

CURVES WITH LARGE TANGENT SPACE

BY

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ABSTRACT. THEOREM. *Let V be a complex analytic variety irreducible at a point $p \in V$. Given any integer l , there exists an analytic curve C_l on V passing through p and irreducible at p such that the germs of C_l and V at p are isomorphic up to order l .*

In [6] Hironaka and Rossi have proven the following result: Let V be a complex analytic variety of pure dimension r and p an isolated singular point of V . Then there exists an integer ν_0 so that if p' is an isolated singular point of a complex analytic variety V' of pure dim r and if $\mathcal{O}_p(V)/m_p^\nu \simeq \mathcal{O}_{p'}(V')/m_{p'}^\nu$, for some $\nu \geq \nu_0$, where the isomorphism is as \mathbb{C} algebras and m_p is the maximal ideal of the local ring $\mathcal{O}_p(V)$ of V at p , then $\mathcal{O}_p(V) \simeq \mathcal{O}_{p'}(V')$. We show here that it is necessary for V and V' to be of the same dimension.

THEOREM. *Let V be a complex analytic variety irreducible at a point $p \in V$. Given any integer l , there exists an analytic curve C_l on V passing through p and irreducible at p such that $\mathcal{O}_p(V)/m_p(V)^l \simeq \mathcal{O}_p(C_l)/m_p(C_l)^l$ as \mathbb{C} algebras. In particular, there exists an irreducible analytic curve C_2 on V having the same tangent space at p as V does, where the tangent space to V at $p = m_p(V)/m_p(V)^2$.*

In §§2 and 3, we generalize this result to complete domains and finally to arbitrary analytically irreducible local Noetherian rings to read as follows: Let R be a local Noetherian ring, whose completion \hat{R} with respect to the maximal ideal M is a domain. Then for every integer $l > 0$, there exists a prime ideal P_l in R , with $P_l \subset M^l$ and $\dim R/P_l = 1$. The results of §§1 and 2 can be deduced from those of §3; we include both proofs to illustrate the various techniques. The argument presented in Theorem 3 is due to M. Hochster, and the authors are grateful for his permission to reproduce it here.

In §4, we give an application of these results to local differentiable

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embeddings of a real or complex analytic variety. We show how to recover several other known results, due to Risler [12] and Merrien [8]. Also these curves have application to the solution of power series equations [5].

1.

PROOF OF THEOREM. We may assume that a neighborhood of p in V is embedded in \mathbb{C}^n , $n = \text{embedding dim of } V \text{ at } p$, and p is the origin in \mathbb{C}^n . We consider the blowing up of the maximal ideal, that is the quadratic transform B of V with center at p ; B is the closure in $V \times \mathbb{P}^{n-1}$ of the set B' in $V \times \mathbb{P}^{n-1}$ where $B' = \{(y_1, \dots, y_n, z_1, \dots, z_n) \in V - \{0\} \times \mathbb{P}^{n-1} : z_j y_j = z_i y_i \text{ for all } 1 \leq i, j \leq n\}$. It is known that the natural projection $\phi: B \rightarrow V$ has the following properties: $F = \phi^{-1}(p) \subset \mathbb{P}^{n-1}$ is the tangent cone to V at p so $\dim F = r - 1$. $\phi: B - \phi^{-1}(p) \rightarrow V - p$ is a biholomorphism, so $B - \phi^{-1}(p)$ is topologically connected and $B = B - \phi^{-1}(p)$ is of pure dim r .

We also consider the normalization X of B . There exists a holomorphic map $\pi: X \rightarrow B$ such that: π is a proper map with finite fibers. If S is the singular locus of B , then $\pi^{-1}(B - S)$ is dense in X , $\pi: \pi^{-1}(B - S) \rightarrow B - S$ is biholomorphic, and X is locally irreducible. The local ring of X at any point x , $\mathcal{O}_x(X)$ is the integral closure of $\mathcal{O}_{\pi(x)}(B)$ in its full quotient ring. If $S(X)$ denotes the singular locus of X , then $\dim_x S(X) < \dim_x X - 2$. $\dim \pi^{-1}(F) = r - 1$ because π preserves dim.

Hence there exist $x \in \pi^{-1}(F)$ such that x is a simple point of X . Now $\mathcal{O}_x(X) \simeq \mathbb{C}\{x_1, \dots, x_r\}$, the convergent power series in r variables so there exist convergent power series $f_1(x_1, \dots, x_r), \dots, f_n(x_1, \dots, x_r)$ all vanishing at the origin such that $\mathcal{O}_p(V) = \mathbb{C}\{y_1, \dots, y_n\} = \mathbb{C}\{f_1, \dots, f_n\}$. We now show that $\mathcal{O}_p(V)$ is a subring of $\mathcal{O}_x(X)$, that is the canonical homomorphism $\mathcal{O}_p(V) \rightarrow \mathcal{O}_x(X)$ is an injection. Let $B_{\pi(x)} = B_1 \cup B_2 \cup \dots \cup B_n$ be a decomposition into germs of irreducible components (which all have the same dimension since the connected manifold $\phi^{-1}(\text{Reg } V)$ is dense in B) such that the germ of X at x is the normalization of B_1 . $\phi|X - \phi^{-1}(F)$ is an open map and $\pi|B_1 - F$ is an open map so $\pi \circ \phi(X)$ contains an open set of V ; hence any analytic function vanishing on $\pi(B_1)$ must vanish identically on the irreducible variety V , so it is an injection.

