INTERPRETATION OF THE $p$-ADIC LOG GAMMA FUNCTION
AND EULER CONSTANTS USING THE BERNOULLI MEASURE

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Abstract. A regularized version of J. Diamond's $p$-adic log gamma
function and his $p$-adic Euler constants are represented as integrals using B.
Mazur's $p$-adic Bernoulli measure.

1. Introduction. Let $\mathbb{Z}_p$, $\mathbb{Z}_p^\times$, $\Omega_p$ denote, respectively, the $p$-adic integers, the
$p$-adic integers not divisible by $p$, and the completion of the algebraic closure
of the field of $p$-adic numbers. Let $| |_p$ be the absolute value on $\Omega_p$ with
$|p|_p = p^{-1}$.

J. Diamond [3] defined a $p$-adic log gamma function

$$G_p(x) = \lim_{k \to \infty} p^{-k} \sum_{0 < n < p^k} (x + n) \log_p(x + n) - (x + n) \quad \text{for } x \in \Omega_p - \mathbb{Z}_p$$

and a closely related function

$$G_p^*(x) = \lim_{k \to \infty} p^{-k} \sum_{0 < n < p^k, p \nmid n} (x + n) \log_p(x + n) - (x + n) \quad \text{for } x \in \Omega_p - \mathbb{Z}_p^\times$$

$$= G_p(x) - G_p(x/p) \quad \text{for } x \in \Omega_p - \mathbb{Z}_p.$$

Our purpose is to define a "regularized" version of $G_p^*$ and show that it can
be represented as a simple integral over $\mathbb{Z}_p^\times$ with respect to Mazur's $p$-adic
Bernoulli measure $\mu_\alpha$, namely, as the "convolution" of $\mu_\alpha$ with the $p$-adic
logarithm (see (7) below). Recall (see [9], or Chapter II of [6]) that for
$\alpha \in 1 + p\mathbb{Z}$, $\alpha \neq 1$, $\mu_\alpha$ is defined by

$$\mu_\alpha(a + p^m\mathbb{Z}_p) = \alpha^{-1} [\alpha a p^{-m}] + (\alpha^{-1} - 1)/2.$$ 

(Actually, any $\alpha \in \mathbb{Z}_p^\times$ not a root of 1 can be used to regularize; but we shall
take $\alpha \in 1 + p\mathbb{Z}$ for simplicity.) The moments of $\mu_\alpha$ give the $p$-adic zeta-function
$\xi_p$ that was first defined by Kubota and Leopoldt [8].

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Our integral formula for the derivative of the regularized log gamma function (see (8) below) is an example of a very general phenomenon noticed by D. Barsky [2]: any Krasner analytic function on \( \Omega_p - Z_p^* \) satisfying a certain growth condition is the "Cauchy transform" of some \( p \)-adic measure on \( Z_p^* \). Theorem 2.1 below shows that, in the case of the regularized \( D \)-log gamma function, this measure is \( \mu_a \).

We shall also express Diamond's generalized \( p \)-adic Euler constants

\[
\gamma_p(a, p^m) = \lim_{k \to \infty} p^{-k} \sum_{0 < n < p^k, n \equiv a \pmod{p^m}} \log_p n \quad \text{if } p \nmid a, \\
\gamma_p = \gamma_p(0, 1) = \frac{p}{p-1} \sum_{a=1}^{p-1} \gamma_p(a, p)
\]

(1)
as integrals.

2. Regularized log gamma function. Among the \( p \)-adic analogues of classical formulas which Diamond [3] derives for \( G_p \) is the "\( p \)-adic Stirling series"

\[
G_p(x) = (x - \frac{1}{2}) \log_p x - x + \sum_{r=1}^{\infty} \frac{B_{r+1}}{r(r+1)x^r} \quad \text{for } |x|_p > 1,
\]

(2)

where \( B_k \) is the \( k \)th Bernoulli number.

