THE COHOMOLOGY OF THE SYMMETRIC GROUPS

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Abstract. Let $S_n$ be the symmetric group on $n$ letters and $SG$ the limit of the sets of degree +1 homotopy equivalences of the $n - 1$ sphere. Let $p$ be an odd prime. The main results of this paper are the calculations of $H^*(S_n, \mathbb{Z}/p)$ and $H^*(SG, \mathbb{Z}/p)$ as algebras, determination of the action of the Steenrod algebra, $\mathcal{A}(p)$, on $H^*(S_n, \mathbb{Z}/p)$ and $H^*(SG, \mathbb{Z}/p)$ and integral analysis of $H^*(S_n, \mathbb{Z}, p)$ and $H^*(SG, \mathbb{Z}, p)$.

0. Introduction. Let $K$ and $L$ be discrete groups with $L$ abelian. The groups $H^*(K, L)$ have been of interest for years. [12] and [11] first considered these cohomology groups algebraically and their relation with topological problems. The algebraic groups $H^*(K, L)$ are isomorphic to $H^*(BK, L)$ where $BK$ is the topological classifying space for the group $K$.

Suppose $K$ is $S_n$, the symmetric group on $n$ letters. Then $H^*(S_n, L)$ is especially important. In the 1950's, work on cohomology operations, [29] and [30], showed the necessity for knowledge of $H^*(S_p, \mathbb{Z}/p)$. The construction of the mod $p$ Steenrod operations depends on properties of $S_p$. Furthermore the Adem relations were derived using the structure of $H^*(S_p, \mathbb{Z}/p)$.

If $L$ is a ring then $H^*(K, L)$ is a graded ring. The homology of symmetric products, [9], [17], [20], [21], and [28], computed the groups $H^*(S_n, \mathbb{Z}/p)$ as $\mathbb{Z}/p$ vector spaces. The graded ring structure, which was not analyzed, becomes important in later problems.

There is an interesting link that ties $S_n$ to $SG$. Recall $Q(S^0) = \text{dir lim } S^n$ is the space of “infinite loops of $S^\infty$” and $SG = \text{dir lim } SG_n$ where $SG_n$ is the space of degree +1 homotopy equivalences of $S^{n-1}$. $SG$ is homotopy equivalent to the +1 component of $Q(S^0)$.

Theorem. (1) There is a canonical map $\omega: B_{S^n} = \text{dir lim } B_{S_n} \to Q(S^0)$ inducing integral and mod $p$ homology isomorphisms.

(2) The inclusions $S_n \times S_m \to S_{n+m}$ give $H_*(S_\infty)$ the structure of an algebra. $\omega_*$ is an algebra isomorphism and a Hopf algebra isomorphism mod $p$ where $H_*(Q(S^0))$ is an algebra under the loop sum product.

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The above theorem is contained in the work of many people including [10], [16], [22], [24], [25].

Thus $B_{S_p}$ properly interpreted is a model for $SG$.

In all that follows let $p$ be an odd prime. We will write $H^*(K)$ for $H^*(K, Z/p)$. $H^*(K, Z, p)$ is, by definition, [5], the $p$-primary component of $H^*(K, Z)$. In [4] the algebra structure of $H^*(S_p)$ is computed but the arguments do not generalize to $S_p$; $i > 3$. The main results of this paper are the calculations of $H^*(S_n)$ and $H^*(SG)$ as algebras, determination of the action of the Steenrod algebra, $\mathcal{A}(p)$, on $H^*(S_n)$ and $H^*(SG)$ and integral analysis of $H^*(S_n, Z, p)$ and $H^*(SG, Z, p)$.

This paper is essentially my Stanford University Ph. D. thesis written under the direction of R. James Milgram, whom I would like to thank for his advice and encouragement. I would also like to thank the referee for his numerous helpful comments including shorter proofs for two of the propositions in §II. In addition after submission of this paper I learned that Benjamin Cooper [35] and Hùynh Mùi [36] have also studied $H^*(S_p)$.

I. Statement of results. It is well known that a $p$-Sylow subgroup $K_p$ of a finite group $K$ contains all the $p$-primary homology information; more precisely, $H^*(K)$ and $H^*(K, Z, p)$ are isomorphic to subrings of $H^*(K_p)$ and $H^*(K_p, Z, p)$ respectively, which are invariant under the action of certain automorphisms. It is also well known, [6], that a $p$-Sylow subgroup of $S_p$, is isomorphic to $\wr^iZ/p$, the $i$-fold wreath product of $Z/p$. In the next section we examine a specific embedding of $\wr^iZ/p$ in $S_p$ and show the existence of an $H^*( )$ detecting family consisting of subgroups of the form $\times^mZ/p$. In fact we have the following subgroups and natural inclusions: $k_{j,i}: T_{j,i} \to S_p$, for $1 < j < i$ and the map $k^*_{j,i} = \prod_{j=1}^{i} k_{j,i}^*: H^*(S_p) \to \prod_{j=1}^{i} H^*(T_{j,i})$, where $T_{j,i} = \times^{p^{i-j}}(\times^{i}Z/p)$.

The first theorems compute the images of $k^*_{j,i}$'s and the map $k^*_{i}$. We show that $k^*_{j,i}$ detects a set of multiplicative generators for $H^*(S_p)$ whose relations are trivial to compute. Hence the map $k^*_{i}$ determines $H^*(S_p)$. Later for simplicity we will want to identify $u \in H^*(S_p)$ with its natural image $k^*_{j,i}(u) \in H^*(T_{j,i})$ but we must wait until Theorems A–D have been stated to avoid possible confusion.

Recall $H^*(\times^kZ/p) = E(e_1, \ldots, e_k) \otimes P(b_1, \ldots, b_k)$ with degree $(e_m) = 1$, degree $(b_m) = 2$ for all $m$. Furthermore $\beta_p(e_m) = b_m$, where $\beta_p$ is the Bockstein operator associated with the exact coefficient sequence $0 \to Z/p \to Z/p^2 \to Z/p \to 0$.

Consider the following classes in $H^*(\times^kZ/p)$: (a matrix cohomology class will always mean the cohomology class given by the formal determinant of that matrix)
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$L_i = \begin{vmatrix}
    b_1^{p-r} & \cdots & b_1^{p-i} \\
    \vdots & \ddots & \vdots \\
    b_1 & \cdots & b_1 \\
\end{vmatrix}$

i.e. the $k, j$ entry of $L_i$

is $b_j^p$ ($0 \leq r \leq i - 1$).

$L_i = \begin{vmatrix}
    b_1^{p-s} & \cdots & b_1^{p-i} \\
    \vdots & \ddots & \vdots \\
    b_1 & \cdots & b_1 \\
\end{vmatrix}$

i.e. the $b_j^{p'}$ row of the numerator is omitted

$(1 \leq j \leq i - 1)$.

$Q_{j,i} = \frac{L_i}{L_i}$

$= \begin{vmatrix}
    b_1^{p-s} & \cdots & b_1^{p-i} \\
    \vdots & \ddots & \vdots \\
    b_1 & \cdots & b_1 \\
\end{vmatrix}$

i.e. $L_i$ is the $L_i$ determinant

with the $b_1 \cdots b_i$ row replaced by the row $e_1 \cdots e_i$.

$M_{j,i} = \begin{vmatrix}
    b_1^{p-s} & \cdots & b_1^{p-i} \\
    \vdots & \ddots & \vdots \\
    b_1 & \cdots & b_1 \\
\end{vmatrix}$

i.e. the $b_j^{p'}$ row is omitted ($1 \leq j \leq i - 1$).

**Note.** (i) If $i = 1$ then $L_1 = b_1$ and $L_1 = e_1$ are the only two classes defined.

(ii) [19] proved $Q_{j,i}$ is integral, not merely rational, mod $p$. See appendix for proof.

$S_p$, can be thought of as the permutations of the point set $\Pi l Z/p$. Let $k_{i,i}$:

$T_{i,i} = \times^{i} Z/p \rightarrow \{\text{permutations of } \Pi l Z/p\}$ be defined by: $k_{i,i}(a_1, \ldots, a_i)$ sends $(b_1, \ldots, b_i)$ to $(a_1 + b_1, \ldots, a_i + b_i)$ where $Z/p$ is written additively.

Then $k_{i,i}$ is seen to be equivalent to the adjoint representation (2.5) and
includes $T_{i,d}$ in $S_p$. The normalizer $N$ of $k_{i,d}(T_{i,d})$ in $S_p'$ maps onto $\text{GL}(i, \mathbb{Z}/p)$ (2.10) and induces an action on $H^*(T_{i,d})$ as follows. If $\cup x$ in $\text{GL}(i, \mathbb{Z}/p)$ represents the coset $\times T_{i,d}$ in $N$ then the homomorphism $\text{ad}_{\times}: H^*(T_{i,d}) \rightarrow H^*(T_{i,d})$ operates as follows: $\text{ad}_{\times}(e_m) = \cup e_m$, $\text{ad}_{\times}(b_m) = \cup b_m$ where $e_m$, $b_m$ are treated as the vectors $(0, \ldots, e, \ldots, 0)$ and $(0, \ldots, b, \ldots, 0)$ in $H^*(T_{i,d})$ with nonzero entries in the $m$th place. Hence $\text{ad}_{\times}$ operates on the above determinant classes via the determinant function; that is, $\text{ad}_{\times}(L_i) = \det(\cup x L_i)$. By 2.13 image $k^*_{i,d}$ is contained in $H^*(T_{i,d})^{\text{GL}(i, \mathbb{Z}/p)}$.

Let $\mathcal{W}_1$ be the algebra $E(L_1 L_i L_i^{-2}) \otimes P(L_i^{-1})$. For $i$ greater than 1 let $\mathcal{W}_i$ be the subalgebra of $H^*(T_{i,d})$ generated by: $1$, $L_i^{-1}$, $Q_{j,i}$, $L_i L_i^{-2}$, $M_{j,i} L_i L_i^{-2}$, $M_{j,i} M_{h,i} L_i L_i^{-3}$, $M_{j,i} M_{h,i} L_i L_i^{-3}$, with $1 < j$, $h < i - 1$ and $j < h$. $\mathcal{W}_i$ is contained in $H^*(T_{i,d})^{\text{GL}(i, \mathbb{Z}/p)}$ (2.12). Then $\mathcal{W}_i$ contains the polynomial algebra $P(L_i^{-1}, Q_{1,i}, Q_{2,i}, \ldots, Q_{i-1,i})$ and all other generators of $\mathcal{W}_i$ are exterior. However the algebra they generate is not an exterior subalgebra as there are zero products. The multiplication of these exterior products is determined by the relations:

1. $L_i^2 = M_j^2 = 0$, $1 < j < i - 1$,
2. $L_i^2 M_{1,i} M_{2,i} \cdots M_{i-1,i} \neq 0$.

For example $(M_{2,i} M_{3,i} L_i L_i^{-2})(M_{2,i} M_{3,i} L_i L_i^{-3}) = 0$.

**Theorem A.** Image $k^*_{i,d} = \mathcal{W}_i$.

**Examples.** (i) If $i = 1$ then $0 \rightarrow H^*(S_p) \rightarrow k^* : H^*(\mathbb{Z}/p)$ where $H^*(\mathbb{Z}/p) = E(L_1) \otimes P(L_1)$ and $H^*(S_p) = E(L_1 L_i L_i^{-2}) \otimes P(L_i^{-1})$. (ii) If $i = 2$ the results of [4] are obtained. (iii) Let $p = 3$, $i = 3$ then $k^*_{3,3}: H^*(S_{27}) \rightarrow H^*(\mathbb{Z}/3 \times \mathbb{Z}/3 \times \mathbb{Z}/3)$ and image $k^*_{3,3}$ is generated by:

1. Polynomial generators

\[ L_3^2 = \begin{bmatrix} b_1^9 & b_2^9 & b_3^9 \\ b_1^6 & b_2^6 & b_3^6 \\ b_1^3 & b_2^3 & b_3^3 \end{bmatrix}, \quad Q_{1,3} = \begin{bmatrix} b_1^{27} & b_2^{27} & b_3^{27} \\ b_1^9 & b_2^9 & b_3^9 \\ b_1 & b_2 & b_3 \end{bmatrix}, \]

\[ Q_{2,3} = \begin{bmatrix} b_1^{27} & b_2^{27} & b_3^{27} \\ b_1^3 & b_2^3 & b_3^3 \\ b_1 & b_2 & b_3 \end{bmatrix}. \]

2. Exterior generators
M_{1,3}M_{2,3} = \begin{vmatrix} b_1 & b_2 & b_3 \\ b_1 & b_2 & b_3 \\ b_1 & b_2 & b_3 \end{vmatrix},
M_{1,3}L_3, M_{1,3}L_3, M_{2,3}L_3, M_{2,3}L_3, L_3L_3.

