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R-SEPARABLE COORDINATES FOR THREE-DIMENSIONAL COMPLEX RIEMANNIAN SPACES

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ABSTRACT. We classify all R-separable coordinate systems for the equations $\sum_{i,j=1}^{3} g^{-1/2}\partial_j(g^{1/2}g^{ij}\partial_i\psi) = 0$ and $\sum_{i,j=1}^{3} g^{ij}\partial_i W\partial_j W = 0$ with special emphasis on nonorthogonal coordinates, and give a group-theoretic interpretation of the results. We show that for flat space the two equations separate in exactly the same coordinate systems.

1. Introduction. We study the problem of R-separation of variables for the equations

$$\text{(a)}\quad \Delta_3\psi = \sum_{i,j=1}^{3} \frac{1}{\sqrt{g}}\,\partial_{x^i}\left(\sqrt{g}\; g^{ij}\partial_{x^j}\psi\right) = 0,$$

$$\text{(b)}\quad \sum_{i,j=1}^{3} g^{ij}\partial_{x^i}W\partial_{x^j}W = 0. \tag{1.1}$$

Here, $dx^2 = \sum g_{ij}dx^i dx^j$ is a complex Riemannian metric, $g = \det(g_{ij}) \neq 0$, $\sum_j g^{ij}g_{jk} = \delta^i_k$, and $g_{ij} = g_{ji}$. Thus (1.1)(a) is the Laplace equation and (1.1)(b) is the associated Hamilton-Jacobi equation on a complex Riemannian space.

We classify all metrics for which equations (1.1) admit solutions via R-separation of variables and we indicate explicitly the group-theoretic significance of each type of variable separation. Special attention is given to nonorthogonal separable systems and to metrics for conformally flat spaces.

It is straightforward to show that (1.1)(b) admits (additive) separation of variables in every coordinate system for which (1.1)(a) admits a product R-separation and that in general (1.1)(b) separates in more systems than does (1.1)(a). However, by making use of some results of Eisenhart [**1**] we shall show explicitly that for flat space these two equations separate in exactly the same coordinate systems. For this case one can choose Cartesian coordinates x, y, z so that $ds^2 = dx^2 + dy^2 + dz^2$ and (1.1) becomes

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$$\text{(a)}\quad \Delta_3\psi = \partial_{xx}\psi + \partial_{yy}\psi + \partial_{zz}\psi = 0,$$
$$\text{(b)}\quad (\partial_x W)^2 + (\partial_y W)^2 + (\partial_z W)^2 = 0. \tag{1.2}$$

It follows from earlier papers by the authors [2]–[6], that all R-separable systems for these equations are characterized by pairs of commuting operators in the enveloping algebra of the conformal symmetry algebra $o(5)$. This fact permits use of the representation theory of $o(5)$ to derive properties of the R-separated (special function) solutions of the complex Laplace equation.

Similar comments hold for the possible distinct real forms of our complex flat space equations, i.e., the real Laplace and wave equations in three space. One need only modify the results obtained here by classifying the possible real metrics with the appropriate signature. These comments also pertain to all real Riemannian spaces, except that the second-order symmetry operators need not belong to the enveloping algebra of the Lie symmetry algebra.

This paper is closely related to two other papers by the authors. In [6] we studied in detail the group-theoretic relation between the real wave equation in three-space and the associated Hamilton-Jacobi equation. Here we fill in a gap left in that article by showing explicitly that these equations separate in exactly the same coordinate systems. In [7] the corresponding separation of variables problem for the equations $\Delta_3\psi = E\psi$ and $\sum g^{ij}\partial_i W \partial_j W = E$, where E is a nonzero constant, is solved. This reference also contains a discussion of related work on separation of variables by other authors.

The present paper is one in a series devoted to uncovering the relationships between group representation theory, separation of variables and special function identities [8], [9].

2. R-separation for Laplace's equation. Here we classify the metrics for which the equation $\Delta_3\psi = 0$, (1.1)(a), admits an R-separation of variables. We will characterize these metrics via pairs of commuting symmetry operators.

Recall that

$$L = \sum_{j=1}^{3} \xi_j(x^i)\partial_{x^j} + \xi(x^i) \tag{2.1}$$

is a *first-order symmetry operator* for (1.1)(a) if $[L, \Delta_3] = \rho(x^i)\Delta_3$ for some analytic function ρ [9]. The set of all symmetries L forms a Lie algebra $\mathcal{G}$ under the commutator bracket $[A, B] = AB - BA$, called the *symmetry algebra* of Δ_3. It is easy to show that the ξ_j satisfy the differential equations for the conformal Killing vectors related to the metric (g_{ij}) and, neglecting the trivial symmetry $L = 1$, $\mathcal{G}$ is a subalgebra of the infinitesimal conformal group of the metric. As is well known [10], when (g_{ij}) is flat then $\mathcal{G} \cong o(5)$, a ten-dimensional complex Lie algebra. In general $\mathcal{G}$ is isomorphic to a subalgebra of $o(5)$.

Similarly,

$$L' = \sum_{j,k=1}^{3} \eta_{jk}(x^i)\partial_{x^jx^k} + \sum_{l=1}^{3} \eta_l(x^j)\partial_{x^l} + \eta(x^i) \tag{2.2}$$

is a *second-order symmetry operator* for Δ_3 if $[L', \Delta_3] = K\Delta_3$ where K is a first-order differential operator of the form (2.1) (but not necessarily a symmetry operator). The second-order symmetries do not form a Lie algebra but they do form a vector space which can be decomposed into orbits under the adjoint action of $\mathcal{G}$. If, acting on the solution space of (1.1)(a), every such L' agrees with a second-order polynomial in the enveloping algebra of $\mathcal{G}$, then equation (1.1)(a) is said to be *class* I [9]. Otherwise (1.1)(a) is *class* II.

We shall first show explicitly that each pure separable solution $\psi = A(x^1)B(x^2)C(x^3)$ of (1.1)(a) is characterized by a commuting pair of second-order symmetries L_1, L_2 such that

$$L_i\psi = \lambda_i\psi, \qquad i = 1, 2, \tag{2.3}$$

where the eigenvalues λ_i are the separation constants. In the following we classify each of the separable systems and list the associated operators L_1, L_2.

As in [7] our analysis is based on the number of ignorable coordinates. Here the coordinate x^k is ignorable if $L = \partial_{x^k}$ is a symmetry operator, i.e., $[\partial_{x^k}, \Delta_3] = \rho\Delta_3$.

In the case of three ignorable coordinates the general form of the contravariant metric can be taken as $g^{ij} = Q(x^1, x^2, x^3)\delta_{ij}$, $i, j = 1, 2, 3$. If we write (1.1)(a) in the form

$$a_{11}\psi_{11} + a_{22}\psi_{22} + a_{33}\psi_{33} + a_1\psi_1 + a_2\psi_2 + a_3\psi_3 = 0$$

we see that the conditions $a_i = QC_i$ (C_i constant), imply that

$$Q = \exp\left(-C_1x^1 - C_2x^2 - C_3x^3\right). \tag{2.4}$$

The corresponding differential form is

$$\begin{aligned} &(1) \quad ds^2 = \exp\left(C_1x^1 + C_2x^2 + C_3x^3\right)\left[(dx^1)^2 + (dx^2)^2 + (dx^3)^2\right], \\ &\quad L_1 = \partial_{x^1x^1}, \qquad L_2 = \partial_{x^2x^2}. \end{aligned} \tag{2.5}$$

The case of two ignorable coordinates can be treated similarly. The differential form is

$$\begin{aligned} &(2) \quad ds^2 = \exp\left(C_1x^1 + C_2x^2\right)\left[\sum_{ij} G_{ij}(x^3)dx^i\,dx^j\right], \\ &\quad L_1 = \partial_{x^1}, \qquad L_2 = \partial_{x^2}. \end{aligned} \tag{2.6}$$