Let \( l(x) = (x - \frac{1}{2}) \log_p x - x \). Define operators \( T_p, T_\alpha \) for \( 0 \neq \alpha \in \Omega_p \), by

\[
T_p f(x) = f(x/p), \quad T_\alpha f(x) = \alpha^{-1} f(\alpha x).
\]

Then

\[
(1 - T_p)G_p(x) = G_p^*(x) \quad \text{for } x \in \Omega_p - Z_p, \\
(1 - T_\alpha)(1 - T_p)l(x) = -(1 - 1/p)x \log_p \alpha,
\]

(3)

(4)

and, if we let \( D = d/dx \),

\[
DT_\alpha = \alpha T_\alpha D, \quad DT_p = p^{-1} T_p D.
\]

(5)

Let \( A_r = \{ x \in \Omega_p \mid |x - a|_p > r \quad \text{for all } a \in Z_p^* \} \). Thus, \( A_1 = \{ x \in \Omega_p \mid |x|_p > 1 \} \). Choose \( \alpha \in 1 + pZ, \alpha \neq 1 \). Define \( G_{p, \alpha} \) on \( A_1 \) by

\[
G_{p, \alpha} = (1 - T_\alpha)(1 - T_p)(G_p - l),
\]

i.e., by (3) and (4),

\[
G_{p, \alpha}(x) = (1 - T_\alpha)G_p^*(x) + (1 - 1/p)x \log_p \alpha \quad \text{for } x \in A_1.
\]

Theorem 2.1. For \( x \in A_1 \),
THE \( p \)-ADIC LOG GAMMA FUNCTION

\[ G_{p,a}(x) = -\int_{\mathbb{Z}_p^x} \log_p(x - t) \mu_a(t); \quad (7) \]

\[ D'G_{p,a}(x) = (-1)^r(r - 1)! \int_{\mathbb{Z}_p^x} \frac{\mu_a(t)}{(x - t)^r} \quad \text{for } r \geq 1 \]

\[ = (1 - a'T_a)(1 - p^{-r}T_p)G_{p,a}^{(r)}(x) \quad \text{for } r \geq 2. \quad (8) \]

**Proof.** Using (6) and (2), we write the left side of (7) as

\[ \sum_{r=1}^{\infty} \frac{1}{r} (\alpha^{-r-1} - 1)(1 - p^r) \left( -\frac{B_{r+1}}{r+1} \right) = \sum_{r=1}^{\infty} \frac{1}{rx^r} \int_{\mathbb{Z}_p^x} t^r \mu_a(t), \]

by the fundamental property of \( \mu_a \), which allows it to be used to interpolate \( \xi(-r) \) [9]. Hence

\[ G_{p,a}(x) = \int_{\mathbb{Z}_p^x} \sum_{r=1}^{\infty} \frac{(t/x)^r}{r} \mu_a(t) = -\int_{\mathbb{Z}_p^x} \log_p(1 - t/x) \mu_a(t) \]

\[ = -\int_{\mathbb{Z}_p^x} \log_p(x - t) \mu_a(t), \]

since \( \mu_a(Z_p^x) = 0 \). The formula for \( D'G_{p,a} \) now follows immediately. Q.E.D.

**Corollary 2.2.** For \( x \in A_1 \),

\[ G'_{p,a}(x) - G'_{p,a}(ax) = -(1 - 1/p)\log_p x - \int_{\mathbb{Z}_p^x} \frac{\mu_a(t)}{x - t} \]

\[ = -\int_{\mathbb{Z}_p^x} \left( \frac{1}{x} + \frac{1}{x - t} \right) \mu_a(t). \]

The first equality in the corollary follows from (4) and (8), and the second follows from formula (12) in §4 below.

We now use (7) to define \( G_{p,a} \) for \( x \in \Omega_p - Z_p^x \). This integral exists for all such \( x \), since with \( x \notin Z_p^x \) fixed, \( \log_p(x - t) \) is continuous in \( t \in Z_p^x \).

**Corollary 2.3.** With \( G_{p,a} \) defined by (7),

\[ G_{p,a}(0) = (1 - \alpha^{-1})L'_p(0, \omega), \]

where \( L_p \) is the \( p \)-adic \( L \)-function [5] and \( \omega \) is the Teichmüller character.