It is clear that $m_p(V) = m_x(X) \cap \mathcal{O}_p(V)$. The Krull topology of $\mathcal{O}_x(X)$ defined by powers of the maximal ideal induces a topology T_2 on $\mathcal{O}_p(V)$ which is clearly stronger than the natural topology T_1 on $\mathcal{O}_p(V)$.

LEMMA 1. $T_1 = T_2$, that is there is an increasing function $h: \mathbb{Z}^+ \rightarrow \mathbb{Z}^+$ such that $m_x(X)^{h(i)} \cap \mathcal{O}_p(V) \subset m_p(V)^i$.

LEMMA 2. Given integers r and $N > 0$, there exist integers n_1, n_2, \dots, n_r all > 0 so that for any formal power series $h(x_1, \dots, x_r)$, $h(t^{n_1}, \dots, t^{n_r}) \equiv 0$ implies $\text{ord } h(x_1, \dots, x_r) \geq N$.

We will finish deducing the theorem before giving the proofs of these lemmas. We will pick suitable convergent power series $x(t) = (x_1(t), \dots, x_r(t))$ in one variable t without constant terms and let C_t be the image of $f(x(t))$ in V ; then the domain

$$T = \mathbf{C}\{f_1(x_1(t), \dots, x_r(t)), \dots, f_n(x_1(t), \dots, x_r(t))\}$$

will be the local ring $\mathcal{O}_p(C_t)$ in the theorem. Let $R = \mathcal{O}_p(V)$, $\psi: R \rightarrow T$ be the canonical surjection given by substitution ($\phi \rightarrow \phi(f(x(t)))$), and I_t denote the kernel of ψ . For $g \in R$, by $\text{ord } g$ we mean the order of g considered as an element of $\mathcal{O}_x(X)$.

Let $N = h(l) + 1$. Now choose the monomials t^{n_1}, \dots, t^{n_r} which satisfy the property in Lemma 2. If $g \in I_t$, then $g(f(t^{n_1}, \dots, t^{n_r})) = 0$ so by Lemma 2, $\text{ord } g > N > h(l)$, so $g \in m_p(V)^l$. Hence $I_t \subset m_p(V)^l$. Now $T = R/I_t \simeq \mathcal{O}(C_t)$ and the maximal ideal $m(T)$ of T is $m(R)/I_t$. Hence

$$\begin{aligned} \mathcal{O}_p(C_t)/m_p(C_t)^l &= T/m(T)^l = (R/I_t)/(m(R)/I_t)^l \\ &\simeq R/(m(R)^l + I_t) = R/m(R)^l \end{aligned}$$

as $I_t \subset m(R)^l$. Q.E.D.

PROOF OF LEMMA 2. Suppose n_1, \dots, n_r are any positive integers and $h(x_1, \dots, x_r)$ is a nonzero formal power series such that $h(t^{n_1}, \dots, t^{n_r}) = 0$. Then there exist two monomials $x_1^{\alpha_1} \cdots x_r^{\alpha_r}$ and $x_1^{\beta_1} \cdots x_r^{\beta_r}$ appearing in h such that $\alpha_1 n_1 + \cdots + \alpha_r n_r = \beta_1 n_1 + \cdots + \beta_r n_r$ and each $\alpha_i < N$. Letting $\gamma_i = \alpha_i - \beta_i$, we have each $\gamma_i < N$ too. So it is enough to choose n_i so that $\sum \gamma_i n_i = 0$ and $\gamma_i < N$ implies each $\gamma_i = 0$.

Let p be a prime $> N$, $l_i, 1 \leq i \leq r$, be primes chosen inductively, $l_1 > p$, so that $rp^2 l_i < l_{i+1}$, and $n_i = p^{r-i} l_i$. Suppose $\sum_{i=1}^r \gamma_i n_i = 0$ and each $\gamma_i < p$, let k be the largest integer for which $\gamma_k \neq 0$. By our choice p^{r-k+1} divides n_1, \dots, n_{k-1} so p^{r-k+1} divides $n_k = p^{r-k} l_k \gamma_k$ so p divides $l_k \gamma_k$; since p and l_k are relatively prime, p divides γ_k . We know $\gamma_k < p$ so a contradiction will follow when we show $\gamma_k > 0$.

Now $n_i = p^{r-i} l_i$, $n_{i+1} = p^{r-i-1} l_{i+1}$, and $l_{i+1} > rp^2 l_i \Rightarrow rp n_i < n_{i+1} \Rightarrow n_i < n_k$ for $i \leq k - 1$. Since $\gamma_i < p$ for all i , we have

$$\begin{aligned} \left| \sum_{i=1}^{k-1} \gamma_i n_i \right| &< \sum_{i=1}^{k-1} |\gamma_i n_i| \leq p(k-1)n_{k-1} \\ &< p(k-1)n_k / rp < n_k. \end{aligned}$$

Hence $\sum_{i=1}^k \gamma_i n_i = 0$ implies that $\gamma_k n_k > 0$ and so $\gamma_k > 0$. Q.E.D.

