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ON THE STABLE DECOMPOSITION OF $\Omega^2 S^{r+2}$

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ABSTRACT. In this paper we show that $\Omega^2 S^{r+2}$ is stably homotopy equivalent to a wedge of suspensions of other spaces C_k^1, and that C_k^1 is homotopy 2-equivalent to the Brown-Gitler spectrum.

1. Introduction. In this paper we show that $\Omega^2 S^{r+2}$ is stably homotopy equivalent to a wedge of suspensions of other spaces C_k^1, that C_k^1 cannot be further decomposed into a wedge, and that C_k^1 is homotopy 2-equivalent to the Brown-Gitler spectrum $B([k/2])$ [3].

Let

$$C_k^r = C_{2,k} \ltimes_{\Sigma_k} \left(\bigwedge^k S^r \right),$$

where $C_{n,k}$ is the space of k distinct points in R^n. Snaith [12] showed that $\bigvee_{k=1}^{\infty} C_k^r$ is stably homotopy equivalent to $\Omega^2 S^{r+2}$, if $r > 0$. F. Cohen, Mahowald, and Milgram [6] showed that

$$C_k^r = \begin{cases} S^{k(r-1)} C_k^1 & \text{if } r \text{ odd}, \\ S^{kr} C_k^0 & \text{if } r \text{ even}. \end{cases}$$

Our two main results are the following.

THEOREM A. *C_k^0 is stably homotopy equivalent to $(\bigvee_{i=1}^{[k/2]} C_i^1) \vee S^0$.*

THEOREM B. *C_k^1 is homotopy 2-equivalent to $S^k B([k/2])$.*

Thus, $\Omega^2 S^{r+2}$ is stably homotopy equivalent to a wedge of suspensions of C_i^1. More precisely, we have the following corollary.

COROLLARY C. *$\Omega^2 S^{r+2}$ is stably homotopy equivalent to $\bigvee_{k=1}^{\infty} S^{k(r-1)} C_k^1$ if r is odd and to*

$$\bigvee_{k=1}^{\infty} \left(S^{kr} \vee \bigvee_{i=1}^{[k/2]} S^{kr} C_i^1 \right)$$

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for r even and positive. For completeness, we note that a component of Ω^2S^2 is homotopy equivalent to Ω^2S^3. Furthermore, each piece of Ω^2S^{r+2} is homotopy 2-equivalent to a Brown-Gitler spectrum.

We note that $C_k^0 = C_{2,k}/\Sigma_k \cup$ (base point) and that $C_{2,k}/\Sigma_k = K(B_k, 1)$, where B_k is the kth braid group [2]. Hence Theorems A and B describe the stable homotopy type of $K(B_k, 1)$.

Finally, we note that C_k^1 cannot be further decomposed into a wedge. However, our low-dimensional computations lead us to believe that $C_{3,k} \ltimes_{\Sigma_k} (\bigwedge S^1)$ can be decomposed into much smaller pieces.

2. Results about the Brown-Gitler spectrum. Let A be the mod two Steenrod algebra, $\chi: A \to A$ the canonical antiautomorphism, and define M_k to be the A-module:

$$M_k = A/A\left\{\chi(\mathrm{Sq}^i)|i > k\right\}.$$

One of the properties of the spectrum $B(k)$ is that

(a) $H^*(B(k); Z_2) \approx M_k$.

In the course of proving that h_1h_i represents a homotopy element [11], Mahowald proves that $H^*(C_k^1; Z_2) \approx S^kM_{[k/2]}$. If it had been known that C_k^1 and $S^kB([k/2])$ were homotopy 2-equivalent, Mahowald's proof could have been simplified. (Unfortunately, our proof of Theorem B does not simplify Mahowald's proof, since we use his technique to prove B.) J. F. Adams noted that property (a) does not characterize $B(k)$. Other properties of $B(k)$ are the following (see [3]):

(b) If $H = K(Z_2)$ is the Eilenberg-Mac Lane spectra and $\alpha: B(k) \to H$ corresponds to $1 \in H^0(B(k); Z_2)$, then $\alpha_*: B(k)_q(X) \to H_q(X; Z_2)$ is an epimorphism for $q < 2k + 2$ and X a CW-complex.

