and is less than 2δ as $n \to \infty$.

Note. Even if F is not continuous, $\langle \phi F_n \rangle_n$ converges to $\langle \phi F \rangle$ as long as F is bounded and $P\{\omega\colon F \text{ is not continuous at } \omega\} = 0$, since then $\int \phi F (dP_n - dP)$ still goes to zero as $n \to \infty$.

(3) *Riemann approximations converge uniformly on compact sets.* Let the family of functions $\{x_\alpha(t)\}$ be contained in a compact subset of $D^G[0, 1]$. Let f be in C_k^∞, and suppose $v\colon [0, 1] \to \mathbf{R}$ is continuous and of bounded variation. Let $\{t_j\}_N$ be partitions of $[0, 1]$ such that $\Delta_N = \sup_{1 \leqslant j \leqslant N} |t_j - t_{j-1}|$ converges to zero as $N \to \infty$.

Then $\sum_j f(x_\alpha(t_{j-1}))[v(t_j) - v(t_{j-1})]$ converges to $\int_0^1 f(x_\alpha(t))\, dv(t)$ uniformly in α as $N \to \infty$.

To prove this, v can be assumed increasing by writing it as the difference of two increasing functions. Furthermore, note that the functions $f(x_\alpha(t))$ are in a compact subset of $D^{\mathbf{R}}[0, 1]$. So,

$$\begin{aligned}&\left|\int_0^1 f(x_\alpha(t))\, dv(t) - \sum_j f\big(x_\alpha(t_{j-1})\big)\big[v(t_j) - v(t_{j-1})\big]\right| \\ &\quad = \left|\sum_j \int_{t_{j-1}}^{t_j} f(x_\alpha(t)) - f\big(x_\alpha(t_{j-1})\big)\, dv(t)\right| \\ &\quad \leqslant 2\omega^*_{f(x_\alpha)}(\Delta_N)(v(1) - v(0)) + 2 \sup_{x \in G} |f(x)| \cdot \sum_j \big(v(b_j) - v(a_j)\big) \\ &\qquad \cup_j (a_j, b_j) \supset J_\delta + \delta \cdot (v(1) - c(0))\end{aligned}$$

where ω^* is the $D^{\mathbf{R}}[0, 1]$ modulus of continuity; and $J_\delta = \{t\colon |f(x_\alpha(t + 0)) - f(x_\alpha(t - 0))| > \delta\}$, consisting of isolated points, is contained in a finite union of arbitrarily small intervals. These terms are small for large N by compactness of $f(x_\alpha(t))$ and continuity of v.

c. *Quasi-martingales* [2], [7], [11], [12]. A real-valued process is a quasi-martingale by definition when it can be expressed in the form $q(t) = \mu(t) +$

$b(t)$, where $\mu(t)$ is a martingale relative to some σ-fields and $b(t)$ is of bounded variation. Meyer's uniqueness property implies the following:

Suppose that $\mu_1(t) + b_1(t) = \mu_2(t) + b_2(t)$ are two representations of $q(t)$, where q, the μ's and the b's are uniformly bounded in t and ω and the b's are continuous.

Then $b_1 \equiv b_2$ and $\mu_1 \equiv \mu_2$.

3. **Results.**

a. *Preliminary*. Let G be a second countable d-dimensional Lie group. Choose a basis $\{\xi_j\}$ for the Lie algebra Γ. Then choose a coordinate system (ϕ, W) at the identity, ε, satisfying:

1. W has compact closure.
2. $\phi = (\phi_1, \ldots, \phi_d)$: $G \to \mathbf{R}^d \cdot \phi(\varepsilon) = 0$.
3. $\xi_i\phi_j(\varepsilon) = \delta_{ij} = \{1 \text{ if } i = j, 0 \text{ if } i \neq j\}$.

Define the numbers ρ_k^{ij} to equal $\xi_i\xi_j\phi_k(\varepsilon)$. (Note below that ρ_k^{ij} occur only in the expression $\rho_k^{ij}dA_{ij}$ where A_{ij} is symmetric. Choosing exponential coordinates, $-\rho_k^{ij} = \rho_k^{ji}$, so these terms disappear.) Furthermore, extend ϕ to a function in C_k^∞-smooth functions on G having compact support. U and V generally will denote neighborhoods of ε.

The basic probability space $(\Omega, \mathfrak{F}, P)$ is as follows:

1. $\Omega = D^G[0, 1] = \{\omega: [0, 1] \to G$ such that $\omega(t + 0)$ and $\omega(t - 0)$ exist for every t, $\omega(t) = \omega(t + 0)$, $t < 1$, and $\omega(1) = \omega(1 - 0)\}$.
2. $\mathfrak{F} = \bigcup_{t \geqslant 0} \mathfrak{F}_t = \bigcup_{t \geqslant 0} \mathfrak{F}\{\omega(s): 0 \leqslant s \leqslant t\}$.
3. P is a probability measure on $(\Omega, \mathfrak{F})$.

Throughout, assume the notational correspondences x's to P's, y's to Q's and z's to R's between the paths of processes and their corresponding measures on $D^G[0, 1]$.

Define a *covariance function* $A(t)$ on $[0, 1]$ to be a continuous family of real $d \times d$ symmetric matrices such that:

1. $A(0) = 0$.
2. $A(t) - A(s)$ is positive semidefinite for all $1 \geqslant t \geqslant s \geqslant 0$.

Define a *Lévy measure function* $M(t, dx)$ to be a (weakly) continuous increasing family of σ-finite measures on G such that:

1. $M(0, dx) = 0$.
2. For any bounded f in $C^\infty(G)$ such that $0 = f(\varepsilon) = \xi_i f(\varepsilon)$, $1 \leqslant i \leqslant d$, $\int f(x)M(t, dx)$ is finite (and continuous in t).

In the following $B(t)$ will generally denote a continuous vector-valued function of bounded variation with components $B_k(t)$, $1 \leqslant k \leqslant d$.

b. Martingale Characterization Theorem. *Let $B(t)$ be a continuous function of bounded variation, $A(t)$ a covariance function and $M(t, dx)$ a Lévy measure function.*

Then there exists a unique process $z(t)$, equivalently a unique measure R on $D^G[0, 1]$, such that for every f in C_k^∞,

$$f(z(t)) - \int_0^t L(ds) f(z(s))$$

is a martingale with respect to R, where

$$L(dt)f(x) = \int_{G-\{\varepsilon\}} f(xz) - f(x) - \sum_j \phi_j(z)\xi_j f(x) \cdot M(dt, dz)$$
$$+ \frac{1}{2} \sum_i \sum_j \xi_i \xi_j f(x) \cdot dA_{ij}(t) + \sum_k \xi_k f(x) \cdot dB_k(t).$$

The process $z(t)$ is Markov and is furthermore a stochastically continuous process with independent increments.

c. Uniqueness of Parameters Theorem.

1. *Let $z(t)$ be a process such that there are continuous functions of bounded variation $B_\alpha(t)$, covariance functions $A_\alpha(t)$ and Lévy measure functions $M_\alpha(t, dx)$, $\alpha = 1, 2$, such that for every f in C_k^∞,*

$$f(z(t)) - \int_0^t L_\alpha(ds) f(z(s))$$

are martingales with respect to the (one) measure R on $D^G[0, 1]$, where L_α is defined as in the characterization theorem.

Then $B_1 \equiv B_2$, $A_1 \equiv A_2$ and $M_1 \equiv M_2$.

2. *Let $z_1(t)$ and $z_2(t)$ be processes, as in Theorem 1, determined by parameters B_1, A_1, M_1 and B_2, A_2, M_2, respectively.*

Suppose that $z_1(t) = z_2(t)b(t)$ in distribution for some continuous function $b(t)$. Set $\beta(t) = b(t)^{-1}$. Then:

1. $M_1(dt, dx) = M_2(dt, b(t)dxb(t)^{-1})$.
2. $dA_1(t) = \mathrm{Ad}_{\beta(t)} dA_2(t) \mathrm{Ad}^*_{\beta(t)}$.
3. *$b(t)$ must be of bounded variation and*

$$dB_{1k}(t) = \sum_j \mathrm{Ad}^{kj}_{\beta(t)} dB_{2j}(t) + db_k(t)$$
$$+ \int \left(\phi_k(z) - \sum_j \phi_j\left(b(t)zb(t)^{-1}\right) \mathrm{Ad}^{kj}_{\beta(t)} \right) M_1(dt, dz).$$

d. Limit theorem for uniformly small variables. *Suppose for each $n > 0$, n independent random elements X_{nj}, $1 \leqslant j \leqslant n$, with corresponding distributions F_{nj} are given. The X_{nj} are assumed to be uniformly small in the sense that*

$$\operatorname*{lt}_{n\to\infty} \max_{1 \leqslant j \leqslant n} P\left(X_{nj} \in \tilde{U}\right) = 0,$$

for every neighborhood U of ε.

Define corresponding means m_{nj} by the equation

$$\phi_k(m_{nj}) = \int \phi_k(x)\, dF_{nj}(x).$$

Define the centered variables Y_{nj} to equal $X_{nj}m_{nj}^{-1}$.

On [0, 1] *define the following*:

1. *The mean functions*: $m_n(0) = \varepsilon$, $m_n(t) = \prod_{j=1}^{[nt]} m_{nj}$.
2. *The covariance step-functions*:

$$A_{ij}^n(t, U) = \sum_{k=1}^{[nt]} \int_U (\phi_i(x) - \phi_i(m_{nk}))(\phi_j(x) - \phi_j(m_{nk})) \cdot dF_{nk}(x).$$

3. *The measure-functions*: $M_n(t, dx) = \sum_{j=1}^{[nt]} F_{nj}(dx)$.
4. *Approximating x processes*: $x_n(t) = \prod_{j=1}^{[nt]} X_{nj}$.
5. *Approximating centered processes*: $y_n(t) = \prod_{j=1}^{[nt]} Y_{nj}$.
6. *Write $x_n(t) = z_n(t)m_n(t)$, the product of a centered process and a mean function. Define Z_{nj} so that $z_n(t) = \prod_{j=1}^{[nt]} Z_{nj}$. The variables Z_{nj} thus equal $m_n((j-1)/n)Y_{nj}m_n((j-1)/n)^{-1}$.*

(Note that any family of partitions $\{t_j^n\}$ such that $\sup_j |t_j^n - t_{j-1}^n|$ goes to zero as $n \to \infty$ may be used, not just those of the form $\{j/n\}$. Also, [0, 1] *is used throughout as a generic fixed finite time-interval.) Observe that $x_n(t)$, $y_n(t)$, and $z_n(t)$ determine corresponding measures P_n, Q_n and R_n on $D^G[0, 1]$.*

THEOREM. *Suppose that the following conditions are satisfied*:

1. *$\{m_n(t)\}$ converges uniformly on* [0, 1] *to $m(t)$.*
2. *$\{M_n(t, dx)\}$ converges weakly, uniformly on* [0, 1], *to a measure function $M(t, dx)$ in the sense that*

$$\int f(x)M_n(t, dx) \to \int f(x)M(t, dx)$$

for any bounded continuous f that is identically zero on some neighborhood of ε.

3. *For some U, a neighborhood of ε, that is an $M(1, dx)$-continuity set, $\{A_n(t, U)\}$ converges uniformly on* [0, 1] *to a function $A(t, U)$.*

Then

1. a. *$m(t)$ is necessarily continuous.*

   b. *$M(t, dx)$ is a Levy measure function.*

   c. *$A(t, U)$ is of the form $A(t) + \{\int_U \phi_i(x)\phi_j(x)M(t, dx)\}$ for a covariance function $A(t)$.*

2. *The families $\{P_n\}$, $\{Q_n\}$, and $\{R_n\}$ are convergent to measures P, Q and R, respectively, corresponding to stochastically continuous processes with independent increments.*

3. *The limiting measure R is characterized by the condition that for every f in C_k^∞:*

$$f(z(t)) - \int_0^t \int_{G-\{\varepsilon\}} f\left(z(s)m(s)zm(s)^{-1}\right) - f(z(s))$$
$$- \sum_j \phi_j(z)\eta_j(s) f(z(s)) \cdot M(ds, dz)$$
$$- \frac{1}{2} \int_0^t \sum_i \sum_j \left(\eta_i(s)\eta_j(s) - \sum_k \rho_k^{ij}\eta_k(s)\right) f(z(s)) \cdot dA_{ij}(s)$$

is a martingale with respect to R, where $\eta_j(s) \equiv \mathrm{Ad}_{m(s)}(\xi_j)$.

4. *For limiting measures R and P corresponding to paths $z(t)$ and $x(t)$, respectively, $x(t) = z(t)m(t)$ in distribution.*

5. *As in #4, $x(t)$ is said to have a representation $z(t)m(t)$ with parameters (m, A, M).*

If $x(t) = z_1(t)m_1(t) = z_2(t)m_2(t)$ are two representations of $x(t)$ with corresponding parameters (m_1, A_1, M_1) and (m_2, A_2, M_2), then $m_1 \equiv m_2$, $A_1 \equiv A_2$, and $M_2 \equiv M_2$.

e. Representation Theorem for Independent Increment Processes. *Let $x(t)$ be any stochastically continuous process with independent increments. Then there exist a continuous function $m(t)$, a covariance function $A(t)$ and a Lévy measure function $M(t, dx)$ such that $x(t)$ has a representation $z(t)m(t)$ with parameters (m, A, M).*

f. The proofs of the above theorems yield the following

Corollary. (1) *Let $x_n(t)$ be step-function processes as in the limit theorem with corresponding step-parameters (m_n, A_n, M_n); or*

(2) *let $x_n(t)$ be processes having representations with parameters (m_n, A_n, M_n).*

Let $x(t)$ be a process having a representation with parameters (m, A, M). Then $x_n(t)$ converges to $x(t)$ if and only if:

1. *$m_n(t)$ converges uniformly on $[0, 1]$ to $m(t)$.*

2. *$M_n(t, dx)$ converges to $M(t, dx)$, away from ε (i.e., as in the limit theorem).*

3. *For U, a neighborhood of ε and continuity set of $M(1, dx)$, $A_n(t, U)$ in case (1) or, in case (2),*

$$A_n(t) + \left\{\int_U \phi_i(x)\phi_j(x) M_n(t, dx)\right\}$$

converges to

$$A(t) + \left\{\int_U \phi_i(x)\phi_j(x) M(t, dx)\right\}.$$

4. The Martingale Characterization Theorem and the Uniqueness of Parameters Theorem.

a. *Martingale Characterization Theorem.* Let $B(t)$, a continuous vector-valued function of bounded variation with components $B_k(t)$, $A(t)$, a covariance function, and $M(t, dx)$, a Lévy measure function, be given. Consider the condition that for every function f in C_k^∞,

$$f(z(t)) - \int_0^t \int_{G-\{\varepsilon\}} f(z(s)z) - f(z(s)) - \sum_j \phi_j(z)\xi_j f(z(s)) \cdot M(ds, dz)$$
$$- \frac{1}{2} \int_0^t \sum_i \sum_j \xi_i \xi_j f(z(s)) \cdot dA_{ij}(s) - \int_0^t \sum_k \xi_k f(z(s)) \cdot dB_k(s)$$

is a martingale with respect to some measure R on $D^G[0, 1]$. The theorem is that R exists and is unique.