(3) The relations that any product of exterior generators is zero except
(a) \((M_{1,3}M_{2,3})(L_3L_3) = -(M_{1,3}L_3)(M_{2,3}L_3) = (M_{2,3}L_3)(M_{1,3}L_3),
(b) \((M_{1,3}L_3)(L_3L_3) = (M_{1,3}L_3)L_3,
(c) \((M_{2,3}L_3)(L_3L_3) = (M_{2,3}L_3)L_3,
(d) \((M_{1,3}L_3)(M_{2,3}L_3) = (M_{1,3}M_{2,3})L_3.

The proof of Theorem A depends, in part, on [17] and a counting argument. As noted above the classes in image \( k^*_{i,i} \) are \( \text{GL}(i, Z/p) \) invariant. A calculation and [8] show \( P(b_1, \ldots, b_i)^{\text{GL}(i,Z/p)} \) is isomorphic to the polynomial subalgebra of image \( k^*_{i,i} \). For \( i = 2 \), [4] shows

\( (E(e_1, e_2) \otimes P(b_1, b_2))^{\text{GL}(2,Z/p)} \cong H^*(Z/p \times Z/p)^{\text{GL}(2,Z/p)} \cong k^*_{2,2}. \)

If \( p > 5, i > 3 \) then \( (E(e_1, \ldots, e_i) \otimes P(b_1, \ldots, b_i))^{\text{GL}(i,Z/p)} \) properly contains image \( k^*_{i,i} \); for example, \( M_1, M_2, L_3, L_3^{-4} \) is not in image \( k^*_{i,i} \). For \( p = 3, i > 3 \), it is unknown if image \( k^*_{i,i} \) equals the ring of invariants.

Consider the inclusion \( \times_{m=1}^{p-1} (S_p')_m \rightarrow \times_{m=1}^{i-1} (S_p')_m \) where \( (S_p')_m \) permutes the \( p^i-1 \) letters \((m - 1)p^i-1 + 1, \ldots, mp^i-1\). Then let \( k_{i-1,j}^*: T_{i-1,j} \rightarrow (S_p')_m \) be the composition \( I_{i-1}(\times_{m=1}^{p-1} (S_p')_m) \). More generally let \( k_{i,j}^*: T_{i,j} \rightarrow (S_p')_m \) be the composition \( I_j(\times_{m=1}^{p-1} (S_p')_m) \) where \( I_j \) is the inclusion \( \times_{m=1}^{p-1} (S_p')_m \rightarrow (S_p')_m \) given by letting \( (S_p')_m \) permute the \( m \)-th block of \( p^j \) letters.

Let \( 1 < j < i \), then \( (S_p')_m \) operates on \( T_{i,j} \) and on the algebra \( \otimes_{m=1}^{p^i-1} ((\times_j^i)_{m}) \) contained in \( H^*(T_{i,j}) \cong \otimes_{m=1}^{p^i-1} (H^*(XZ/p))_m \) by permuting the \( p^j \) copies of \( XZ/p \).

**Theorem B.** For \( 1 < j < i \) image \( k^*_{i,j} \) is isomorphic to the algebra of \( (S_p')_m \) invariant classes of \( \otimes_{m=1}^{p^i-1} ((\times_j^i)_{m}) \).

**Notation.** Let \( u_m \in ((\times_j^i)_{m}) \) then \( S \langle u_1, u_2, \ldots, u_{p^i-1} \rangle \) is the \( (S_p')_m \) invariant class generated by \( u_1 \cdot u_2^2 \cdot \cdots \cdot u_{p^i-1} \) \((u_m \) is allowed to be \( 1 \in H^0(XZ/p) \)). If \( u_1 \) is odd dimensional then \( S \langle u_1, u_2, \ldots, u_{p^i-1} \rangle = 0. \)

**Examples.** (i) image \( k^*_{1,1} \) is generated by:

\[ A_{k,i} = \sum_{m=1}^{p^i-1} (L_1 L_1^{(p^i - 2) + k(p^i - 1)})_m \]

\[ = S \langle (L_1 L_1^{(p^i - 2) + k(p^i - 1)}), 1, \ldots, 1 \rangle, \text{ for } 0 < k < p^i-1 - 1, \]

and

\[ B_{k,i} = \sum (L_1^{p^i-1})_{m_1} (L_1^{p^i-1})_{m_2} \cdots (L_1^{p^i-1})_{m_k} \]

where \( 1 < k < p^i-1 \) and the sum runs over all sequences \( 1 < m_1 < m_2 \).
< • • • < mk < p'•-

Thus \( B_{k,d} = S \langle L_{p}^{-1}, L_{p-1}^{-1}, \ldots, L_{p-1}^{-1}, 1, \ldots, 1 \rangle \)

where \( L_{p}^{-1} \) appears \( k \) times.

(ii) Let \( p = 3 \), then \( k_{2,3}^{*} : H^{*}(\mathbb{S}_{27}) \to H^{*}(T_{23}) \) and image \( k_{2,3}^{*} \) is generated by:

\[
\begin{align*}
S \langle \text{ext}, 1, 1 \rangle & \quad S \langle \text{poly}, 1, 1 \rangle & \quad S \langle \text{ext}, \text{poly}, 1 \rangle \\
S \langle M_{1,2} L_{2}, M_{1,2} L_{2}, M_{1,2} L_{2} \rangle & \quad S \langle \text{ext}, \text{poly}, \text{poly} \rangle \\
S \langle \text{poly}, \text{poly}, 1 \rangle & \quad S \langle \text{poly}, \text{poly}, \text{poly} \rangle & \quad S \langle \text{ext}, \text{ext}, \text{poly} \rangle
\end{align*}
\]

where

(a) \text{ext} runs through \( M_{1,2} L_{2}, M_{1,2} L_{2}, \) and \( L_{2} L_{2}. \)
(b) \text{poly} runs through \( L_{2} \) and \( Q_{1,2}. \)
(c) As \( M_{1,2} L_{2} \) and \( L_{2} L_{2} \) are odd dimensional neither can appear twice in

any \( S \langle -, -, - \rangle. \) For example \( S \langle L_{2} L_{2}, L_{2} L_{2}, 1 \rangle = 0. \) Note that \( S \langle M_{1,2} L_{2}, 1, 1 \rangle \) has height 3 while \( S \langle M_{1,2} L_{2}, 1, 1 \rangle \) is exterior.

(iii) In image \( k_{s}^{*} \), the classes

\[
S \langle M_{1,2} L_{2} L_{p}^{-3}, 1, \ldots, 1 \rangle
\]

and

\[
S \langle (M_{1,2} L_{2} L_{p}^{-3})_{1}, \ldots, (M_{1,2} L_{2} L_{p}^{-3})_{p}, 1, \ldots, 1 \rangle
\]

have height \( p \) while \( S \langle M_{1,2} L_{2} L_{p}^{-3}, \ldots, M_{1,2} L_{2} L_{p}^{-3} \rangle \) is exterior. This pattern generalizes to image \( k_{s}^{*} \), \( 3 < j < i - 1, \) in the obvious way.

**Note.** Example (iii) shows how all even dimension exterior generators in \( \mathfrak{W}_{j} \) build classes in \( H^{*}(T_{j,i}) \) which are the images under \( k_{j,i}^{*} \) of classes \( u \in H^{*}(\mathbb{S}_{n}) \) where each \( u \) generates a truncated polynomial algebra of height \( p \) in \( H^{*}(\mathbb{S}_{n}). \) These are the truncated polynomial algebras described in [22].

Let \( u \in H^{*}(\mathbb{S}_{p}) \) then \( k_{j,i}^{*}(u) = (k_{j,i}^{*}(u), \ldots, k_{j,i}^{*}(u)) \) and the algebra structure restricted to these detecting groups is compatible with component-wise projection. Clearly to calculate \( H^{*}(\mathbb{S}_{p}) \) we must know when a class \( u \in H^{*}(\mathbb{S}_{p}) \) has nontrivial image under more than one \( k_{j,i}^{*}. \)

**Definition.** \( u \in H^{*}(\mathbb{S}_{p}) \) is a multiple image class if and only if \( k_{j,i}^{*}(u) \neq 0 \)

for at least two different values of \( j. \)

Given \( u_{1}, u_{2} \in H^{*}(\mathbb{S}_{p}) \) with \( u_{1} \) detected only by \( k_{j_{1},i}^{*} \) and \( u_{2} \) detected only by \( k_{j_{2},i}^{*} \), with \( j_{1} \neq j_{2} \) then \( u_{1} + u_{2} \) is a multiple image class. However this type of multiple image class is decomposable as a sum of classes and thus is a “trivial” multiple image class. The next three definitions and following theorem give all “nontrivial”; i.e., indecomposable, multiple image classes.

**Definition.** \( \mathfrak{W}_{j} \) is the subalgebra contained in \( \mathfrak{W}_{i} \) generated by \( 1, M_{g,i}, L_{p}^{-1}, L_{h,i}, 1 < g, h < i - 1, g < h. \)

**Definition.** Given \( x_{m_{j}} \in \mathfrak{W}_{j} \) we define \( x_{m_{j},-1} \in \mathfrak{W}_{j-1} \) as follows:

(a) If \( x_{m_{j}} = 1 \) then \( x_{m_{j},-1} = 1. \)
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(b) If \( x_{m,j} = Q_{h,j} \) then \( x_{m,j-1} = Q_{h-1,j-1} \), for \( 2 \leq j \leq i \) and \( 1 \leq h \leq j - 1 \) with the convention \( Q_{0,j} = L_{j}^{-1} \).
(c) If \( x_{m,j} = M_{g,j} M_{h,j} L_{j}^{-3} \) then \( x_{m,j-1} = -M_{g,j-1} M_{h,j-1} L_{j}^{-3} \), for \( 3 \leq j \leq i, 0 \leq g, h \leq j \) and \( g \leq h \) with the convention \( M_{g,j} = L_{j}^{-1} \).
(d) If \( x_{m,j} = x_{m,j}^r x_{m,j}^r \) then \( x_{m,j-1} = x_{m-1,j-1}^r x_{m-1,j-1} \).

Note. (a) through (d) define a unique class \( x_{m,j} \) for every \( x_{m,j} \in \mathbb{M}_j \).

**Definition.** \( u \in H^* (S_n) \) is sum indecomposable if and only if \( u = u_1 + u_2 \) for \( u_1, u_2 \in H^* (S_n) \).

**Theorem C.** Suppose \( u \in H^* (S_n) \) is both sum indecomposable and a multiple image class. Further suppose \( j \) is the largest integer such that \( k_j^*(u) \neq 0 \). Then

\[
k_j^*(u) = \langle x_{1,j}, \ldots, x_{p^{-1},j} \rangle
\]

with \( x_{m,j} \in \mathbb{M}_j \) for \( 1 \leq m \leq p^{-1,j} \), and

\[
k_{j-1}^*(u) = \langle x_{1,j-1}, \ldots, x_{1,j-1}, \ldots, x_{p^{-1,j-1}}, \ldots, x_{p^{-1,j-1}} \rangle
\]

where each \( x_{m,j-1} \) is as defined above and appears \( p \) times in \( k_{j-1}^*(u) \). If \( j - 1 \geq 2 \) and each \( x_{m,j-1} \in \mathbb{M}_{j-1} \) (not just \( \mathbb{M}_j \)) then \( k_{j-2}^*(u) \neq 0 \) and may be obtained from \( k_{j-1}^*(u) \) precisely as \( k_{j-1}^*(u) \) was obtained from \( k_{j}^*(u) \). In fact this iteration continues \( r \) times until either \( j - r = 2 \) or \( x_{m,j-r} \in \mathbb{M}_{j-r} \) when \( k_{j-(r+1)}^*(u) = 0 \) for all \( t > 0 \). Thus \( u \) has \( r + 1 \) nontrivial images in the detecting groups: \( k_{j-x}^*(u) \) for \( 0 \leq s < r \).

**Example.** For \( H^* (S_{27}, \mathbb{Z}/3) \) the only sum-indecomposable multiple image classes of \( k_3^* \) occurring as generators in the examples after Theorems A and B are:

\[
\begin{align*}
(B_9, & (Q_{1,2})(Q_{1,2}) (Q_{1,2}), Q_{2,3}), \\
(0, & (L_{2}^2)(L_{2}^2)(L_{2}^2), Q_{1,3}), \\
(0, & (M_{1,2}L_{2}^2)(M_{1,2}L_{2}^2)(M_{1,2}L_{2}^2), -M_{1,3}M_{2,3}), \\
(B_3, & \langle Q_{1,2}, 1, 1 \rangle, 0), \\
(B_6, & \langle Q_{1,2}, Q_{1,2} \rangle, 0).
\end{align*}
\]

Consider \( u_1, u_2 \in H^* (S_p) \) where \( k_3^*(u_1) = \langle L_{1}^p, 1, \ldots, 1 \rangle, 0, 0 \rangle, k_3^*(u_2) = \langle 0, \langle L_{3}^p, 1, \ldots, 1 \rangle, 0 \rangle, 0 \rangle, \) Then \( k_3^*(u_1 u_2) = 0 \) but in fact \( u_1 u_2 \neq 0 \) in \( H^* (S_p) \) and \( u_1 u_2 \) is detected by subgroups of the form \( T_1 \times T_2 \times \cdots \times T_p \) where \( T_n = T_{1,2} \) or \( T_{2,2} \) and both \( T_{1,2} \) and \( T_{2,2} \) must occur at least once. These detecting groups are included in \( \mathbb{S}_p \) through \( \times^p (S_p) \). More generally a nonsymmetric detecting group, \( \times_{n=1}^p (T_{r_{n,1},r_{n,2}}) \) of \( \mathbb{S}_p \) is a product of detecting groups of \( \mathbb{S}_p \) included in \( \mathbb{S}_p \) through \( \times^p (S_{p^{-1}}) \) where \( T_{r_{n,1},r_{n,2}} \neq T_{r_{n,2},r_{n,1}} \) for some \( r_{1,1}, r_{2,1}, s_1, s_2 \). These nonsymmetric detecting groups detect all classes \( u \in H^* (S_p) \) not detected by the map \( k_1^* \) as stated in Theorem D.