If there is only one ignorable coordinate x^1 and the remaining two separation equations are second-order then the contravariant metric assumes the form

$$g^{11} = Q[R(x^2) + S(x^3)], \qquad g^{22} = g^{33} = Q,$$
$$g^{12} = QH(x^2), \qquad g^{13} = QI(x^3), \qquad g^{23} = 0. \tag{2.7}$$

The resulting Laplace equation appears as

$$a_{11}\psi_{11} + a_{22}\psi_{22} + a_{33}\psi_{33} + a_{12}\psi_{12} + a_{13}\psi_{13} + a_1\psi_1 + a_2\psi_2 + a_3\psi_3 = 0. \tag{2.8}$$

If $\bar{R} = R - H^2$ and $\bar{S} = S - I^2$ are both nonconstant then the requirement $a_2 = Qf(x^2)$ implies

$$F(x^2) = \partial_{x^2}\ln\left[(\bar{R} + \bar{S})Q\right] + \partial_{x^1}\ln Q \tag{2.9}$$

with a similar constraint arising from $a_3 = Qh(x^3)$ and a condition on a_1. It follows that $Q = h(x^2)k(x^3)e^{-C_1x^1}[\bar{R} + \bar{S}]^{-1}$. The corresponding metric is

$$(3) \quad ds^2 = e^{C_1X^1}G(x^2)J(x^3)\Big[\big(U(x^2) + V(x^3)\big) \cdot \big((dx^2)^2 + (dx^3)^2\big) + (dX^1)^2\Big],$$

$$L_1 = \partial_{X^1}, \qquad L_2 = G^{-1/2}\partial_{x^2}\big(G^{1/2}\partial_{x^2}\big) + U\big(\tfrac{1}{2}C_1\partial_{X^1} + \partial_{X^1X^1}\big) \tag{2.10}$$

where $dX^1 = dx^1 - Hdx^2 - Idx^3$. If either $\bar{R}_2 = 0$ or $\bar{S}_3 = 0$ then $Q = h(x^2)k(x^3)e^{-C_1x^1}$ and the metric is a special case of (3).

If x^1 is ignorable, the separation equation in x^2 is first-order while that in x^3 is second-order, then the contravariant metric tensor becomes

$$g^{33} = g^{12} = Q, \qquad g^{13} = QH(x^3),$$
$$g^{11} = Q[R(x^2) + S(x^3)], \qquad g^{22} = g^{33} = 0, \tag{2.11}$$

and the resulting Laplace equation appears as

$$a_{11}\psi_{11} + a_{33}\psi_{33} + a_{13}\psi_{13} + a_{12}\psi_{12} + a_1\psi_1 + a_2\psi_2 + a_3\psi_3 = 0. \tag{2.12}$$

The conditions $a_2 = Qf(x^2)$, $a_3 = Qh(x^3)$ imply $Q = k(x^2)l(x^3)e^{-C_1x^1}$. There are no further restrictions and the metric is

$$ds^2 = e^{C_1x^1}F(x^2)G(x^3)\Big[-\big(R(x^2) + S(x^3)\big)(dx^2)^2 + \big(H(x^3)dx^2 + dx^3\big)^2 + 2dx^1dx^2\Big]. \tag{2.13}$$

However, introduction of the new ignorable variable X^1 where $dX^1 = dx^1 - \frac{1}{2}R(x^2)\,dx^2$ reduces (2.13) to a metric with two ignorable variables, a special case of (2).

If there are no ignorable variables then the coordinates must be orthogonal and the conditions on the metric are well known to be

$$(4) \quad g_{ii} = \frac{SQ}{M_{i1}}, \qquad \frac{g^{1/2}}{QS} = \prod_{i=1}^{3} f_i(x^i) \tag{2.14}$$

where M_{i1} is the indicated cofactor of the Stäckel matrix whose determinant is S, and the f_i are arbitrary. The operators are those listed in (2.27) with $M \equiv 1$.

This completes the list of coordinates for which $\Delta_3\psi = 0$ admits pure separation. Note that the only possible nonorthogonal systems are of type (2).

We now turn our attention to the metrics which are R-separable. For these there is a fixed function $R = R(x^1, x^2, x^3)$ such that for $\psi = e^R\phi$ the resulting differential equation obeyed by ϕ is purely separable in x^1, x^2, x^3. (In case $R = S_1(x^1) + S_2(x^2) + S_3(x^3)$ then R-separation reduces to pure separation.) As before, we will characterize each R-separable solution $\psi = e^R A(x^1)B(x^2)C(x^3)$ by listing a commuting pair of second-order symmetries L_1, L_2 such that (2.3) holds.

Upon extraction of the modulation factor e^R, equation (1.1)(a) assumes the form

$$\Delta_3\psi = e^R\left(\sum_{i,j=1}^{3} b_{ij}\phi_{ij} + \sum_{i=1}^{3} b_i\phi_i + b_0\phi\right) = 0,$$

$$b_{ij} = g^{ij}, \qquad b_i = \sum_{j=1}^{3}\left[g^{-1/2}\partial_{x^j}\left(g^{1/2}g^{ij}\right) + 2g^{ij}R_j\right],$$

$$b_0 = \sum_{i,j=1}^{3}\left[\left(R_{ij} + R_iR_j\right)g^{ij} + R_j g^{-1/2}\partial_{x^i}\left(g^{1/2}g^{ij}\right)\right]. \tag{2.15}$$

We proceed by considering the number of ignorable variables in the equation for ϕ. Clearly, x^l is an ignorable variable in this equation if and only if $L = \partial_{x^l} + \rho(x^j)$ is a symmetry operator for (1.1)(a), where $\rho = -\partial_{x^l}R$.

If extraction of the factor $M = e^R$ leads to a transformed equation with three ignorable coordinates then we can take the contravariant metric tensor to be of the form $g^{ij} = Q\delta^{ij}$. The coefficient b_i of the reduced equation is

$$b_i = Q\partial_{x^i}\ln\left(M^2Q^{-1/2}\right).$$

The condition $b_i = QC_i$, C_i constant, then leads to

$$M^2Q^{-1/2} = \exp\left(C_1x^1 + C_2x^2 + C_3x^3\right). \tag{2.16}$$

The remaining condition for separation is $b_0 = QC_0$ which implies

$$\sum_{i=1}^{3}\left(M_{ii} - \tfrac{1}{2}Q_iQ^{-1}M_i\right) = C_0,$$

$$(1)' \quad L_1 = \partial_{x^1} - \tfrac{1}{4}Q^{-1}Q_1 - \tfrac{1}{2}C_1, \qquad L_2 = \partial_{x^2} - \tfrac{1}{4}Q^{-1}Q_2 - \tfrac{1}{2}C_2. \tag{2.17}$$

If the transformed equation has two ignorable variables x^1, x^2 then the components of the contravariant metric assume the form $g^{ij} = QG^{ij}(x^3)$. If we take $g^{1/2} = h(x^3)Q^{-3/2}$ then the coefficient b_1 has the form

$$b_1 = Q\big[G^{11}\partial_{x^1}\ln(M^2Q^{-1/2}) + G^{21}\partial_{x^2}\ln(M^2Q^{-1/2}) + G^{31}\partial_{x^3}\ln(M^2Q^{-1/2}) + h^{-1}\partial_{x^3}(hG^{31})\big].$$

From the condition $b_1 = Qf(x^3)$ and the similar conditions $b_2 = Qg(x^3)$, $b_3 = Qi(x^3)$ we deduce that

$$\partial_{x^i}\ln(M^2Q^{-1/2}) = F_i(x^3).$$

Thus,

$$M^2Q^{-1/2} = H(x^3)\exp\big[C_1x^1 + C_2x^2\big]. \tag{2.18}$$

The condition on the constant term and the defining operators are

$$\Delta_3 M = QMF(x^3), \tag{2.19}$$

$$(2)' \quad L_1 = \partial_{x^1} - \tfrac{1}{4}Q^{-1}Q_1 - \tfrac{1}{2}C_1, \qquad L_2 = \partial_{x^2} - \tfrac{1}{4}Q^{-1}Q_2 - \tfrac{1}{2}C_2.$$