In fact, the right-hand side equals

\[ \lim_{k \to \infty} \frac{1 - \alpha^{-1}}{- (p - 1)p^k} L_p \left( - (p - 1)p^k, \omega \right) \]

\[ = -\lim_{k \to \infty} \frac{1 - \alpha^{-1}}{(p - 1)p^k} \cdot \int_{\mathbb{Z}_p^x} \exp \left\{ (p - 1)p^k \log_p t \right\} \mu_a(t) \]

\[ = -\int_{\mathbb{Z}_p^x} \log_p t \mu_a(t) = G_{p,a}(0). \]
3. Analytic continuation. Diamond [3] proved that $G_p'' = D^2G_p$ is analytic in the sense of Krasner [7] on $\Omega_p - Z_p$, but he noted that $G_p$ and $G_p'$ are not. However, for our regularized $G_{p,\alpha}$ already the first derivative is Krasner analytic.

THEOREM 3.1. $DG_{p,\alpha}$ is Krasner analytic on $\Omega_p - Z_p ^\times$.

PROOF. Since $\Omega_p - Z_p ^\times = \cup_{m=1}^{\infty} A_{p^{-m}}$, it suffices to show that for $m$ fixed, $f(x) = - DG_{p,\alpha}(x)$ is a uniform limit of rational functions on $A_{p^{-m}}$ without poles there. Since

$$f = \sum_{0 < a < p^{m+1}} f_a, \quad \text{where} \quad f_a(x) = \int_{a + p^{m+1}Z_p} \frac{\mu_a(t)}{x - t},$$

it suffices to show this for $f_a$. For $t = a + p^{m+1}s$, $s \in Z_p$, we have

$$\frac{1}{x - t} = \frac{1}{x - a} \sum_{j=0}^{\infty} \left( \frac{p^m}{x - a} \right)^j p^{js},$$

so that

$$f_a(x) = \frac{1}{x - a} \sum_{j=0}^{\infty} \left( \frac{p^m}{x - a} \right)^j p^{js} \mu_a(a + p^{m+1}s).$$

Since $|p^m/(x - a)|_p < 1$ on $A_{p^{-m}}$ and $|f|_p < 1$, it follows that $f_a$ is in fact a uniform limit of rational functions on $A_{p^{-m}}$. Q.E.D.

COROLLARY 3.2. For all $x \in \Omega_p - Z_p ^\times$ and all $r \geq 2$,

$$(1 - \alpha' T_a) G_p^{**}(x) = (-1)^r (r - 1)! \int_{Z_p} \frac{\mu_a(t)}{(x - t)^r}. $$

In fact, Diamond’s argument proving Krasner analyticity of $G_p''$ on $\Omega_p - Z_p$ will also prove Krasner analyticity of $G_p^{**}$ and all higher derivatives on $\Omega_p - Z_p ^\times$. Since both sides of the equality are Krasner analytic on $\Omega_p - Z_p ^\times$ and agree on $A_1$, they must be equal on all of $\Omega_p - Z_p ^\times$.

COROLLARY 3.3 (see [4]). $D' G_p^*(0) = -(r-1)! L_p(r, \omega^{1-r})$ for $r \geq 2$.

In fact, by Corollary 3.2, the left side equals

$$-(\alpha' - 1)^{-1}(r - 1)! \int_{Z_p} t^{-r} \mu_a(t) = -(r-1)! L_p(r, \omega^{1-r}).$$

Questions. 1. In [4, Propositions 4 and 5], Diamond proved the following relation for $L_p(r, \chi \omega^{1-r})$ when $\chi$ is a Dirichlet character mod $p^m$, $m > 1$, $r \geq 2$:

$$L_p(r, \chi \omega^{1-r}) = \frac{(-1)^r}{p^{mr} (r - 1)!} \sum_{0 < \chi \geq p^m, \chi \equiv a} \chi(a) D' G_p(a/p^m). \quad (9)$$
Can this expression be derived using the integral formulas? The difficulty comes when trying to express $D^r G_p(a/p^m)$ in terms of $D^r G_{p,a}$ when $a \neq 0$.