First we need
Definition. Let \( u \in H^*(\tilde{S}_p) \) and \( n < p^i \). Then we have the natural inclusion \( I_{p^i,n} : \tilde{S}_n \to \tilde{S}_p \). We say \( u \) restricts nonzero to \( \tilde{S}_n \) if and only if \( I_{p^i,n}^*(u) \neq 0 \). For notational convenience we write \( u \) for both the class in \( H^*(\tilde{S}_p) \) and the restriction in \( H^*(\tilde{S}_n) \).

Theorem D. (1) The classes in \( H^*(\tilde{S}_p) \) not detected by \( k_i^* \) are products of classes that are detected by \( k_i^* \).

(2) Let \( u_m \in H^*(\tilde{S}_p) \). Suppose \( k_i^*(u_m) \neq 0 \), \( \prod_{m=1}^{m} k_i^*(u_m) = 0 \) and let \( n_m \) be the smallest power of \( p \) such that \( u_m \) restricts nonzero to \( H^*(\tilde{S}_{n_m}) \). Then \( \prod_{m=1}^{m} n_m \neq 0 \) in \( H^*(\tilde{S}_p) \) unless:

(a) \( u_m = u_m \) is an odd dimensional exterior class in \( H^*(\tilde{S}_{n_m}) \), for some \( 1 < m_1 < m_2 < m_3 \).

(b) \( u_m = u_m = \cdots = u_m \) is an even dimensional exterior class in \( H^*(\tilde{S}_{n_m}) \) for some \( 1 < m_1 < m_2 < \cdots < m_p < r \) or \( \prod_{m=1}^{m} n_m \) is not contained in \( \tilde{S}_p \).

Note. The classes \( u_m \) appearing in condition (b) are the generators for the truncated polynomial algebras described in example (iii) after Theorem B.

Thus every \( u \in H^*(\tilde{S}_p) \) is expressible as a sum of monomials \( \Sigma a(u_1, \ldots, u_r)u_1 \otimes \cdots \otimes u_r \) where \( a(u_1, \ldots, u_r) \in Z/p \), \( u_i \in H^*(\tilde{S}_p) \) with \( k_i(u_i) \neq 0 \) for all \( i \).

Definition. \( u \in H^*(\tilde{S}_p) \) is proper if and only if \( u = \Sigma a(u_1, \ldots, u_r)u_1 \otimes \cdots \otimes u_r \) with \( k_i^*(u_1 \otimes \cdots \otimes u_r) \neq 0 \) for each monomial in the sum.

Thus Theorems A through D compute \( H^*(\tilde{S}_p) \) and from this point on we will identify elements of \( H^*(\tilde{S}_p) \) with their image under \( k_i^* \). That is \( L_i^{-1}Q_{i,i} \in H^*(\tilde{S}_p) \) is the unique proper class \( u \in H^*(\tilde{S}_p) \) such that \( k_i^*(u) = (0, \ldots, 0, L_i^{-1}Q_{i,i}) \). Care must be taken with multiple image classes under this identification. Notice, by Theorem C, that \( Q_{i,i} \in H^*(\tilde{S}_p) \) is the unique proper class \( u \in H^*(\tilde{S}_p) \) such that \( k_i^*(u) = (0, \ldots, 0, \Sigma \langle L_i^{-1}, \ldots, L_i^{-1} \rangle, Q_{i,i}) \).

Since

\[
\mathcal{P}(b^k) = \begin{cases} 
  b^k & \text{if } j = 0, \\
  b^{k+1} & \text{if } j = p, \\
  0 & \text{otherwise},
\end{cases}
\]

it is easy to determine the action of the Steenrod algebra \( \mathcal{O}(p) \) on \( H^*(\tilde{S}_p) \).

Consider \( M_{1,3}L_3 \) in \( H^{47}(\tilde{S}_{27}, Z/3) \). Then

\[
\mathcal{P} \left[ \begin{array}{cccc}
  b_1^9 & b_2^9 & b_3^9 \\
  b_1 & b_2 & b_3 \\
  e_1 & e_2 & e_3 & b_1 & b_2 & b_3
\end{array} \right] = \left[ \begin{array}{ccc}
  b_1^9 & b_2^9 & b_3^9 \\
  b_1 & b_2 & b_3 \\
  e_1 & e_2 & e_3 & b_1 & b_2 & b_3
\end{array} \right] = L_3L_3.
\]
This computation involved use of the Cartan formula; however, all terms except the first are zero. The next theorem describes the \( \mathfrak{a} (p) \) action on \( \mathfrak{W}_i \). Note the polynomial subalgebra of \( \mathfrak{W}_i \) is closed under the \( \mathfrak{a} (p) \) action while a class in the ideal generated by the exterior generators of \( \mathfrak{W}_i \) may be “bocksteined” into the polynomial algebra; e.g., \( \beta \mathfrak{P}^1 (M_{i,d} L_i^{p-2}) = L_i^{p-1} \) for \( i > 1 \). Using the Cartan formula and the following theorem it is trivial to compute the \( \mathfrak{a} (p) \) action on all the detecting groups.

**Theorem E.** The following relations and the Cartan formula describe the \( \mathfrak{a} (p) \) action on \( \mathfrak{W}_i \).

1. \( \mathfrak{P}^q (M_{j,d}^h L_i^{p-3}) = M_{j,d}^h M_{h-1,i} L_i^{p-3}, \) \( j > h \) and \( M_{0,i} = L_i \),
2. \( \mathfrak{P}^q (M_{j,d}^h L_i^{p-3}) = M_{j-1,i} L_i^{p-3}, \) \( j > h \) and \( M_{0,i} = L_i \),
3. \( \beta (L_i) = L_i \),
4. \( \mathfrak{P}^q (Q_h) = Q_{h-1}, \) with \( Q_{0,i} = L_i^{p-1} \),
5. \( \mathfrak{P}^q (L_i^{p-1}) = - Q_{i-1,i} L_i^{p-1} \) for \( i > 1 \) while \( \mathfrak{P}^j (L_i^{p-1}) = (p-1) L_i^{p-1-j} \) for \( j < p-1 \),
6. \( \mathfrak{P}^q (M_{i-1,i} L_i^{p-2}) = (p-2)(M_{i-1,i} L_i^{p-2})(Q_{i-1,i}) \\
\mathfrak{P}^q (M_{i-1,i} L_i^{p-2}) = (p-2)(M_{i-1,i} L_i^{p-2})(Q_{i-1,i}).

The following diagram is conceptually helpful.

**Examples.** (i) Consider \( A = (0, S \langle M_{1,d} L_2, M_{2,d} L_2, M_{1,d} L_2 \rangle, -M_{2,d} M_{1,d} \rangle \) in \( H^{30}(S_2, Z/3) \). Then

\[ \mathfrak{P}^1 (A) = (0, -S \langle L_2 L_2, M_{1,d} L_2, M_{2,d} L_2 \rangle, 0) \]

while

\[ \beta \mathfrak{P}^1 (A) = (0, 0, M_{2,d} L_3). \]
Let $n$ be an arbitrary integer. Then $n$ may be written uniquely as follows: $n = \sum_{j=0}^{i} a_j p^j$ with $0 \leq a_j < p - 1$, $a_i \neq 0$. A $p$-Sylow subgroup $K_p$ of $S_n$ is isomorphic to

$$K_p = \mathbb{X} \left( \mathbb{X} \left( \mathbb{X} \cdots \mathbb{X} \left( \mathbb{X} \left( \mathbb{X} \mathbb{X} \cdots \mathbb{X} \mathbb{X} \right) \cdots \mathbb{X} \mathbb{X} \right) \cdots \mathbb{X} \mathbb{X} \right) \cdots \mathbb{X} \mathbb{X} \right) \mathbb{X} \left( \mathbb{X} \mathbb{X} \cdots \mathbb{X} \mathbb{X} \right) \cdots \mathbb{X} \mathbb{X} \right).$$

To compute $H^*(\mathcal{S}_n)$ consider the following diagram of inclusions

$\xymatrix{ & \mathcal{S}_n \ar[dr]^{I_{p^{i+1},n}} \ar[dl]_{J_n} & \\
\mathcal{S}_{p^{i+1}} & \mathcal{S}_{p^{i+1}} \ar[ur]_{I_{2p^{i+1},2p^i}} & & \mathcal{S}_{p^{i+1}} \ar[ll]_{i+1 \text{ wt } Z/p} \\
\mathcal{S}_{p^i} \times \mathcal{S}_{p^i} \ar[ur]_{i \text{ wt } Z/p \times \text{ wt } Z/p} & & \mathcal{S}_{p^i} \ar[ll]_{i+1 \text{ wt } Z/p} & & \mathcal{S}_{p^i} \ar[ll]_{i \text{ wt } Z/p} & & \mathcal{S}_{p^i} \ar[ll]_{i+1 \text{ wt } Z/p}}$

**Theorem F.** (1) $I_{p^{i+1},n}^*$ is surjective.

(2) $J_n^*$ is injective.

(3) $v \in \text{Image } J_n^*$ if and only if there exists a $u \in H^*(\mathcal{S}_{p^{i+1}})$ such that

$$(I_{p^{i+1},n} \circ J_n)^*(u) = v$$

$$= \sum \mathcal{S} \langle u_{i,1}, \ldots, u_{i,a_i} \rangle \otimes \cdots \otimes \mathcal{S} \langle u_{1,1}, \ldots, u_{a_1,1} \rangle \in H^*(\mathcal{K}_p)$$

with $u_{i,r} \in H^*(\mathcal{S}_{p^i})$ for each $r$.

**Important Example.** Let $n = 2p^i$. We have

$\xymatrix{ & \mathcal{S}_{2p^i} \ar[dr]^{I_{2p^{i+1},2p^i}} & \\
\mathcal{S}_{p^i} \times \mathcal{S}_{p^i} \ar[ur]_{i \text{ wt } Z/p \times \text{ wt } Z/p} & & \mathcal{S}_{p^i} \ar[ll]_{i+1 \text{ wt } Z/p} & & \mathcal{S}_{p^i} \ar[ll]_{i \text{ wt } Z/p} & & \mathcal{S}_{p^i} \ar[ll]_{i+1 \text{ wt } Z/p}}$
$p^i < k < p^{i+1}$, \( I^*_{p^{i+1}}(A_{k,i+1}) = A'_{i,k} \otimes 1 + 1 \otimes A'_{k,i} \) where \( A'_{k,i} \) is expressible in terms of \( A_{r,i} \) and \( B_{r,i} \) for \( r < p^i \).

\[
I^*_{p^{i+2p}}(B_{k,i+1}) = \sum_{n=m=k} B_{n,i} \otimes B_{m,i} = \sum_{n=0}^{p^i} S \langle B_{n,i}, B_{2p^i-n,i} \rangle,
\]

where \( 0 < n, m < p^i, 0 \leq k < 2p^i \), and \( B_{0,i} = 1 \). Similar restrictions occur on the other detecting groups. Thus the natural inclusions \( S_n \to S_{n+1} \to \cdots \to \dir lim S_n \) are easily analyzed. Clearly

\[
S_{p^i} \to S_{p^{i+1}} \to \cdots \to \dir lim S_p,
\]

is a cofinal direct limit and we have \( H^*(\dir lim S_n) \cong H^*(\dir lim S_p) \cong \inv lim H^*(S_p) \). Notice Theorem F implies \( \inv lim H^*(S_p) \) is attained for each \( t \) at a finite stage.