It is convenient if $M.(t, dx)$, $A(t)$, and $B(t)$ have densities with respect to t. By a nonrandom rescaling of time this can be achieved. Such a rescaling, inducing a one-to-one correspondence from $D^G[0, 1]$ to $D^G[0, T]$, for some finite T, preserves existence and uniqueness of measures.

LEMMA 1. *Absolutely continuous version of parameters. Given a continuous function of bounded variation, $B(t)$, a covariance function $A(t)$, and a Lévy measure function $M(t, dx)$, there is a continuous strictly increasing function ψ from $[0, 1]$ onto some finite interval $[0, T]$ and functions $b_k(t)$, $a(t)$, and $m(t, dx)$, densities corresponding to a function of bounded variation, a covariance function and a Lévy measure function, respectively, such that the martingales*

$$f(Z(\psi(t))) - \int_0^{\psi(t)} \int_{G-\{\varepsilon\}} f(Z(s)z) - f(Z(s))$$
$$- \sum_j \phi_j(z)\xi_j f(Z(s)) \cdot m(s, dz)\, ds$$
$$- \frac{1}{2} \int_0^{\psi(t)} \sum_i \sum_j a_{ij}(s)\xi_i \xi_j f(Z(s))\, ds$$
$$- \int_0^{\psi(t)} \sum_k b_k(s)\xi_k f(Z(s))\, ds$$

determine a unique process $Z(t)$ on $D^G[0, T]$ if and only if a unique z process is determined on $D^G[0, 1]$ by the martingales

$$f(z(t)) - \int_0^t \int_{G-\{\varepsilon\}} f(z(s)z) - f(z(s))$$
$$- \sum_j \phi_j(z)\xi_j f(z(s)) \cdot M(ds, dz)$$
$$- \frac{1}{2}\int_0^t \sum_i \sum_j \xi_i\xi_j f(z(s)) \cdot dA_{ij}(s)$$
$$- \int_0^t \sum_k \xi_k f(z(s)) \cdot dB_k(s).$$

The processes are related as $z(t) = Z(\psi(t))$, $Z(t) = z(\psi^{-1}(t))$.

PROOF. Choose f_0 in C^∞ satisfying:

1. $f_0(\varepsilon) = \xi_i f_0(\varepsilon) = 0$, $1 \leqslant i \leqslant d$.

2. $\{D_{ij} f_0(\varepsilon)\}$, $1 \leqslant i, j \leqslant d$, is a positive definite matrix, where $D_{ij} = \xi_i\xi_j - \sum_k \rho_k^{ij}\xi_k$.

3. $0 < f_0(x) \leqslant 1$, $x \neq \varepsilon$.

Since $f_0(x)M(dt, dx)$ is, modulo a multiplicative constant, a probability distribution of the two variables t and x, it factors into the conditional distribution of x given t, $m_0(t, dx)$, times the marginal distribution of t,

$$F(dt) = \int f_0(x)M(dt, dx).$$

Define

$$a(t) = \frac{dA}{d\psi}(\psi^{-1}(t)), \quad b_k(t) = \frac{dB_k}{d\psi}(\psi^{-1}(t))$$

and

$$m(t, dx) = \frac{1}{f_0(x)} m_0(\psi^{-1}(t), dx) \frac{dF}{d\psi}(\psi^{-1}(t)),$$

where

$$\psi(t) = \operatorname{tr} A(t) + \int f_0(x)M(t, dx) + \sum_{k=1}^d \int_0^t |dB_k(s)| + t.$$

Substitute into the expression for the $z(t)$ martingales and rescale the time by ψ.

Note. The definition of ψ implies that

$$\operatorname{tr} a(t) + \int f_0(x)m(t, dx) + \sum_{k=1}^d |b_k(t)| \leqslant 1.$$

In the following, then, replacing Z by z and $[0, T]$ by the generic $[0, 1]$, consider a process $z(t)$ corresponding to a measure R on $D^G[0, 1]$ with respect to which for every f in C_k^∞, $f(z(s)) - \int_0^t L(s)f(x(s))\,ds$ is a martingale,

where

$$L(t) = \int_{G-\{\varepsilon\}} \left(r_z - 1 - \sum_j \phi_j(z)\xi_j \right) m(t, dz)$$
$$+ \frac{1}{2} \sum_i \sum_j a_{ij}(t)\xi_i\xi_j + \sum_k b_k(t)\xi_k,$$

where r_z denotes right translation by z, that is, $r_z f(x) = f(xz)$.

Now, for existence, approximating processes will be determined by approximating $L(t)$ by operators generating Poisson processes.

Before proceeding, define the following:

1. For $E \subset G$, $C_b^\infty(E) = \{g \in C^\infty(E)\colon g$ is bounded on $E\}$. Denote $C_b^\infty(G)$ by just C_b^∞. For g in $C_b^\infty(E)$, $\|g\|_E = \sup_{x\in E}|g(x)|$ and $\|g\|_{Ep}$ is a bound on E for g and all (right) derivatives up through order p, when these are uniformly bounded on E (for example when E is compact).

2. For suitable $v(t, x)$ on $[0, 1] \times G$, and any set $E \subset G$, define

$$\|v\|^E = \int_0^1 \sup_{x\in E} |v(t, x)|\, dt = \left\| \sup_{x\in E} |v(t, x)| \right\|_{L^1(dt)}.$$

Convergence of functions v_n to v in $K - L^1$ means that $\|v_n - v\|^C \to 0$ for every compact $C \subset G$.

3. For suitable functions $v(t, x)$ on $[0, 1] \times G$ that are smooth in x, define, for $E \subset G$,

$$\|v\|_E = \sup_{t,x\in E} |v(t, x)|, \quad \|v\|_{E1} = \|v\|_E + \sum_i \|\xi_i v\|_E$$

and

$$\|v\|_{E2} = \|v\|_{E1} + \sum_i \sum_j \|\xi_i\xi_j v\|_E.$$

PRIMARY LEMMA. *Let $v_n(t, x)$ be measurable in t and x and smooth in x. Suppose that, for $n > 0$,*

1. *$\|v_n\|_G$ is finite.*
2. *$\|v_n\|_{K2}$ is finite for every compact $K \subset G$.*

Then there are constants $\gamma_1(m, C)$ and $\gamma_2(\phi, b, a, m)$, depending as indicated, where C is any compact set containing a neighborhood of the origin, such that:

1. *For each compact K,*

$$\|L(t)v_n(t, x)\|^K \leq \gamma_1\|v_n\|_G + \gamma_2\|v_n\|_{KC2}$$

where $KC = \{kc\colon k \in K, c \in C\}$.

2. *γ_1 decreases to zero and γ_2 is bounded as C increases to G, so that if, in addition, $\sup_{n>0}\|v_n\|_G < \infty$ and $\operatorname{lt}\sup_{n\to\infty}\|v_n\|_{K2} = 0$ for every compact $K \subset G$, then $L(t)v_n(t, x)$ converges to zero in $K - L^1$ as $n \to \infty$.*

PROOF.

$$\sup_{x\in K}|L(t)v_n(t,x)| = \sup_{x\in K}\Bigg|\int_{G-\{\varepsilon\}} v_n(t,xz) - v_n(t,x)$$

$$- \sum_j \phi_j(z)\xi_j v_n(t,x)\cdot m(t,dz)$$

$$+ \frac{1}{2}\sum_i\sum_j a_{ij}(t)\xi_i\xi_j v_n(t,x) + \sum_k b_k(t)\xi_k v_n(t,x)\Bigg|$$

$$\leqslant \|v_n\|_{K2}\Big(\operatorname{tr} a(t) + \sum_k |b_k(t)|\Big) + (2\|v_n\|_G + \|\phi\|_G\|v_n\|_{K1})m(t,\tilde{C})$$

$$+ (2+\|\phi\|_G)\|v_n\|_{KC2}\cdot\Big(m(t,\tilde{U}) + \int_U\sum_j \phi_j^2(z)m(t,dz)\Big),$$

where U is a neighborhood of ε and C is a large compact set containing U. Integrating in t yields

$$\|L(t)v_n(t,x)\|^K \leqslant \|v_n\|_G\cdot 2M(1,\tilde{C})$$

$$+ \|v_n\|_{KC2}\Bigg[\operatorname{tr} A(1) + \sum_k\int_0^1 |b_k(t)|\,dt + (2+2\|\phi\|_G)M(1,\tilde{U})$$

$$+ (2+\|\phi\|_G)\int_U\sum_j \phi_j^2(z)M(1,dz)\Bigg]$$

$$= \gamma_1\|v_n\|_G + \gamma_2\|v_n\|_{KC2}.$$

In the case $\|v_n\|_G$ uniformly bounded and $\|v_n\|_{K2}$ converges to zero, choose C large to control the first term and then choose n large to make the sum arbitrarily small.

Define the following classes of generators:

$$\mathfrak{L} = \{L(t) \text{ corresponding to any appropriate set of functions } a(t), b_k(t), m(t,dx)\},$$

$$\mathfrak{L}_1 = \Big\{L(t)\in\mathfrak{L}: L(t) = \int(r_z-1)m(t,dz) + \sum_i\sum_j a_{ij}(t)\xi_i\xi_j + \sum_k b_k(t)\xi_k \text{ with } \operatorname*{ess\,sup}_{0<t<1} m(t,G) \text{ finite}\Big\},$$

$$\mathfrak{L}_2 = \Big\{L(t)\in\mathfrak{L}_1: L(t) = \int(r_z-1)m(t,dz)\Big\}$$

and

$$\mathfrak{L}_3 = \{L(t) \in \mathfrak{L}_2 \colon m(t, dz) \text{ is a step function}\}.$$

LEMMA 2. *Corresponding to any generator L in $\mathfrak{L}_3$ and every starting pair (t_0, x_0), there is a measure R on $D^G[0, 1]$ such that:*

1. $R\{z(t_0) = x_0\} = 1$.
2. *For $t \geqslant t_0$, f in C_k^∞,*

$$f(z(t)) - \int_{t_0}^{t} L(s) f(z(s))\, ds$$

is a martingale relative to R.

PROOF. Since L is a step-function of bounded operators, it generates a (stochastically continuous) Poisson process having the required properties. The transition probabilities for the process are determined by solutions to $\partial u/\partial t + Lu = 0$, $u(T, x) = f(x)$; T in $[0, 1]$, f in C_k^∞ given; which is solved explicitly below (Lemma 9).

LEMMA 3. *Let R be a process on $D^G[0, 1]$ corresponding to a generator L in $\mathfrak{L}$. Let U and V be neighborhoods of ε with compact closure, $\overline{U} \subset V$. Let $f = f_U$ and $g = g_{U,V}$ in C_k^∞ satisfy:*

f1. $f(\varepsilon) = 1$.
f2. $0 \leqslant f \leqslant 1$ *on* U.
f3. $f \equiv 0$ *off* U.
g1. $g \equiv 1$ *on* $\overline{U}$.
g2. $0 \leqslant g \leqslant 1$ *on* V.
g3. $g \equiv 0$ *off* V.

Then, for σ any stopping time $\leqslant 1$ a.s. R,

1.

$$R(\tau^{\sigma}_{z(\sigma)U} < \delta \| \mathfrak{F}_\sigma) \leqslant \sup_{0 < T < 1} \int_T^{(T+\delta)\wedge 1} \sup_{x \in \overline{U}} |L(s) f(x)|\, ds, \quad a.s.\ R$$

for any $0 < \delta \leqslant 1$.

2.

$$R\left(z(\sigma)^{-1} z(\tau^{\sigma}_{z(\sigma)U}) \in \tilde{V};\ (\tau^{\sigma}_{z(\sigma)U}) \leqslant 1 \| \mathfrak{F}_\sigma\right) \leqslant \int_0^1 \sup_{x \in \overline{U}} |L(s) g(x)| ds \quad a.s.\ R.$$

PROOF. Referring to Theorems 2.1 and 3.1 of [8a], the martingales $f(z(t)) - \int_T^t L(s) f(z(s))\, ds$ and $g(z(t)) - \int_T^t L(s) g(z(s))\, ds$ relative to R are martingales also with respect to the conditional distribution given $\mathfrak{F}_\sigma$, the σ-field of events up to time σ. So assume $T = 0$ (by translation), $\sigma = 0$ (by the remark) and $z(\sigma) = \varepsilon$ (by translation). Then

$$R(\tau^{\sigma}_{z(\sigma)U} < \delta \| \mathfrak{F}_{\sigma}) = R(\tau^0_U < \delta) \leqslant \left\langle \left(1 - f\left(z\left(\tau^0_U \wedge \delta\right)\right)\right)\left(\tau^0_U < \delta\right)\right\rangle$$

$$\leqslant \left\langle 1 - f\left(z\left(\tau^0_U \wedge \delta\right)\right)\right\rangle \leqslant \int_0^{\delta} \sup_{x \in \bar{U}} |L(s) f(x)|\, ds,$$

since by Doob's stopping theorem,

$$\left\langle 1 - f\left(z\left(\tau^0_U \wedge \delta\right)\right)\right\rangle = -\left\langle \int_0^{\tau} L(s) f(z(s))\, ds \right\rangle,$$

where $\tau = \tau^0_U \wedge \delta \wedge 1$. Similarly,

$$R\left(z(\sigma)^{-1} z(\tau^{\sigma}_{z(\sigma)U}) \in \tilde{V};\ (\tau^{\sigma}_{z(\sigma)U} \leqslant 1) \| \mathfrak{F}_{\sigma}\right)$$

$$= \left\langle \chi_{\tilde{V}}\left(z\left(\tau^0_U\right)\right);\ \left(\tau^0_U \leqslant 1\right)\right\rangle \leqslant \int_0^1 \sup_{x \in \bar{U}} |L(s) g(x)|\, ds,$$

as above.

LEMMA 4. *Given L in $\mathfrak{L}$, let R_n be measures on $D^G[0, 1]$, all starting at (t_0, x_0), corresponding to generators L_n in $\mathfrak{L}$ such that for every f in C_k^{∞}, $\|L_n f - Lf\|^G \to 0$ as $n \to \infty$.*

Then the measures R_n are compact and any limit point R satisfies the condition that $f(z(t)) - \int_{t_0}^t L(s) f(z(s))\, ds$ is a martingale relative to R for every f in C_k^{∞}.

PROOF. For compactness, apply Lemma 3 and the Compactness Lemma of part 2b, #1.

For condition 1:

$$\psi_U(\delta) = \operatorname*{lt\,sup}_{n\to\infty} \psi_U(n, \delta) = \operatorname*{lt\,sup}_{n\to\infty} \sup_{0 \leqslant T \leqslant 1} \int_T^{(T+\delta)\wedge 1} \sup_x |L_n(s) f_U(x)|\, ds$$

$$\leqslant \operatorname*{lt}_{n\to\infty} \int_0^1 \sup_x |(L_n(s) - L(s)) f_U(x)|\, ds$$

$$+ \sup_T \int_T^{T+\delta} \sup_x |L(s) f_U(x)|\, ds$$

$$\leqslant \text{constant} \cdot \|f\|_{G2} \cdot \sup_T \left| \int_T^{T+\delta} \left(m(s, \tilde{U}) + \int_U \sum_j \phi_j^2(z) m(s, dz) \right.\right.$$

$$\left.\left. + \sum_k |b_k(s)| + \operatorname{tr} a(s) \right) ds \right|,$$

which decreases to zero as $\delta \downarrow 0$.