The third possibility is that the reduced equation has one ignorable variable x^1. Then we have the following cases:

(a) The separation equations are both second-order in the nonignorable coordinates. The contravariant tensor must assume the form

$$g^{22} = g^{33} = Q, \qquad g^{13} = QI(x^3), \qquad g^{12} = QH(x^2),$$
$$g^{11} = Q\big(S(x^2) + T(x^3)\big), \qquad g^{23} = 0. \tag{2.20}$$

However, introducing a new ignorable variable X^1 such that $dx^1 = dX^1 + Hdx^2 + Idx^3$ we can obtain a new metric for which $H \equiv I \equiv 0$. The condition $b_2 = Qf(x^2)$ becomes

$$\partial_{x^2}\ln\big(M^2/[Q(S+T)]^{1/2}\big) = F(x^2).$$

There is a similar condition arising from the requirement $b_3 = Qg(x^3)$. These conditions together with $b_1 = Q(U(x^2) + V(x^3))$ imply

$$M^2Q^{-1/2} = \exp(C_1x^1)[S+T]^{1/2}\exp\big(j(x^2) + k(x^3)\big) \tag{2.21}$$

and the condition on the constant term is

$$\Delta_3 M = MQ\big[W(x^2) + Z(x^3)\big]. \tag{2.22}$$

Thus,

$$(3)' \quad ds^2 = Q^{-1}(S+T)^{-1}\Big[(dX^1)^2 + (S+T)\big((dx^2)^2 + (dx^3)^2\big)\Big],$$
$$L_1 = \partial_{X^1} - \tfrac{1}{4}Q^{-1}Q_1 - \tfrac{1}{2}C_1,$$
$$L_2 = M\big(\partial_{x^2x^2} + S\partial_{X^1X^1} + C_1S\partial_{X^1} + j'\partial_{x^2} + W\big)M^{-1}. \tag{2.23}$$

(b) If one of the separation equations is of first order and the other of second order then the contravariant metric must have the form

$$g^{33} = g^{12} = Q, \qquad g^{13} = QH(x^3),$$
$$g^{11} = Q\big(P(x^2) + S(x^3)\big), \qquad g^{22} = g^{23} = 0. \tag{2.24}$$

However, the change of ignorable variable $dx^1 = dX^1 + \frac{1}{2} P\, dx^2$ reduces the metric to one with two ignorable variables, a special case of (2)′.

Finally we treat the case where there are no ignorable variables. This corresponds to the most general type of orthogonal coordinates and is well known, e.g., **[11]**. The required conditions are

$$g_{ij} = \frac{S}{M_{i1}} Q\delta_{ij}, \qquad \frac{\sqrt{g}}{QS} = M^{-2} \prod_{l=1}^{3} f_l(x^l) \tag{2.25}$$

where M_{i1} is the indicated cofactor of the Stäckel matrix with determinant S. There is an additional condition

$$\sum_{i=1}^{3} \frac{M_{i1}}{f_i S} \partial_{x^i}\left(\frac{f_i}{M^2} \partial_{x^i} M \right) = \frac{\alpha}{M}, \qquad \alpha \in \mathbf{C}. \tag{2.26}$$

The operators are

$$(4)' \quad L_1 = \frac{M}{S}\left[\left(\frac{\phi_2 q_3 - \phi_3 q_2}{f_1} \right) \partial_{x^1}(f_1 \partial_{x^1}) + \left(\frac{\phi_3 q_1 - \phi_1 q_3}{f_2} \right) \partial_{x^2}(f_2 \partial_{x^2}) \right.$$
$$\left. + \left(\frac{\phi_1 q_2 - \phi_2 q_1}{f_3} \right) \partial_{x^3}(f_3 \partial_{x^3}) \right] M^{-1},$$

$$L_2 = \frac{M}{S}\left[\left(\frac{\phi_3 - \phi_2}{f_1} \right) \partial_{x^1}(f_1 \partial_{x^1}) + \left(\frac{\phi_1 - \phi_3}{f_2} \right) \partial_{x^2}(f_2 \partial_{x^2}) \right.$$
$$\left. + \left(\frac{\phi_2 - \phi_1}{f_3} \right) \partial_{x^3}(f_3 \partial_{x^3}) \right] M^{-1} \tag{2.27}$$

where, without loss of generality, we have chosen the Stäckel matrix in the form

$$\begin{bmatrix} \phi_1 & q_1 & 1 \\ \phi_2 & q_2 & 1 \\ \phi_3 & q_3 & 1 \end{bmatrix}$$

with $\phi_i = \phi_i(x^i)$, $q_i = q_i(x^i)$.

To this point we have investigated R-separation in which the modulation function M depends on the variables x^i alone and not on the separation constants. However, there is considerable freedom in the definition of coordinates x^i such that R-separation is maintained. Consider for example the case of three ignorable variables as treated above. The separable solution has the form

$$\psi = \exp\left[R(x^1, x^2, x^3) + \sum_{i=1}^{3} C_i x^i\right]. \tag{2.28}$$

However, we could also define

$$x^i = \sum_{j=1}^{3} a_j^i X^j + F^i(X^1, X^2, X^3) \tag{2.29}$$

to produce new variables X^i, and the corresponding solution would be

$$\psi = \exp\left[\bar{R}(X^1, X^2, X^3) + \sum_{i=1}^{3} C_i F^i + \sum_{j=1}^{3}\left(\sum_{i=1}^{3} C_i a_j^i\right) X^j\right], \tag{2.30}$$

which again admits R-separation with modulation function $\bar{M} = \exp[\bar{R} + \Sigma_{i=1}^3 C_i F^i]$. This last separation is the most general possible which yields three ignorable variables. However we do not regard this more general separation as essentially different from that given in (2.17) and always choose modulation functions which are independent of the separation parameters. We can argue in much the same way for separable systems with one and two ignorable variables.

This completes our classification of separable coordinates for (1.1)(a). Note that the only possible nonorthogonal separable coordinate systems are from classes (2) and (2)′. However, all systems from these classes correspond to commuting pairs of first-order symmetries. Hence, their existence and classification is purely a group-theoretic phenomenon.

3. Separable systems for the Hamilton-Jacobi equation. Next we give the analogous classification of separable systems for equation (1.1)(b). Recall that R-separation of variables for (1.1)(b) means that $W = R(x^1, x^2, x^3) + \Sigma_{i=1}^3 W_i(x^i)$ where the separation equations for the functions W_i are second degree first-order nonlinear equations. If the fixed function R is zero then (1.1)(b) admits a (pure) separation of variables.

To define the symmetry operators for (1.1)(b) we employ a phase space formalism. The coordinates in this space are (x^j, p_j) where $p_j = \partial_{x^j} W$, $j = 1, 2, 3$. The Poisson bracket of two functions F, G on phase space is the function

$$\{F(x,p), G(x,p)\} = \sum_{j=1}^{3} \left(\partial_{x^j} G \partial_{p_j} F - \partial_{x^j} F \partial_{p_j} G\right). \tag{3.1}$$

A *first-order symmetry* of (1.1)(b) is a function

$$\mathfrak{L} = \sum_{i=1}^{3} \xi_i(x) p_i \tag{3.2}$$

such that $\{\mathfrak{L}, \Sigma_{ij} g^{ij} p_i p_j\} = \rho(x)(\Sigma_{ij} g^{ij} p_i p_j)$ for some analytic function ρ. Note that the $(\xi_i(x))$ are just the conformal Killing vector fields for the metric (g_{ij}).

The first-order symmetries form a Lie algebra $\mathcal{K}$ under the Poisson bracket. Here dim $\mathcal{K} \leq 10$ and the maximum dimension is achieved if and only if (g_{ij}) is conformally flat, in which case $\mathcal{K} \cong o(5)$ **[10]**.