Recall the theorem stated in the introduction that \( \dir lim S_i \to Q(S^0) = \dir lim Q(S^i) \). Furthermore, if \( G_n \) is the set of homotopy equivalences of \( S^n \) then \( G = \dir lim G_n \) is homotopy equivalent to the union of the +1 and -1 components of \( Q(S^0) \). Thus \( \dir lim B_{S_n} \) properly interpreted is a model for \( G \) and we have:

\[
\inv lim H^*(S_p) \cong H^*(Q(S^0)_0) \cong H^*(SG)
\]
as algebras. Thus \( H^*(SG) \) can be identified with “infinite symmetric sums” in the \( S_i \) algebras with the proper identifications; i.e., \( S \langle Q_{j_0}, 1, \ldots \rangle \leftrightarrow S \langle Q_{j_0-1, j_1, \ldots}, Q_{j_1-1, \ldots}, 1, \ldots \rangle \). The \( \mathbb{Z}(p) \) action on \( H^*(SG) \) restricts to that on \( B_{S_p} \) for each \( i \) and there is a unique action which has this property. Theorem E describes the restriction of this action. Recall, [22] and [24], \( H^*(\dir lim S_p) \) is a Hopf algebra isomorphic to \( H^*(Q(S^0)_0) \) with the coalgebra product on \( H^*(\dir lim S_p) \) induced by the inclusions \( S_{p^i} \times S_{p^i} \to S_{2p^i} \). Thus Theorem F gives the loop sum coalgebra map on \( H^*(Q(S^0)_0) \).

As \( Q(S^0)_0 \) is an \( H \)-space it is possible to obtain integral information about \( H^*(SG, Z, p) \) on \( H^*(S_p, Z, p) \) (see [14]). [2] gives a Hopf algebra Bockstein spectral sequence with

\[
E_1 \cong H^*(\dir lim S_{p^i}, Z/p),
\]
\[
E_\infty \cong H^*(\dir lim S_{p^i}, Z, p)/\text{Tor}.
\]

Let \( x, y \in S_{p^i} \) and let

\[
L_{n,j}(x; y_n+1, \ldots, y_m, 1, \ldots) = S \langle xL_{j}^{p-1}, \ldots, xL_{j}^{p-1}, y_{n+1}, \ldots, y_m, 1, \ldots \rangle
\]

and

\[
L_{n,j}(x; y_n+1, \ldots, y_m, 1, \ldots) = S \langle xL_{j}^{p-2}, xL_{j}^{p-1}, \ldots, xL_{j}^{p-1}, y_{n+1}, \ldots, y_m, 1, \ldots \rangle
\]

where \( y_r \neq xL_{j}^{p-1} \) or \( xL_{j}^{p-2} \). Note a class in \( H^*(\dir lim S_p) \) may have
more than one representation as \( L_{nj}(\cdots) \) or \( L_{n\mu j}(\cdots) \); for example,
\[
\langle xL_j^{f-1}, xL_j^{f-1}, yL_j^{f-1}, 1, \ldots \rangle = L_{2j}(x: y, 1, \ldots) = L_{1j}(y: x, 1, \ldots).
\]

**Theorem G.** Let \( k_{j,\infty}^* = \lim \text{dir lim} k_{j,\mu}^* \) and let \( u \in H^*(\text{dir lim} S_{p^j}) \) be a proper class. Then there exists a smallest positive integer \( j \) such that \( k_{j,\infty}^*(u) \neq 0 \). Then
\[
k_{j,\infty}^*(u) = \langle x_1, \ldots, x_m, 1, \ldots \rangle \quad \text{and}
\]

(1) If some \( x_n \) contains an odd number of \( M_g \) factors or if \( k_{j,\infty}^*(u) = L_{nj}(\cdots) \) or \( L_{n\mu j}(\cdots) \) for \( n \) not divisible by \( p \) then \( u \) is in the image or domain of \( \beta_p \).

(2) Let \( r > 2 \). If \( d_{r-1}(v) = u \) in \( E_{r-1} \) of the Bockstein spectral sequence and \( k_{j,\infty}^*(u) = \langle x_1, \ldots, x_m, 1, \ldots \rangle \) with no \( x_n \) containing an odd number of \( M_g \) terms or the factor \( L_j \) then there exist \( v' \) and \( u' \) such that \( d_r(v') = u' \) where \( k_{j,\infty}^*(u') = \langle x_1, \ldots, x_1, \ldots, x_m, \ldots, x_m, 1, \ldots \rangle + \Sigma u^\prime \). Each \( x_h \) appears \( p \) times in \( \langle x_1, \ldots, x_1, \ldots, x_m, \ldots, x_m, 1, \ldots \rangle \) and each \( u^\prime = \langle x_1, \ldots, x_1, 1, \ldots \rangle \) with \( t < pm \).

**Corollary 1.** Let \( r > 2 \) then
\[
d_r(L_{p^{r-1}}(x: 1, \ldots)) = L_{p^{r-1}}(x: 1, \ldots)
\]
where \( x \) satisfies the same conditions as the \( x_n \)'s in (2) of Theorem G.

Let \( R_p \) be the inclusion \( S_{p^j} \to \text{dir lim} S_{p^j} \) then \( R_p^* \) gives the Bockstein structure of \( H^*(S_{p^j}, Z, p) \).

**Corollary 2.** \( Q_{j, i} \in H^*(S_{p^j}, Z, p) \) has order \( p^{i+1} \).

**Examples.** (i) \( L_{p^r}(M_{1, j}M_{2, j}L_j^{p-3}: 1, \ldots) \) is a class of order \( p \) in \( H^*(S\Sigma, Z, p) \), while \( L_{p^r}(M_{1, j}M_{2, j}L_j^{p-3}: 1, \ldots) \) is a class of order \( p^{r+1} \).

(ii) \( (B_6, S \langle Q_{1, 2}, Q_{1, 2}, 1 \rangle, 0) \in H^{24}(S_{27}, Z, 3) \) has order 9.

Finally the results of this paper have an application to cobordism theory. Although [3], [13] and [18] completely compute the PL and TOP cobordism ring at the prime 2, the odd case still has unanswered questions, notably the odd torsion in \( \Omega^{PL} \). Using results of [3], [15], [26], [27], [32], [34], [37], [38], [39] and this paper one may calculate the \( E^2 \) term of the Adams spectral sequence converging to \( \Omega^{PL} \otimes Z_p \). Current joint work with H. Ligaard, J. P. May and R. J. Milgram computes this \( E^2 \) term and gives infinite families of nontrivial differentials of all orders in the spectral sequence.

**II. The embedding and the detecting family.**

2.1. **Definition.** Let \( K \) be a finite group and \( L \) a subgroup of \( S_n \) then \( K \wr L \) is defined to be the group whose elements are
\[
\{(f, g): f \text{ is a mapping of } (1, 2, \ldots, n) \text{ into } K, g \in L\}
\]
and whose multiplication is given by \( (f, g)(f', g') = (ff'_g, gg') \), where \( f'_g(g(i)) = f(i) \) and \( ff'(i) = f(i)f'(i) \).
2.2. Definition. Let $X$ be a space and $\{A_i\}$ a collection of subspaces of $X$. $\{A_i\}$ is a $Z/p$ cohomology detecting family for $X$ if the inclusion map $H^*(X) \to \amalg H^*(A_i)$ is an injection.

2.3. Lemma. Let $K_p$ be a $p$-Sylow subgroup of $K$, then the transfer $t(K, K_p): H^*(K_p) \to H^*(K)$ is an epimorphism and the inclusion $i(K_p, K): H^*(K) \to H^*(K_p)$ is a monomorphism whose image consists of stable elements of $H^*(K_p)$. Furthermore we have the direct sum decomposition $H^*(K_p) \cong \text{Im } i(K_p, K) \oplus \text{Ker } t(K, K_p)$.

Proof. See [5, Chapter XII, p. 257] for the definition of stable and p. 259 for a proof of the lemma.

Recalling that a $p$-Sylow subgroup of $S_p$, is isomorphic to $wr^iZ/p$, [6] gives

2.4. Corollary. If $\{A_j\}$ is a $Z/p$ detecting family for $wr^iZ/p$ then it is one for $S_p$, also.

2.5. Definition. Let $G$ be a finite group of order $n$. Then the adjoint representation $A: G \to S_n$ is defined as follows: Let $A(g)$ be the permutation $\{g_i \mapsto g_{g_i}\}$ where $S_n$ is thought of as the permutations on the $n$ elements of $G$.

The adjoint representation is obviously a monomorphism and includes $G$ in $S_n$. Let $G = \times'Z/p$, then the adjoint representation of $\times'Z/p$ in $S_p'$ is clearly equivalent to the map $\tilde{k}_{i,l}: \times'Z/p \to S_p'$ defined in §1. (The two maps differ by at most a reordering of the elements of $\times'Z/p$; that is, an inner automorphism of $S_p'$.)

Again considering $S_p'$ as the permutations on the set $\amalg Z/p$ the map $I_{l-1}: \times'_{m=1}(S_{p^{l-1}})_m \to S_p'$ defined in the introduction is realized by letting $(S_{p^{l-1}})_m$ permute the set $\amalg_{l-1}Z/p \times \{m\}$ contained in $\amalg Z/p$.

Note that under the specific embeddings $k_{i,l}$ and $I_{l-1}$ the subgroup $\times'_{m=1}(S_{p^{l-1}})_m \to \times_{l-1}Z/p \to k_{i,l}S_p'$, is contained in the subgroup $\times'_{m=1}(S_{p^{l-1}})_m \to \times_{l-1}S_p'$. Any $p$-Sylow subgroup of $\times'_{m=1}(S_{p^{l-1}})_m$ that contains $\times_{l-1}Z/p \times \{0\}$ is isomorphic to $\times'_{m=1}(wr^{l-1}Z/p)_m$. Then $\times'_{m=1}(wr^{l-1}Z/p)_m$ and $\times Z/p$ generate a $p$-Sylow subgroup of $S_p'$, which must be isomorphic to $wr^iZ/p$. Thus we have the following commutative diagram with the above mentioned inclusions:
where \( \tilde{k}_{i-1,i} = \prod_{m=1}^{p-1}(k_{i-1,i-1})_m. \) The specific form of \( k_{i,j} \) and \( \tilde{k}_{i-1,i} \) guarantees \( \prod_{m=1}^{p-1}(\times^{i-1}(Z/p)_m \times \times (Z/p)_m \times \times ) \) factors through \( \times \prod_{m=1}^{p-1}(w^{i-1}(Z/p)_m. \)

More generally if \( I_{m_1, \ldots, m_i} : \mathbb{S}_{p_{m_1}} \times \times \times \mathbb{S}_{p_{m_i}} \rightarrow \mathbb{S}_{p^i} \) is defined by letting \( \mathbb{S}_{p_{m}} \) permute the \( p_{m_i} \) letters \( (p_{m_1} + \times \times \times + p_{m_{i-1}} + 1, \ldots, p_{m_i} + \times \times \times + p_{m_i}) \) then the map \( I_{m_1, \ldots, m_i} \circ (\prod_{r=1}^{i-1}k_{m_r,m_i}) \) includes \( \prod_{r=1}^{i-1}(\times (Z/p)) \) in \( \mathbb{S}_{p^i} \).

If \( m_1 = m_2 = \times \times = m_{p-1} = j \) then \( \prod_{r=1}^{i-1}(\times (Z/p)) \rightarrow \mathbb{S}_{p^i} \), has the form

\[
k_{j,i} = I_{j,i, \ldots, j} \prod_{r=1}^{p-1} (k_{j,i})_r \times (\times (Z/p)) \rightarrow \mathbb{S}_{p^i}.
\]

2.6. DEFINITION. Let \( T_{j,i} = \times (\times (Z/p)). \) Let \( k_{j,i} : T_{j,i} \rightarrow \mathbb{S}_{p^i} \) be the above inclusion. Then \( T_{j,i} \) is called a totally symmetric detecting group.

Notice \( T_{j,i} \) and \( k_{j,i} \) are defined for \( 1 < j < i \). The following lemmas are established in the proofs of Theorems A through D:

2.7. LEMMA. The set \( \{ I_{m_1, \ldots, m_i} \circ (\prod_{r=1}^{i-1}(k_{m_r,m_i})) : \prod_{r=1}^{i-1}(Z/p) \rightarrow \mathbb{S}_{p^i} \} \)

forms a \( p \)-detecting family for \( \mathbb{S}_{p^i} \).

2.8. LEMMA. The totally symmetric detecting groups \( T_{j,i}, 1 < j < i \), detect a set of multiplicative generators for \( H^*(\mathbb{S}_{p^i}). \) (This is the first part of Theorem D.)

2.9. LEMMA. In \( Z/p \) cohomology, \( \ker k_{j,i} \cap \ker I_{j,i-1} = 0. \)

These lemmas may be proved directly using [27], induction on \( i \), and 3.1.

We now examine the normalizers of the detecting subgroups in \( \mathbb{S}_{p^i}. \)

Consider \( k_{j,i} : T_{j,i} \rightarrow \mathbb{S}_{p^i}. \) Let \( a_i \in \mathbb{S}_{p^i} \), generate \( k_{j,i}(0 \times 0 \times \times (Z/p)_r \times \times \times ) = 0 \) and let \( N_i \) be the normalizer of \( k_{j,i}(T_{j,i}) \) in \( \mathbb{S}_{p^i}. \) Define a homomorphism \( \psi : N_i \rightarrow GL(i, Z/p) \) as follows: If \( x \in N_i \) then \( xa_{r}x^{-1} = a_1^{a_2^{a_3^{a_4^{a_5^{a_6}}}}} \). Then let \( \psi(x) \) be the matrix whose \( (m, n) \)th entry is \( s_{m,n}. \) Clearly \( \psi(x) \) is nonsingular.