PROOF OF LEMMA 1. Let $q = \pi(x)$, $S' = \mathcal{O}_{\pi(x)}(B)$, and $S'_1 = \mathcal{O}_{\pi(x)}(B_1)$. Since $\mathcal{O}_x(X)$ is the integral closure of $\mathcal{O}_{\pi(x)}(B_1)$ in its field of quotients, $\mathcal{O}_x(X)$ is a finite module over $\mathcal{O}_{\pi(x)}(B_1)$. Hence the Krull topology of S'_1 is induced by the Krull topology of $\mathcal{O}_x(X)$. Therefore it suffices to show that T_1 is induced by

the Krull topology of S'_1 . We now recall:

(A) ZARISKI SUBSPACE THEOREM [2]. *Let R be an analytically irreducible domain (completion is a domain) and S a local ring birational with R (means same quotient field) and dominating R , and S a spot over R (means S is the localization of a finite algebra over R), then R is a subspace of S .*

(B) *The ring of convergent power series at an irreducible point of a complex analytic variety is analytically irreducible. [7, p. 89].*

(C) [3, Lemma 1.11]. *Let R be an analytic local ring and S a spot over R . Then there is a smallest analytic ring S' containing S and $\hat{S} = \hat{S}'$, where $\hat{}$ denotes completion in the maximal ideal topology.*

Now let $(p; a_1, a_2, \dots, a_n)$ denote the coordinates of the point $\pi(x)$ on B . By linear change of coordinates, we may assume $a_1 = 1$. Let

$$S = \mathcal{O}_p(V)[y_2/y_1, \dots, y_n/y_1](z_1 - a_1, \dots, z_n - a_n),$$

where $z_i = y_i/y_1$. We first show that $\mathcal{O}_{\pi(x)}(B)$, the local analytic ring of B at the point $\pi(x)$, is the smallest analytic ring S' containing S . We have a commutative diagram:

$$\begin{array}{ccc} S \subset \mathcal{O}_{\pi(x)}(B) & & \\ \cap & \nearrow \phi & \\ S' & & \end{array}$$

Since $y_1, \dots, y_n, z_1 - a_1, \dots, z_n - a_n$ generate the maximal ideal of both S and $\mathcal{O}_{\pi(x)}(B)$ we have that $\phi(M_S) = M_{\mathcal{O}_{\pi(x)}(B)}$. Since S' and $\mathcal{O}_{\pi(x)}(B)$ are both analytic rings, it follows [9] that $\phi(S') = \mathcal{O}_{\pi(x)}(B)$.

Now S is a spot over $\mathcal{O}_p(V)$ so by (B) and (A) above, $\mathcal{O}_p(V)$ is a subspace of S . Since S is obviously a subspace of \hat{S} , we have $\mathcal{O}_p(V)$ is a subspace of \hat{S} . Next by part (C), we have $\hat{S} = \hat{S}'$. Hence $\mathcal{O}_p(V)$ is a subspace of \hat{S}' . But $\mathcal{O}_p(V) \subset S'$ so it follows that $\mathcal{O}_p(V)$ is a subspace of S' . By similar techniques we could show that $\mathcal{O}_p(V)$ is a subspace of S'_1 .

REMARK. Alternately we could go directly to the normalization and show $\mathcal{O}_p(V)$ is a subspace of $\mathcal{O}_x(X)$ by applying (A), (B), and (C). This proof would proceed by using the fact that $\mathcal{O}_x(X)$ is a finite $\mathcal{O}_{\pi(x)}(B)$ module and constructing a spot over $\mathcal{O}_p(V)$ for which $\mathcal{O}_x(X)$ is the smallest analytic ring containing the spot. Q.E.D.

2. By techniques similar to those employed in §1, one can prove a formal version of the theorem.

THEOREM 2. *If k is a field of characteristic zero and $R = k[[y_1, \dots, y_n]]/I$ is a complete domain, and m is the maximal ideal of R , then for every integer*

$l > 0$, there is a prime ideal P_l of depth one with $P_l \subset m^l$.

Before starting on the proof of this we need to establish some preliminaries.

For an arbitrary Noetherian ring S and field $k \subset S$, let $D^1(S/k)$ be the module of Kahler differentials of S over k . If S is local with max ideal m , let

$$\hat{D}(S/k) = D^1(S/k) / \bigcap_{i=1}^{\infty} m^i D^1(S/k).$$

It is well known that for a local ring S containing its residue field of characteristic zero, with $\hat{D}(S/k)$ a finite S module, we have $\hat{D}(S/k)$ is a free S module if and only if S is regular. If S is finitely generated over k (respectively complete) then $D(S/k)$ (respectively $\hat{D}(S/k)$) is a finite module over S . If M is a finitely generated S module, generated by say e_1, \dots, e_m , let (a_{ij}) be a matrix with entries in S such that for each i , $\sum_{j=1}^m a_{ij} e_j = 0$, and such that any row vector (a_1, a_2, \dots, a_m) for each $\sum a_j e_j = 0$ is a linear combination with coefficients in S of the rows of (a_{ij}) ; in other words the rows of (a_{ij}) generate the module of relations of e_1, \dots, e_m . For an integer $p \geq 0$, the $(n - p) \times (n - p)$ minors of (a_{ij}) generate an ideal $I_p(M)$; by convention $I_p(M) = S$ for $p \geq n$. The ideal does not depend on either the choice of basis $\{e_1, \dots, e_m\}$ or the relation matrix (a_{ij}) ; $I_p(M)$ is the p th fitting of M . We have $I_0 \subseteq I_1 \subseteq \dots \subseteq I_n = S$. By the jacobian ideal J of S we will mean the first nonzero fitting ideal of $D(S/k)$. One knows that S is regular if and only if J is the unit ideal. We will denote $\hat{D}(S/k)$ by $\Omega(S/k)$.