The (strictly) *second-order symmetries* of (1.1)(b) are the functions

$$\mathcal{L}' = \sum_{i,j=1}^{3} \eta_{ij}(x) p_i p_j, \qquad \eta_{ij} = \eta_{ji} \tag{3.3}$$

such that $\{\mathcal{L}', \Sigma g^{ij}p_ip_j\} = \mu(x,p)(\Sigma g^{ij}p_ip_j)$ where μ is a linear function of the p_l. The vector space of second-order symmetries can be decomposed into orbits under the adjoint action of $\mathcal{K}$.

We will show explicitly that every class of separable solutions W of (1.1)(b) is characterized by a pair of first- or second-order symmetries $\mathcal{L}_1$, $\mathcal{L}_2$ which are in involution: $\{\mathcal{L}_1, \mathcal{L}_2\} = 0$. The exact characterization is

$$\mathcal{L}_1 = \lambda_1, \qquad \mathcal{L}_2 = \lambda_2 \tag{3.4}$$

where λ_1, λ_2 are the separation constants **[6]**. In the following we classify the separable systems and list the associated functions $\mathcal{L}_1$, $\mathcal{L}_2$.

The classification is analogous to that of §2 and is straightforward. There are 4 cases:

(1) 3 ignorable variables

$$ds^2 = Q\sum_{i=1}^{3} (dx^i)^2,$$
$$\mathcal{L}_1 = p_1^2, \qquad \mathcal{L}_2 = p_2^2. \tag{3.5}$$

(2) 2 ignorable variables

$$dx^2 = Q \sum_{i,j=1}^{3} G_{ij}(x^3)dx^i dx^j,$$
$$\mathcal{L}_1 = p_1, \qquad \mathcal{L}_2 = p_2. \tag{3.6}$$

(3) 1 ignorable variable

$$dx^2 = Q\left[\left(U(x^2) + V(x^3)\right)\left((dx^2)^2 + (dx^3)^2\right) + (dx^1)^2\right],$$
$$\mathcal{L}_1 = p_1^2, \qquad \mathcal{L}_2 = (U+V)^{-1}\left(Vp_2^2 - Up_3^2\right). \tag{3.7}$$

(4) no ignorable variables

$$dx^2 = Q\left[\frac{(dx^1)^2}{q_2 - q_3} + \frac{(dx^2)^2}{q_3 - q_1} + \frac{(dx^3)^2}{q_1 - q_2}\right],$$

$$\mathfrak{L}_1 = [(q_1 - q_2)(q_1 - q_3)(q_2 - q_3)]^{-1} \times [(q_3^2 - q_2^2)p_1^2 + (q_1^2 - q_3^2)p_2^2 + (q_2^2 - q_1^2)p_3^2],$$

$$\mathfrak{L}_2 = [(q_1 - q_2)(q_1 - q_3)(q_2 - q_3)]^{-1} \times [q_2q_3(q_2 - q_3)p_1^2 + q_1q_3(q_3 - q_1)p_2^2 + q_1q_2(q_1 - q_2)p_3^2],$$

$$q_i = q_i(x^i),\ i = 1, 2, 3. \quad (3.8)$$

We see, furthermore, that if we look for an R-separable solution of the Hamilton-Jacobi equation then the conditions on the contravariant metric imply that the differential forms be of one of the types (1)–(4) listed above. For these types, equation (1.1)(b) is purely separable, so we encounter no strictly R-separable solutions. We can of course produce R-separable solutions by an appropriate redefinition of coordinates.

Note that the only possible nonorthogonal separable coordinates are of type (2), hence they can be classified by group-theoretic methods.

4. Flat space coordinates. We now show that every separable coordinate system for equation (1.1)(b) in flat space admits R-separation for the corresponding Laplace equation (1.1)(a). (The converse of this statement already follows from the results developed thus far.) This will be accomplished by showing that the metrics of types (1)–(4) reduce for flat space to metrics which R-separate (1.1)(a). We first treat types (1), (3) and (4), which metrics correspond to orthogonal coordinates.

The method for finding all such metrics has been developed by Eisenhart [1]. In his article Eisenhart classifies all flat space metrics of the form

$$d\bar{s}^2 = \sum_{i=1}^{3} \bar{H}_i^2 (dx^i)^2 = e^{2\sigma} \sum_{i=1}^{3} H_i^2 (dx^i)^2 \quad (4.1)$$

where the metric

$$ds^2 = \sum_{i=1}^{3} H_i^2 (dx^i)^2 \quad (4.2)$$

is in Stäckel form, i.e., $H_i^2 = S/M_{i1}$, and also obeys the Robertson condition $S = \prod_{i=1}^{3} H_i f_i(x^i)$. We can see by a suitable redefinition of the functions Q that the flat space metrics of types (1), (3) and (4) can be written in the form (4.1). Conversely, each metric (4.1) permits separation in (1.1)(b). Thus Eisenhart's classification can be viewed as a listing of the possible orthogonal separable systems for (1.1)(b). We will check each of the entries on this list to see that it also R-separates (1.1)(a).

First we summarize Eisenhart's results and give the corresponding forms of the function σ. The condition that the metric ds^2 be in Stäckel form is

$$\partial^2_{x^j x^i} \ln H_i^2 - \partial_{x^j} \ln H_i^2 \partial_{x^k} \ln H_i^2 + \partial_{x^j} \ln H_i^2 \partial_{x^k} \ln H_j^2$$
$$+ \partial_{x^i} \ln H_i^2 \partial_{x^j} \ln H_k^2 = 0, \qquad i, j, k \neq, \tag{4.3}$$

and the Robertson condition is equivalent to the requirement $R_{ij} = 0$ $(i \neq j)$ where R_{ij} is the Ricci tensor. From this last requirement and the condition that $d\bar{s}^2$ be a flat space metric we have

$$2\partial^2_{x^j x^k} \ln H_i^2 + \partial_{x^j} \ln H_i^2 \partial_{x^k} \ln H_i^2 - \partial_{x^k} \ln H_i^2 \partial_{x^j} \ln H_k^2 = 0,$$
$$\partial^2_{x^i x^j} \sigma - \partial_{x^i} \sigma \partial_{x^j} \sigma - \tfrac{1}{2} \partial_{x^i} \sigma \partial_{x^j} \ln H_i^2 - \tfrac{1}{2} \partial_{x^j} \sigma \partial_{x^i} \ln H_j^2 = 0,$$
$$i, j, k \neq. \tag{4.4}$$

It follows from equations (4.3), (4.4) that

$$\partial^2_{x^i x^j} \ln H_i^2 = 0,$$
$$\partial_{x^j} \ln H_i^2 \partial_{x^k} \ln H_i^2 - \partial_{x^j} \ln H_i^2 \partial_{x^k} \ln H_j^2 - \partial_{x^k} \ln H_i^2 \partial_{x^j} \ln H_k^2 = 0. \tag{4.5}$$

Thus the H_i^2 satisfy the same conditions as the coefficients of a metric for which the related Helmholtz equation admits a separation of variables. The metrics for which this is so can assume the following normal forms:

1. $H_i^2 = 1, \quad i = 1, 2, 3,$ (4.6)
2. $H_1^2 = 1, \quad H_2^2 = \phi(x^1), \quad H_3^2 = \psi(x^1),$ (4.7)
3. $H_1^2 = 1, \quad H_2^2 = X_2\sigma_1(\sigma_2 + \sigma_3), \quad H_3^2 = X_3\sigma_1(\sigma_2 + \sigma_3),$
   $\sigma_i = \sigma_i(x^i), X_i = X_i(x^i),$ (4.8)
4. $H_1^2 = H_3^2 = \sigma_1 + \sigma_3, \quad H_2^2 = \sigma_1\sigma_3,$ (4.9)
5. $H_i^2 = X_i(x^i - x^j)(x^i - x^k), \quad i, j, k \neq.$ (4.10)