2.10. PROPOSITION. The sequence \( 1 \rightarrow k_{i,j}(T_{j,i}) \rightarrow N_i \rightarrow \mathbb{S}_{p^i} \rightarrow 1 \) is exact.

PROOF. Preceding \( k_{i,j} \) by any automorphism \( \varphi : T_{i,j} \rightarrow T_{j,i} \) is just a reordering of the underlying set of \( T_{i,j}. \) This reordering, considered as an element of \( \mathbb{S}_{p^i}, \) conjugates \( k_{i,j} \) to \( k_{j,i} \circ \varphi. \) This implies \( \psi \) is onto. The remainder of the proposition follows trivially.

For \( x \in \mathbb{S}_{p^i} \), the homomorphism \( \text{ad}_x : H^*(T_{i,j}) \rightarrow H^*(xT_{i,j}x^{-1}) \) is induced by the inner automorphism \( y \rightarrow xyx^{-1}. \) Let \( E = \sum_{m=1}^{n}a_me_m \) and \( B = \sum_{m=1}^{n}a_m^e_m \) in \( H^*(T_{i,j}) \) then it follows directly from the definition of \( \psi \) that

2.11. PROPOSITION. For \( x \in N_i, \) \( \text{ad}_x(E) = \psi(x)E \) and \( \text{ad}_x(B) = \psi(x)B. \)

Since \( \text{ad}_x \) is a ring homomorphism 2.11 determines \( \text{ad}_x \) on all of \( H^*(T_{i,j}). \)
Since the $p$th power homomorphism, $a \mapsto a^p$, is the identity on $\mathbb{Z}/p$ we have $P(x_1^p, \ldots, x_f^p) = (P(x_1, \ldots, x_f))^p$ for all polynomials $P$. This fact and direct computation yield

2.12. Proposition. $ad_x$ operates on the classes $L_i$, $Q_{i,j}$, $M_{j,i}$, $\lambda_q$ via multiplication by the determinant function.

2.13. Corollary. The algebra $\mathcal{D}_i$ is contained in $H^*(T_{i,j})^{\text{GL}(i,\mathbb{Z}/p)}$.

2.14. Lemma. If $G$ is a finite group, $K$ a subgroup, and $N_{K,G}$ the normalizer of $K$ in $G$ then the image of $H^*(G)$ in $H^*(K)$ is contained in $H^*(K)^{N_{K,G}}$.

Proof. Any inner automorphism of $G$ induces the identity on $H^*(G)$. Hence we have the following commutative diagram:

\[
\begin{array}{cccccc}
H^*(G) & \xrightarrow{id} & H^*(G) \\
\downarrow & & \downarrow \\
H^*(K) & \xrightarrow{ad_x} & H^*(xKx^{-1})
\end{array}
\]

Allowing $x$ to run through $N_{K,G}$ gives the lemma.

2.15. Corollary. Let $u \in H^*(S_{p^i})$ then $k_{j,i}^*(u) \in H^*(T_{i,j})^{\text{GL}(i,\mathbb{Z}/p)}$.

Proof. Immediate from 2.10 and 2.14.

Let $N_{j,i}$ be the normalizer of $k_{j,i}$: $T_{j,i} \rightarrow S_{p^i}$ in $S_{p^i}$.

2.16. Proposition. The sequence

$$1 \rightarrow \times p^{-1} N_j \rightarrow N_{j,i} \xrightarrow{\psi} S_{p^{-1}} \rightarrow 1$$

is exact.

Proof. Both $N_{j,i}$ and $\times p^{-1} N_j$ act on $T_{j,i}$ via conjugation. But $x \in N_{j,i}$ permutes the $p^{-1}$ orbits of $\times p^{-1} N_j$. This gives a homomorphism $\psi: N_{j,i} \rightarrow S_{p^{-1}}$, which is clearly onto and has an obvious section $\psi$. Notice $\psi((p(x)^{-1}) \cdot x \in \times p^{-1} N_j$ as $\psi(p(x)^{-1}) \in N_{j,i}$ and $\psi(p(x)^{-1}) \cdot x \in \times p^{-1} S_{p^i}$. The proposition follows.

Let $N_{m_1, \ldots, m_r}$ be the normalizer of $I_{m_1, \ldots, m_r}(\text{GL}(\mathbb{Z}/p))$: $\text{GL}(\mathbb{Z}/p)$ in $S_{p^i}$, and let $S_{m_1, \ldots, m_r}$ be the subgroup of $S_{m_1}$ generated by the transpositions $(a, c)$ where $m_a = m_c$. Minor modification of 2.16 yields the following three propositions.

2.17. Proposition. The sequence $1 \rightarrow \times r=1 N_{m_r} \rightarrow N_{m_1, \ldots, m_r} \xrightarrow{\psi} S_{m_1, \ldots, m_r} \rightarrow 1$ is exact.

2.18. Proposition. Let $\overline{N_j}$ be the normalizer of $I_j$: $\times p^{-1} S_{p^i} \rightarrow S_{p^i}$ in $S_{p^i}$. Then the sequence $1 \rightarrow \times p^{-1} S_{p^i} \rightarrow \overline{N_j} \xrightarrow{\psi} S_{p^{-1}} \rightarrow 1$ is exact.
2.19. PROPOSITION. Let \( \bar{N}_{m_1, \ldots, m_n} \) be the normalizer of \( I_{m_1, \ldots, m_n} : \times_{r=1}^{n} S_{m_r} \to S_{p'} \) in \( S_{p'} \). Then the sequence \( 1 \to \times_{r=1}^{n} S_{m_r} \to \bar{N}_{m_1, \ldots, m_n} \to S_{m_1, \ldots, m_n} \to 1 \) is exact.

2.20. LEMMA. If \( G \) is a finite group and \( K \) a subgroup then \( i(K, G)^* t(G, K) = \sum_{x \in G/K} i_x \cdot \text{ad}_x \) where \( \text{ad}_x : H^*(K) \to H^*(xKx^{-1}) \) is the homomorphism induced by \( y \mapsto xyx^{-1} \) for \( y \in K \), \( i_x \) is the inclusion map \( H^*(xKx^{-1}) \to H^*(xKx^{-1} \cap K) \) and \( t_x \) is the transfer \( H^*(xKx^{-1} \cap K) \to H^*(K) \).

PROOF. [5, XII. 9.1, p. 257].

2.21. PROPOSITION. If \( K \) is a proper subgroup of \( \times^m Z/p \) then the transfer \( t : H^*(K) \to H^*(\times^m Z/p) \) is zero.

PROOF. [4, I.2.1].

III. Some properties of \( \overline{Q}(p) \) and the proof of Theorem E. In this section we state facts about the Steenrod algebra needed to prove Theorems A through D and give a proof of Theorem E.

First recall the construction of the Steenrod \( p \)th powers ([31] gives the complete treatment and we quote it frequently in what follows). Let \( X \) be a finite regular cell complex then we have the following spaces and maps:

\[
X_p \xrightarrow{j} W_{Z/p} \times_{Z/p} X^p \xleftarrow{1 \times \Delta} W_{Z/p} \times_{Z/p} X = B_{Z/p} \times X
\]

where \( j \) is the inclusion and \( \Delta \) is the diagonal map. Given any \( u \in H^*(X) \) there exists a unique natural class \( \overline{Q}(u) \) in \( H^*(W_{Z/p} \times_{Z/p} X_p) \) such that:

1. \( j^*(\overline{Q}(u)) = u \otimes \cdots \otimes u = u^{\otimes p} \).
2. \((1 \times \Delta)^*(\overline{Q}(u)) \) in \( H^*(B_{Z/p} \times X) \) can be expanded by the Künneth theorem. \((1 \times \Delta)^*(\overline{Q}(u)) = \sum w_k \otimes D_k(u) \) where \( w_k \) generates \( H^k(Z/p) \) and \( D_k : H^q(X) \to H^{q-k}(X) \) are homomorphisms which define the elements of \( \overline{Q}(p) \).
3. \( \beta D_{2k}(u) = D_{2k-1}(u), \beta D_{2k-1}(u) = 0 \) and \( \beta D_0(u) = 0 \).

3.1. THEOREM [31]. If \( z \in H^*(W_{Z/p} \times_{Z/p} X_p) \), then \( z \) is of the form \( z = t z_1 + z_2 \cdot \overline{Q}(z_3) \) with \( z_1 \in H^*(X_p), z_2 \in H^*(B_{Z/p}) \) and \( z_3 \in H^*(X) \), where \( t \) is the transfer. Furthermore the sequence

\[
H^*(X_p) \xrightarrow{(1 \times \Delta)^*} H^*(W_{Z/p} \times_{Z/p} X_p) \xrightarrow{(1 \times \Delta)^*} H^*(B_{Z/p} \times X)
\]

is exact.

PROOF. [31, VII. 4.1, p. 104 and VIII. 3.6, p. 126].

3.2. DEFINITION [31]. Let \( u \in H^q(X) \) then

\[
\overline{Q}^i(u) = a_{i,q} D_{(q-2i)(p-1)}(u),
\]

\[
\beta \overline{Q}^i(u) = a_{i,q} D_{(q-2i)(p-1)-1}(u),
\]
where $a_{j,q}$ is a nonzero constant in $\mathbb{Z}/p$ dependent on $j$ and $q$. If $k \neq (q - 2j)(p - 1)$ or $(q - 2j)(p - 1) - 1$ for some $j$ then $D_k(u) = 0$.

3.3. Proposition. If $q$ is even, say $q = 2n$, then $a_{j,2n} = (-1)^{j+n}$.

Proof. Follows directly from [31, VII. 6.1 and VII. 6.3] (note correction of the formula in VII. 6.1 on the first page of the appendix to [31]).

The following is well known:

3.4. Lemma. I. Let $p$ be a prime and $a = \sum_{i=0}^{m} a_i p^i$, $c = \sum_{i=0}^{m} c_i p^i$ ($0 \leq a_i, c_i < p - 1$). Then

$$\left( \frac{c}{a} \right) \equiv \prod_i \left( \frac{c_i}{a_i} \right) \pmod{p}.$$  

II. $\mathbb{P}^j(e) = 0$ for all $j > 0$.

III. $\mathbb{P}^j(b^k) = \binom{j}{k} b^{k+(p-1)/j}$.

IV. (Cartan formula) $\mathbb{P}^j(uv) = \sum_{m+n=j} \mathbb{P}^m(u) \mathbb{P}^n(v)$.

V. $\mathbb{P}^j(b^m) = \begin{cases} \binom{p^m}{j} b^{m+(p-1)/j} & \text{if } j = 0, \\ \binom{p^m}{j+1} b^{m+1} & \text{if } j = p^m, \\ 0 & \text{otherwise}. \end{cases}$

Proof. [31, see I.2.6, V. 1, VII. 2.2 and VI. 2.3].

The proof of Theorem E follows from direct calculation and Lemma 3.4. Note: To prove relation (4) of Theorem E, just expand $\mathbb{P}^{p-1}(Q_{k,i} L_i^{-1})$.

IV. Symmetric products and image $k_i$. In this chapter we summarize results of [17] which give $H^*(S_n)$ as $\mathbb{Z}/p$ vector spaces and give an upper bound on the size of $k_i$.

Recall the monomial $\mathbb{P}^l = \beta_{s_1} \beta_{s_2} \cdots \beta_{s_i} \in \mathbb{P}(p)$ is called admissible if $s_i > ps_i - 1 + e_{i-1}$ for each $i > 1$, and the excess of $\mathbb{P}^l = 2s_k + e_k - \Sigma_{j=1}^{k-1}(2s_i(p - 1) + e_i)$. The excess of any admissible monomial is nonnegative. Let $\mathbb{P}(p)_n$ be the subvector space of $\mathbb{P}(p)$ spanned by those monomials of excess $< n$.

Let $SP^k(S^{2n})$ be the $k$ symmetric product of $S^{2n}$ (see [17] for the definition and properties of the symmetric products of a space).

4.1. Theorem [17]. (1) $H_*(SP^k(S^{2n})) = \sum_{m=1}^{k} H_*(SP^m(S^{2n}), SP^{m-1}(S^{2n})).$

(2) $\mathbb{R}(S^{2n}, \mathbb{Z}/p) = \sum_{m=1}^{\infty} H_*(SP^m(S^{2n}), SP^{m-1}(S^{2n}))$ is isomorphic to $H_*(K(Z, 2n))$.

There is a bigrading of $\mathbb{R}(S^{2n}, \mathbb{Z}/p)$ given by

$$\mathbb{R}_{i,m}(S^{2n}, \mathbb{Z}/p) = H_i(SP^m(S^{2n}), SP^{m-1}(S^{2n})).$$

(3) For $\mathbb{R}(S^{2n}, \mathbb{Z}/p)$ the generators $q_i$ in homology are in 1-1 correspondence with admissible monomials $\mathbb{P}^l = \beta_{s_1} \beta_{s_2} \cdots \beta_{s_i} \in \mathbb{P}(p)_2$ and the bidegree
of this generator is \(|q_i, q_i\rangle = 1\) under the isomorphism in (2).