2. Does Corollary 3.2 hold when $r = 0, 1$, i.e., do we have

$$G_p^*(x) - \alpha^{-1} G_p^*(ax) = -(1 - 1/p) \alpha \log \alpha - \int_{\mathbb{Z}_p^*} \log_p(x - t) \mu_p(t),$$

$$G_p^*(x) - G_p^*(ax) = -\int_{\mathbb{Z}_p^*} \left( \frac{1}{t} + \frac{1}{x - t} \right) \mu_p(t)$$

for all $x \in \Omega_p - \mathbb{Z}_p^*$. Is $G_p^*(x) - G_p^*(ax)$ Krasner analytic on $\Omega_p - \mathbb{Z}_p^*$? In particular, can Corollary 2.3 be rewritten simply: $G_p^*(0) = L_p^r(0, \omega)$? Note that when $x \in p\mathbb{Z}$, the left side of (10) can be rewritten

$$\lim_{k \to \infty} p^{-k} \sum_{0 < n < p^k, p \nmid n} \log_p(n + x) - \log_p(n + ax) = -\sum_{|x| < n < |ax|, p \nmid n} \frac{1}{n}$$

(here $||$ means ordinary archimedean absolute value of an integer).

We obtain a partial affirmative answer to the second question in the following

**Theorem 3.4.** $G_p^*(x) - G_p^*(ax)$ is Krasner analytic on $A_{|a - 1|_p}$, and for $x \in A_{|a - 1|_p}$,

$$G_p^*(x) - G_p^*(ax) = -(1 - 1/p) \alpha \log \alpha - \int_{\mathbb{Z}_p^*} \frac{\mu_p(t)}{x - t}$$

$$= -\int_{\mathbb{Z}_p^*} \left( \frac{1}{t} + \frac{1}{x - t} \right) \mu_p(t).$$

**Proof.** Since $A_{|a - 1|_p} = \bigcup_{r > |a - 1|_p} A_r$, it suffices to write $f(x) = G_p^*(x) - G_p^*(ax)$ as a uniform limit of rational functions on $A_r$ for fixed $r > |a - 1|_p$. We have

$$f(x) = \lim_{k \to \infty} p^{-k} \sum_{0 < n < p^k, p \nmid n} \log_p(x + n) - \log_p(ax + n)$$

$$= -(1 - 1/p) \alpha \log_p \alpha - \lim_{k \to \infty} p^{-k} \sum_{0 < n < p^k, p \nmid n} \log_p \frac{x + n/\alpha}{x + n}.$$

Thus, if we let $\alpha' = 1 - 1/\alpha$ and $f_n(x) = \log_p(1 - \alpha' n/(x + n))$, we have

$$f(x) = -(1 - 1/p) \alpha \log_p \alpha - \lim_{k \to \infty} p^{-k} \sum_{0 < n < p^k, p \nmid n} f_n(x),$$

where the limit is uniform on $A_r$. But, since $|\alpha' n/(x + n)|_p < |\alpha - 1|_p/r < 1$ for $x \in A_r$, it follows that each

$$f_n(x) = -\sum_{j=1}^\infty \frac{1}{j} \left( \frac{\alpha' n}{x + n} \right)^j.$$
is a uniform limit of rational functions. Q.E.D.

**Corollary 3.5.** For $|x|_p < 1$,

$$G_p^n(x) - G_p^n(ax) = \int_{\mathbb{Z}_p^+} \sum_{j=1}^{\infty} \frac{x^j}{j+1} \mu_a(t) = \sum_{j=1}^{\infty} (1 - \alpha^j)(1 - p^{-j-1})L_p(j + 1, \omega^{-j})x^j.$$  

**4. Euler constants.** In [3] Diamond defined generalized $p$-adic Euler constants by (1) above and proved that

$$\frac{1}{p^{m-1}(p-1)} \sum_{x \neq x_0} \tilde{\chi}(a)L_p(1, \chi) = \frac{\gamma_p(a, p^m) - p^{-m}\gamma_p}{p} \quad \text{for } p \nmid a. \quad \text{(11)}$$  

Once we express $\gamma_p$ in terms of the $p$-adic zeta-function $\zeta_p$—actually, it will equal the Euler constant one would expect from a zeta-function—we can express $\gamma_p$ and, hence, $\gamma_p(a, p^m)$ as integrals.