It follows immediately from the Serre criteria for normality that if S is a normal Noetherian ring and $k \subset S$, then every minimal prime of $\text{Jac}(\Omega(S/k))$ has height ≥ 2 . (We will denote this by saying $\text{ht } J \geq 2$.)

LEMMA 3. *Let S be a Noetherian ring, $k \subset S$, $J = \text{Jac}(\Omega(S/k))$, M a maximal ideal of S with $J \not\subset M$, $S/M = F$, $F \subset S$, and F a finite field extension of k , then S_M is regular.*

PROOF. $\text{Jac}(\Omega(S_M/k)) = \text{Jac}(\Omega(S/k)) \cdot S_M = JS_M = S_M$ as $J \not\subset M$. In char 0 all extensions are separable and for a finite separable field extension, $\Omega(F/k) = 0$. So by the exact sequence $0 \rightarrow S \otimes_F \Omega(F/k) \rightarrow \Omega(S/k) \rightarrow \Omega(S/F) \rightarrow 0$ we see that $\Omega(S/k) = \Omega(S/F)$ so $\text{Jac}(\Omega(S_M/F)) = S_M$. Hence S_M is regular.

LEMMA 4. *Let $k \subset S \subset F[[t]]$, where $k \subset F$ are fields, $[F : k] < \infty$, $S \not\subset F$, and S is a Noetherian complete ring. Then $\text{Krull dim } S \leq 1$.*

PROOF. We will show that $F[[t]]$ is a finite extension over S and hence integral over S so by the going up and down theorem for integral extensions, $\text{Krull dim } S = \text{Krull dim } F[[t]]$.

Because $[F : k] < \infty$, it is easy to see that the map

$$v: S \rightarrow F[[t]] \xrightarrow{\text{evaluation}} F$$

is not injective: Let $g \in S, g \notin F$, and $v(g) \neq 0$, then $v(g)$ satisfies some minimal polynomial over $k, v(g)^r + k_1v(g)^{r-1} + \dots + k_r = 0$ with $k_i \in k$ and $k_r \neq 0$. So $f = g^r + k_1g^{r-1} + \dots + k_{r-1}g + k_r \in S, v(f) = 0$, and $f \neq 0$. To see the last statement, let $g = \sum a_i t^i, a_i \in F$, and a_m be the first nonzero term with $m > 0$; then the m th coefficient of f is $a_m(k_r + 2k_{r-1}a_0 + 3k_{r-2}a_0^2 + \dots + rk_0a_0^{r-1})$ which is nonzero since the polynomial $\sum_{i=1}^r i k_{r-i} a_0^{i-1}$ has degree less than the minimal polynomial for a_0 .

Since $1 \in S, m(F[[t]]) \cap S \subset m(S)$. Let g be an element of the kernel of $v, g = t^n u(t), n > 0, u$ a unit in $F[[t]]$. We can subject $F[[t]]$ to an F isomorphism $x = \phi(t)$ so that $\phi(g) = x^n$; hence we may assume $k \subset S \subset F[[x]]$ and $x^n \in S$. Now $x^n \in m(F[[x]])$ and $x^n \in S$ so $x^n \in m(S)$; since S is complete, $k[[x^n]] \subset S$. Let f_1, \dots, f_r be a basis of F over k . Then $\{f_j x^i | 1 \leq j \leq r, 1 \leq i \leq n\}$ is a basis of $F[[x]]$ over $k[[x^n]]$. Since $k[[x^n]] \subset S, F[[x]]$ is also a finite S module.

REMARK. The hypothesis of complete is critical here as the following example shows. There exist in $k[[t]]$ an infinite set $\{f_i\}_{i=1}^\infty$ of algebraically independent elements [1]; like $e^x - 1, e^{x^2} - 1, \dots$ for example. Then $k[f_1, f_2, f_3, \dots]$ is not Noetherian and has infinite Krull dim. And $S' = k[f_1, \dots, f_n]_{(f_1, \dots, f_n)}$ is local, Noetherian, not complete, and has Krull dim n . Note that $k[[f_1, \dots, f_n]]$ has Krull dim one. (Any two power series in one variable are analytically dependent.) The problem here is that the completion of $k[f_1, \dots, f_n]_{(f_1, \dots, f_n)}$ does not inject into $k[[f_1, \dots, f_n]]$.

PROOF OF THEOREM 2. Let B be the blowing up of the maximal ideal in R , and N be the integral closure of B in its quotient field. Clearly B and N are contained in the quotient field of R , and B is a finite algebra over N . That N is a finite B module follows from the below fact.

Fact D [9, 36.1, 36.5]. A ring A is pseudogeometric if A is Noetherian and if for every prime ideal p of A and ring $E, A/p \subset E$, with the quotient field of E a finite extension of the quotient field of A/p , and E integral over A/p , we have that E is a finite A/p module. In characteristic zero, complete rings are pseudogeometric. If A is pseudogeometric, then every localization of a finite algebra over A is pseudogeometric.