For condition 2:

Let K be a compact set containing U, and set

$$V = UKU = \{u_1 k u_2 \colon u_1, u_2 \in U, k \in K\}.$$

Then, choosing $g_{UV} \equiv 1$ on $\bar{U}K = \{\bar{u}k \colon \bar{u} \in \bar{U}, k \in K\}$:

$$\operatorname*{lt\,sup}_{n\to\infty} \varepsilon_n = \operatorname*{lt\,sup}_{n\to\infty} \int_0^1 \sup_{x\in\bar{U}} |L_n(s)\, g(x)|\, ds \leqslant \int_0^1 \sup_{x\in\bar{U}} |L(s)\, g_{UV}(x)|\, ds.$$

Since $g \equiv 1$ on $\bar{U}$, the derivatives of g are zero there and, furthermore,

$$\int_0^1 \sup_{x\in\bar{U}} |L(s)\, g(x)|\, ds = \int_0^1 \sup_{x\in\bar{U}} \left(\int_{xz \notin \bar{U}K} (1 - g(xz)) m(s, dz) \right) ds$$
$$\leqslant M(1, \tilde{K}) = \varepsilon(K),$$

decreasing to zero as $K \uparrow G$.

Let R be any limit point of R_n. Restrict to a subsequence of R_n converging to R. The above conditions imply stochastic continuity of R since

$$R\left(z(s)^{-1} z(t) \notin \bar{U}, |s - t| < \delta\right) \leqslant \operatorname*{lt\,inf}_{n\to\infty} R_n\left(z(s)^{-1} z(t) \in \tilde{\bar{U}}, |s - t| < \delta\right)$$
$$\leqslant \operatorname*{lt\,sup}_{n\to\infty} \sup_s R_n(\tau^s_{z(s)U} < \delta \| \mathcal{F}_s) \leqslant \psi_U(\delta)$$

as above.

Now consider the martingales

$$\mu_n(t) = f(z_n(t)) - \int_{t_0}^t L_n(s) f(z_n(s))\, ds$$

for a general f in C_k^∞. These are uniformly bounded functionals on $D^G[0, 1]$ as seen in the above estimate for $\psi_U(\delta)$ (take $T = 0$, $\delta = 1$).

As a functional, $f(z_n(t)) = f(\omega(t))$ relative to R_n and is independent of n. The term

$$I_n = \int_0^t L_n(s) f(z_n(s))\, ds = \int_0^t L_n(s) f(\omega(s))\, ds$$

(relative to R_n) and

$$\int_0^t \sup_x |L_n(s) f(x)|\, ds \to \int_0^t \sup_x |L(s) f(x)|\, ds$$

so that

$$I_n \to I = \int_0^t L(s) f(z(s))\, ds$$

as $n \to \infty$, uniformly on compact sets of $D^G[0, 1]$.

PROPOSITION. *For every bounded continuous functional ϕ on $D^G[0, 1]$, $\langle \phi(\omega)\, \mu_n(t)(\omega) \rangle$ converges to $\langle \phi(\omega)\, \mu(t)(\omega) \rangle$.*

PROOF. To apply part 2b, #2, check that $R\{z \colon f(z(t))$ is not continuous at $z\} = 0$. Convergence of z_n to z in $D^G[0, 1]$ means that there are continuous

monotone increasing transformations λ_n of $[0, 1]$ such that $z(\lambda_n(t))^{-1}z_n(t)$ converges to ε and $\lambda_n(t) - t$ converges to 0 uniformly in t as $n \to \infty$. And

$$\begin{aligned}&\left|f(z_n(t)) - f(z(t))\right| \\ &\qquad \leqslant \left|f(z_n(t)) - f\big(z(\lambda_n(t))\big)\right| + \left|f\big(z(\lambda_n(t))\big) - f(z(t))\right|,\end{aligned}$$

which for n large, $U \subset W$ (coordinate neighborhood) is

$$\leqslant d\|f\|_{W1}\operatorname{diam}\phi(U) + \left|f\big(z(\lambda_n(t))\big) - f(z(t))\right|,$$

and this last term goes to zero in probability by stochastic continuity of z.

The martingale property $(\mu(t)\|\mathfrak{F}_s) = \mu(s)$, for $s \leqslant t$, is equivalent to $\langle\phi_s\mu(t)\rangle = \langle\phi_s\mu(s)\rangle$ for every $\mathfrak{F}_s$-measurable bounded continuous functional ϕ_s. Since $\mu_n(t)$ is a martingale,

$$\langle\phi_s\mu_n(t)\rangle_n = \langle\phi_s\mu_n(s)\rangle_n,$$

and now apply the proposition, letting $n \to \infty$, to get the martingale property for $\mu(t)$ relative to R.

LEMMA 5. *Let L be a generator in $\mathfrak{L}_2$. Then there exists a sequence $\{L_n\} \subset \mathfrak{L}_3$ such that*:

1. *For every g in C_b^∞, $L_n g$ converges to Lg in $K - L^1$.*
2. *For every f in C_k^∞, $\|L_n f - Lf\|^G$ converges to zero.*

PROOF. $L(t)g(x) = \int g(xz) - g(z)\cdot m(t, dz)$. Take π_N to be the dyadic partition of $[0, 1]$ with spacing 2^{-N}. And set $L_N(t) = L(t)$ conditioned on $\pi_N = \sum_j \chi_{I_j}(t)\cdot 2^N\int_{I_j} L(s)\,ds$, where $I_j = [(j-1)2^{-N}, j2^{-N}]$.

Then for t in $I_j(t)$, K compact,

$$\begin{aligned}&\int dt \sup_{x\in K}\left|L(t)g(x) - L_N(t)g(x)\right| \\ &\qquad = \int dt \sup_{x\in K}\left|\int\left[g(xz) - g(x)\right]\left[m(t, dz) - 2^N\int_{I_j(t)} m(s, dz)\,ds\right]\right| \\ &\qquad \leqslant \int dt\Bigg[\|g\|_G\left|m(t, G) - 2^N\int_{I_j(t)} m(s, G)\,ds\right| \\ &\qquad\qquad + \sup_{x\in K}\left|\int g(xz)m(t, dz) - 2^N\int_{I_j(t)}\int g(xz)m(s, dz)\,ds\right|\Bigg].\end{aligned}$$

Set

$$\lambda(t, x) = \int g(xz)m(t, dz),$$

$$\bar{m}_N(t, dz) = 2^N \int_{I_j(t)} m(s, dz)\, ds, \quad \text{and} \quad \bar{\lambda}_N(t, x) = \int g(xz)\bar{m}_N(t, dz).$$

Then

$$\|L_N g - Lg\|^K \leqslant \|g\|_G \int dt |m(t, G) - \bar{m}_N(t, G)| + \|\lambda - \bar{\lambda}_N\|^K.$$

The first term goes to zero since the averages of an L^1 function converge to the original function in L^1. For the second term, let U be any neighborhood of ε with compact closure and let $\{Ux_1, \ldots, Ux_n\}$ be a finite cover of K. Then

$$\|\lambda - \bar{\lambda}_N\|^K \leqslant \int dt \max_{1 \leqslant j \leqslant n} |\lambda(t, x_j) - \bar{\lambda}_N(t, x_j)|$$

$$+ \int dt \max_{1 \leqslant j \leqslant n} \sup_{u \in U} \left(|\lambda(t, ux_j) - \lambda(t, x_j)| + |\bar{\lambda}_N(t, ux_j) - \bar{\lambda}_N(t, x_j)|\right).$$

The first term again goes to zero as above. The second term is

$$\leqslant 2 \int dt \int \sup_{x \in K} \sup_{u \in U} |g(uxz) - g(xz)| m(t, dz).$$

And, choosing C compact such that $M(1, \tilde{C}) \leqslant \delta/2\|g\|_G$,

$$\int dt \int \sup_{x \in K} \sup_{u \in U} |g(uxz) - g(xz)| m(t, dz)$$

$$\leqslant \int dt \int_C \sup_{x \in K} \sup_{u \in U} |g(uxz) - g(xz)| m(t, dz)$$

$$+ 2\|g\|_G M(1, \tilde{C}) \leqslant 2\delta$$

as $U \downarrow \{\varepsilon\}$ by uniform continuity of g on $\bar{U}KC$. #1 follows. Similarly, for f in C_k^∞,

$$\|L_N f - Lf\|^G \leqslant \|f\|_G \int dt |m(t, G) - \bar{m}_N(t, G)| + \|\lambda_f - \bar{\lambda}_{f,N}\|^G.$$

But, for K compact, F = support of f,

$$\|\lambda_f - \bar{\lambda}_{f,N}\|^G = \int dt \sup_{x \in G} \left| \int f(xz)[m(t, dz) - \bar{m}_N(t, dz)] \right|$$

$$\leqslant \int dt \sup_{x \in G} \left| \int_K f(xz)[m(t, dz) - \bar{m}_N(t, dz)] \right|$$

$$+ \int dt \sup_{x \in G} \left| \int_{\tilde{K}} f(xz)[m(t, dz) - \bar{m}_N(t, dz)] \right|$$

$$\leqslant \int dt \sup_{x \in FK^{-1}} \left| \int_K f(xz)[m(t, dz) - \bar{m}_N(t, dz)] \right| + 2\|f\|_G M(1, \tilde{K})$$

so that #2 follows just like #1.

LEMMA 6. *Any L in $\mathfrak{L}_1$ can be approximated by generators $\{L_n\} \subset \mathfrak{L}_2$ as in Lemma* 5.

PROOF. Let $g_n(t, dx)$ be a Gaussian measure on $\mathbf{R}^d$ with mean $m = (m_1, \ldots, m_d)$ and covariance (c_{ij}) where

$$m_k = \frac{1}{n}\left(b_k(t) + \sum_i \sum_j \rho_k^{ij} a_{ij}(t)\right) \quad \text{and} \quad c_{ij} = \frac{1}{n} a_{ij}(t).$$

Recall that $b_k(t)$ and tr $a(t)$ are bounded.

For (ϕ, W) the standard coordinate system at ε, let $p_n(t, dx) = g_n|_{\phi(W)}$ and set $G_n(t, A) = p_n(t, \phi(A))$, for $A \subset W$.

Define

$$L_n(t) = \int (r_z - 1)(m(t, dz) + nG_n(t, dz)).$$

Then for a neighborhood of ε, $N \subset W$, by Taylor's expansion,

$$\begin{aligned}
&\sup_{x \in K} |L(t)g(x) - L_n(t)g(x)| \\
&\quad \leqslant \sup_{x \in K} \Bigg| \frac{1}{2} \sum_i \sum_j a_{ij}(t)\xi_i\xi_j g(x) + \sum_k b_k(t)\xi_k g(x) \\
&\qquad - \int_N \Bigg(\sum_k \phi_k(z)\xi_k g(x) + \frac{1}{2} \sum_i \sum_j \phi_i(z)\phi_j(z) \\
&\qquad\qquad \times \left[\xi_i\xi_j - \sum_k \rho_k^{ij}\xi_k\right] g(x)\Bigg) nG_n(t, dz)\Bigg| \\
&\qquad + \|g\|_{KN3} \operatorname{diam} \phi(N) \cdot \int_N \sum_k \phi_k^2(z) nG_n(t, dz) \\
&\qquad + 2\|g\|_G nG_n(t, W\tilde{N}).
\end{aligned}$$

The boundedness of $b_k(t)$ and tr $a(t)$ implies that $g_n(t, \widetilde{\phi(N)})$ goes to zero exponentially as $n \to \infty$. So,

$$\begin{aligned}
&\operatorname{lt\,sup}_{n\to\infty} \|L(t)g(x) - L_n(t)g(x)\|^K \\
&\quad \leqslant \operatorname{lt}_{n\to\infty} \text{constant}\,(d) \cdot \|g\|_{KN3} \\
&\qquad \cdot \Bigg| \frac{1}{n} \sup_{\substack{0 \leqslant t \leqslant 1 \\ 1 \leqslant i \leqslant d}} \left(b_i(t) + \sum_p \sum_q \rho_i^{pq} a_{pq}(t)\right)^2 \\
&\qquad\qquad \cdot (\operatorname{diam} \phi(N) + 1) + \operatorname{diam} \phi(N) \cdot \sup_{0 \leqslant t \leqslant 1} \operatorname{tr} a(t)\Bigg| \\
&\quad = \text{constant}\,(d) \cdot \|g\|_{KN3} \cdot \operatorname{diam} \phi(N) \cdot \sup_{0 \leqslant t \leqslant 1} \operatorname{tr} a(t).
\end{aligned}$$

Since N is arbitrary, $\mathrm{lt}_{n\to\infty}\|L_n g - Lg\|^K = 0$. For f in C_k^∞, replace the bound $\|g\|_{KN3}$ by $\|f\|_{G3}$.

LEMMA 7. *Any L in $\mathfrak{L}$ may be approximated by generators $\{L_n\} \subset \mathfrak{L}_1$ as in Lemma 5.*

PROOF. Set

$$\begin{aligned} L_n(t) &= \int_{\tilde{U}_n}\left(r_z - 1 - \sum_j \phi_j(z)\xi_j\right)m(t, dz) \\ &\quad + \sum_i \sum_j a_{ij}(t)\xi_i\xi_j + \sum_k b_k(t)\xi_k \\ &= \int (r_z - 1)m_n(t, dz) + \sum_i \sum_j a_{ij}(t)\xi_i\xi_j \\ &\quad + \sum_k \left(b_k(t) - \int_{\tilde{U}_n} \phi_k(z)m(t, dz)\right)\xi_k, \end{aligned}$$

where U_n are neighborhoods shrinking to ε and $m_n(t, dz) = \chi_{\tilde{U}_n}(z)m(t, dz)$. Note that $m_n(t, G) = m(t, \tilde{U}_n)$ is finite since it is bounded by $(1/u_{0,n})\int f_0(x)m(t, dx)$ where $u_{0,n} = \inf_{x\in\tilde{U}_n} f_0(x) > 0$. And

$$\|L(t)g(x) - L_n(t)g(x)\|^K \leqslant (\text{constant}) \cdot \|g\|_{KU_n2}\left(\int_{U_n} \sum_j \phi_j^2(z)M(1, dz)\right)$$

by Taylor's expansion. For f in C_k^∞, the bound $\|f\|_{G2}$ replaces $\|g\|_{KU_n2}$.

Finally,

LEMMA 8. *Existence of R. Let L be in $\mathfrak{L}$. Then, for every starting pair (t_0, x_0), there exists a measure R on $D^G[0, 1]$ such that:*

1. $R\{z(t_0) = x_0\} = 1$.
2. *For f in C_k^∞, $t \geqslant t_0$,*

$$f(z(t)) - \int_{t_0}^t L(s)f(z(s))\, ds$$

is a martingale relative to R.

PROOF. By Lemma 2, there exists a process for every L in $\mathfrak{L}_3$. Apply Lemma 4 to generators L in $\mathfrak{L}_3$ that approximate L. This is possible by Lemmas 5–7.