**Proof.** [17].

**Remarks.** (1) is due to N. E. Steenrod. [8] and [21] also studied (1) and (2).

The next theorem follows from the fact that the singular locus of \((S^{2n})^{p'}\) under \(\mathcal{S}_{p'}\) has dimension \(2n(p' - 1)\).

4.2. **Theorem** [17]. For \(k < 2n - 1\), \(H^k(\mathcal{S}_{p'}) \cong H_{2n(p') - k}(\mathcal{S}^{p'}(S^{2n}))\).

Since \(H_j(\mathcal{S}^{p'}(S^{2n})) \cong H_j(\mathcal{S}^{p'}(S^{2n}), \mathcal{S}^{p' - 1}(S^{2n}))\) for \(j > 2n(p' - 1) + 1\)
we may identify \(H^k(\mathcal{S}_{p'})\) with elements in \(\mathcal{R}(S^{2n}, \mathbb{Z}/p)\) of bidegree \((2n(p'), p')\).

Thus for \(k < 2n - 1\) classes in \(H^k(\mathcal{S}_{p'})\) correspond to classes \(\Sigma a\); with each \(a \in \mathcal{R}(S^{2n}, \mathbb{Z}/p)\) having bidegree \((-p')\). This gives \(H^k(\mathcal{S}_{p'})\) as \(\mathbb{Z}/p\) vector spaces. Recall there are two types of classes in \(\mathcal{R}(S^{2n}, \mathbb{Z}/p)\)

with bidegree \((-p')\):

1. \(a\) corresponds to \(q_i\) of bidegree \(|q_i| + 2n, p'\),
2. \(a = \Pi b_k\) where \(b_k\) has bidegree \((-p')\), for some \(j < i\) and occurs in \(H_{\ast}(\mathcal{S}^{p'}(S^{2n}), \mathcal{S}^{p' - 1}(S^{2n}))\).

On the other hand the multiplication map \(M: \mathcal{S}^{p'}(S^{2n}) \times \cdots \times \mathcal{S}^{p'-1}(S^{2n}) \to \mathcal{S}^{p'}(S^{2n})\) and 4.2 give a map \(m: \bigotimes^p H^\ast(\mathcal{S}_{p'},(\mathcal{S}_{p'})) \to H^\ast(\mathcal{S}_{p'})\).

4.3. **Lemma** [21]. \(m\) is the transfer map induced by the inclusion

\[ I_{i-1}: \times S_{p'-1} \to S_{p'} \]

**Proof.** [21].

4.4. **Lemma.** Let \(u \in H^\ast(\mathcal{S}_{p'})\) correspond to \(a \in \mathcal{R}(S^{2n}, \mathbb{Z}/p)\). If \(a\) is of type 2 then \(k_{i\ast}(u) = 0\).

**Proof.** Suppose \(a\) is of type 2 then \(a\) is in the image of \(M_{\ast}\). By 4.3, \(u\) is in the image of the transfer \(t: H^\ast(\times^p S_{p'-1}) \to H^\ast(\mathcal{S}_{p'})\). But 3.1 implies \(k_{i\ast}t = 0\).

Hence \(k_{i\ast}(u) = 0\).

Let \(\mathcal{R}_{2n(p') - k,p'}(S^{2n}, \mathbb{Z}/p)\) be the subspace of \(\mathcal{R}_{2n(p') - k,p'}(S^{2n}, \mathbb{Z}/p)\)
spanned by elements of type 1. Then 4.4 yields:

4.5. **Theorem** [17]. As \(\mathbb{Z}/p\) vector spaces

\[ \dim((\text{image } k_{i\ast})) < \dim(\mathcal{R}_{2n(p') - k,p'}(S^{2n}, \mathbb{Z}/p)). \]

**V. The proof of Theorem A.** We now proceed with the proof of Theorem A.

5.1. **Lemma.** \(\mathcal{W}_i\) is contained in image \(k_{i\ast}\).

**Proof.** By induction on \(i\). The lemma is classically true for \(i = 1\) and [4] proves the lemma for \(i = 2\). Assume \(\mathcal{W}_{i-1}\) is contained in image \(k_{i-1,i-1}\). The next four lemmas establish 5.1.
5.2. Lemma. There exists \( u \in H^*(\mathbb{S}_p) \) such that
\[
k_{i-1,1}(u) = (M_{i-2,i-1}M_{i-3,i-1}L_{i-1}^{p-3}) \otimes p \in H^*(T_{i-1,i}).
\]

Proof. Recall the following commutative diagram containing the construction of the Steenrod powers on \( \mathbb{S}_{p-1} \):

\[
\begin{array}{c}
B_{\mathbb{S}_{p-1}^{p-1}} \\
\uparrow \\
B_{\mathbb{S}_{p-1}^{p-1}} \\
\uparrow \\
B_{\mathbb{T}_{i-1,i}} \\
\uparrow \\
W_{Z/p} \times Z/p \\
\uparrow \\
1 \times \Delta \\
\uparrow \\
B_{\mathbb{T}_{i-1,i}} \\
\end{array}
\]

Of course the composition \( B_{\mathbb{T}_{i-1,i}} \rightarrow B_{\mathbb{S}_p} \) is \( Bk_{i-1,i} \) and the composition \( B_{\mathbb{T}_{i-1,i}} \rightarrow B_{\mathbb{S}_p} \) is \( Bk_{i-1,i} \).

Let \( u' \in H^*(\mathbb{S}_{p-1}) \) be such that \( k_{i-1,1}(u') = M_{i-2,i-1}M_{i-3,i-1}L_{i-1}^{p-3} \) then \( \mathfrak{P}(u') = u'' \in H^*(\mathbb{S}_{p-1} \text{ wr } Z/p) \). Let \( A = \mathbb{S}_{p-1} \text{ wr } Z/p \). Then 2.20 gives
\[
i(A, \mathbb{S}_p)*i(\mathbb{S}_p, A) = \sum_{x \in \mathbb{S}_p/A} t_x i_x \text{ad}_x
\]
and we have the following commutative diagram:

\[
\begin{array}{c}
H^*(A) \xrightarrow{\text{ad}_x} H^*(xAx^{-1}) \xrightarrow{i_x} H^*(xAx^{-1} \cap A) \xrightarrow{t_x} H^*(A) \\
\downarrow \Sigma \text{ad}_x \downarrow \Sigma i_x \downarrow \Sigma \text{ad}_x \downarrow \Sigma i_x \\
\Sigma H^*(T') \xrightarrow{\text{ad}_x} \Sigma H^*(xT'x^{-1}) \xrightarrow{i_x} \Sigma H^*(xT'x^{-1} \cap T_{i-1,i}) \xrightarrow{t_x} \Sigma H^*(T_{i-1,i})
\end{array}
\]

where \( T' \) runs through all inclusions \( \times^m Z/p \) in \( A \). (The last square commutes by 2.21 and [31, V. 7.2], as \( xT_{i-1,i}x^{-1} \subset A \) implies \( x \in A \).)

Thus 2.16, 2.18 and 2.21 show
\[
k_{i-1,1}(A, \mathbb{S}_p)(u'') = \sum_x (M_{i-2,i-1}M_{i-3,i-1}L_{i-1}^{p-3}) \otimes p
\]

where the sum runs over a coset representation \( \overline{N}_{i-1} = N_{x \in \mathbb{S}_{p-1} \text{ wr } \mathbb{S}_p} \) mod. As
A contains a $p$-Sylow subgroup of $\mathbb{S}_{p^i}$, $[N_{i-1}: A] = c \equiv 0 \pmod{p}$. Let $u = t(A, \mathbb{S}_{p^i})(c^{-1}u')$; then $k_{i,j}^*(u) = (M_{i-2,i-1}M_{i-3,i-1}L_{i-1}^{p-3})^{\otimes p}.

5.3. Lemma. There exists $u \in H^*(\mathbb{S}_{p^i})$ such that

$$k_{i-1,j}^*(u) = (Q_{i-2,i-1})^{\otimes p} \in H^*(T_{i-1,i}).$$

Proof. Identical to that of 5.2.

5.4. Lemma. There exists $u \in H^*(\mathbb{S}_{p^i})$ such that $k_{i,i}^*(u) = M_{i-1,i}M_{i-2,i}L_{i}^{p-3}$.

Proof. Let $u' \in H^*(\mathbb{S}_{p^i})$ be such that $k_{i-1,i-1}^*(u') = M_{i-2,i-1}M_{i-3,i-1}L_{i-1}^{p-3}$ and $u \in H^*(\mathbb{S}_{p^i})$ be such that $k_{i,i}^*(u) = (M_{i-2,i-1}M_{i-3,i-1}L_{i-1}^{p-3})^{\otimes p} \in H^*(T_{i-1,i})$. Recall 3.1 implies image $k_{i,i}^*$ is contained in the $H^*(\mathbb{Z}/p)$ module generated by image$(1 \times \Delta)^*\varphi$. A simple dimension check shows that the only classes in $H^*(\mathbb{S}_{p^i} \wr \mathbb{Z}/p)$ that could project to $k_{i-1,j}^*(u)$ are $\varphi(u')$ and $b^x_i + \varphi(u')$, where $x = \frac{1}{2}$ dimension$(u)$. By 2.15, $k_{i,i}^*(u)$ is $GL(i, \mathbb{Z}/p)$ invariant. As $(u')^p = 0$ in $H^*(\mathbb{S}_{p^i})$ the class $b^x_i + \varphi(u')$ is not $GL(i, \mathbb{Z}/p)$ invariant (there cannot be a pure $b^x_i$ term in $(1 \times \Delta)^*\varphi(u')$ for $r > 1$). Hence $k_{i,i}^*(u) = (1 \times \Delta)^*\varphi(u')$. It is easy to see that dimension$(u') = 2(p^{i-1} - p^{i-2} - p^{i-3}) = 2n.$ Thus

$$k_{i,i}^*(u) = (1 \times \Delta)^*\varphi(u') = \sum_k w_k \otimes D_k \left( M_{i-2,i-1}M_{i-3,i-1}L_{i-1}^{p-3} \right)$$

$$= (-1)^n \left[ \sum_j w_{(2n-2)j}(p-1) \otimes (-1)^j \varphi \left( M_{i-2,i-1}M_{i-3,i-1}L_{i-1}^{p-3} \right) + \sum_j w_{(2n-2)j(p-1)-1} \otimes (-1)^j \beta \varphi \left( M_{i-2,i-1}M_{i-3,i-1}L_{i-1}^{p-3} \right) \right].$$

Consider $M_{i-1,i}M_{i-1,i}L_{i}^{p-3}$. Expanding along the $e_1, b_1$ columns we have

$$M_{i-1,i}M_{i-2,i}L_{i}^{p-3} = \sum_{A,B,C_k} (-1)^\varphi b^x_i \left( ABC_1 \cdots C_{p-3} \right)$$

$$+ \sum_{D,E,C_k} (-1)^\varphi e_1 b^x_i \left( DEC_1 \cdots C_{p-3} \right)$$

where $A$ runs over all $i-1 \times i-1$ minors of $M_{i-1,i}$ eliminating the $b^x_i$ ($0 < u < i-2$) row and column, $B$ runs over all $i-1 \times i-1$ minors of $M_{i-2,i}$ eliminating the $b^x_i$ ($0 < v < i-3$, or $v = i-1$) row and column, $C_k$ ($k = 1, \ldots, p-3$) is any $i-1 \times i-1$ minor of $L_{i}$ eliminating the $b^x_i$ ($0 < z_k < i-1$) row and column, $r$ satisfies the relation $\dim(M_{i-1,i}M_{i-2,i}L_{i}^{p-3}) = 2r + \dim(A) + \dim(B) + \sum_{k=1}^{p-3} \dim(C_k)$, and $\varphi \equiv u + v + \sum_{k=1}^{p-3} z_k \pmod{2}$ if $v \neq i-1$, and $\equiv (i-u) + \sum_{k=1}^{p-3} z_k \pmod{2}$ if
\[ v = i - 1. \] D and E are \( i - 1 \times i - 1 \) minors of \( M_{i-1,i} \) and \( M_{i-2,i} \) respectively with exactly one minor eliminating the \( e_1 \) row and column, the other eliminating a \( b^p \) row and column.

If \( C_k \) is the minor eliminating the \( b^p \) row and column then \( C_k = \begin{pmatrix} m_x(L_{i-1}) \end{pmatrix} \) where \( m_x = p^x + p^{x+1} + \cdots + p^{i-2} \) (if \( z_x = i - 1 \)).