Let $M(R) = (y_1, \dots, y_n)R, B = R[y_2/y_1, \dots, y_n/y_1]$, and $I = (y_1, \dots, y_n)B = y_1B$ and IN be the extension of I to N . N is a finite B module so N/IN is a finite B/I module. However B/I does not inject into N/IN .

EXAMPLE. $R = k[[x, y, z]]/(x^4 + y^4 - z^2)$, let $z = xv, y = xu, B = k[[x]][[u, v]]/(x^2(1 + u^4) - v^2)$. Then v/x is in the quotient field of $B, (v/x)^2 - (1 + u^4) = 0$ so $v/x \in N$. However $v/x = \sqrt{1 + u^4}$ is in \hat{B} but not in B .

Hence x divides v in N , but x does not divide v in B . So v is an element of the kernel $B/I \rightarrow N/IN$.

Now I is principal and a proper ideal. Furthermore it can be shown that $\dim N/IN = \dim N - 1$. (It should be pointed out that B and N have problems with their Krull dimension. In opposition to the case of local rings or affine rings, B does not have the property that every maximal chain of primes has the same length.)

EXAMPLE. Let $R = k[[x, y]]$ and $B = k[[x, y]][y/x]$. Then $(0) \subset (x) \subset (x, y/x)$ is a chain of primes of length 2 in B . But $(0) \subset I = (1 - y(y/x))B$ is a maximal chain of primes of length one in B : I is principal so of height one; it suffices to show I is a maximal ideal. Now $x - y^2 \in I$ so $B/I = k[[y^2, y]][1/y] = k[[y]][1/y] = k((y))$ which is a field.

Let J be the jacobian ideal of N . Now $\text{ht } J > 2$, and $\text{ht } IN = 1$ so $J \not\subset IN$; hence $J' = J/IN$ is not zero in $N' = N/IN$. Clearly N' is an affine ring, so there exists a maximal ideal M' of N/IN with $J' \not\subset M'$. (This is easily seen by tensoring N' with the algebraic closure \bar{k} of k , finding a maximal ideal M_0 of $N' \otimes_k \bar{k}$ with $J' \otimes_k \bar{k} \not\subset M_0$ via the Hilbert Nullstellensatz, and letting $M' = M_0 \cap N'$.) Now let M be the contraction of M' to N and $S = \hat{N}_M$. Clearly the contraction of M to R is the maximal ideal of R . Since N_M is a spot over R , the residue field F of S is finite over the residue field k of R . By Lemma 3, S is regular. By Cohen's structure theorem for complete local rings, $S = F[[t_1, \dots, t_r]]$, where $r = \dim R$. Let $\eta: S \rightarrow F[[t]]$ be a homomorphism as in Lemma 2 with $\ker \eta \subset M(S)^l$. Let i be the inclusion $R \rightarrow B \rightarrow N \rightarrow S$, and $\psi = \eta(i)$. By Lemma 4, $\dim \psi(R) = 1$; hence $\dim R/\ker \psi = 1$. The theorem now follows from the Chevalley subspace theorem:

Fact E [13, p. 255]. Let $R \subset S$ be complete local rings with $m(S) \cap R = m(R)$. Then R is a subspace of S . Q.E.D.

3. We would now like to improve the results of §§1 and 2 to yield the following:

PROPOSITION 1. *If \mathbf{R} is the field of real numbers and $A = \mathbf{R}\{x_1, \dots, x_n\}/I$ is the local ring of germs of convergent power series over \mathbf{R} at a point p on a real analytic variety V , and $l > 0$, then there is a curve C_l lying on V and passing through p such that V and C_l are isomorphic up to order l at p . This curve is given by a \mathbf{R} algebra homomorphism $A \rightarrow \mathbf{R}\{t\}$.*

Of course the ideal I must be the ideal of a real analytic variety, that is the real Hilbert Nullstellensatz must hold for I . That is every $f \in A$ which vanishes on the locus of I belongs to I . It is known that this is equivalent to the following condition on I : for every $f_1, \dots, f_p \in A$ with $f_1^2 + \dots + f_p^2 \in I$, we have each $f_i \in A$.

Unfortunately this does not seem to follow immediately because when

blowing up and normalizing the residue field is likely to change from \mathbf{R} to \mathbf{C} , yielding a complex analytic curve on the complexification, $V \otimes_{\mathbf{R}} \mathbf{C}$ of V , rather than a real analytic curve on V . For this reason, we give a different proof of Theorem 2 which works for a more general class of rings.

THEOREM 3. *Let S be an analytically irreducible domain with residue field of characteristic zero. Then for every $l > 0$, there is a prime P_l with $P_l \subset M^l$ and $\dim S/P_l = 1$.*

REMARK. Actually this theorem holds in mixed characteristic as well, but we are not interested in that here.