For uniqueness, consider the problem TP: Given an L in $\mathfrak{L}$, T in $[0, 1]$ and f in C_k^∞, does there exist $u(t, x)$ on $[0, 1] \times G$ such that:

1. $\|u\|_G$ is finite.
2. $u(t, x)$ and its derivatives in x are jointly continuous in t and x.
3. $\|L(t)u(t, x)\|^K$ is finite for every compact $K \subset G$.

4. For $0 \leqslant s, t \leqslant T$,

$$u(t, x) - u(s, x) = -\int_s^t L(\sigma)u(\sigma, x)\, d\sigma; \quad u(T, x) = f(x).$$

LEMMA 9. *TP is solvable for L in* $\mathcal{L}_3$.

PROOF. On $[0, T]$, $L(t) = \sum_{j=1}^N \chi_{I_j}(t)L_j$, where L_j's are bounded operators on C_b^∞, since

$$\|L_j g(x)\|_G = \left|\int g(xz) - g(x)\cdot m_j(dz)\right| \leqslant 2\|g\|_G \sup_j m_j(G),$$

where $m_j = \chi_{I_j} m$. For $I_j = (t_{j-1}, t_j]$, $\Delta_j = t_j - t_{j-1}$, define, for t in I_k, $k(t) = k$ and $\delta(t) = t_k - t$ (or $T - t$ if t is in I_N).

Then

$$v(t, x) = e^{\delta(t)L_{k(t)}} \cdots e^{\Delta_{N-1}L_{N-1}} e^{\Delta_N L_N} f(x)$$

satisfies property #4. (For integrability of $L(\sigma)v(\sigma, x)$, see below.)

PROPOSITION. *Each L_j generates a contraction semigroup on C_b^∞.*

PROOF. Let L be an L_j and let g be in C_b^∞. Then

$$Lg(x) = \int g(xz) - g(x)\cdot m(dz) = \lambda \int g(xz) - g(x)\cdot \mu(dz),$$

where $\int \mu(dz) = 1$. So $|\int g(xz)\,\mu(dz)| \leqslant \|g\|_G$, and

$$\begin{aligned} |e^{tL}g(x)| &= \left|\exp \lambda t\left[\int r_z \mu(dz) - 1\right]\cdot g(x)\right| \\ &= e^{-\lambda t}\sum_{k=0}^\infty \frac{\lambda^k t^k}{k!}\left|\left(\int r_z \mu(dz)\right)^k g(x)\right| \leqslant \|g\|_G. \end{aligned}$$

Property #1 follows, $\|v(t, x)\|_G \leqslant \|f\|_G$. Since L acts on the right, L commutes with left derivatives, ξ_L, of any order. And the semigroups e^{tL_j} commute with ξ_L. Thus, $u(t, x) = \xi_L v(t, x)$ exists and satisfies property #4 with $u(T, x) = \xi_L f(x)$, and $\|\xi_L v\|_G \leqslant \|\xi_L f\|_G$ as above. On compact subsets of G, right derivatives can be bounded in terms of left derivatives. Thus, $\|v\|_{K2}$ is finite for every compact $K \subset G$. Property #3 now follows by the Primary Lemma. And, for any v satisfying #1, #3 and #4, by the estimate in the Primary Lemma and uniform boundedness of the space derivatives of v, for x and y in a compact set,

$$\begin{aligned} |v(t, x) - v(s, y)| &= |v(t, x) - v(s, x) + v(s, x) - v(s, y)| \\ &\leqslant \int_s^t L(\sigma)v(\sigma, x)\, d\sigma + |v(s, x) - v(s, y)| \end{aligned}$$

is bounded by a constant times $(t - s)(\|v\|_G + \|v\|_{KC2}) + \varepsilon_s(x, y)$ where y is

in K, a compact set containing a neighborhood of x, C is a compact set containing a neighborhood of ε, and $\varepsilon_s(x, y) \to 0$ as $x^{-1}y$ converges to ε uniformly in s, depending only on the boundedness of the first derivatives in x of v on $[0, 1] \times K$.

This is property #2.

LEMMA 10. *TP is solvable for L in $\mathfrak{L}$.*

PROOF. Suppose L is in $\mathfrak{L}_2$. Take L_n converging to L as in Lemma 5.

Let u_n be the solution corresponding to L_n. Then as in proof of Lemma 9, $\|u_n\|_G$ and $\|u_n\|_{K2}$ are uniformly bounded in n. The proof then shows that, furthermore, the u_n are equicontinuous. Apply the Ascoli-Arzela Theorem to u_n and a similar argument to the left derivatives $\xi_L u_n$ to conclude that there exists a smooth u satisfying properties #1, #2 and #3 such that $\sup_{n>0}\|u_n - u\|_G$ is finite and $\|u_n - u\|_{K3}$ converges to zero as $n \to \infty$.

Finally,

$$\|L_n u_n - Lu\|^K \leqslant \|(L_n - L)(u_n - u)\|^K + \|L_n u - Lu\|^K + \|L(u_n - u)\|^K.$$

The bounds in the proofs of Lemmas 5 (and 6 and 7) of the type $\|g\|_G$, $\|g\|_{K2}$, and $\|g\|_{K3}$ for the functions $u_n - u$ and u are uniform in t so that the corresponding estimates are valid. The first two terms thus go to zero, and the Primary Lemma applies to the third term.

So TP is solvable for L in $\mathfrak{L}_2$. By applying Lemmas 6 and 7 in similar arguments, it follows that TP is solvable for all L in $\mathfrak{L}$.

The existence of solutions to TP will yield the uniqueness of R for a given L. Assume an L in $\mathfrak{L}$ is given. R is any measure corresponding to L, as in Lemma 8.

LEMMA 11. *Let $u(t, x)$ be jointly C^∞ in t and x such that $u(t, x) \equiv 0$ for x outside of some compact set. Let R be as in Lemma 8, starting at (t_0, x_0). Then*

$$u(t, z(t)) - \int_{t_0}^{t} \frac{\partial u}{\partial t}(\sigma, z(\sigma)) + L(\sigma)u(\sigma, z(\sigma)) \cdot d\sigma$$

is a martingale relative to R for $t \geqslant t_0$.

PROOF. For $t \geqslant s \geqslant t_0$,

$$\begin{aligned}
&\big(u(t, z(t)) - u(s, z(s))\|\mathfrak{F}_s\big) \\
&\quad = \big(u(t, z(t)) - u(t, z(s))\|\mathfrak{F}_s\big) + u(t, z(s)) - u(s, z(s)) \\
&\quad = \int_s^t L(w)u(t, z(w)) + \frac{\partial u}{\partial w}(w, z(s)) \cdot dw
\end{aligned}$$

$$= \int_s^t \frac{\partial u}{\partial w}(w, z(w)) + L(w)u(w, z(w)) \cdot dw$$

$$+ \int_s^t L(w)u(t, z(w)) - L(w)u(w, z(w)) \cdot dw$$

$$+ \int_s^t \frac{\partial u}{\partial w}(w, z(s)) - \frac{\partial u}{\partial w}(w, z(w)) \cdot dw.$$

Since $\partial u/\partial w$ is in $C_k^\infty(G)$ for each w and $L(w)$ commutes with differentiation in time, the 'error' terms

$$\int_s^t \int_w^t L(w) \frac{\partial u}{\partial \sigma}(\sigma, z(w))\, d\sigma\, dw - \int_s^t \int_s^w L(\sigma) \frac{\partial u}{\partial w}(w, z(\sigma))\, d\sigma\, dw$$

vanish since the first integral is over $\{s \leqslant w \leqslant \sigma \leqslant t\}$ and the second is over $\{s \leqslant \sigma \leqslant w \leqslant t\}$ and σ and w are switched in the second integral.

LEMMA 12. *Let $u(t, x)$ be a solution of TP for L. Then, relative to any R for any starting pair (t_0, x_0), $u(t, z(t))$ is a martingale, for $t_0 \leqslant t \leqslant T$.*

PROOF. Translate time so $t_0 = 0$.

Let $\psi_K(x)$ be in $C^\infty(G)$ such that:

1. $\psi_K(x) = 1$, for $x \in K$, a compact set.
2. $0 \leqslant \psi_K \leqslant 1$.
3. The support of ψ_K is in K', a compact set containing K.

Let, on $\mathbf{R}$, $\rho_\varepsilon(t)$ be an approximate δ-function with support that shrinks to zero as ε goes to zero.

Now define

$$u_{K\varepsilon}(t, x) = \psi_K(x) \int u(t - s, x)\rho_\varepsilon(s)\, ds.$$

$u_{K\varepsilon}$ is smooth in t and x and has compact support in x. Lemma 11 implies that

$$u_{K\varepsilon}(t, z(t)) - \int_0^t \frac{\partial u_{K\varepsilon}}{\partial t}(s, z(s)) + L(s)u_{K\varepsilon}(s, z(s)) \cdot ds$$

is a martingale.

PROPOSITION. *Set $u_K(t, x) = \psi_K(x)u(t, x)$. Then*

$$u_K(t, z(t)) + \int_0^t \psi_K(z(s))L(s)u(s, z(s)) - L(s)u_K(s, z(s)) \cdot ds$$

is a martingale.

PROOF. The martingale property will hold if

$$\operatorname*{lt}_{\varepsilon\downarrow 0} \sup_x |u_{K\varepsilon}(t,x) - u_K(t,x)| + \left\| \frac{\partial u_{K\varepsilon}}{\partial t}(t,x) + \psi_K(x)L(t)u(t,x) \right\|^G$$
$$+ \|L(t)u_{K\varepsilon}(t,x) - L(t)u_K(t,x)\|^G$$

equals zero.

First term.

$$\sup_x |u_{K\varepsilon} - u_K| \leqslant \int \sup_{x\in K'} |u(t-s,x) - u(t,x)|\rho_\varepsilon(s)\,ds$$
$$\leqslant \sup_{\substack{x\in K'\\ |s|<\delta}} |u(t-s,x) - u(t,x)| + 2\|u\|_G \cdot \int_{|s|>\delta} \rho_\varepsilon(s)\,ds$$

which goes to zero as $\varepsilon\downarrow 0$ by uniform continuity of u on $[0, 1] \times K'$, choosing δ appropriately. The second term is identically zero for small enough ε such that support$(\rho_\varepsilon) \subset [-\delta, \delta]$.

Third term. The above argument applies to $u_{K\varepsilon} - u_K$ and derivatives up to second order. The estimates in the proof of the Primary Lemma therefore apply, in this case to $\| \ \|^G$.

Second term.

$$-\partial u_{K\varepsilon}/\partial t = \psi_K(x) \int L(t-s)u(t-s,x)\rho_\varepsilon(s)\,ds.$$

And the second term equals

$$\int \sup_x |\psi_K(x)| \left| \int (L(t-s)u(t-s,x) - L(t)u(t,x))\rho_\varepsilon(s)\,ds \right| dt$$
$$\leqslant \int \int_{|s|<\delta} \sup_{x\in K'} |L(t-s)u(t-s,x) - L(t)u(t,x)|\rho_\varepsilon(s)\,ds\,dt,$$

when support$(\rho_\varepsilon) \subset [-\delta, \delta]$.

For each s, consider

$$\lambda_s(t) = \sup_{x\in K'} |L(t-s)u(t-s,x) - L(t)u(t,x)|.$$

This may be written

$$\sup_{x\in K'} |L(t+h)u(t+h,x) - L(t)u(t,x)|,$$

which is bounded by

$$\sup_{x\in K'}\left|\int u(t+h,xz)-u(t+h,z)-\sum_j \phi_j(z)\xi_j u(t+h,z)\cdot m(t+h,dz)\right.$$

$$\left.-\int u(t,xz)-u(t,z)-\sum_j \phi_j(z)\xi_j u(t,z)\cdot m(t,dz)\right|$$

$$+\sum_i\sum_j |a_{ij}(t+h)-a_{ij}(t)|\cdot \|u\|_{K'2}$$

$$+\sum_i\sum_j |a_{ij}(t+h)|\,\|u(t+h,x)-u(t,x)\|_{K'2}$$

$$+\sum_k |b_k(t+h)|\,\|u(t+h,x)-u(t,x)\|_{K'1}$$

$$+\sum_k |b_k(t+h)-b_k(t)|\cdot \|u\|_{K'1}.$$

For each x after integrating in t, these latter terms go to zero with h, since L^1 functions are continuous in the L^1 norm, and by uniform continuity of $u(t, x)$ and its derivatives in x on $[0, 1] \times K'$. The first term can be written, writing U_x for the integrands,

$$\sup_{x\in K'}\left|\int U_x(t+h,z)m(t+h,dz)-U_x(t,z)m(t,dz)\right|,$$

which goes to zero in $L^1(dt)$ if

$$\int |U_x(t,z)-U_y(t,z)|m(t,dz)$$

goes to zero in $L^1(dt)$ uniformly for $x^{-1}y$ near ε (see the proof of Lemma 5 for $\lambda - \bar{\lambda}_N$). And

$$\int dt \int |U_x(t,z)-U_y(t,z)|m(t,dz)$$

$$\leqslant \int dt \int_C |U_x(t,z)-U_y(t,z)|m(t,dz) + 2\sup_{x,t,z} U_x(t,z)M(1,\tilde{C}).$$

Choose C to be a large compact set containing a neighborhood of ε; the U term is small by uniform continuity of u and its derivatives on $[0, 1] \times K'C$.

Thus, $\|\lambda_s(t)\|_{L^1}$ goes to zero as $s \to 0$. Then

$$\int dt \int_{|s|<\delta} \lambda_s(t)\rho_\varepsilon(s)\,ds \leqslant \sup_{|s|<\delta} \|\lambda_s(t)\|_{L^1}$$

goes to zero with δ. So,

$$u_K(t,z(t)) + \int_0^t \psi_K(z(s))L(s)u(s,z(s)) - L(s)u_K(s,z(s))\cdot ds$$

is a martingale.

PROPOSITION. *Fix H, a compact set, and let τ_H equal the first exit time of $z(t)$ from H. Then $u(t \wedge \tau_H, z(t \wedge \tau_H))$ is a martingale.*

PROOF.

$$u_K(t \wedge \tau_H, z(t \wedge \tau_H)) + \int_0^{t \wedge \tau_H} \psi_K(z(s)) L(s) u(s, z(s)) - L(s) u_K(s, z(s)) \cdot ds$$

is a martingale. And the martingale property will be preserved as K increases up to G if

$$\operatorname*{lt}_{K \uparrow G} \left\langle \left| u_K(t \wedge \tau_H, z(t \wedge \tau_H)) - u(t \wedge \tau_H, z(t \wedge \tau_H)) \right| \right\rangle + \left\| \psi_K(x) L(t) u(t, x) - L(t) u_K(t, x) \right\|^H = 0.$$

The first term goes to zero as $K \uparrow G$, since it is bounded by $\|u\|_G \cdot R(z(\tau_H) \notin K)$. And, for $K \supset H$,

$$\int dt \sup_{x \in H} \left| \psi_K(x) L(t) u(t, x) - L(t) u_K(t, x) \right| = \left\| L(t)(u(t, x) - u_K(t, x)) \right\|^H.$$

As K increases up to G, $u - u_K$ satisfies the hypotheses of the Primary Lemma, since for any compact set C,

$$\|u - u_K\|_{C2} \equiv 0 \quad \text{for } K \supset C.$$

Now, $u(t \wedge \tau_H, z(t \wedge \tau_H))$ is a martingale for every compact H. Since the z's are paths in $D^G[0, 1]$, $\sup_H \tau_H = \infty$. Letting $H \uparrow G$, the uniform boundedness of u implies that $u(t, z(t))$ is a martingale on $[0, 1]$.