Let

$$\tilde{\gamma}_p = \frac{p}{p-1} \lim_{\varepsilon \to 0} \left( \zeta_p(1 + \varepsilon) - (1 - 1/p)/\varepsilon \right) = \frac{p}{p-1} \lim_{N \to \infty} \left( \zeta_p(1 - (p-1)p^N) + (1 - 1/p)/(p-1)p^N \right) = \frac{p}{p-1} \lim_{N \to \infty} \left( 1 - (p-1)p^{N-1} \right) \left( -\frac{B(p-1)p^N}{p^N} + p^{-N-1} \right) = \frac{p}{p-1} \lim_{N \to \infty} \left( \frac{B(p-1)p^N}{(p-1)p^N} - p^{-N-1} \right).$$

We claim that $\tilde{\gamma}_p = \gamma_p$. In fact, Kubota and Leopoldt [8, §3] prove that if $A(u) = \sum_{n=0}^{\infty} a_n(u-1)^n$ converges for $|u-1|_p < 1/p \ (< 1/4 \text{ if } p = 2)$, and if we let

$$M^k(A) \overset{\text{def}}{=} \sum_{0 \leq i < p^k} A(i/\omega(i)), \quad M(A) = \lim_{k \to \infty} M^k(A), \quad L(u) = \sum_{n=1}^{\infty} (1)^{-1}(u-1)^n/n, \quad \text{then}$$

$$\tilde{\gamma}_p(s) = \frac{1}{s-1} \sum_{n=0}^{\infty} \frac{M(L^n)}{n!} (1-s)^n = \frac{1}{s-1} \left( 1 - \frac{1}{p} \right) + \frac{1}{s-1} \sum_{n=1}^{\infty} \frac{M(L^n)}{n!} (1-s)^n.$$
Using this, we have

\[
(1 - \frac{1}{p}) \gamma_p = \lim_{\varepsilon \to 0} \left( \gamma_p(1 + \varepsilon) - \left(1 - \frac{1}{p}\right)/\varepsilon \right) = \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \sum_{n=1}^{\infty} \frac{M(L^n)}{n!} (-\varepsilon)^n
\]

\[
= -M(L) = - \lim_{k \to \infty} p^{-k} \sum_{0 < i < p^k} \log_p i = \left(1 - \frac{1}{p}\right)^{\gamma_p}.
\]

This proves the claim. (The above proof, which is shorter and neater than my original proof, was kindly given me by the referee.)

Thus, Diamond's \(p\)-adic Euler constant agrees with the one from the Kubota-Leopoldt zeta-function.

We now derive integral formulas for \(\gamma_p\) and \(\gamma_p(a, p^m)\). In what follows we now suppose \(\alpha \in 1 + p^m \mathbb{Z}\), \(\alpha \neq 1\), and \(\chi\) is a Dirichlet character mod \(p^m\).

For small \(s\) we use

\[
L_p(1 + s, \chi) = (\alpha^s - 1)^{-1} \int_{Z_p^s} t^{-s-1} \omega^s(t) \chi(t) \mu_\alpha(t)
\]

\[
= (\alpha^s - 1)^{-1} \int_{Z_p^s} \exp\left\{-s \log_p t\right\} \frac{\chi(t)}{t} \mu_\alpha(t).
\]

Note that \((\alpha^s - 1)/s \to \log_p \alpha\) as \(s \to 0\). Let \(\varepsilon(\chi) = 1\) if \(\chi = \chi_0\), 0 otherwise. Then

\[
\frac{1}{\log_p \alpha} \int_{Z_p} \frac{\chi(t)}{t} \mu_\alpha(t) = \lim_{s \to 0} sL_p(1 + s, \chi) = (1 - 1/p)\varepsilon(\chi). \quad (12)
\]