We first recall [13, pp. 86–92, Vol. I] the properties of the norm map. Let K be a finite algebraic extension of a field k of degree n and $y \in K$. Fixing a basis w_1, w_2, \dots, w_n of K over k , we write $yw_i = \sum_{j=1}^n a_{ij}w_j$, $a_{ij} \in k$, $1 \leq i \leq n$, or in matrix notation $y\Omega = A\Omega$, where $A = (a_{ij})$ is an $n \times n$ matrix and Ω is the column matrix with entries w_i . The characteristic polynomial $\det(A - XI) = X^n + a_1X^{n-1} + \dots + a_n$ has y as root. It is not necessarily the minimal polynomial of y over k . It is not hard to see that the polynomial does not depend on the choice of basis of K over k . Note that $a_n = (-1)^n \det A$. We set the norm of y , $N_{K/k}(y) = (-1)^n a_n$. The following properties are well known:

- (a) $N_{K/k}(xy) = N_{K/k}(x)N_{K/k}(y)$.
- (b) If $y \in k$, then $N_{K/k}(y) = y^n$.
- (c) If $R \subset S$ are domains with quotient fields $k \subset K$ respectively, with S finite over R , R normal, and $y \in S$, then $N_{K/k}(y) \in R$ (because the norm is just the product of all the conjugates of x in some normal extension of K). Also if p is an ideal of S , then $N_{K/k}(p) \subset R \cap p$.

Also recall [11] Pfister and Poperin have generalized Artin approximation to show the following: Let (R, m) be a complete local ring. Let $f_1, \dots, f_m \in R[x] = R[x_1, \dots, x_n]$ be m polynomials in n variables over R . Then $\forall c \in \mathbf{N}$, $\exists N_c \in \mathbf{N}$ (depending on c and f_1, \dots, f_m) such that if $y = (y_1, \dots, y_n) \in R^n$ and each $f_i(y) = 0 \pmod{m^{N_c}}$, then $\exists(\lambda_1, \dots, \lambda_n) \in R^n$ such that $\lambda_i = y_i \pmod{m^c}$ for each i , and each $f_i(\lambda) = 0$.

LEMMA 5. *Let (R, m) be analytically irreducible local domain (i.e. \hat{R} is a domain) and let $a_1, \dots, a_n \in \mathbf{N}$ be given. Then there exist $N \in \mathbf{N}$ such that*

$$(R - m^{a_1}) \cdots (R - m^{a_n}) \subset R - m^N.$$

PROOF. Pick $c = \max\{a_1, \dots, a_n\}$ and apply the above to the equation $X_1 \cdot X_2 \cdots X_n = 0$ over \hat{R} , i.e. let $h = 1$ and $f_1(X) = X_1 \cdots X_n$. Choose $N = N_c$ as guaranteed by the theorem. Now suppose $y_i \in R - m^{a_i}$. It then follows that $y_1 \cdot y_2 \cdots y_m \in m^N$. For if not $f_1(y) = 0 \pmod{m^{N\hat{R}}}$ and we could choose $\lambda_1, \dots, \lambda_n \in \hat{R}$ such that $\lambda_i = y_i \pmod{m^c\hat{R}}$ and $\lambda_1 \cdots \lambda_n = 0$.

Now $\lambda_i = y_i \bmod m^c \hat{R}$, $y_i \notin m^a$, and $c \geq a_i$ implies $\lambda_i \notin m^a \hat{R}$ which implies $\lambda_i \neq 0$. But \hat{R} is a domain. This is a contradiction. Q.E.D.

LEMMA 6. *Let (S, n) be a complete local domain, (R, m) be a normal local domain, with $R \subset S$, and S a finite R module. Let the quotient fields of R and S be F and G respectively. Let $\mathcal{U} = \text{Norm}_{G/F}$. Then $\mathcal{U}(S) \subset R$ and for all $c \in \mathbb{N}$, there exist $N_c \in \mathbb{N}$ such that $\mathcal{U}^{-1}(m^{N_c}) \subset n^c$.*

PROOF. We first assume that G is a normal field extension of F , that is the fixed field of the Galois group of G over F is precisely F . Let $\{\phi_1, \dots, \phi_\nu\} = \text{Gal}(G/F)$, say $\phi_1 = \text{identity}$. Each ϕ_i induces an R automorphism of S . For all $s \in S$, $\mathcal{U}(s) = \prod_{i=1}^\nu \phi_i(s)$. Hence if $s = \phi_1(s) \notin n^c$, we have $\phi_i(s) \notin n^c$ for each i and by Lemma 5, there exist N_c such that $(S - n^c)^n \subset S - n^{N_c}$. But then $\mathcal{U}(s) \notin n^{N_c}$. Hence $\mathcal{U}^{-1}(m^{N_c}) \subset n^c$.

In the general case, let H be a finite normal field extension of F containing G and let $(T, q) = \text{integral closure of } R \text{ in } H$. Since T is finite over S , by the Artin Reese lemma, the natural Krull topology on S is the same as the topology induced from T . Hence for every c there exists d such that $q^d \cap S \subset n^c$. By the special case, there exists N so $\mathcal{U}_{T/R}^{-1}(m^N) \subset q^d$. This N also works for S : If $s \in S$ and $\mathcal{U}_{S/R}(s) \in m^N$, then

$$\begin{aligned} \mathcal{U}_{T/R}(s) &= \mathcal{U}_{S/R}(\mathcal{U}_{T/S}(s)) = \mathcal{U}_{S/R}(s^{[H:G]}) \\ &= (\mathcal{U}_{S/R}(s))^{[H:G]} \in m^{N[H:G]} \subset m^N \Rightarrow s \in q^d \cap S \subset n^c. \end{aligned}$$