LEMMA 13. *$u(t, x)$ is unique. That is, if $u(t, x)$ and $v(t, x)$ are solutions of TP for a given T and f, then $u \equiv v$.*

PROOF. The difference $\Delta = u - v$ is a solution of TP on $[0, T] \times G$ with $f \equiv 0$. So $\Delta(t, z(t))$ is a martingale for any measure R corresponding to L, as in Lemma 8. For $R(z(s) = x) = 1$,

$$\Delta(s, x) = \langle \Delta(0, z(0)) \rangle = \langle \Delta(T, z(T)) \rangle = \langle f(z(T)) \rangle = 0.$$

LEMMA 14. *Let R_1 and R_2 be two measures such that for some starting pair $(s, x), j = 1, 2$:*

1. *$R_j\{z(s) = x\} = 1$.*
2. *For every f in C_k^∞,*

$$f(z(t)) - \int_s^t L(s) f(z(s)) \cdot ds$$

is a martingale relative to (both of the) R_j.

Then

$$\langle f(z(T))\rangle_1 = \langle f(z(T))\rangle_2,$$

for every f in C_k^∞ and $s \leqslant T \leqslant 1$.

PROOF. As in Lemma 13,

$$\langle f(z(T))\rangle_j = \langle u(T, z(T))\rangle_j = \langle u(0, z(0))\rangle_j = u(s, x).$$

Since compact sets generate the Borel sets of G,

$$\langle \chi_A(z(T))\rangle_{s,x} = \pi(s, x, T, A)$$

is well defined for every s, x, T and A. The theorem will follow from the fact that R determines a Markov process with uniquely determined transition probabilities $\pi(s, x, T, A)$.

THEOREM. *R is unique.*

PROOF. Theorems 2.1 and 3.1 of [8a] imply that for any σ in [0, 1] and $u(t, x)$ a solution of TP, there is a regular conditional probability λ_ω^σ on [0, 1]. This implies that $z(t)$ is a Markov process: for $T \geqslant \sigma$, $\lambda_\omega^\sigma(z(T) \in S)$ should equal $\pi(\sigma, z(\sigma), T, S)$. For f in C_k^∞, approximating compact S, as in Lemmas 13 and 14,

$$\langle f(z(T))\rangle_{\lambda_\omega^\sigma} = u(\sigma, z(\sigma)) = \langle f(z(T))\rangle_{\sigma, z(\sigma)}, \quad \text{a.s.}$$

Take f's, $0 \leqslant f \leqslant 1$, converging to χ_S; then

$$\langle \chi_S(z(T))\rangle_{\lambda_\omega^\sigma} = R(z(T) \in S \| \mathcal{F}_\sigma) = \langle \chi_S(z(T))\rangle_{\sigma, z(\sigma)} = \pi(\sigma, z(\sigma), T, S)$$

as required.

REMARK. All of the above processes determined by B, A, M martingales have independent increments, as is immediately seen since these are Markov processes where the transition probabilities $\pi(s, x, t, xA)$ are independent of x; equivalently, the generator $L(t)$ is spatially homogenous.

b. *Uniqueness of Parameters Theorem.*

THEOREM 1. *Let $z(t)$ be a process as in the characterization theorem such that for every f in C_k^∞, $\alpha = 1, 2$,*

$$f(z(t)) - \int_0^t L_\alpha(ds) f(z(s))$$

are martingales relative to the (one) measure R on $D^G[0, 1]$.

$$L_\alpha(dt) = \int \left(r_z - 1 - \sum_j \phi_j(z)\xi_j\right) M_\alpha(dt, dz)$$
$$+ \sum_i \sum_j \xi_i \xi_j dA_{\alpha,ij}(t) + \sum_k \xi_k dB_{\alpha,k}(t).$$

Then $B_1 \equiv B_2$, $A_1 \equiv A_2$ and $M_1 \equiv M_2$.

PROOF. By Lemma 1 of §a (the characterization theorem), the generators can be assumed to be of the form $L_\alpha(t)$ with corresponding densities b_α, a_α and m_α; a time change valid for both L's at once can be effected by adding the ψ's for the individual generators together (see proof of Lemma 1, §a).

By Meyer's uniqueness property for quasimartingales (cf. part 2c, #1),

$$\int_0^t L_1(s)f(z(s))\,ds \equiv \int_0^t L_2(s)f(z(s))\,ds \quad \text{a.s. identically in } t.$$

Pick $\zeta(t)$ such that $R(N_{\zeta(t)}) > 0$ for every neighborhood $N_{\zeta(t)}$ of $\zeta(t)$ in $D^G[0, 1]$. Since $\int_0^t (L_1 - L_2)(s)f(\omega(s))\,ds$ is a bounded continuous functional on $D^G[0, 1]$,

$$\int_0^t L_1(s)f(\zeta(s))\,ds \equiv \int_0^t L_2(s)f(\zeta(s))\,ds$$

for every f in C_k^∞, identically in t.

By Lebesgue's Theorem on differentiating integrals,

$$L_1(t)f(\zeta(t)) = L_2(t)f(\zeta(t)) \quad a.e.\ t$$

where the null set depends on f.

Since C_k^∞ is separable, a single null set E of t's can be chosen that works for all f in C_k^∞.

For $t \notin E$, choose f in C_k^∞ such that $f(x) = g(\zeta(t)^{-1}x)$ with g identically zero on a neighborhood of ε. Then

$$L_1(t)f(\zeta(t)) = \int g(z)m_1(t, dz) = L_2(t)f(\zeta(t)) = \int g(z)m_2(t, dz).$$

Thus, $m_1(t, dx) = m_2(t, dz)$.

Now take

$$f_j(x) = \phi_j\big(\zeta(t)^{-1}x\big) \quad \text{and} \quad f_{pq}(x) = \phi_p\big(\zeta(t)^{-1}x\big)\phi_q\big(\zeta(t)^{-1}x\big).$$

Then

$$(L_1(t) - L_2(t))f_{pq}(\zeta(t)) = 2(a_{1,pq}(t) - a_{2,pq}(t)) = 0.$$

And

$$\begin{aligned}(L_1(t) - L_2(t))&f_j(\zeta(t))\\ &= b_{1,j}(t) + \sum_p \sum_q \rho_{pq}^j a_{1,pq}(t) - b_{2,j}(t) - \sum_p \sum_q \rho_{pq}^j a_{2,pq}(t)\\ &= b_{1,j}(t) - b_{2,j}(t) = 0.\end{aligned}$$

LEMMA 1. *Let $z(t)$ be a process determined by martingales of the form*

$$f(z(t)) - \int_0^t L(ds)f(z(s))$$

as in Theorem 1.

Let $b(t)$ be a continuous function of bounded variation.
Then for any f in C_k^∞,

$$f(z(t)b(t)) - \int_0^t \mathrm{Ad}_{\beta(s)} L(ds) f(z(s)b(s)) - \int_0^t \sum_k \xi_k f(z(s)b(s))\, db_k(s)$$

is a martingale, where $\beta(t) = b(t)^{-1}$, and

$$\begin{aligned}\mathrm{Ad}_{\beta(s)} L(ds) f(xb(s)) = \int & f(xzb(s)) - f(xb(s)) \\ & - \sum_j \phi_j(z) \mathrm{Ad}_{\beta(s)}(\xi_j) f(xb(s)) \cdot M(ds, dz) \\ & + \frac{1}{2} \sum_i \sum_j \mathrm{Ad}_{\beta(s)}(\xi_i)\, \mathrm{Ad}_{\beta(s)}(\xi_j) f(xb(s)) \cdot dA_{ij}(s) \\ & + \sum_k \mathrm{Ad}_{\beta(s)}(\xi_k) f(xb(x)) dB_k(s).\end{aligned}$$

PROOF. $\{b(t), 0 \leqslant t \leqslant 1\}$ is compact so that $f(xb(t)) = u(t, x)$ satisfies the hypotheses of Lemma 11 of the preceding section in the case $b(t)$ is smooth. Approximate $b(t)$ uniformly on [0, 1] by smooth b_n such that the b_n converge weakly as measures on [0, 1] to db (cf. part 2a, #3). Martingales are preserved in the limit.

Following Lemma 11, the term $(\partial/\partial t)\ f(z(s)b(s))$ becomes $\sum_k \xi_k f(z(s)b(s)) db_k(s)$ since the time differentiation applies just to the 'b' variable.

In applying $L(ds)$, r_z and the right derivatives apply just to the 'x' variable. Setting $g(x) = f(xb)$ yields

$$\xi g(x) = \frac{d}{dt}\bigg|_0 f(xbe^{t\mathrm{Ad}_\beta \xi}) = \mathrm{Ad}_\beta \xi f(xb), \qquad \beta = b^{-1}.$$

LEMMA 2. *Let $z_1(t)$ and $z_2(t)$ be processes as in Lemma 1 corresponding to functions B_1, A_1, M_1 and B_2, A_2, M_2, respectively. Suppose that $z_1(t) = z_2(t)b(t)$ in distribution for some continuous function $b(t)$. Then $b(t)$ is of bounded variation.*

PROOF. Let R_2 denote the measure on $D^G[0, 1]$ corresponding to $z_2(t)$. For any f in C_k^∞,

$$\begin{aligned}&\big(f(z_2(t)b(t)) - f(z_2(s)b(s)) \| \mathcal{F}_s\big) \\ &\qquad = \big(f(z_2(t)b(t)) - f(z_2(s)b(t)) \| \mathcal{F}_s\big) + f(z_2(s)b(t)) - f(z_2(s)b(s)).\end{aligned}$$

Then (see Lemma 1)

$$\begin{aligned}\int_s^t L_1(d\sigma) f(z_2(\sigma)b(\sigma))\, d\sigma = & \int_s^t \mathrm{Ad}_{\beta(t)} L_2(d\sigma) f(z_2(\sigma)b(t)) \\ & + f(z_2(s)b(t)) - f(z_2(s)b(s)) \quad \text{a.s. } R_2.\end{aligned}$$

For U a neighborhood of ε, and

$$\lambda_b = (2 + \|\phi\|_G)\Big(\sup_{0<t<1}\ \sup_{1<i,j<d} \big|\mathrm{Ad}^{ij}_{\beta(t)}\big| + 1\Big)^2,$$

$$\big|f(z_2(s)b(t)) - f(z_2(s)b(s))\big| \leqslant C\|f\|_{G2}(\Phi(t) - \Phi(s))$$

for some constant C depending only on d and λ_b with Φ a bounded increasing function. This holds a.s. R_2. Now consider $E_t = \{x\colon R_2(z(t) \in N_x) > 0$ for every neighborhood of x, $N_x\}$. By continuity of f,

$$\sup_{x\in E_s} \big|f(xb(t)) - f(xb(s))\big| \leqslant C\|f\|_{G2}(\Phi(t) - \Phi(s)).$$

Since G is second countable, E_t is nonempty for every t.

Given f in C_k^∞, and $0 \leqslant s \leqslant t \leqslant 1$, take x in E_s. Then

$$\begin{aligned}\big|f(b(t)) - f(b(s))\big| &= \big|f\big(x^{-1}xb(t)\big) - f\big(x^{-1}xb(s)\big)\big| \\ &= \big|f_x(xb(t)) - f_x(xb(s))\big| \leqslant C\cdot \|f_x\|_{G2}(\Phi(t) - \Phi(s)) \\ &= C\cdot \|f\|_{G2}(\Phi(t) - \Phi(s)),\end{aligned}$$

where $f_x(z) \equiv f(x^{-1}z)$ is in C_k^∞.

Thus, $b(t)$ is of bounded variation (see part 2a, #3).

THEOREM 2. *Let $z_1(t)$ and $z_2(t)$ be processes determined by martingales corresponding to parameters B_1, A_1, M_1 and B_2, A_2, M_2, respectively. Suppose that $z_1(t) = z_2(t)b(t)$ in distribution for some continuous function $b(t)$.*

Then:

1. $M_1(dt, dx) = M_2(dt, b(t)dxb(t)^{-1})$.
2. $dA_1(t) = \mathrm{Ad}_{\beta(t)}dA_2(t)\mathrm{Ad}^*_{\beta(t)}$.
3. *$b(t)$ must be of bounded variation and*

$$\begin{aligned}dB_{1k}(t) &= \sum_j \mathrm{Ad}^{kj}_{\beta(t)}dB_{2,j}(t) + db_k(t) \\ &\quad + \int\Big(\phi_k(z) - \sum_j \phi_j\big(b(t)zb(t)^{-1}\big)\mathrm{Ad}^{kj}_{\beta(t)}\Big)\cdot M_1(dt, dz).\end{aligned}$$

PROOF. Apply Lemma 2, Lemma 1 and Theorem 1.

1. Note that, substituting $z \to b(s)zb(s)^{-1}$,

$$\begin{aligned}&\int f(xzb(s)) - f(xb(s)) - \sum_j \phi_j(z)\mathrm{Ad}_{\beta(s)}(\xi_j)f(xb(s))\cdot M_2(ds, dz) \\ &\quad = \int f(xb(s)z) - f(xb(s)) \\ &\qquad - \sum_j \phi_j\big(b(s)zb(s)^{-1}\big)\mathrm{Ad}_{\beta(s)}(\xi_j)f(xb(s))\, M_2\big(ds, b(s)dzb(s)^{-1}\big).\end{aligned}$$

2.

$$\xi_i\xi_j dA_{1,ij}(t) = \sum_k \sum_l \mathrm{Ad}^{ik}_{\beta(t)}\mathrm{Ad}^{jl}_{\beta(t)} dA_{2,kl}(t).$$

3. The last term is the correction for #1. Observe that the integrand in this last term is uniformly bounded and is of second order in z at ε.

5. Limit theorem for uniformly small variables.

a. *The set-up*. For each $n > 0$, n random elements X_{nj} with corresponding distributions F_{nj} are given. The family X_{nj} satisfies:

1. Independence–for fixed n, X_{nj}, $1 \leqslant j \leqslant n$ are independent.
2. Uniform smallness–for every neighborhood of ε, U,

$$\operatorname*{lt}_{n\to\infty} \max_{1\leqslant j\leqslant n} P\left(X_{nj} \in \tilde{U}\right) = 0.$$

Define the means m_{nj} by the equation

$$\phi_k(m_{nj}) = \int \phi_k(x)\, dF_{nj}(x).$$

They are well-defined elements of G, in fact of W, for n sufficiently large, since

$$\max_{\substack{1\leqslant j\leqslant n\\ 1\leqslant k\leqslant d}} \left|\phi_k(m_{nj})\right| \leqslant \max_{u\in U}\left|\phi(u)\right| + \max_{x\in G}\left|\phi(x)\right| \cdot P\left(X_{nj} \in \tilde{U}\right).$$

Thus $\phi(m_{nj}) \in \phi(W)$ for all large n; and m_{nj} are uniformly close to ε as $n \to \infty$. Define mean functions

$$m_n(t) = \prod_{j=1}^{[nt]} m_{nj}, \qquad m_n(0) = \varepsilon,$$

and covariance step-functions

$$A^n_{ij}(t, U) = \sum_{k=1}^{[nt]} \int_U \left(\phi_i(x) - \phi_i(m_{nk})\right)\left(\phi_j(x) - \phi_j(m_{nk})\right) \cdot dF_{nk},$$

$$A^n_{ij}(0, U) = 0.$$

Note that $A^n(t, U) - A^n(s, U)$ are positive semidefinite matrices for all $0 \leqslant s \leqslant t \leqslant 1$, and that

$$\begin{aligned}&\left|A^n_{ij}(t, U) - A^n_{ij}(s, U)\right|\\ &\qquad \leqslant \tfrac{1}{2}\left(A^n_{ii}(t, U) + A^n_{jj}(t, U) - A^n_{ii}(s, U) - A^n_{jj}(s, U)\right)\end{aligned}$$

for $s \leqslant t$. The mean and covariance determine the Gaussian part of the limit process.