Define \(\gamma_p(\chi)\) by

\[
(1 - 1/p)\gamma_p(\chi) = \begin{cases} 
\lim_{s \to 0} \left( L_p(1 + s, \chi) - \frac{1 - 1/p}{s} \varepsilon(\chi) \right) \\
(1 - 1/p)\gamma_p \quad \text{if } \chi = \chi_0 \\
L_p(1, \chi) \quad \text{otherwise}.
\end{cases}
\]

Then

\[
(1 - 1/p)\gamma_p(\chi) = \lim_{s \to 0} \int_{Z_p} \frac{1}{\alpha^s - 1} \frac{e^{-s \log_p t} \chi(t)}{t} - \frac{\chi(t)}{s \log_p \alpha} \mu_\alpha(t)
\]

\[
= \lim_{s \to 0} \int_{Z_p} \left( \frac{e^{-s \log_p t}}{e^{s \log_p \alpha} - 1} - \frac{1}{s \log_p \alpha} \right) \frac{\chi(t)}{t} \mu_\alpha(t)
\]

\[
= \lim_{s \to 0} \frac{1}{s \log_p \alpha} \int_{Z_p} \left( \frac{1 - s \log_p t}{1 + (s/2) \log_p \alpha} - 1 \right) \frac{\chi(t)}{t} \mu_\alpha(t)
\]

\[
= - \int_{Z_p} \left( \frac{\log_p t}{\log_p \alpha} + \frac{1}{2} \right) \frac{\chi(t)}{t} \mu_\alpha(t).
\]
Since (11) gives us

\[
(1 - 1/p)\gamma_p(a, p^m) = p^{-m} \left( (1 - 1/p)\gamma_p + \sum_{x \neq x_0} \chi(x) L_p(1, \chi) \right)
\]

\[
= p^{-m} (1 - 1/p) \sum_{\chi} \chi(a) \gamma_p(\chi)
\]

\[
= -p^{-m} \int_{\mathbb{Z}_p^*} \left( \frac{\log_p t}{\log_p \alpha} + \frac{1}{2} \right) \frac{1}{t} \sum_{\chi} \chi(t/a) \mu_\alpha(t),
\]

we may conclude

**Theorem 4.1.**

\[
(1 - 1/p)\gamma_p(a, p^m) = -\int_{a + p^m \mathbb{Z}_p} \left( \frac{\log_p t}{\log_p \alpha} + \frac{1}{2} \right) \frac{\mu_\alpha(t)}{t}.
\]

**Remark.** Note that

\[
G'_p(a/p^m) = \lim_{k \to \infty} p^{-k} \sum_{\ell < n < p^k} \log_p \left( \frac{a}{p^m + n} \right) = -p^{-m}\gamma_p(a, p^m),
\]

and, similarly,

\[
G_p^*(0) = -(1 - 1/p)\gamma_p.
\]

Using Diamond's relation (9) in the same way as we used (11) to prove Theorem 4.1, we see that for \( r > 2 \),

\[
G_p^{(r)}(a/p^m) = p^{-mr} (r - 1)! (-1)^r (\alpha^{r-1} - 1)^{-1} \int_{a + p^m \mathbb{Z}_p} \frac{\mu_\alpha(t)}{t^r}.
\]

Namely, replace the left-hand side of (9) by \((\alpha^{r-1} - 1)^{-1} \int_{\mathbb{Z}_p} \chi(t) t^{-r} \mu_\alpha(t)\).

Then let \( \chi \) run over all characters mod \( p^m \), for each \( \chi \) multiply (9) by \( \overline{\chi}(a) \), and take the sum. One obtains

\[
(p^m - p^{m-1}) (\alpha^{r-1} - 1)^{-1} \int_{a + p^m \mathbb{Z}_p} t^{-r} \mu_\alpha(t)
\]

\[
= \frac{(-1)^r}{p^{-mr} (r - 1)!} (p^m - p^{m-1}) D^r G_p(a/p^m).
\]

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**References**


4. _____, *On the values of p-adic L-functions at positive integers* (to appear).


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