**Case 1.** Suppose \( v = i - 1 \). Then the minor of \( M_{i-2,i} \) eliminating the \( b^p \) row and column is \( M_{i-2,i-1} \). If \( A \) is an \( i - 1 \times i - 1 \) minor of \( M_{i-1,i} \) eliminating the \( b^p \) row and column and \( AM_{i-2,i-1} \neq 0 \) then \( u \neq i - 2 \). Thus \( A = \begin{pmatrix} j_1 \\ j_2 \end{pmatrix} \) where \( j_1 = p^u + p^{u+1} + \cdots + p^{i-2} \) (if \( u = i - 3 \) then \( j_1 = 0 \)). Thus if \( v = i - 1 \) we have

\[
ABC_1 \cdots C_{p-3} = (-1)^{\begin{pmatrix} j_1 \\ j_2 \end{pmatrix}}(M_{i-2,i-1})^{-1}(M_{i-3,i-1})^{-1}(M_{i-1,i-1})^{-1}(L_{i-1})^{-1} \cdots (L_{i-1})^{-1}.
\]

**Case 2.** Suppose \( 0 < v < i - 3 \). Then \( A = \begin{pmatrix} j_1 \\ j_2 \end{pmatrix} \) where \( j_2 \) is an \( i - 1 \times i - 1 \) minor of \( M_{i-1,i} \) eliminating the \( b^p \) row and column and \( A^2 M_{i-2,i-1} \neq 0 \) unless \( v = i - 3 \). Thus \( A = \begin{pmatrix} j_1 \\ j_2 \end{pmatrix} \) where \( j_2 = p^v + p^{v+1} + \cdots + p^{i-4} + p^{i-2} \) unless \( v = i - 3 \) in which case \( j_2 = p^{i-2} \). Then we have

\[
ABC_1 \cdots C_{p-3} = \begin{pmatrix} j_1 \\ j_2 \end{pmatrix}(M_{i-2,i-1})^{-1}(M_{i-3,i-1})^{-1}(M_{i-1,i-1})^{-1}(L_{i-1})^{-1} \cdots (L_{i-1})^{-1}.
\]

**Note.** In Case 1 we have terms involving \((-1)^{\begin{pmatrix} j_1 \\ j_2 \end{pmatrix}}(M_{i-2,i-1})^{-1}(M_{i-3,i-1})^{-1}(M_{i-1,i-1})^{-1} \cdots (L_{i-1})^{-1} \cdots (L_{i-1})^{-1} \begin{pmatrix} j_1 \\ j_2 \end{pmatrix} \) but it is clear that \( j_1 \) can never equal \( j_2 \) in these cases.

Thus if \( ABC_1 \cdots C_{p-3} \neq 0 \) we have written \( ABC_1 \cdots C_{p-3} \) uniquely as

\[
\begin{pmatrix} j_1 \\ j_2 \end{pmatrix}(M_{i-2,i-1})^{-1}(M_{i-3,i-1})^{-1}(M_{i-1,i-1})^{-1}(L_{i-1})^{-1} \cdots (L_{i-1})^{-1} \begin{pmatrix} j_1 \\ j_2 \end{pmatrix} \]

for certain \( j_1, j_2, m_{i-1}, \ldots, m_{p-3} \). Clearly shows if

\[
Y = \begin{pmatrix} j_1 \\ j_2 \end{pmatrix}(M_{i-2,i-1})^{-1}(M_{i-3,i-1})^{-1}(M_{i-1,i-1})^{-1}(L_{i-1})^{-1} \cdots (L_{i-1})^{-1} \begin{pmatrix} j_1 \\ j_2 \end{pmatrix} \neq 0
\]

then \( Y = ABC_1 \cdots C_{p-3} \) for a suitable choice of \( A, B, C_1, \ldots, C_{p-3} \) and is thus analyzed in Case 1 or Case 2 above.

Let \( j = j_1 + j_2 + \sum_{k=3}^{p} m_k \). For both \( v = i - 3 \) and \( v < i - 3 \) it is trivial to see that \( \varphi = j \) (mod 2). Hence the Cartan formula and the above facts yield the following decomposition of \( M_{i-1,i}M_{i-2,i}L^{p-3}_1 \) where the first sum runs over all integers \( j \).

\[
M_{i-1,i}M_{i-2,i}L^{p-3}_1 = \sum_j b_1^{(n-j)(p-1)}(-1)^{\begin{pmatrix} j_1 \\ j_2 \end{pmatrix}}(M_{i-2,i-1}M_{i-3,i-1}L^{p-3}_1) + \sum_{\begin{pmatrix} D \\ E \end{pmatrix}} (-1)^{\begin{pmatrix} j_1 \\ j_2 \end{pmatrix}} \otimes DEC_1 \cdots C_{p-3}.
\]

Let \( U = k^{n}_1(u) - (-1)^{\begin{pmatrix} j_1 \\ j_2 \end{pmatrix}}(M_{i-1,i}M_{i-2,i}L^{p-3}_1) \). \( U \) is clearly \( GL(i, \mathbb{Z}/p) \)-invariant. Any monomial term in \( U \) must contain the factor \( e_1e_j \) (\( j \neq 1 \)) but as there is no monomial in \( U \) with an \( e_2e_3 \) factor symmetry implies \( U = 0 \). As
In the expression $n = p^{i-1} - p^{i-2} - p^{i-3}$, we have

$$k_i^\ast(u) = -M_{i-1,i}M_{i-2,i}L_i^{p-3}.$$  

This proves 5.4.

**Note.** By keeping careful track of $D$, $E$, and $\beta(\mathbb{S}^j(M_{i-2,i-1}^j, M_{i-3,i-1}))$, it is possible to see directly that

$$\sum_{D} (-1)^j e_1 b_i^j \otimes DEC_1 \cdots C_{p-3}$$

$$= -\sum_{j} e_1 b_i^{(n-j)(p-1)-1} \otimes (-1)^j \beta(\mathbb{S}^j(MML^{p-3}))$$

where $MML^{p-3} = M_{i-2,i-1}M_{i-3,i-1}L_i^{p-3}$.

**5.5. Lemma.** There exists $u \in H^\ast(\mathbb{S}_p)$ such that $k_i^\ast(u) = Q_{i-1,i}$.

**Proof.** The proof is similar to that of 5.4. We let $u' \in H^\ast(\mathbb{S}_{p-1})$ be such that $k_i^\ast(u') = Q_{i-1,i-1}$ and $u \in H^\ast(\mathbb{S}_p)$ be such that $k_i^\ast(u) = (Q_{i-2,i-1})^{tp} \in H^\ast(T_{i-1,i})$. Then $k_i^\ast(u)$ is the $GL(i, Z/p)$ invariant class containing $(1 \times \Delta)^\ast(\mathbb{S}(u'))$. But [8] proved $Q_{i-1,i}$ is the only $GL(i, Z/p)$ invariant polynomial in this dimension. Thus $k_i^\ast(u) = c Q_{i-1,i}$, where $c$ is a constant. Note $(1 \times \Delta)^\ast(\mathbb{S}(u'))$ contains the term $w_0 \otimes D_0(Q_{i-2,i-1}) = (Q_{i-2,i-1})^p \neq 0$. Hence $c \neq 0$.

The naturality of the Steenrod algebra implies image $k_i^\ast$ contains $\mathbb{S}(p)(M_i, M_{i-1}, L_i^{p-3}, Q_{i-1,i})$. By Theorem E any generator $\mathbb{S}_i$ is contained in $\mathbb{S}(p)(M_i, M_{i-1}, L_i^{p-3}, Q_{i-1,i})$ (see the diagram after Theorem E). This completes the proof of Lemma 5.1.

By 4.5, to complete the proof of Theorem A it suffices to construct a 1-1 correspondence between nonzero monomials in $\mathbb{S}_i$ and admissible monomials in $\mathbb{S}(p)$.

**5.6. Lemma.** $M_{i-1,i}M_{i-2,i} \cdots M_{i,i-1} \neq 0$.

**Proof.** The term $e_1 e_2 \cdots e_i (b_1^{p^{i-1}})^{-1} (b_2^{p^{i-2}})^{-1} \cdots (b_i)^{-1}$ appears with coefficient 1 in the term-by-term expansion of $M_{i-1,i}M_{i-2,i} \cdots M_{i,i-1}$.

The only admissible monomials of length 1 in $\mathbb{S}(p)$ are $\mathbb{S}^{n-j}(u_2n)$ and $\beta(\mathbb{S}^{n-j}(u_2n))$ which correspond to $(L_i^{p-1})^j$ and $(L_i^{p-2})^j(L_i^{p-1})^j$ in $\mathbb{S}_i$. Thus we may assume, by induction, that an $i-1$ length admissible monomial in $\mathbb{S}(p)$ starting with $\mathbb{S}^{n-j}(u_2n)$ corresponds to a $j$-fold product monomial in $\mathbb{S}_{i-1}$ ($j < n$). Let $A$ be an admissible monomial in $\mathbb{S}(p)_2n$.

**Case 1.** $e_1 = 0$; that is, $A = \beta e_2 \mathbb{S} \cdots \beta e_2 \mathbb{S} \mathbb{S}^{n-j}(u_2n)$. The dimension of $\mathbb{S}^{n-j}(u_2n)$ is $2p(n-j) + 2j$ and hence $s_2 = p(n-j) + k$, $0 < k < j$, if $A(u_2n)$ is nonzero and admissible. Consider
A' = \beta^s \beta^s \cdots \beta^{s_2} \beta^{s_2}(u_{2(p(n-j)+j)}) \quad \text{where} \quad \bar{u}_{2(p(n-j)+j)} = \mathbb{P}^{n-j}(u_{2n}).

A' is an admissible monomial of length \( i - 1 \) and \( s_2 = (p(n-j)+j) - (j-k) \). Thus \( A' \) corresponds to a \((j-k)\)-fold product monomial in \( \mathbb{W}_{i-1} \), call it \( U_{j-k} \). Identify \( A \) with \( \bar{U}_{j-k}(Q_{i-1,i})^k \) in \( \mathbb{W}_i \). \( \bar{U}_{j-k} \) comes from \( U_{j-k} \) by changing the detecting index from \( i-1 \) to \( i \); i.e., \( Q_{m,i-1} \rightarrow Q_{m,i} \).

**Case 2.** \( e_1 = 1 \); that is, \( A = \beta^s \beta^s \cdots \beta^{s_2} \beta^{s_2}(u_{2n}) \). Then consider that part of \( A \) until a second Bockstein occurs.

\[ U_{j-k} \] comes from \( U_{j-k} \) by changing the detecting index from \( i-1 \) to \( i \); i.e., \( Q_{m,i-1} \rightarrow Q_{m,i} \). If \( k = i \) or no second Bockstein occurs assign to \( A \) the monomial

\[ (M_{i-1,i,L_{p-3}^i}(L_{p-3}^i))^{m_{i-1}}(Q_{i-1,i})^{m_{i-1}-1}(Q_{i-1,i-1})^{m_{i-2}} \cdots (Q_{i-2,i})^{m_2}(Q_{i-1,i})^{m_1-1} \]

where \( \bar{U}_{A'} \) comes from \( U_{A'} \) by changing the detecting index from \( i-k \) to \( i \); i.e., \( Q_{m,i-k} \rightarrow Q_{m,i} \). If \( k = i \) or no second Bockstein occurs assign to \( A \) the monomial

\[ (M_{i-1,i,L_{p-3}^i}(L_{p-3}^i))^{m_{i-1}}(Q_{i-1,i})^{m_{i-1}-1} \cdots (Q_{i-2,i})^{m_2}(Q_{i-1,i})^{m_1-1} \]

and \( \mathbb{P}^a \cdots \beta^{-1}(u_{2n}) \) has dimension \( 2p^k(n-j) + 2p^{k-1}m_1 + 2p^{k-2}m_2 + \cdots + 2pm_{k-1} + 2(j-m_1-m_2-\cdots-m_{k-1}+1) \). For \( A \) to be admissible and nonzero we must also have \( j-m_1-m_2-\cdots-m_{k-1}+1 > 0 \). Then

\[ A' = \beta^s \beta^s \cdots \beta^{s_2} \beta^{s_2}(u_{2n}) = A''(\beta^s \beta^s \cdots \beta^{s_2} \beta^{s_2}(u_{2n})) \]

and \( A'' \) corresponds to a \((j-m_1-m_2-\cdots-m_{k+1})\)-fold product monomial in \( \mathbb{W}_{i-k} \), call it \( U_{A''} \). Identify \( A \) with the monomial

\[ (M_{i-1,i,L_{p-3}^i}(L_{p-3}^i))^{m_{i-1}}(Q_{i-1,i})^{m_{i-1}-1} \cdots (Q_{i-2,i})^{m_2}(Q_{i-1,i})^{m_1-1} \]

where \( U_{A''} \) comes from \( U_{A''} \) by changing the detecting index from \( i-k \) to \( i \); i.e., \( Q_{m,i-k} \rightarrow Q_{m,i} \). If \( k = i \) or no second Bockstein occurs assign to \( A \) the monomial