The measure functions

$$M_n(t, dx) = \sum_{j=1}^{[nt]} F_{nj}(dx)$$

determine the jumps of the limit process.

The centered variables

$$Y_{nj} = X_{nj}m_{nj}^{-1} \quad \text{and} \quad Z_{nj} = m_n\left(\frac{j-1}{n}\right)Y_{nj}m_n\left(\frac{j-1}{n}\right)^{-1}$$

correspond to 'infinitesimal' and 'macro' centerings, respectively. Denote the distribution of Y_{nk} by G_{nk}.

Approximating processes are

$$x_n(t) = \prod_{j=1}^{[nt]} X_{nj}, \quad y_n(t) = \prod_{j=1}^{[nt]} Y_{nj}$$

and

$$z_n(t) = x_n(t)m_n(t)^{-1} = \prod_{j=1}^{[nt]} Z_{nj}.$$

These correspond, respectively, to measures P_n, Q_n, and R_n on $D^G[0, 1]$.

Hypotheses for convergence are the following:

1. $m_n(t)$ converges to $m(t)$ uniformly on $[0, 1]$.

2. $M_n(t, dx)$ converges weakly, uniformly on $[0, 1]$, to $M(t, dx)$, a measure function, in the sense that

$$\int f(x)M_n(t, dx) \to \int f(x)M(t, dx)$$

for any bounded continuous f that vanishes identically on a neighborhood of ε.

3. $A_{ij}^n(t, U)$ converges to $A_{ij}(t, U)$ uniformly on $[0, 1]$ for some neighborhood of ε, U, that is a continuity set of $M(1, dx)$ and which *a fortiori* is a continuity set for all $M(t, dx)$, $0 \leqslant t \leqslant 1$.

PRELIMINARY LEMMAS.

1. *$m(t)$ is continuous.*

2. *$M(t, dx)$ is a Lévy measure function.*

3. *$A(t, U)$ is of the form*

$$A(t) + \left\{\int_U \phi_i(x)\phi_j(x)M(t, dx)\right\}$$

for a covariance function $A(t)$.

PROOF. 1. Since the increments m_{nk} are uniformly near ε as $n \to \infty$, and $m_n(t)$ converges uniformly to $m(t)$, $m(t)$ is continuous.

2. Let U be any neighborhood of ε; $U \subset V$ a fixed neighborhood. Set $s(x) = \sum_{j=1}^d \phi_j^2(x)$. Then

$$\int_{\tilde{U}} s(x)M_n(t, dx) \leqslant \int_{\tilde{U}} s(x)M_n(1, dx) = \left(\int_{\tilde{U}V} + \int_{\tilde{U}\tilde{V}}\right)(s(x)M_n(1, dx))$$

$$\leqslant \operatorname{tr} A_n(1, V) + 2\left(\max_{\substack{1 \leqslant k \leqslant n \\ 1 \leqslant j \leqslant d}} |\phi_j(m_{nk})|\right)$$

$$\cdot \left(1 + \max_{x \in G} |\phi(x)|\right) M_n(1, \tilde{U}) + \int_{\tilde{V}} s(x)M_n(1, dx).$$

By uniform smallness of m_{nk} and convergence of $M_n(t, dx)$, letting $n \to \infty$ first,

$$\operatorname*{lt}_{U \downarrow \{\varepsilon\}} \int_{\tilde{U}} s(x)M(t, dx) \leqslant \operatorname{tr} A(1, V) + \int_{\tilde{V}} s(x)M(1, dx),$$

since the right-hand side is independent of $U \subset V$.

Continuity of $M(t, dx)$ follows by uniform convergence of $M_n(t, dx)$, similar to the proof for $m(t)$.

3. Continuity of $A(t, U)$ follows by uniform convergence, similar to the proof for $m(t)$. Then continuity of $A(t)$ will follow from that of $A(t, U)$ and $M(t, dx)$. Let $N \subset U$, N an $M(1, dx)$ continuity neighborhood of ε.

$A_n(t, U) - A_n(t, N)$ converges to $\{\int_{U\tilde{N}} \phi_i(x)\phi_j(x)M(t, dx)\}$. Thus, the limit

$$A_n(t, U) - \left\{\int_U \phi_i(x)\phi_j(x)M(t, dx)\right\}$$

is independent of U for all sufficiently small continuity neighborhoods U of ε. Call the common limit $A(t)$. Then since $M(t, dx)$ is a Lévy measure function, $A(t) = \operatorname{lt}_{U \downarrow \{\varepsilon\}} A(t, U)$ is a covariance function, positive-definiteness being preserved in the limits $n \to \infty$, $U \downarrow \{\varepsilon\}$.

b. *Compactness for* $\{y_n(t)\}$.

LEMMA 1. *Let f be in C_k^∞. Define*

$$j_n(t) = \begin{cases} t, & nt \text{ is an integer}, \\ ([nt] + 1)/n, & nt \text{ not an integer}. \end{cases}$$

Then, with respect to Q_n, the process

$$\mu_n(t) = f(y_n(j_n(t)))$$
$$- \sum_{k=1}^{nj_n(t)} \int f\left(y_n\left(\frac{k-1}{n}\right)y\right) - f\left(y_n\left(\frac{k-1}{n}\right)\right) \cdot dG_{nk}(y)$$

is a martingale relative to the σ-fields $\mathcal{F}_{j_n(t)}$, where $\mathcal{F}_t = \{\omega(s), 0 \leqslant s \leqslant t, \omega \in D^G[0, 1]\}$.

PROOF. For $(j-1)/n < t \leqslant j/n$, $(j-2)/n < s \leqslant (j-1)/n$,

$$(\mu_n(t) - \mu_n(s)\|\mathcal{F}_{j_n(s)}) = f\left(y_n\left(\frac{j}{n}\right)\right)\|\mathcal{F}_{(j-1)/n} - f\left(y_n\left(\frac{j-1}{n}\right)\right)$$
$$- \int f\left(y_n\left(\frac{j-1}{n}\right)y\right) - f\left(y_n\left(\frac{j-1}{n}\right)\right) \cdot dG_{nj}(y) = 0$$

by independence of Y_{nj} and $Y_{n,j-1}$. For $(j-1)/n < s < t \leqslant j/n$, $\mu_n(t) = \mu_n(s)$ so their difference is zero.

LEMMA 2. *Compactness inequality for* $\{y_n\}$. *Let* $V \subset W$ *be a neighborhood of* ε. *Then, as in Lemma* 3 *of part* 4,

$$Q_n(\tau^{\sigma}_{y(\sigma)V} < \delta \|\mathcal{F}_\sigma) \leqslant \sup_{T,x} \sum_{nT_n+1}^{nT_n+n\delta_n} \sup_Y \left| \int f_{xV}(Yy) - f_{xV}(Y) \cdot dG_{nk}(y) \right|$$

for every stopping time $\sigma \leqslant 1$, *where* $T_n = j_n(T)$, $\delta_n = j_n(\delta)$, $f_{xV}(z) \equiv f_V(x^{-1}z)$.

PROOF. Same as in Lemma 3 of part 4.

Denote the expression in the right-hand side of the compactness inequality for $x = \varepsilon$ by $\Sigma(f_V, n, \delta, T)$. The compactness lemma of part 2b, #1 shows that the following imply compactness.

1. $\mathrm{lt}_{\delta\downarrow 0}\mathrm{lt}\sup_{n\to\infty}\Sigma(f_V, n, \delta, T) = 0$.
2.

$$Q_n\left(y(\sigma)^{-1}y(\tau^{\sigma}_{y(\sigma)U}) \in \tilde{C};\ (\tau^{\sigma}_{y(\sigma)U} \leqslant 1)\|\mathcal{F}_\sigma\right) \leqslant \varepsilon_n(C),$$

where $\mathrm{lt}_{C\uparrow G}\ \mathrm{lt}\sup_{n\to\infty}\varepsilon_n(C) = 0$.

In 1, uniformity in x is assured since f_{xV} are translates of f_V, so estimates for f_{xV} at x correspond to those for f_V at ε.

LEMMA 3. *Compactness for* $y_n(t)$. *The family* $\{y_n(t)\}$ *is weakly compact.*

PROOF. For a function f in C_k^∞, denote a bound for f and its derivatives through third order by $\|f\|$. Here $f = f_V$.

Consider a typical term in $\Sigma(f_V, n, \delta, T)$ of the form

$$\sup_Y \left| \int f(Yy) - f(Y) \cdot dG_{nk}(y) \right|.$$

For clarity denote m_{nk} by m, dG_{nk} and dF_{nk}, respectively, by dG and dF, and introduce the function $g(u) = f(Yu)$. The estimates below will hold uniformly in Y.

Then

$$\int g(y) - g(\varepsilon) \cdot dG(y) = \int g(xm^{-1}) - g(\varepsilon) \cdot dF(x).$$

Using Taylor's expansion (part 2a, #2) for $x, m \in U$, expand around m and rearrange to get

$$\sum_i \sum_j \int_U \frac{1}{2} \alpha_{ij}[\phi_i(x) - \phi_i(m)][\phi_j(x) - \phi_j(m)] \cdot dF(x)$$

$$+ \int_U R(x, m) \sum_i [\phi_i(x) - \phi_i(m)]^2 \cdot dF(x)$$

$$+ \int_{\tilde{U}} g(xm^{-1}) - g(\varepsilon) - \sum_i \alpha_i(\phi_i(x) - \phi_i(m)) \cdot dF(x)$$

$$+ \int \sum_i \alpha_i(\phi_i(x) - \phi_i(m)) dF(x)$$

$$= (1) + (2) + (3) + \text{zero} \quad \text{by definition of } m.$$

Note that

$$\alpha_i = \frac{\partial}{\partial \phi_i} g(xm^{-1})|_{x=m}, \quad \alpha_{ij} = \frac{\partial^2}{\partial \phi_i \partial \phi_j} g(xm^{-1})|_{x=m};$$

these α's being bounded in absolute value by $\|f\|$. U is taken to be a continuity set of $M(1, dx)$.

Now consider a sum of such terms from $K = nT_n + 1$ to $nT_n + n\delta_n$. As $n \to \infty$, (2), the remainder term, is bounded by

$$\left(\sup_{x,m \in U} |R(x, m)|\right)\left(\sum_i A_{ii}(T + \delta, U) - A_{ii}(T, U)\right)$$

$$\leqslant \left(\sup_{x,m \in U} |R(x, m)|\right)\left(\operatorname{tr} A(1) + \sum_i \int_U \phi_i^2(x) M(1, dx)\right),$$

which goes to zero as U shrinks to $\{\varepsilon\}$. Choose U small to control these terms. For the (1) terms, lt sup as $n \to \infty$ is bounded by

$$\|f\| \cdot \left(\sum_i A_{ii}(T + \delta, U) - A_{ii}(T, U)\right) = \|f\| \cdot (\operatorname{tr} A(T + \delta) - \operatorname{tr} A(T))$$

$$+ \int_U \sum_j \phi_j^2(x)[M(T + \delta, dx) - M(T, dx)],$$

which goes to zero with δ uniformly in T.

The (3) terms, as $n \to \infty$, are bounded by

$$(2\|f\| + 2d\|\phi\| \, \|f\|)[M(T + \delta, \tilde{U}) - M(T, \tilde{U})],$$

which goes to zero with δ uniformly in T.

So condition 1 holds. For condition 2, take $C = \bar{U}K$ for K a compact set containing U.

Then

$$\begin{aligned} &Q_n\left(y(\sigma)^{-1}y(\tau^{\sigma}_{y(\sigma)U}) \in \tilde{C};\ (\tau^{\sigma}_{y(\sigma)U} \leqslant 1) \| \mathcal{F}_{\sigma}\right) \\ &\qquad \leqslant Q_n\left(y(t) \text{ has a jump} > \text{size of } K\right) \\ &\qquad \leqslant M_n(1, \tilde{K}). \end{aligned}$$

Taking lt sup as $n \to \infty$, K can be taken large enough to make the right-hand side arbitrarily near zero, by weak convergence of the $M_n(1, dx)$.

Now to see what the martingales associated with $y(t)$ are.

LEMMA 4. *Martingales for $y(t)$. Let $y(t)$ be any limit point of $\{y_n(t)\}$. For any $f \in C_k^{\infty}$,*

$$\begin{aligned} \mu(t) = f(y(t)) &- \int_0^t \int_{G-\{\varepsilon\}} f(y(s)y) - f(y(s)) \\ &- \sum_j \phi_j(y)\xi_j f(y(s)) \cdot M(ds, dy) \\ &- \frac{1}{2}\int_0^t \sum_i \sum_j \left(\xi_i\xi_j - \sum_k \rho_k^{ij}\xi_k\right) f(y(s)) \cdot dA_{ij}(s) \end{aligned}$$

is a martingale.

PROOF. Referring to the above proof, now with an arbitrary f in C_k^{∞}, the martingales $\mu_n(t)$ equal

$$\begin{aligned} f(y_n(t_n)) &- \sum_{k=1}^{nt_n} \int f\left(y_n\left(\frac{k-1}{n}\right)y\right) - f\left(y_n\left(\frac{k-1}{n}\right)\right) \cdot dG_{nk}(y) \\ &= f(y_n(t_n)) - \sum_{k=1}^{nt_n} \left[\int_{\tilde{U}} f\left(y_n\left(\frac{k-1}{n}\right)xm_{nk}^{-1}\right) - f\left(y_n\left(\frac{k-1}{n}\right)\right)\right. \\ &\qquad - \sum_i \alpha_{ij}^n(\phi_i(x) - \phi_i(m_{nk})) \cdot dF_{nk}(x) \\ &\qquad + \frac{1}{2}\int_U \sum_i \sum_j \alpha_{ijk}^n(\phi_i(x) - \phi_i(m_{nk})) \\ &\qquad \cdot (\phi_j(x) - \phi_j(m_{nk})) \cdot dF_{nk}(x) \\ &\qquad \left. + \int_U R(x, m_{nk}) \sum_i (\phi_i(x) - \phi_i(m_{nk}))^2 \cdot dF_{nk}(x)\right] \\ &= f(y_n(t_n)) - [B_n(t, U) + C_n(t, U) + R_n(t, U)], \end{aligned}$$

where

$$t_n = j_n(t), \quad \alpha_{ik}^n = \frac{\partial}{\partial \phi_i} f\left(y_n\left(\frac{k-1}{n}\right) x m_{nk}^{-1}\right)\Bigg|_{x = m_{nk}}$$

and

$$\alpha_{ijk}^n = \frac{\partial^2}{\partial \phi_i \partial \phi_j} f\left(y_n\left(\frac{k-1}{n}\right) x m_{nk}^{-1}\right)\Bigg|_{x = m_{nk}}.$$

First let $n \to \infty$, then let U shrink to $\{\varepsilon\}$ via $M(1, dx)$-continuity sets. The above proof shows that lt $\sup_{n\to\infty} R_n(t, U)$ goes to zero as U shrinks. Note that the B and C terms are Riemann sums approximating integrals. $B_n(t, U)$ becomes

$$\int_0^t \int_{\tilde{U}} f(y(s)y) - f(y(s)) - \sum_i \phi_i(y)\xi_i f(y(s)) \cdot M(ds, dy)$$

as U shrinks to $\{\varepsilon\}$, the y-integral extends over the set $G - \{\varepsilon\}$. And (see part 2a, #2) $C_n(t, U)$ converges to

$$\frac{1}{2}\int_0^t \sum_i \sum_j \left(\xi_i \xi_j - \sum_k \rho_k^{ij} \xi_k\right) f(y(s)) \cdot \left[dA_{ij}(s) + \int_U \phi_i(x)\phi_j(x) M(ds, dx)\right].$$

The M term goes to zero as U shrinks to $\{\varepsilon\}$.