PROOF OF THEOREM 3. Let the Krull dimension of $S = d$. We will prove that for all i , $0 \leq i < d$, and all c , there exists a prime p in S with $\text{ht } p = i$ and $p \subset m^c$. By a trivial induction on i , we may assume $i = 1$. Let X_1, \dots, X_d be a system of parameters for S , then \hat{S} is a finite extension of $R = k[[X_1, \dots, X_d]]$. Let $\mathcal{U} = \text{Norm}_{\hat{S}/R}$. Pick N so large that $\mathcal{U}^{-1}(m^N) \subset \hat{n}^c$. Let P_0 be the prime in R generated by $X_1^{p_1} - X_2^{p_2}$, where p_1, p_2 are large prime integers; $P_0 \subset m^N$. Let \tilde{p} be a prime of \hat{S} lying over P_0 . Then $\tilde{p} \subset \hat{n}^c$, for $s \in \tilde{p} \Rightarrow \mathcal{U}(s) \in \tilde{p} \cap R \subset P_0 \subset m^N \Rightarrow s = \mathcal{U}^{-1}\mathcal{U}(s) \subset \hat{n}^c$. Now let $p = \tilde{p} \cap S$. Clearly $p \subset n^c$, and $p \cap S \neq (0)$ because $X_1^{p_1} - X_2^{p_2} \in p \cap S$.

4. It is clear that by the techniques of the last section, we can prove Proposition 1. (The projection of the local parametrization of a real analytic variety is semianalytic so one just picks a high order curve in this semi-analytic set not lying in the discriminant locii and lifts it to a real analytic curve on V . Details are omitted and left to the reader. See for instance [8].)

We now show how Proposition 1 can be used to study the ring of C^∞ and C^k functions on an irreducible real analytic set. Let X be a real analytic set in \mathbb{R}^n , irreducible at x , and $C_x^\infty(X)$ be the ring of germs at x of infinitely differentiable functions on X . Let $T: C_x^\infty(X) \rightarrow \mathbb{R}[[x_1, \dots, x_n]]/I(X)$ be the Taylor map, taking a C^∞ function to its Taylor series. Then T is clearly an \mathbb{R}

algebra homomorphism. It is well known that T is surjective. Let m_1 be the maximal ideal of $\mathcal{F} = \mathbf{R}[[x_1, \dots, x_n]]/I(X)$ and m_2 be the maximal ideal of $C_x^\infty(X)$. Let $p_i \subset m_1^i$ be as in Proposition 1. Since there is a one to one correspondence between primes of $C_x^\infty(X)$ containing $\ker T$ and primes of \mathcal{F} , $q_i = T^{-1}(p_i)$ is a prime of $C_x^\infty(X)$ of depth one with $q_i \subset m_2^i$.

COROLLARY 1. *Let X be germ of a real analytic subvariety in \mathbf{R}^n , irreducible at x , and $C_x(X)$ the ring of germs at x of real valued infinitely differentiable functions on X . Then there is an irreducible real analytic curve C in X passing through x so that their C^∞ tangent spaces are the same at x , that is $T(V, C_x^\infty) = T(C, C_x^\infty)$, where $T(V, C_x^\infty) = \{r \in \mathbf{R}^n \mid \sum_{i=1}^n r_i \partial_i f(x) = 0 \ \forall f \in I(V, C_x^\infty)\}$.*

REMARK. We have not made use of the fact that $T(V, \mathcal{Q}_x) = T(V, C_x^\infty)$ in the above corollary, and in fact our theorem can be used to prove results of this nature.

COROLLARY 2. *Let X be a germ of a real analytic subvariety in \mathbf{R}^n , irreducible at x , $C_x^k(X)$ the ring of germs at x of real valued k times continuously differentiable functions on X , and $\mathcal{Q}_x(X)$ the ring of germs of real analytic functions on X . Then there exist $k > 0$ so that $T(X, C_x^k) = T(X, \mathcal{Q}_x)$, where*

$$T(X, C_x^k) = \left\{ r \in \mathbf{R}^n \mid \sum_{i=1}^n r_i \partial_i f(x) = 0 \ \forall f \in I(X, C_x^k) \right\}$$

and

$$T(X, \mathcal{Q}_x) = \left\{ r \in \mathbf{R}^n \mid \sum_{i=1}^n r_i \partial_i f(x) = 0 \ \forall f \in I(X, \mathcal{Q}_x) \right\}.$$

PROOF. Pick a curve C in X as in Corollary 1. Then clearly:

$$\begin{aligned} T(C, C_x^k) &\subset T(X, C_x^k) \\ \cap &\qquad \qquad \cap \\ T(C, \mathcal{Q}_x) &= T(X, \mathcal{Q}_x) \end{aligned}$$

so it suffices to see that these exist $k > 0$ so that $T(C, C_x^k) = T(C, \mathcal{Q}_x)$. We sketch the proof of this fact below. Assume the point x is the origin. Let $\phi: \mathbf{R} \rightarrow C$ be the desingularization of the irreducible curve C (obtained by complexifying C , normalizing the new complex curve and then restricting to the real part). Without loss of generality, one may assume that C is embedded real analytically in minimal dimension. One may make a real analytic change of coordinates so that ϕ has the form $\phi(t) = (t^{q_1}u_1(t), \dots, t^{q_n}u_n(t))$, where the u_i are units $q_1 < q_2 < \dots < q_n$, and there is no polynomial in $\phi_1(t), \dots, \phi_{k-1}(t)$ whose order is precisely q_k . (If this condition is not