The metric $d\bar{s}^2$ corresponds to flat space if and only if

$$R_{ii,k} - R_{ik,i} - \tfrac{1}{4} g_{ii} R_{,k} = 0 \tag{4.11}$$

where $R_{ij,k}$ denotes the covariant derivative of R_{ij} with respect to x^k. Eisenhart has shown that these equations are equivalent to the conditions

$$\partial_{x^k}(A_{ik} + A_{ki} - A_{jk} - A_{kj}) + (\partial_{x^k} \ln H_i)(A_{ij} + A_{ji} - A_{jk} - A_{kj})$$
$$- (\partial_{x^k} \ln H_j)(A_{ij} + A_{ji} - A_{ik} - A_{ki}) = 0,$$
$$A_{ij} = \frac{1}{H_j^2}\left[2\partial^2_{x^j x^j} \ln H_i^2 + \partial_{x^j} \ln H_i^2 \partial_{x^j} \ln\left[\frac{H_i^2}{H_j^2 H_k^2}\right]\right]. \tag{4.12}$$

From [12, p. 92] and the second of equation (4.4) the conditions for σ are

$$\partial^2_{x^i x^i} \lambda - \tfrac{1}{2} \partial_{x^i} \lambda \partial_{x^i} \ln H_i^2 + \tfrac{1}{2} H_j^{-2} \partial_{x^i} \lambda \partial_{x^j} H_i^2$$
$$+ \tfrac{1}{2} H_k^{-2} \partial_{x^k} \lambda \partial_{x^k} H_i^2 = \lambda\left[R_{ii} - \tfrac{1}{2} H_i^2\left(\tfrac{1}{2} R - \Delta\lambda\right)\right] \tag{4.13}$$

where $\sigma = -\ln\lambda$ and

$$\Delta\lambda = \lambda^{-2}\sum_{i=1}^{3} H_i^{-2}(\partial_{x^i}\lambda)^2,$$

as well as

$$\partial^2_{x^i x^j}\lambda = \tfrac{1}{2}\,\partial_{x^i}\lambda\partial_{x^j}\ln H_i^2 + \tfrac{1}{2}\,\partial_{x^j}\lambda\partial_{x^i}\ln H_j^2. \qquad (4.14)$$

We now evaluate the possible metrics (4.6)–(4.10) and the corresponding function λ.

1. In this case either $\lambda = 1$ or $\lambda = (x^1)^2 + (x^2)^2 + (x^3)^2$, and we have the forms

$$(1)\quad d\bar{s}^2 = (dx^1)^2 + (dx^2)^2 + (dx^3)^2, \qquad (4.14)$$

$$(1)'\quad d\bar{s}^2 = \frac{(dx^1)^2 + (dx^2)^2 + (dx^3)^2}{(x^1)^2 + (x^2)^2 + (x^3)^2}, \qquad (4.15)$$

respectively, where (1) and (1)′ indicate the types of R-separation for the Laplace equation as listed in §2. Here the coordinates are given by $x^1 = \lambda x$, $x^2 = \lambda y$, $x^3 = \lambda z$. (From the point of view of group theory these two systems are conformally equivalent since one is obtained from the other by an inversion, a symmetry of the Laplace equation.)

2. For forms of this type we can assume $\psi(x') = 1$ by suitable redefinition of x^1 and absorption of ψ into $e^{2\sigma}$. The only condition on ϕ then becomes

$$2\left(\frac{\phi'}{\phi}\right)' + \left(\frac{\phi'}{\phi}\right)^2 = a^2, \qquad a \in \mathbf{C}. \qquad (4.16)$$

There are three kinds of solutions to this equation:

$$\text{(a)}\quad \phi = (x^1)^2, \qquad \text{(b)}\quad \phi = e^{2x^1}, \qquad \text{(c)}\quad \phi = \sin^2 x^1. \qquad (4.17)$$

Choice (a) yields $\lambda = 1$ or $\lambda = (x^1)^2 + (x^3)^2$ with corresponding forms

$$(2)\quad d\bar{s}^2 = (dx^1)^2 + (x^1)^2(dx^2)^2 + (dx^3)^2, \qquad (4.18)$$

$$(2)'\quad d\bar{s}^2 = \frac{(dx^1)^2 + (x^1)^2(dx^2)^2 + (dx^3)^2}{\left[(x^1)^2 + (x^3)^2\right]^2}, \qquad (4.19)$$

both of which are known to yield R-separation for the Laplace equation. The connection with Cartesian coordinates is

$$\lambda x = x^1\cos x^2, \qquad \lambda y = x^1 \sin x^2, \qquad \lambda z = x^3.$$

(Again, these two systems are conformally equivalent.)

We can treat cases (b) and (c) simultaneously because in both cases the metric ds^2 can be written in the form $ds^2 = (dx^3)^2 + (dz^1)^2 + (dz^2)^2 + (dz^3)^2$ where $(z^1)^2 + (z^2)^2 + (z^3)^2 = 1$. It is now easy to see that cases (b) and (c) correspond to the two possible distinct subgroup coordinate systems on the two dimensional complex sphere [13].

$$\text{(b)}\quad z^1 = -\frac{i}{2}\left(e^{-ix^1} - \left[1 + (x^2)^2\right]e^{ix^1}\right), \qquad z^2 = x^2 e^{ix^1},$$
$$z^3 = \frac{1}{2}\left(e^{-ix^1} + \left[1 - (x^2)^2\right]e^{ix^1}\right),$$
$$\text{(c)}\quad z^1 = \sin x^1 \sin x^2, \qquad z^2 = \sin x^1 \cos x^2, \qquad z^3 = \cos x^1. \tag{4.20}$$

The function λ corresponding to either choice of z^i above is given by $\lambda = (i \cos x^3 + z^3)$ and the Cartesian coordinates are

$$\lambda x = \sin x^3, \qquad \lambda y = z^1, \qquad \lambda z = z^2.$$

Coordinate systems of this type are known to correspond to R-separable solutions of the Laplace equation for all five choices of separable coordinates on the two-dimensional complex sphere [14]. The systems (b), (c) above are of type (2)′.

3. Without loss of generality we can assume $\sigma_1 = 1$. There is then only the case $H_1^2 = 1$, $H_2^2 = X_2(x^2 - x^3)$, $H_3^2 = X_3(x^3 - x^2)$ to consider, for if σ_2 or σ_3 are constants we are reduced to case 2. Substituting this form of ds^2 into (4.12) we obtain the equations

$$(x^2 - x^3)^2\left(\frac{1}{X_2}\right)'' + 4(x^3 - x^2)\left(\frac{1}{X_2}\right)' + 2(x^3 - x^2)\left(\frac{1}{X_3}\right)' + 6\left(\frac{1}{X_2} - \frac{1}{X_3}\right) = 0,$$
$$(x^3 - x^2)^2\left(\frac{1}{X_3}\right)'' + 4(x^2 - x^3)\left(\frac{1}{X_3}\right)' + 2(x_2 - x_3)\left(\frac{1}{X_2}\right)' + 6\left(\frac{1}{X_3} - \frac{1}{X_2}\right) = 0. \tag{4.21}$$

Differentiating the first equation twice with respect to x^2 we find $(1/X_2)^{(4)} = 0$. Consequently the above conditions reduce to

$$1/X_i = 4(x^i - a)(x^i - b)(x^i - c), \qquad i = 2, 3. \tag{4.22}$$

The coordinates and metrics have exactly the same form as for type 2 above except that now one obtains the three elliptic separable systems on the complex sphere [13], [14] (according as some of a, b, c become equal):

$$\text{(d)}\quad z^1 = \frac{1}{k'}\, dn(x^1, k) dn(x^2, k), \qquad z^2 = \frac{ik}{k'}\, cn(x^1, k) cn(x^2, k),$$
$$z^3 = k\, sn(x^1, k)\, sn(x^2, k),$$

$$\text{(e)}\quad z^1 = \frac{1}{2}\left(\frac{\cosh x^2}{\cosh x^1} + \frac{\cosh x^1}{\cosh x^2}\right), \qquad z^2 = \tanh x^1 \tanh x^2,$$

$$z^3 = \frac{i}{\cosh x^1 \cosh x^3} - \frac{i}{2}\left(\frac{\cosh x^2}{\cosh x^1} + \frac{\cosh x^1}{\cosh x^2}\right),$$

$$\text{(f)}\quad z^1 = \frac{-i}{8x^1x^2}\left\{\left[(x^2)^2 - (x^1)^2\right]^2 + 4\right\}, \qquad z^2 = \frac{(x^2)^2 + (x^1)^2}{2x^1x^2},$$

$$z^3 = \frac{1}{8x^1x^2}\left\{-\left[(x^2)^2 - (x^1)^2\right]^2 + 4\right\}. \tag{4.23}$$

Here, *sn*, *cn*, *dn* are Jacobi elliptic functions [15]. These coordinates are of type (3)′. Note that all five separable systems on the complex sphere have now appeared.