To prove the martingale property, proceed as in the proof of Lemma 4 of part 4. For each n, consider $\mu_n(t)$ as a functional of the paths in $D^G[0, 1]$. These are uniformly bounded, since as $n \to \infty$ they are bounded by

$$\|f\| + \left(\|f\| \int_V \sum_i \sum_j \phi_i(y)\phi_j(y) \cdot M(1, dy) + (2\|f\| + d\|f\| \, \|\phi\|) M(1, \tilde{V})\right)$$
$$+ 2d^2 \|f\| \cdot \left(1 + \max_{i,j,k} |\rho_k^{ij}|\right) \cdot \operatorname{tr} A(1).$$

PROPOSITION. *For any bounded continuous functional ϕ on $D^G[0, 1]$, $\langle \phi\mu_n(t)\rangle_n$ converges to $\langle \phi\mu(t)\rangle$ along any subsequence such that the corresponding measures converge.*

PROOF. $\mu_n(t)$ consists of two parts, $f(y_n(j_n(t)))$ and the integral terms. $f(y(j_n(t)))$ converges to $f(y(t))$ uniformly on any set of functions in $D^G[0, 1]$ that are continuous at t and contained in some compact set of $D^G[0, 1]$. So it is sufficient to check that $Q(y\colon f(y(t))$ is not continuous at $y) =$ zero. This follows by stochastic continuity of y as in the proof of Lemma 4, part 4. The integral terms converge uniformly on compact sets of $D^G[0, 1]$, first as $n \to \infty$ (part 2b, #4), and then as U shrinks to $\{\varepsilon\}$.

To apply the proposition, again note that the martingale property $(\mu(t)\|\mathcal{F}_s) = \mu(s)$ is equivalent to $\langle \mu(t)\phi_s\rangle = \langle \mu(s)\phi_s\rangle$ for every bounded, continuous $\mathcal{F}_s$-measurable ϕ_s, while $\mathcal{F}_s \subset \mathcal{F}_{j_n(s)}$ so that $\langle \mu_n(t)\phi_s\rangle_n = \langle \mu_n(s)\phi_s\rangle_n$ since μ_n is in fact a martingale.

By the proposition, $\langle \mu_n(t)\phi_s \rangle_n$ converges to $\langle \mu(t)\phi_s \rangle$ and $\langle \mu_n(s)\phi_s \rangle_n$ to $\langle \mu(s)\phi_s \rangle$ so that these are equal too.

The processes $\{y_n\}$ are compact and any limit point has associated martingales. The Martingale Characterization Theorem now implies there is a unique limit that is a stochastically continuous process with independent increments. (See end of the next section.)

c. *Compactness for* $\{z_n(t)\}$. The approximating processes $z_n(t) = x_n(t)m_n(t)^{-1}$ are related to $y_n(t)$ by the equations

$$Z_{nj} = m_n\left(\frac{j-1}{n}\right)Y_{nj}m_n\left(\frac{j-1}{n}\right)^{-1} \quad \text{and} \quad z_n(t) = \prod_{j=1}^{[nt]} Z_{nj}.$$

Like Y_{nj}, Z_{nj} are independent, $1 \leqslant j \leqslant n$, for each n so that the following is immediate (see Lemma 1).

LEMMA 5. *Let f be in C_k^∞. Then, for every n,*

$$v_n(t) = f(z_n(j_n(t))) - \sum_{k=1}^{nj_n(t)} \int f\left(z_n\left(\frac{k-1}{n}\right)z\right)$$
$$-f\left(z_n\left(\frac{k-1}{n}\right)\right)\cdot dG_{nk}\left(m_n\left(\frac{k-1}{n}\right)^{-1} z m_n\left(\frac{k-1}{n}\right)\right)$$

is a martingale.

The proof of compactness for $\{z_n\}$ is exactly parallel to that for $\{y_n\}$ except that the typical term to be estimated is of the form

$$\int f\left(z_n\left(\frac{k-1}{n}\right)z\right) - f\left(z_n\left(\frac{k-1}{n}\right)\right)\cdot dG_{nk}\left(m_n^{-1}\left(\frac{k-1}{n}\right)zm_n\left(\frac{k-1}{n}\right)\right)$$
$$= \int f(\zeta mzm^{-1}) - f(\zeta)\cdot dG(z),$$

denoting $z_n((k-1)/n)$ by ζ, $m_n((k-1)/n)$ by m and G_{nk} by G. It is important to note here that m denotes the cumulative mean function up to time $(k-1)/n$ and so may not become infinitesimal as $n \to \infty$.

Note also the following:

LEMMA 6. *Preservation of neighborhoods. Let (ϕ, W) be the standard coordinate patch. Let K be any compact set. For any set S, denote the set $\{k^{-1}sk\colon k \in K, s \in S\}$ by $K^{-1}SK$. Let V be a given neighborhood of ε. Then:*

1. *There is a neighborhood of ε, N, such that $K^{-1}NK \subset W$. And $K^{-1}N'K \subset W$ for any $N' \subset N$; these shrink to $\{\varepsilon\}$ with N'.*
2. *There is a neighborhood of ε, N'', such that $N'' \subset K^{-1}VK$.*

PROOF. 1. Suppose $x_n \to \varepsilon$ and there are points k_n in K such that $k_n^{-1}x_nk_n$ is not in W. By compactness of K, choose a subsequence of k_n converging to k.

Then along that subsequence, $k_n^{-1}x_nk_n$ converges to ε, a contradiction.

Once in W, the shrinking of these neighborhoods can be controlled in terms of coordinates.

2. In fact, $K^{-1}VK = \bigcup_{k\in K}k^{-1}Vk$ is open and contains ε since V does.

Note particularly that $K = \{m_n(t),\ m(t)\}_{n,0\leqslant t\leqslant 1}$ is compact by uniform convergence of $m_n(t)$.

Recalling the $\{y_n\}$ case, the following can be taken together.

LEMMA 7. *Compactness of* $\{z_n(t)\}$. *The family* $\{z_n(t)\}$ *is weakly compact.*

LEMMA 8. *Martingales for* $z(t)$. *For* $z(t)$ *any limit point of* $\{z_n(t)\}$ *and* $f \in C_k^\infty$,

$$\begin{aligned}\nu(t) = f(z(t)) &- \int_0^t\int_{G-\{\varepsilon\}} f\big(z(s)m(s)zm(s)^{-1}\big)\\ &-f(z(s)) - \sum_j \phi_j(z)\eta_j(s)f(z(s))\cdot M(ds, dz)\\ &-\frac{1}{2}\int_0^t\sum_i\sum_j\left(\eta_i(s)\eta_j(s) - \sum_k \rho_k^{ij}\eta_k(s)\right)f(z(s))\cdot dA_{ij}(s)\end{aligned}$$

is a martingale, where $\eta_i(s) = \mathrm{Ad}_{m(s)}\xi_i$.

PROOF. The proofs are essentially the same as for $\{y_n\}$ noting the following:

1. The auxiliary function $g(u) = f(\zeta mum^{-1})$. To compute its derivatives in terms of f, use the adjoint mappings as follows:

$$\begin{aligned}\xi g(\varepsilon) &= \frac{d}{dt}\bigg|_0 f(\zeta mze^{t\xi}m^{-1})|_{z=\varepsilon} = \frac{d}{dt}\bigg|_0 f(\xi me^{t\xi}m^{-1})\\ &= \frac{d}{dt}\bigg|_0 f(\zeta e^{t\mathrm{Ad}_m\xi}) = \mathrm{Ad}_m\xi f(\zeta).\end{aligned}$$

Similarly,

$$\xi_1\xi_2 g(\varepsilon) = \mathrm{Ad}_m(\xi_1)\mathrm{Ad}_m(\xi_2)f(\zeta).$$

So all derivatives 'ξ' become '$\mathrm{Ad}_m\xi$'.

Recall that

$$\mathrm{Ad}_m\xi_j = \sum_{k=1}^d \mathrm{Ad}_m^{kj}\xi_k.$$

These numbers are uniformly bounded in all indices since the m's are in a compact set and Ad_m depends smoothly on m.

2. Estimates for $\{z_n\}$ are thus the same as for $\{y_n\}$ since the Ad terms introduce finite sums with uniformly bounded coefficients. Lemma 6 allows validity of the Taylor expansion and precludes extimates depending on U

from being affected by the inner automorphisms involved.

3. Referring to Lemma 4, 'B_n' and 'C_n' terms determined the limiting martingales. For $z(t)$: B_n terms:

$$\sum_k \int f\left(z_n\left(\frac{k-1}{n}\right)m_n\left(\frac{k-1}{n}\right)zm_{nk}^{-1}m_n\left(\frac{k-1}{n}\right)^{-1}\right) - f\left(z_n\left(\frac{k-1}{n}\right)\right) - \sum_{i=1}^{d} \alpha_{ik}^n \phi_i(z)\cdot dF_{nk}(z)$$

becomes

$$\int_0^t \int_{G-\{\varepsilon\}} f\left(z(s)m(s)zm(s)^{-1}\right) - f(z(s)) - \sum_{j=1}^{d} \phi_j(z)\eta_j(s)f(z(s))\cdot M(ds, dz).$$

C_n terms:

$$\sum_k \int \frac{1}{2}\sum_i \sum_j \alpha_{ijk}^n\left[\phi_i(x) - \phi_i(m_{nk})\right]\left[\phi_j(x) - \phi_j(m_{nk})\right]\cdot dF_{nk}(x)$$

becomes

$$\int_0^t \frac{1}{2}\sum_i \sum_j \left(\eta_i(s)\eta_j(s) - \sum_k \rho_k^{ij}\eta_k(s)\right) f(z(s))\cdot dA_{ij}(s).$$

Writing the Ad's as matrices and rearranging second order terms yields the following expression for $\nu(t)$:

$$\begin{aligned} f(z(t)) &- \int_0^t \int_{G-\{\varepsilon\}} f\left(z(s)m(s)zm(s)^{-1}\right) - f(z(s)) \\ &- \sum_k \sum_i \phi_i(z)\mathrm{Ad}_{m(s)}^{ki}\xi_k f(z(s))\cdot M(ds, dz) \\ &- \frac{1}{2}\int_0^t \sum_i \sum_j \xi_i\xi_j f(z(s))\cdot dC_{ij}(s) \\ &+ \frac{1}{2}\int_0^t \sum_i \sum_j \sum_k \sum_l \rho_k^{ij}\mathrm{Ad}_{m(s)}^{lk}\xi_l f(z(s))\cdot dA_{ij}(s), \end{aligned}$$

where $dC(t) = \mathrm{Ad}_{m(t)}dA(t)\mathrm{Ad}_{m(t)}^*$.

The characterization theorem implies convergence of $z_n(t)$ to $z(t)$ which is stochastically continuous and has independent increments. For observe that the limiting processes $y(t)$ and $z(t)$ are exactly of the type considered in that theorem. The 'y' martingales are a special case of the 'z' martingales with $m(t)$ identically equal to ε, and for $z(t)$ with parameters (m, A, M):

$$f(z(t)) - \int_0^t \int_{G-\{\varepsilon\}} f(z(s)z) - f(z(s))$$
$$- \sum_j \phi_j(z)\xi_j f(z(s)) \cdot N(ds, dz)$$
$$- \frac{1}{2} \int_0^t \sum_i \sum_j \xi_i \xi_j f(z(s)) \cdot dC_{ij}(s)$$
$$- \int_0^t \sum_k \xi_k f(z(s)) \cdot dB_k(s)$$

is a martingale for every f in C_k^∞, where since

$$N(dt, dz) = M\big(dt, m(t)^{-1} dz m(t)\big),$$

then $N(t, dz)$ is a Lévy measure function,

$$dC(t) = \mathrm{Ad}_{m(t)} dA(t) \mathrm{Ad}^*_{m(t)},$$

implies $C(t)$ is a covariance function, and

$$dB_k(t) = \frac{1}{2} \sum_i \sum_j \sum_l \rho_l^{ij} \mathrm{Ad}^{kl}_{m(t)} dA_{ij}(t)$$
$$+ \int_{G-\{\varepsilon\}} \left(\phi_k(z) - \sum_j \phi_j\big(m(t)^{-1} z m(t)\big) \mathrm{Ad}^{kj}_{m(t)} \right) N(dt, dz)$$

yields $B(t)$ as a function of bounded variation; note that the integrand is uniformly bounded in t and z and is of second order in z at ε.

The following is now immediate.

LEMMA 9. *Convergence of $x_n(t)$. The family $\{x_n(t)\}$ is compact and converges to the process $x(t)$ corresponding to the limit $z(t)$ of $\{z_n(t)\}$ such that $x(t) = z(t)m(t)$, in distribution. $x(t)$ is a stochastically continuous process with independent increments.*

LEMMA 10. *Uniqueness of the representation $x(t) = z(t)m(t)$. Suppose $x(t) = z_1(t)m_1(t)$ and also $x(t) = z_2(t)m_2(t)$, in distribution, where z_1 and z_2 are as above, with parameters (m_1, A_1, M_1) and (m_2, A_2, M_2), respectively. Then $m_1 \equiv m_2$, $A_1 \equiv A_2$ and $M_1 \equiv M_2$.*

PROOF. Since $z_1(t)m_1(t) = z_2(t)m_2(t)$,

$$z_1(t) = z_2(t)m_2(t)m_1(t)^{-1} = z_2(t)b(t)$$

where $b(t)$ is continuous. As seen above, in standard form, the z martingales are determined by functions B, C, and N where

1. $N(dt, dx) = M(dt, m(t)^{-1} dx m(t))$;
2. $dC(t) = \mathrm{Ad}_{m(t)} dA(t) \mathrm{Ad}^*_{m(t)}$;

3.

$$dB_k(t) = \frac{1}{2}\sum_i \sum_j \sum_l \rho_l^{ij} \mathrm{Ad}_{m(t)}^{kl} dA_{ij}(t)$$
$$+ \int \left(\phi_k(z) - \sum_j \phi_j\big(m(t)^{-1} z m(t)\big) \mathrm{Ad}_{m(t)}^{kj} \right) N(dt, dz).$$

These B, C, N are the 'B, A, M' of the Uniqueness of Parameters Theorem. Thus:

1. $N_1(dt, dx) = N_2(dt, b(t)dxb(t)^{-1})$.
2. $dC_1(t) = \mathrm{Ad}_{\beta(t)} dC_2(t) \mathrm{Ad}_{\beta(t)}^*$.
3. $b(t)$ is of bounded variation and

$$dB_{1k}(t) = \sum_j \mathrm{Ad}_{\beta(t)}^{kj} dB_{2j}(t) + db_k(t)$$
$$+ \int \left(\phi_k(z) - \sum_j \phi_j\big(b(t) z b(t)^{-1}\big) \mathrm{Ad}_{\beta(t)}^{kj} \right) N_1(dt, dz).$$

Let $\mu(t) = m(t)^{-1}$.