satisfied, make inductively a sequence of transformations until it is.) Pick $k > q_n/q_1 + 1$. If $f \in I(X, C_x^k)$ and T_k is the k th order Taylor series about 0, then $f - T_k = o(|x|^k)$ on C . Comparing with $\phi(t)$, we get $\sum_{|\alpha| \leq k} a_\alpha \phi(t)^\alpha = o(t^{q_1 k})$ where $T_k = \sum_{|\alpha| \leq k} a_\alpha x^\alpha$. Then T_k can have no linear terms because each $q_i < kq_1$ and no polynomial in the $\phi_j(t), j \neq i$, can have order q_i .

For additional details of the argument above, see [4].

Straightforward generalization of these arguments yields the following result claimed by the author in [4] and proven by Risler in [12]: Let X be the germ of a real analytic set in \mathbb{R}^n and $I(X)$ the ideal of X in $\mathbb{R}\{x_1, \dots, x_n\}$. Then there is a function $\lambda: \mathbb{N} \rightarrow \mathbb{N}$, with $\lim_{r \rightarrow \infty} \lambda(r) = r$, so that if F is a function of class C^r on \mathbb{R}^n vanishing on X , then $T_{\lambda(r)} F \in I(X) + m^{\lambda(r)}$.

COROLLARY 3. *In [8] Merrien has proven that if I is an ideal of $\mathcal{O}_n = \mathbb{R}\{x_1, \dots, x_n\}$ with $I = \text{ideal}(\text{locus}(I))$, then $I = \bigcap_{\gamma \in \Gamma} \ker \gamma$, where Γ is the set of all \mathbb{R} algebra homomorphisms $\mathcal{O}_n \rightarrow \mathbb{R}\{t\}$ with $I \subset \ker \gamma$. This follows immediately from Proposition 1 because the condition on the ideal is precisely that I is the ideal of a real analytic variety; hence for every $i > 0$, there exists $\gamma_i: \mathcal{O}_n \rightarrow \mathbb{R}[[t]]$ with $I \subset \ker \gamma_i \subset I + m^i$. Hence*

$$I \subset \bigcap_{i=1}^{\infty} \ker \gamma_i \subset \bigcap_{i=1}^{\infty} (I + m^i) = I$$

as \mathcal{O}_n is a local Noetherian ring.

Now suppose $\phi: X \rightarrow Y$ is a map of irreducible analytic varieties and $\phi: \mathcal{O}(Y, q) \rightarrow \mathcal{O}(X, p)$ the induced map on the associated local analytic rings. It is not necessarily true that the image of ϕ is closed in $\mathcal{O}(X, p)$ in the Krull topology. (The map $(x, y) \rightarrow (x, xy, xe^y)$ provides an easy counterexample.)

DEFINITION. Let $\phi: R \rightarrow S$ be a local algebra homomorphism of analytic domains. By a curve in S , we mean a local algebra hom $\gamma: S \rightarrow \mathcal{O}_1$. Clearly a curve in S induces a curve in R . We say that ϕ is nice if for every $f \in S - \phi R$, there is a nonzero curve γ in S such that $\gamma(f) \in \mathcal{O}_1 - \gamma\phi R$. That is a function factors through ϕ if and only if it factors through every curve.

COROLLARY 4. *ϕ is nice if and only if $\phi(R)$ is closed in S in the Krull topology.*

PROOF. Suppose $\phi(R)$ is closed in S . Since the closure of $\phi(R)$ in S is $\bigcap_{k=1}^{\infty} (\phi(R) + M(S)^k)$, for every $f \in S - \phi(R)$, there exists a $k > 0$ so $f \notin \phi(R) + M(S)^k$. Now let γ be a curve in S so that $\ker \gamma \subset M(S)^k$. We have a commutative diagram:

$$\begin{array}{ccc}
 R & \xrightarrow{\phi} & S \\
 \downarrow \pi_1 & & \downarrow \pi_2 \\
 R/\ker \gamma\phi & \xrightarrow{\bar{\phi}} & S/\ker \gamma
 \end{array}$$

It is clear that $\pi_2(f) \notin \bar{\phi}(R/\ker \gamma\phi) = \pi_1\phi(R)$.

Conversely assume ϕ is nice. Let $f \in S - R$ and γ curve in S with $\gamma(f) \notin \gamma\phi R$. Now mappings of one dim analytic rings are always finite of zero, so by the Artin Rees lemma $\gamma\phi R$ is closed in γS . Since γ is continuous, $\gamma^{-1}\gamma\phi R$ is closed in S . Also $f \notin \gamma^{-1}\gamma\phi R$ and $\phi R \subset \gamma^{-1}\gamma\phi R$. Hence $\phi R = \bigcap_{\gamma} \gamma^{-1}\gamma\phi R$ is closed in S .

For additional applications of these ideas, see [5].

ADDED IN PROOF. After this paper was accepted for publication, the authors discovered that Theorem 3 had been proven by R. Berger (*Zur Idealtheorie analytisch normaler Stellenringe*, J. Reine Angew. Math. **201** (1958), 172–177) by different techniques.

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