4. Here equations (4.12) impose the single condition

$$\frac{1}{\sigma_1}\left[2\sigma_1''(\sigma_1 - \sigma_3) - \frac{\sigma_1'^2}{\sigma_1}(3\sigma_1 - \sigma_3)\right]$$
$$- \frac{1}{\sigma_2}\left[2\sigma_3''(\sigma_1 - \sigma_3) - \frac{\sigma_3'^2}{\sigma_3}(3\sigma_1 - \sigma_3)\right] = a(\sigma_1 - \sigma_3)^2, \qquad a \in \mathbf{C}. \tag{4.24}$$

Differentiating successively with respect to x^1 and x^3 we obtain the conditions

$$\frac{2\sigma_1''}{\sigma_1} - \frac{\sigma_1'^2}{\sigma_1^2} = 3a\sigma_1^2 + 2b\sigma_1 + 2c,$$

$$\frac{2\sigma_3''}{\sigma_3} - \frac{\sigma_3'^2}{\sigma_3^2} = -3a\sigma_3^2 - 2b\sigma_3 + 2d. \tag{4.25}$$

For consistency we must have $d = -c$ and

$$(\sigma_1')^2 = 4(\sigma_1 - a)(\sigma_1 - b)(\sigma_1 - c)\sigma_1,$$

$$(\sigma_3')^2 = -4(\sigma_3 - a)(\sigma_3 - b)(\sigma_3 - c)\sigma_3. \tag{4.26}$$

Choosing new variables $\hat{x}^1 = \sigma_1$, $\hat{x}^3 = \sigma_3$ we have the metric (dropping the hats)

$$ds^2 = \frac{(x^1 - x^3)}{4}\left[\frac{(dx^1)^2}{(x^1 - a)(x^1 - b)(x^1 - c)x^1} - \frac{(dx^3)^2}{(x^3 - a)(x^3 - b)(x^3 - c)x^3}\right] + x^1x^3(dx^2)^2. \tag{4.27}$$

The coordinate surfaces corresponding to this metric are cyclides of revolution. A typical choice of coordinates for a, b, c not equal is

$$\lambda x = \left[\frac{(x^1 - c)(x^3 - c)}{(b - c)(a - c)c} \right]^{1/2}, \qquad \lambda y = \left[\frac{x^1 x^3}{abc} \right]^{1/2} \cos x^2,$$

$$\lambda z = \left[\frac{x^1 x^3}{abc} \right]^{1/2} \sin x^2 \tag{4.28}$$

where

$$\lambda = \left[\frac{(x^1 - a)(x^3 - a)}{(b - a)(c - a)a} \right]^{1/2} + \left[\frac{-(x^1 - b)(x^3 - b)}{(c - b)(a - b)b} \right]^{1/2}.$$

Such metrics are of type (3)′ and are known to R-separate the Laplace equation **[2]**, **[4]**, **[16]**.

5. The conditions (4.12) reduce to equations of the form

$$\begin{aligned}(x^2 - x^3)^3&\left[\left(\frac{1}{X_1}\right)' - \frac{2}{X_1}\left(\frac{1}{x^1 - x^2} + \frac{1}{x^1 - x^3}\right)\right]\\ &+ (x^3 - x^1)^2\left[\left(\frac{1}{X_2}\right)' - \frac{2}{X_2}\left(\frac{1}{x^2 - x^1} + \frac{1}{x^2 - x^3}\right)\right]\\ &+ (x^1 - x^2)^2\left[\left(\frac{1}{X_3}\right)' - \frac{2}{X_3}\left(\frac{1}{x^3 - x^1} + \frac{1}{x^3 - x^2}\right)\right]\\ &= a(x^1 - x^2)^2(x^2 - x^3)^2(x^3 - x^1)^2, \qquad a \in \mathbf{C}.\end{aligned} \tag{4.29}$$

Dividing by $(x^1 - x^2)^2$ and differentiating with respect to x^1 we get $(1/X_1)^{(5)} = 120a$. These conditions lead to

$$1/X_i = (x^i - e_1)(x^i - e_2)(x^i - e_3)(x^i - e_4)(x^i - e_5), \qquad i = 1, 2, 3. \tag{4.30}$$

This corresponds to the choice of general cyclidic coordinates in three space. These coordinates are known to R-separate the Laplace equation and they are of type (4)′. A suitable choice of three space coordinates for the e_i all different is

$$\lambda x = \left[\frac{(x^1 - e_2)(x^2 - e_2)(x^3 - e_2)}{(e_2 - e_1)(e_2 - e_3)(e_2 - e_4)(e_2 - e_5)} \right]^{1/2},$$

$$\lambda y = \left[\frac{(x^1 - e_3)(x^2 - e_3)(x^3 - e_3)}{(e_3 - e_1)(e_3 - e_2)(e_3 - e_4)(e_3 - e_5)} \right]^{1/2},$$

$$\lambda z = \left[\frac{(x^1 - e_4)(x^2 - e_4)(x^3 - e_4)}{(e_4 - e_1)(e_4 - e_2)(e_4 - e_3)(e_4 - e_5)} \right]^{1/2},$$

$$\lambda = \left[\frac{(x^1 - e_1)(x^2 - e_1)(x^3 - e_1)}{(e_1 - e_2)(e_1 - e_3)(e_1 - e_4)(e_1 - e_5)} \right]^{1/2} + \left[\frac{-(x^1 - e_5)(x^2 - e_5)(x^3 - e_5)}{(e_5 - e_1)(e_5 - e_2)(e^5 - e_3)(e^5 - e_4)} \right]^{1/2}. \tag{4.31}$$

The corresponding coordinates in the cases when some of the e_i are equal can be found in [2], [16].

This completes our list of orthogonal R-separable coordinates for the flat-space equations (1.1). It follows immediately that (1.1)(a) and (1.1)(b) separate in exactly the same coordinate systems.

In [22], Levinson, Bogert and Redheffer already classified R-separable systems for the Laplace equation in real Euclidean three-space. However, in addition to restricting themselves to real coordinates, thus omitting all systems associated with the wave equation, these authors adopted a rather restrictive definition of separation of variables in which they required that the separated variables could be split off from the Laplace equation one at a time. Thus their classification omits all of the most general and complicated systems, i.e., the possible ellipsoidal, paraboloidal and cyclidic types. The definition of variable separation adopted in the present paper is the usual one and, to the authors' knowledge, the computations of this section constitute the first demonstrably exhaustive classification of orthogonal R-separable coordinates for the flat-space Laplace equation.