\[ (M_{i-1,i,L_{p-3}^i}(L_{p-3}^i))^{m_{i-1}}(Q_{i-1,i})^{m_{i-1}-1} \cdots (Q_{i-2,i})^{m_2}(Q_{i-1,i})^{m_1-1} \]

and \( \mathbb{P}^a \cdots \beta^{-1}(u_{2n}) \) has dimension \( 2p^k(n-j) + 2p^{k-1}m_1 + 2p^{k-2}m_2 + \cdots + 2pm_{k-1} + 2(j-m_1-m_2-\cdots-m_{k-1}+1) \). For \( A \) to be admissible and nonzero we must also have \( j-m_1-m_2-\cdots-m_{k-1}+1 > 0 \). Then

\[ A' = \beta^s \beta^s \cdots \beta^{s_2} \beta^{s_2}(u_{2n}) = A''(\beta^s \beta^s \cdots \beta^{s_2} \beta^{s_2}(u_{2n})) \]

and \( A'' \) corresponds to a \((j-m_1-m_2-\cdots-m_{k+1})\)-fold product monomial in \( \mathbb{W}_{i-k} \), call it \( U_{A''} \). Identify \( A \) with the monomial

\[ (M_{i-1,i,L_{p-3}^i}(L_{p-3}^i))^{m_{i-1}}(Q_{i-1,i})^{m_{i-1}-1} \cdots (Q_{i-2,i})^{m_2}(Q_{i-1,i})^{m_1-1} \]

and \( A'' \) comes from \( U_{A''} \) by changing the detecting index from \( i-k \) to \( i \); i.e., \( Q_{m,i-k} \rightarrow Q_{m,i} \). If \( k = i \) or no second Bockstein occurs assign to \( A \) the monomial

\[ (M_{i-1,i,L_{p-3}^i}(L_{p-3}^i))^{m_{i-1}}(Q_{i-1,i})^{m_{i-1}-1} \cdots (Q_{i-2,i})^{m_2}(Q_{i-1,i})^{m_1-1} \]

where \( m_i = j - m_1 - m_2 - \cdots - m_{i-1} \).

Let \( U_{A(u_{2n})} \) be the above constructed monomial in \( \mathbb{W}_i \) corresponding to \( A(u_{2n}) \). It is routine to verify that for \( U_{A'} \) in \( \mathbb{W}_{i-k} \) and \( \bar{U}_{A'} \) in \( \mathbb{W}_i \) constructed above we have \( \dim(U_{A'}) + 2j(p^i - p^{i-k}) = \dim(U_{A(u_{2n})}) \). This fact and induction on \( i \) show that if \( A(u_{2n}) \) has dimension \( 2n(p^i) - k \) then \( U_{A(u_{2n})} \) has dimension \( k \). Lemma 5.6 shows \( U_{A(u_{2n})} \neq 0 \). Hence, by Theorem 4.5, \( \mathbb{P} \mathbb{W}_i \) must fill out \((\text{image } k_x^*)_k \) for \( k < n \). This finishes the proof of Theorem A.

**VI. Proof of Theorems B, C, D, and F.** Consider the following commutative diagram:
6.1. **Proposition.** Let $u \in H^*(\mathcal{S}_p)$. If $k^*_p(u) = 0$ then there exists $z \in H^*(\times^p\mathcal{S}_{p-1})$ such that $t(z) = u$.

**Proof.** By 4.4 and Theorem A, $k^*_p(u) = 0$ implies $\tilde{k}^*_p(u) = 0$. Hence $(1 \times \Delta)^* h^*(u) = 0$ and $h^*(u) \in \ker(1 \times \Delta)^*$. By 3.1 there exists $z \in H^*(\times^p(\mathcal{S}_{p-1}))$ such that $t''(z) = h^*(u)$. Then $t(z) = t'(t''(u)) = [\mathcal{S}_p : \mathcal{S}_{p-1} \text{ wr } Z/p] u = u \text{ (mod } p)$. 

Let $u_{s,i-1} \in H^*(\mathcal{S}_{p-1})$, then, by induction, $u_{s,i-1}$ pulls back to a $\mathcal{S}_{p-1}$ detecting subgroup $\prod_{i=1}^s T_{x_i} \rightarrow \mathcal{S}_{p-1}$ (recall §11 gives these subgroups and their inclusions into $\mathcal{S}_{p-1}$). Thus to complete the computation of $H^*(\mathcal{S}_p)$ it suffices to compute the map $I_{r-1} t$. First consider the maps $\Phi_{m_1, \ldots, m_r} = (I_{m_1, \ldots, m_r} \circ (\prod_{i=1}^r (k_{m_i,m_i})))^* t_{m_1, \ldots, m_r}: H^*(\times^r_{i=1} \mathcal{S}_{p-1}) \rightarrow H^*(\mathcal{S}_p) \rightarrow \otimes_{i=1}^r H^*(T_{m_i,m_i})$ for all $(m_1, \ldots, m_r)$ such that $\Sigma_{i=1}^r p^{m_i} = p^r$, with $n > 2$ and $m_1, \ldots, m_r$ the transfer $H^*(\times^r_{i=1} \mathcal{S}_{p-1}) \rightarrow H^*(\mathcal{S}_p)$. 

6.2. **Lemma.** Let $u = u_{1,m_1} \otimes \cdots \otimes u_{n,m_n} \in H^*(\times^r_{i=1} \mathcal{S}_{p-1})$ and $k_{m_1,m_2}^*(u_{m_1}) = v_s$. Then

$$\Phi_{m_1, \ldots, m_r}(u) = \sum_{\sigma \in S_{\sigma(1)} \otimes \cdots \otimes S_{\sigma(n)}} v_{\sigma(1)} \otimes \cdots \otimes v_{\sigma(n)}.$$ 

**Proof.** As in the proof of 5.2, 2.16 through 2.21 and the following commutative diagram give the proposition:

\[
\begin{array}{ccccccc}
H^*(A) & \xrightarrow{\text{ad}_x} & H^*(xAx^{-1}) & \xrightarrow{i_x} & H^*(xAx^{-1} \cap A) & \xrightarrow{t_x} & H^*(A) \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
\Sigma H^*(T') & \xrightarrow{\Sigma \text{ ad}_x} & \Sigma H^*(xT'x^{-1}) & \xrightarrow{\Sigma i_x} & \Sigma H^*(xT'x^{-1} \cap T) & \xrightarrow{\Sigma t_x} & H^*(T) \\
\end{array}
\]

where $A = \times_{r=1}^n \mathcal{S}_{p-1}$, $T'$ runs through all inclusions of $\times^r Z/p$ in $A$ and $T = \times_{r=1}^n T_{m_i,m_i}$.

The only $S_{\sigma(1)} \otimes \cdots \otimes S_{\sigma(n)}$ invariant classes not in image $\Phi_{m_1, \ldots, m_r}$ are classes $u' = u_{1,m_1} \otimes \cdots \otimes u_{n,m_n}$ containing $(u_{r,m_1})^{(p)} \in \otimes^p H^*(T_{m_1,m_1})$ as a factor. Recall $u^{(p)} \leftrightarrow \mathcal{S}_{p}(u) \rightarrow (1 \times \Delta)^*(\mathcal{S}_{p}(u))$. Thus $u'$ is in the
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Hence we have

6.3. Lemma. Image\((I_{m_1},\ldots,m_n)\circ (\prod_{r=1}^n k_{m_r,m_r})\)\* \(\cong S(m_1,\ldots,m_n)\) invariant classes of \(\bigotimes H^*(T, m_1)\).

This proves Theorems B and D. A trivial modification of 6.2 and 6.3 proves Theorem F. As 3.1 shows the only multiple image classes are generated by the \(P()\)’s, Theorem C follows, up to constants. Using the notation of Theorem C if \(x_{m_1,\ldots, m_n} = M_{i-2,i-1}M_{i-3,i-1}L_{j-3}^p\) then 5.4 gives \(x_{m_1} = -1(M_{i-1,i-2}L_{j-2}^p)\). If \(x_{m_1} = Q_{i-2,i-1}\) then direct computation shows the constant \(c\) in 5.5 is 1 hence \(x_{m_1} = Q_{i-1,i}\). It is easy to see that application of the Steenrod \(p\)th powers or direct computation yield that the constant is +1 for multiple image polynomial generators and −1 for even dimensional multiple image exterior generators.

VII. Proof of Theorem G.

Proof of (1). Let \(k_{j,\infty}(u) = \Sigma\langle x_1,\ldots,x_m, 1,\ldots\rangle\). As \(j\) is the smallest integer such that \(k_{j,\infty}(u) \neq 0\) it follows that at least one \(x_h\) contains a factor equal to \(L_j^p - L_j^p - 2, L_j^p M_j, L_j^p - 3, \) or \(M_j L_j^p - 2\). If \(k_{j,\infty}(u)\) has at least one representative of the form \(\Sigma M_{i,j}^p (\cdots)\) with \(p\) not dividing \(n\) then \(\beta_p(k_{j,\infty}(u)) = \Sigma nL_{n,j}^p (\cdots) + B \neq 0\) (where \(B\) cannot contain terms in the first sum). Similarly if some \(x_h = M_{i,j}^p L_j^p - 3 Y\) and no \(x_h = M_{i,j}^p L_j^p - 2 Y\) then \(\beta_p(k_{j,\infty}(u)) \neq 0\). Suppose every time the term \(M_{i,j}^p L_j^p - 2 Y\) also appears; then if \(k_{j,\infty}(u) \neq L_{n,j}^p (\cdots) Y\) must be a product of \(Q_{h,j}\)’s. It is then easy to construct a class \(u’\) such that \(\beta_p(u’) = u\) (just replace one \(M_{i,j}^p L_j^p - 2 Y\) by \(M_{i,j}^p L_j^p - 3 Y\)). If \(\beta_p(u) = 0\) and \(M_{i,j}^p L_j^p - 2 Y\) appears a similar construction yields \(u’\) such that \(\beta_p(u’) = u\). The only possibility left is \(\beta_p(u) = 0\), and \(k_{j,\infty}(u) = L_{n,j}^p (\cdots)\). Then \(\beta_p(u’ = u\) where \(k_{j,\infty}(u’) = L_{n,j}^p\).

Proof of (2). We need the following

THEOREM [2]. Let \(r \geq 2\). In homology with the loop sum multiplication if \(d^{r-1}(a) = b\) then \(d^r(a^p) = a^p b\).

Proof. Theorem 5.4 of [2].

The homology and cohomology Bockstein spectral sequences are Hopf algebra duals and Theorem F gives the loop sum coalggebra map in cohomology. If \(a, b\) in \(H_*(Q(S^0))\) are dual to \(u, v\) respectively then Theorem F gives \(\langle u’, a^p \rangle = 1\). Now \(u’\) is not dual to \(a^p\) on the \(E_1\) level; in fact \((u’)^* = a^p + \Sigma a_i\). It is easy to see however that the \(a_i\) are all dual to classes \(u”\) where \(k_{j,\infty}(u”) = \Sigma\langle x_1,\ldots,x_t, 1,\ldots\rangle\) with \(t < pm\).
Many times it is easy to see that the $a_i$ classes do not live to $E_r$. Such is the case with Corollary 1 as induction on $r$ and the fact that $\{L_{p^r-1}(x:1,\ldots)\}$ generate the subalgebra $\{L_{n_1}(x:1,\ldots)\}$ (where $n = 1,\ldots, p^r - 1$) prove the corollary.

**Proof of Corollary 2.** The reduction homomorphism $j_r: H^*(\mathbb{Z}/p^r) \to E_r$ is onto and if $k^*_j(u) = Q_{j,i}$ then $k^*_j(u) = R^*_i(L_{p^r-1}(1:1,\ldots))$.

**Appendix.** We give a proof that the quotient determinants, $Q_{j,i} \in \mathcal{W}_i$, are integral mod $p$. $L_j$ has an explicit factorization first discovered by E. H. Moore in 1896

**Lemma [19].** $L_j = \prod_{(m_1,\ldots,m_i)}(m_1b_1 + \cdots + m_ib_i)$ where $(m_1,\ldots,m_i)$ runs over all elements of $T_j$ with first nonzero coefficient equal to one.

**Proof.** (Compare with [8, p. 76].) $L_j$ is invariant under the special linear group $SL(i,\mathbb{Z}/p)$ which acts transitively on the nonzero elements of $T_j$. Since $b_1$ is a factor of $L_j$ it follows that $\alpha(b_1) = m_1b_1 + \cdots + m_ib_i$ is a factor as well. Hence the product above divides $L_j$ (the factors are all relatively prime). But both sides have the same degree, hence they differ only up a constant factor. But the diagonal term $b_1^e b_2^{e-2} \cdots b_i$ is a factor in both sides only once and each time with coefficient 1.

More generally $b_1$ is a factor of the numerator of $Q_{j,i}$ for every $j$, so $L_j$ is also a factor of the numerator of $Q_{j,i}$ by the above argument. This gives:

**Lemma.** $Q_{j,i}$ is a nontrivial polynomial invariant under $GL(i,\mathbb{Z}/p)$.

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