1. $M_1(dt, dx) = M_2(dt, dx)$.
2.

$$\begin{aligned} dA_1(t) &= \mathrm{Ad}_{\mu_1(t)} dC_1(t) \mathrm{Ad}_{\mu_1(t)}^* \\ &= \mathrm{Ad}_{\mu_1(t)} \mathrm{Ad}_{\beta(t)} dC_2(t) \mathrm{Ad}_{\beta(t)}^* \mathrm{Ad}_{\mu_1(t)}^* \\ &= \mathrm{Ad}_{\mu_2(t)} dC_2(t) \mathrm{Ad}_{\mu_2(t)}^* = dA_2(t). \end{aligned}$$

3.

$$\begin{aligned} dB_{1k}(t) &= \frac{1}{2}\sum_j \mathrm{Ad}_{\beta(t)}^{kj} \sum_q \sum_r \sum_s \rho_s^{qr} \mathrm{Ad}_{m_2(t)}^{js} dA_{2,qr}(t) \\ &\quad + \sum_j \int \left(\mathrm{Ad}_{\beta(t)}^{kj} \phi_j(z) - \sum_q \phi_q\big(m_2(t)^{-1} z m_2(t)\big) \mathrm{Ad}_{\beta(t)}^{kj} \mathrm{Ad}_{m_2(t)}^{jq} \right) \\ &\quad \cdot M_2\big(dt, m_2(t)^{-1} dz m_2(t)\big) + db_k(t) \\ &\quad + \int \left(\phi_k(z) - \sum_j \phi_j\big(b(t) z b(t)^{-1}\big) \mathrm{Ad}_{\beta(t)}^{kj} \right) N_2\big(dt, b(t) dz b(t)^{-1}\big) \\ &= dB_{1k}(t) + db_k(t). \end{aligned}$$

So $db_k(t) = 0$. b, thus being constant, identically equals $b(0) = m_2(0)m_1(0)^{-1} = \varepsilon$. And $m_2(t) \equiv m_1(t)$. This completes the proof of the limit theorem.

6. **Representation theorem for processes with independent increments.** Let $z(t)$ be a given stochastically continuous process with independent increments. To get a representation $z(t)m(t)$ for $x(t)$, define

$$X_{nj} = x\left(\frac{j-1}{n}\right)^{-1} x\left(\frac{j}{n}\right).$$

Since $x(t)$ is stochastically continuous, uniformly on [0, 1], the X_{nj} are uniformly small variables and the approximating processes $x_n(t)$ are weakly convergent to $x(t)$. What is needed is the convergence of the mean functions, covariance step-functions and measure-functions to limiting parameters (m, A, M). (Notation will be as in the limit theorem.)

Since these approximating parameters are step-functions, compactness will follow from the equicontinuity conditions:

1. $m_n(0) = \varepsilon$. For every U_1, there are $\delta > 0$ and $N > 0$ such that $n \geqslant N$ and $|s - t| < \delta$ implies $m_n(s)^{-1}m_n(t)$ is in U_1.

2. $M_n(0, dx) = 0$. $M_n(t, dx)$ restricted to $\tilde{U}_2$ satisfies:

$$\operatorname*{lt}_{\delta\downarrow 0} \operatorname*{lt\,sup}_{n\to\infty} \sup_{|s-t|<\delta} \left|M_n(t, \tilde{U}_2) - M_n(s, \tilde{U}_2)\right| = 0. \tag{1}$$

$$\operatorname*{lt}_{K\uparrow G} \operatorname*{lt\,sup}_{n\to\infty} \sup_{0\leqslant t\leqslant 1} M_n(t, \tilde{K}) = 0; \quad K\text{'s compact.} \tag{2}$$

3. $A_n(0, U_3) = 0$.

$$\operatorname*{lt}_{\delta\downarrow 0} \operatorname*{lt\,sup}_{n\to\infty} \sup_{|s-t|<\delta} \left|A_{n,ij}(t, U_3) - A_{n,ij}(s, U_3)\right| = 0,$$

where U_1, U_2 and U_3 are given neighborhoods of ε.

A symmetric neighborhood of ε, U, is of the form $V^{-1}V = \{v^{-1}w\colon v, w \in V\}$ for some neighborhood of ε, V. And any neighborhood of ε contains a symmetric neighborhood of ε.

Note that it is sufficient for the above to consider the limits with $t > s$ and the U's symmetric neighborhoods.

LEMMA 1. *Let C be a neighborhood of ε. For $M_n(t, dx)$ restricted to $\tilde{U}$:*

1. *$M_n(t, dx)$ are equicontinuous.*
2. $\operatorname{lt}_{K\uparrow G}\operatorname{lt\,sup}_{n\to\infty} \sup_{0\leqslant t\leqslant 1} M_n(t, \tilde{K}) = 0$.
3. *Along a subsequence, $\int f(x)M_n(t, dx)$ converges for every bounded continuous f that is identically zero on U.*

PROOF. 1. U can be taken to be of the form $V^{-1}V$, as noted above. Then

$$\begin{aligned}
&\left|M_n(t+h, \tilde{U}) - M_n(t, \tilde{U})\right| \\
&\quad = \sum_{t<j/n\leqslant t+h} P\left(X_{nj} \in \tilde{U}\right) \leqslant -\sum \log\left(1 - P\left(X_{nj} \in \tilde{U}\right)\right) \\
&\quad = -\log \prod_j P(X_{nj} \in U) = -\log P(\text{all } X_{nj} \in U;\ t < j/n \leqslant t+h) \\
&\quad \leqslant \sup_T \left(-\log P\left(\tau_{x(T)V}^T \geqslant h \| \mathcal{F}_T\right)\right),
\end{aligned}$$

since if all $x(k/n)$ are in $x(j/n)V$, $k > j$, then all increments $x((k-1)/n)^{-1}x(k/n)$ are in U,

$$= \sup_T \left(-\log\left(1 - P\left(\tau_{x(T)V}^T < h \| \mathcal{F}_T\right)\right)\right),$$

which goes to zero with h by stochastic continuity of $x(t)$.

2. Similarly, assuming K is of the form $C^{-1}C$ for a compact set C,

$$\begin{aligned}
\sup_{0\leqslant t\leqslant 1} M_n(t, \tilde{K}) \leqslant M_n(1, \tilde{K}) &= \sum_{j=1}^n P\left(X_{nj} \in \tilde{K}\right) \\
&\leqslant -\log\left(1 - P\left(\tau_C^0 < 1\right)\right),
\end{aligned}$$

which goes to zero as $C \uparrow G$.

3. For $t \geqslant s$,

$$\left|\int f(x)(M_n(t, dx) - M_n(s, dx))\right| \leqslant \sup_{x\in G} |f(x)|\ \left|M_n(t, \tilde{U}) - M_n(s, \tilde{U})\right|,$$

so $\int f(x) M_n(t, dx)$ are equicontinuous. Choosing a countable dense set of f's, diagonalization yields a universal subsequence for convergence.

LEMMA 2. 1. *$m_n(t)$ are equicontinuous.*

2. *$k_n(t) = A_n(t, U)$ are equicontinuous for any neighborhood of ε, U.*

PROOF. First, by the remark preceding Lemma 1, translating the parameters so that any starting time can be assumed to be zero is sufficient.

Observe that regardless of the boundedness of $m_n(t)$, the increments m_{nk} are uniformly small (as in the limit theorem). Define T_V^n to be the first time (nonrandom) that $m_n(t)$ exits from V, for V a neighborhood of ε with $\bar{V}$ compact. Then a subsequence of $m_n(T_V^n)$ converges to an element of $\bar{V}$ (this holds even if $m_n(t)$ does not exit from V, i.e., $T_V^n = \infty$). It can be assumed also that, for n large, $m_n(T_V^n)$ is in V^2.

Next note that equicontinuity of $A_n(t, U)$ is equivalent to that of $\operatorname{tr} A_n(t, U)$. And set $k_n(t) = \operatorname{tr} A_n(t, U)$. For $\delta > 0$, denote by S_δ^n the first time $k_n(t) \geqslant \delta$; setting $S_\delta^n = \infty$ if $k_n(t) < \delta$, $0 \leqslant t \leqslant 1$. Also, the increments of $k_n(t)$ are uniformly small, so a subsequence of $k_n(S_\delta^n)$ converges to δ as $n \to \infty$. And it can be assumed that $k_n(S_\delta^n) < 2\delta$ for n large.

Suppose $m_n(t)$ and/or $k_n(t)$ are not equicontinuous. Then there are V, a symmetric neighborhood of ε and/or $\delta > 0$ such that $\sigma_n = T_V^n \wedge S_\delta^n$ converges to zero. Infinitely often, either:

(a) $\sigma_n = T_V^n$, or

(b) $\sigma_n = S_\delta^n$.

Case (a). $k_n(t) \leqslant 2\delta$ on $0 \leqslant t \leqslant \sigma_n = T_V^n \leqslant S_\delta^n$. $k_n(t)$ is a nonnegative, increasing step-function. Choose an increasing mapping $\psi_n(t)$; $[0, 1] \to [0, \sigma_n]$ such that, for $s \leqslant t$, $k_n(\psi_n(t)) - k_n(\psi_n(s)) \leqslant 4\delta(t - s)$. Then $k_n \circ \psi_n$ are equicontinuous on $[0, 1]$. Along a subsequence, the measure functions $M_n(t, dx)$ converge to an $M(1, dx)$ (away from ε); in fact, since $\psi_n(1) \to 0$, by Lemma 2, $M_n(\psi_n(t), dx) \leqslant M_n(\psi_n(1), dx)$ converges to zero uniformly on $[0, 1]$. The estimates for compactness of $z_n(\psi_n(t))$ hold since $\{m_n(t)\}_{n>0, 0\leqslant t\leqslant 1}$ is contained in a compact set. Choose a subsequence such that $z_n(\psi_n(t))$ converges to $z(t)$. Then $x_n(\psi_n(t)) = z_n(\psi_n(t))m_n(\psi_n(t))$ implies that $m_n(\psi_n(t))$ is compact (as a set of functions) since $x_n(\psi_n(t))$ and $z_n(\psi_n(t))$ are weakly compact. And along an appropriate subsequence, $x_n(\psi_n(t)) \to x(0) = z(t)m(t)$, where $m(t)$ is continuous by stochastic continuity of $z(t)$. Furthermore, $x(0)$ has the unique representation with parameters $(\varepsilon, 0, 0)$. So $m(t) \equiv \varepsilon$. But $m(1) = \mathrm{lt}_{n\to\infty} m_n(T_V^n)$ is on ∂V, contradiction.

Case (b). Just as for case (a), ψ_n: $[0, 1] \to [0, \sigma_n]$ can be chosen such that $k_n(\psi_n(t))$ are equicontinuous yielding a representation $x(0) = z(t)m(t)$ with parameters $(m, A, M) = (\varepsilon, 0, 0)$. But $\operatorname{tr} A(1, U) = \mathrm{lt}_{n\to\infty} k_n(S_\delta^n) = \delta > 0$, contradiction.

THEOREM. *$x(t)$ has a unique representation $z(t)m(t)$ with parameters (m, A, M).*

PROOF. For U a neighborhood of ε, Lemma 2 implies that a subsequence of $M_n(t, dx)$ converges on U. Diagonalization yields a subsequence of $M_n(t, dx)$ converging on every U_j, for U_j a sequence of neighborhoods decreasing to $\{\varepsilon\}$. This determines $M(t, dx)$. Lemmas 1 and 2 imply that along a subsequence as $n \to \infty$, $m_n(t)$, $A_n(t, U)$, and $M_n(t, dx)$ converge uniformly on $[0, 1]$; U is chosen to be an $M(1, dx)$-continuity neighborhood. The limit theorem applies to yield $x(t) = z(t)m(t)$ with parameters (m, A, M), since $x_n(t) = z_n(t)m_n(t)$ converges to $x(t)$ by construction.

REMARK. Note that if $m(t)$ is of bounded variation, the Uniqueness of Parameters Theorem 2 implies that

$$f(x(t)) - \int_0^t \mathrm{Ad}_{\mu(s)} L(ds) f(x(s)) - \int_0^t \sum_k \xi_k f(x(s))\, dm_k(s)$$

is a martingale for every f in C_k^∞, where $L(ds)$ is the generator for $z(t)$ and $\mu(t) = m(t)^{-1}$

$$
\begin{aligned}
f(x(t)) - \int_0^t \int_{G-\{\varepsilon\}} & f\big(z(s)m(s)zm(s)^{-1}m(s)\big) - f(x(s)) \\
& - \sum_j \phi_j(z)\mathrm{Ad}_{\mu(s)}\eta_j(s) f(x(s)) \cdot M(ds, dz) \\
& - \frac{1}{2}\int_0^t \sum_i \sum_j \mathrm{Ad}_{\mu(s)}\eta_i(s)\mathrm{Ad}_{\mu(s)}\eta_j(s) f(x(s)) \cdot dA_{ij}(s) \\
& + \int_0^t \sum_k \left(\sum_i \sum_j \rho_k^{ij} dA_{ij}(s) - dm_k(s) \right) \mathrm{Ad}_{\mu(s)}\eta_k(s) f(x(s)) \\
= f(x(t)) - \int_0^t \int_{G-\{\varepsilon\}} & f(x(s)x) - f(x(s)) \\
& - \sum_j \phi_j(x)\xi_j f(x(s)) \cdot M(ds, dx) \\
& - \frac{1}{2}\int_0^t \sum_i \sum_j \left(\xi_i\xi_j - \sum_k \rho_k^{ij}\xi_k \right) f(x(s)) \cdot dA_{ij}(s) \\
& - \int_0^t \sum_k \xi_k f(x(s)) dm_k(s)
\end{aligned}
$$

is a martingale for every f in C_k^∞, since

$$\mathrm{Ad}_{\mu(t)}\eta(t) = \mathrm{Ad}_{\mu(t)}\mathrm{Ad}_{m(t)}\xi = \mathrm{Ad}_{\mu(t)m(t)}\xi = \xi.$$

COROLLARY. *Every stochastically continuous process with independent increments is the weak limit of processes determined by martingales.*

PROOF. Let $x(t)$ have the representation $z(t)m(t)$. Approximate $m(t)$ uniformly on [0, 1] by smooth $m_k(t)$, so that $x_k(t) = z(t)m_k(t)$ are determined by martingales and converge to $x(t) = z(t)m(t)$.

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