5. Nonorthogonal coordinates in flat space. We have classified all separable systems for equations (1.1) in flat space except the nonorthogonal systems of types (2) and (2)′. All systems of these types are characterized in terms of commuting pairs of symmetry operators from the symmetry algebra $o(5)$. By regarding two separable systems as equivalent if one can be obtained from the other by a conformal symmetry transformation we can reduce the classification of such systems to the determination of all orbits of two-dimensional subspaces of commuting symmetries in $o(5)$ under the adjoint action of $o(5)$. To be specific we first consider the complex Laplace equation

$$(\partial_{xx} + \partial_{yy} + \partial_{zz})\psi = 0. \tag{5.1}$$

Recall that the symmetry algebra of (5.1) is ten-dimensional with basis [4]

$$\begin{aligned}
&J_1 = z\partial_y - y\partial_z, \quad J_2 = x\partial_z - z\partial_x, \quad J_3 = y\partial_x - x\partial_y,\\
&P_1 = \partial_x, \quad P_2 = \partial_y, \quad P_3 = \partial_z, \quad D = -\left(\tfrac{1}{2} + x\partial_x + y\partial_y + z\partial_z\right),\\
&K_1 = x + (x^2 - y^2 - z^2)\partial_x + 2xy\partial_y + 2xz\partial_z,\\
&K_2 = y + (y^2 - x^2 - z^2)\partial_y + 2yx\partial_x + 2yz\partial_z,\\
&K_3 = z + (z^2 - x^2 - y^2)\partial_y + 2zx\partial_x + 2zy\partial_y.
\end{aligned} \tag{5.2}$$

This algebra is isomorphic to the Lie algebra $o(5)$ of all 5×5 complex skew-symmetric matrices. Indeed, a basis for $o(5)$ is provided by the matrices $\Gamma_{ij} = \mathcal{E}_{ij} - \mathcal{E}_{ji} = -\Gamma_{ji}$, $1 \leq i < j \leq 5$, where $\mathcal{E}_{ij}$ is the matrix with the entry 1 in row i, column j, and 0 everywhere else. The identifications

$$\begin{aligned} &J_1 = \Gamma_{43}, \quad J_2 = \Gamma_{24}, \quad J_3 = \Gamma_{32} \quad D = -i\Gamma_{15}, \\ &P_1 = \Gamma_{12} - i\Gamma_{25}, \quad P_2 = \Gamma_{13} - i\Gamma_{35}, \quad P_3 = \Gamma_{14} - i\Gamma_{45}, \\ &K_1 = \Gamma_{12} + i\Gamma_{25}, \quad K_2 = \Gamma_{13} + i\Gamma_{35}, \quad K_3 = \Gamma_{14} + i\Gamma_{45} \end{aligned} \tag{5.3}$$

determine the isomorphism. In addition, the inversion I and reflection R are symmetries of (5.1) which do not belong to the connected component of the identity of the corresponding conformal local Lie symmetry group of (5.1):

$$I\psi(\mathbf{x}) = \frac{1}{\sqrt{\mathbf{x} \cdot \mathbf{x}}}\, \psi(\mathbf{x}/\mathbf{x} \cdot \mathbf{x}), \qquad R\psi(\mathbf{x}) = \psi(-x, y, z). \tag{5.4}$$

We first determine the orbits of symmetry operators Γ under the adjoint action of $o(5)$, i.e., the canonical forms under the similarity transformation $\Gamma \to O\Gamma O^{-1}$, $O \in O(5, \mathbb{C})$. This classification is straightforward and we list only the results. Our classification is in terms of the possible eigenvalues of the matrices $\Phi \in o(5)$. For each such matrix we list a canonical form $\Gamma \in o(5)$ such that $\Gamma = O\Phi O^{-1}$ for some $O \in O(5)$, the complex orthogonal group. It is easy to show that $\lambda = 0$ is always an eigenvalue of Φ and, if $\lambda \neq 0$ is an eigenvalue, then so is $-\lambda$. We use the notation $\lambda(n)$, $n = 2, 3, 5$, to signify that λ corresponds to a generalized eigenvector u of rank n, i.e., n is the smallest integer m such that $(\Phi - \lambda E)^m u = 0$ where E is the 5×5 identity matrix.

TABLE 1. One-dimensional subalgebras of $o(5)$

	possible eigenvalues	*canonical form*
1.	$\pm\alpha, \pm\beta, 0\ (\alpha \neq 0)$	$i\alpha\Gamma_{12} + i\beta\Gamma_{34}$
2.	$\pm\alpha, 0, 0, 0$	$i\alpha\Gamma_{12}$
3.	$\alpha(2), -\alpha(2), 0$	$i\alpha(\Gamma_{12} - \Gamma_{34}) + \frac{1}{2}(i\Gamma_{13} - \Gamma_{14} - \Gamma_{23} - i\Gamma_{24})$
4.	$\pm\alpha, 0(3)$	$i\alpha\Gamma_{12} + (i\Gamma_{35} - \Gamma_{34})/\sqrt{2}$
5.	$0(5)$	$\frac{1}{2}\left(\Gamma_{14} + i\Gamma_{15} - i\Gamma_{24} + \Gamma_{25} - \sqrt{2}\,\Gamma_{34} + i\sqrt{2}\,\Gamma_{35}\right)$

From this table we can determine the conjugacy classes of two-dimensional abelian subalgebras of $o(5)$. Let $\mathcal{V}$ be such a subalgebra. By performing a conjugacy transformation if necessary, we can assume that $\Gamma \in \mathcal{V}$ where Γ is one of the five operators listed in Table 1. Let $\mathcal{G}_\Gamma$ be the centralizer of Γ in $o(5)$. We determine the $\mathcal{G}_\Gamma$-conjugacy classes of one-dimensional subalgebras of $\mathcal{G}_\Gamma$. Choosing a representative Γ_i from each such class we obtain the distinct abelian subalgebras $\{\Gamma, \Gamma_i\}$. Repeating this computation for each Γ on Table 1 and eliminating duplications we obtain Table 2 in which we have listed a representative from each of the six conjugacy classes of two-dimensional abelian subalgebras.

TABLE 2. Two-dimensional abelian subalgebras of $o(5)$

	coordinates	*canonical basis*	*eigenvalues*
1.	Cartesian	$i\Gamma_{35} - \Gamma_{34}$	0(3), 0, 0
		$i\Gamma_{14} + \Gamma_{15}$	0(3), 0, 0
2.	Cartesian′	$i\Gamma_{35} - \Gamma_{34}$	0(3), 0, 0
		$\Gamma_{15} - \Gamma_{24} + i(\Gamma_{14} + \Gamma_{25})$	0(2), 0(2), 0
3.	cylindrical	Γ_{12}	$\pm i$, 0, 0, 0
		$i\Gamma_{35} - \Gamma_{34}$	0(3), 0, 0
4.	spherical	Γ_{12}	$\pm i$, 0, 0, 0
		Γ_{34}	$\pm i$, 0, 0, 0
5.	oscillator	$\Gamma_{12} - \Gamma_{34}$	$\pm i$, $\pm i$, 0
		$i(\Gamma_{13} - \Gamma_{24}) - \Gamma_{14} - \Gamma_{23}$	0(2), 0(2), 0
6.	linear	$i(\Gamma_{14} - \Gamma_{25}) + \Gamma_{15} + \Gamma_{24}$,	0(2), 0(2), 0
		$\frac{1}{\sqrt{2}}(\Gamma_{14} + i\Gamma_{15} - i\Gamma_{24} + \Gamma_{25}$ $\sqrt{2}\,\Gamma_{34} + i\sqrt{2}\,\Gamma_{35})$	0(5)

The separable coordinate systems associated with these abelian subalgebras are termed *split* [6]. We first discuss the orthogonal split systems, all of which have been obtained in §4.

The standard form of the Cartesian coordinates is (4.14)′ and the basis operators for this form are P_1, P_2. The standard form of the cylindrical coordinates is (4.18) and the basis operators are J_3, P_3. Coordinates (4.19) and (4.20)(b) are conformal equivalent to cylindrical coordinates.

The standard form for spherical coordinates is

$$x = x^3 \sin x^1 \cos x^2, \qquad y = x^3 \sin x^1 \sin x^2, \qquad z = x^3 \cos x^1 \tag{5.5}$$

with basis operators J_3, D. However, the toroidal system (4.20)(c) with operators J_1, $K_1 - P_1$ is conformal equivalent to the spherical system. This completes the list of orthogonal coordinates of types (2) and (2)′.

The remaining three systems are nonorthogonal. They correspond to the imbedding of the heat equation into the Laplace equation via the change of coordinates

$$\begin{gathered} x = x^3, \qquad y - iz = x^1, \qquad y + iz = 2x^2, \\ ds^2 = (dx^3)^2 + 2dx^1dx^2, \\ \Delta_3\psi = (\partial_{x^3x^3} + 2\partial_{x^1x^2})\psi = 0. \end{gathered} \tag{5.6}$$

Diagonalization of the operator $L_1 = \frac{1}{2}(P_2 + iP_3) = \partial_{x^1}$ reduces the Laplace equation to

$$(\partial_{x^3x^3} + \lambda\partial_{x^2})\Theta = 0. \tag{5.7}$$

The first type of nonorthogonal coordinates can be characterized by the operators $P_2 + iP_3, J_2 + iJ_3$ and coordinates

$$x = x^2x^3, \qquad y = x^1 + \frac{i}{2}(x^2)^2x^3, \qquad z = ix^1 + \frac{1}{2}(x^2)^2x^3 - x^3,$$
$$ds^2 = (x^3)^2(dx^2)^2 + (dx^3)^2 - 2idx^1dx^3. \tag{5.8}$$

However, a transformation similar to that for (2.13) reduces this system to Cartesian coordinates. Thus we term the system Cartesian′.

System 5 can be characterized by the operators $P_2 + iP_3$, $iJ_1/2 - D$, and coordinates

$$x = x^3\sqrt{x^2}, \qquad y - iz = x^1 - \tfrac{1}{4}(x^3)^2, \qquad y + iz = 2x^2,$$
$$ds^2 = x^2(dx^3)^2 + 2dx^1dx^2 + \frac{1}{4}\frac{(x^3)^2}{x^2}(dx^2)^2. \tag{5.9}$$

We call these *oscillator* coordinates because they transform (5.7) to the time-dependent (complex) Schrödinger equation for the harmonic oscillator [17].

System 6 can be characterized by the operators $P_2 + iP_3$, $K_2 + iK_3 - 8P_1$ and coordinates

$$x = x^3x^2 + 2/x^2, \qquad y - iz = x^1 - \tfrac{1}{2}(x^3)^2x^2 + 2x^3/x^2 + 2/3(x^2)^3,$$
$$y + iz = 2x^2,$$
$$ds^2 = (x^2)^2(dx^3)^2 + 2dx^1dx^2 - \frac{8x^3}{(x^2)^2}(dx^2)^2. \tag{5.10}$$

We call these *linear* coordinates because they transform (5.7) to the time-dependent (complex) Schrödinger equation with a linear potential [17].

We see that the oscillator and linear systems are the only nonorthogonal R-separable coordinate systems for equation (5.1). In [7] it was shown that the flat space Helmholtz equation $\Delta_3\psi = E\psi$ separates in exactly five nonorthogonal systems, three of which are nonsplit. It is straightforward to show that when $E = 0$ these systems all become split. Furthermore, two of the systems are conformal equivalent to oscillator coordinates, two to linear coordinates, and one to (5.8).

The Lie algebraic characterization of the corresponding separable systems for

$$W_x^2 + W_y^2 + W_z^2 = 0$$

is explained in detail in [6].

6. Separable systems for conformally flat spaces. In [1] Eisenhart showed how to find all orthogonal separable systems for the equation $\Delta_3\psi = E\psi$, $E \neq 0$, where g_{ij} is a conformally flat metric. The construction is essentially the same as that employed in §4 to find orthogonal separable systems for the Laplace equation in flat space. Here we describe a method for determining explicitly all R-separable systems for $\Delta_3\psi = 0$ in a conformally flat space C_3.

In such a space we can always find a coordinate system $\{x, y, z\}$ such that [10]

$$ds^2 = Q^2(x, y, z)(dx^2 + dy^2 + dz^2). \tag{6.1}$$

The function Q is determined up to a conformal transformation generated by the operators (5.2) and (5.4). Thus the Laplace equation takes the form

$$\Delta_3\psi = Q^{-3}\left[\partial_x(Q\psi_x) + \partial_y(Q\psi_y) + \partial_z(Q\psi_z)\right] = 0. \tag{6.2}$$

Let $\mathcal{G}$ be the symmetry algebra of (6.2), consisting of symmetry operators L, (2.1). Setting $\psi = f\Theta$ where $f = Q^{-1/2}$ we transform (6.2) to

$$\Theta_{xx} + \Theta_{yy} + \Theta_{zz} - \Phi\Theta = 0 \tag{6.3}$$

where

$$\Phi = Q^{-1/2}(\partial_{xx} + \partial_{yy} + \partial_{zz})Q^{1/2}. \tag{6.4}$$

The symmetry algebra of (6.3) is isomorphic to $\mathcal{G}$ but now the symmetry operators L' take the form $L' = L + Q^{1/2}(LQ^{-1/2})$.

The symmetry group of equation (6.3) has been studied by Ovsjannicov [18]. When $\Phi \equiv 0$, i.e., when $Q^{1/2}$ is harmonic, then (6.3) admits the symmetry algebra $o(5)$ and (6.1) is equivalent to (5.1). If $\Phi \not\equiv 0$ the symmetry algebra is of dimension six or less. If the dimension is six then $\mathcal{G}$ is isomorphic to $\mathfrak{E}(3)$, the Lie algebra of the complex Euclidean group, or to $o(4)$. In the first case Q can be chosen so that $\Phi \equiv E \neq 0$ and (6.3) becomes the flat space Helmholtz equation

$$\Theta_{xx} + \Theta_{yy} + \Theta_{zz} = E\Theta. \tag{6.5}$$

The separable coordinate systems for this equation have been discussed from a group-theoretic viewpoint in [7]. If $\mathcal{G} \cong o(4)$ then Q can be chosen so that $\Phi = E/(1 + r^2)^2$ where $E \neq 0$ and $r^2 = x^2 + y^2 + z^2$. Then (6.3) is equivalent to the Laplace-Beltrami eigenvalue equation for a space of nonzero constant curvature. The separable coordinate systems for this equation are determined in [7] and [13]. In all other cases $\dim \mathcal{G} < 6$.

For *real* conformally flat spaces with positive definite metric one is led to equation (6.3) for x, y, z real. Eisenhart [19] has determined all choices of Φ

in this case for which (6.3) admits *pure* separable solutions in some coordinate system. (It is easy to see that in this special case the only possible separable systems are the eleven orthogonal systems for the real flat space Helmholtz equation. For each of these eleven systems one then need only determine the possible functions Φ which permit variable separation.) In [20] each of the separable systems is characterized by a pair of second-order commuting symmetry operators for equation (6.3), see also [21]. In each case the operators take the form $L_i' = L_i + \rho_i$, $i = 1, 2$, where the L_i are commuting second-order symmetry operators in the enveloping algebra of $\mathfrak{E}(3, R)$, generated by the operators $P_1, P_2, P_3, J_1, J_2, J_3$, (5.2) and the ρ_i are scalar functions. For purely separable solutions of the complex equations (6.3) one could obtain a similar characterization by employing the results of [2] and [4] but this computation has not been carried out.

More generally, every R-separable system for (6.3) also R-separates (5.1). Thus, to find all R-separable systems for (6.3) one need only take each of the R-separable systems for (5.1) as classified in this paper and determine all functions Φ which still permit R-separation. The resulting systems will be characterized by a pair of second-order commuting operators $L_i' = L_i + \rho_i$, $i = 1, 2$, where the L_i belong to the enveloping algebra of $o(5)$ and the ρ_i are scalar-valued functions.

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