(c) If M^n is a smooth n-manifold, ν the normal bundle, and $T(\nu)$ the Thom spectrum of ν, then $\alpha_*: B(k)^q(T(\nu)) \to H^q(T(\nu); Z_2)$ is an epimorphism if $n - q < 2k + 2$.

(d) $\pi_i(B(k)) \approx (\Lambda^k)_i$ for $i \leqslant 2k$, where Λ^k is the graded vector space with basis the symbols λ_I, $I = (i_1, \ldots, i_l)$, $2i_j \geqslant i_{j+1}$, $i_l \geqslant k$, $\dim \lambda_I = \Sigma i_j$.

Along the way to proving Theorem B, we prove the following characterization of $B(k)$.

THEOREM 2.1. *If Y is a spectrum which is trivial at odd primes and Y satisfies properties* (a) *and* (b), *then Y is homotopy equivalent to $B(k)$.*

One may easily verify that the following sequence of A-modules is exact:

$$0 \to M_{[k/2]} \xrightarrow{\alpha} M_k \xrightarrow{\beta} M_{k-1} \to 0,$$

where $\alpha(1) = \chi(\mathrm{Sq}^k)$ and $\beta(1) = 1$.

THEOREM 2.2. *The maps α and β may be realized by a cofibration $B(k-1) \to B(k) \to S^k B([k/2])$ and hence there is a map h: $S^{k-1}B([k/2]) \to B(k-1)$ such that*

$$B(k) \simeq B(k-1) \cup_h C\left(S^{k-1}B([k/2])\right).$$

In §3 we recall some results of [3], prove a lemma characterizing the k-invariants of $B(k)$ and prove Theorems 2.1 and 2.2. In §4 we make some calculations in the Adams spectral sequence of $C_k^1 \wedge K(Z_2, 1)$. In making these calculations we utilize the following results of Mahowald [11].

Let f: $\Omega S^3 \to \Omega S^5$ be the first James-Hopf invariant map. Then Ωf: $\Omega^2 S^3 \to \Omega^2 S^5$ defines a stable map g: $C_{2k}^1 \to C_k^3 = S^{2k} C_k^1$.

THEOREM 2.3. *C_{2k}^1 and C_{2k+1}^1 satisfy property* (a). *Furthermore, there is a commutative diagram*

$$\begin{array}{ccc} H^*(S^{2k}C_k^1) & \xrightarrow{g^*} & H^*(C_{2k}^1) \\ \wr\wr & & \wr\wr \\ M_{[k/2]} & \xrightarrow{\alpha} & M_k \end{array}$$

In §5 we prove Theorem B and in §6 and §7 we prove Theorem A.

3. The k-invariants of $B(k)$. Throughout this section, k is a fixed integer. In [3, (5.1)], a collection of spectra E_q and L_q and maps e_q: $L_q \to E_{q-1}$ were constructed. Also a functor χ on spectra was defined. Let $Y_q = \chi(E_q)$, $K_q = \chi(L_q)$ and $\gamma_q = \chi(e_q)$. Suppose N is a smooth, closed, compact, n-manifold, ν is its normal bundle, $T(\nu)$ is the Thom spectrum of ν (the Thom class is in $H^0(T(\nu))$) and $v \in H^p(T(\nu))$. We will say that (N, v) is adapted to M_k if $n - p < 2k + 2$ and

$$0 \to A\left\{\chi(\mathrm{Sq}^i) | i > k\right\} \to A \xrightarrow{v^*} H^*(T(\nu))$$

is exact, where $v^*(a) = av$. In §4 we describe an A-free acyclic resolution of M_k,

$$\to C_q \xrightarrow{d_q} C_{q-1} \to \cdots \to C_0 \xrightarrow{\varepsilon} M_k \to 0.$$

PROPOSITION 3.1. (i) *$Y_0 = K_0$ and $K_0, K_1, K_2, \ldots$ are generalized Eilenberg-Mac Lane spectra with $\pi_*(K_q)$ a graded Z_2 vector space. Also $K_0 = H$.*

(ii) *Y_q may be taken as a fibration over Y_{q-1} with fibre K_q and k-invariant γ_q (γ_q has degree $+1$). $H^*(K_q) = C_q$ and d_q: $C_q \to C_{q-1}$ is realized by the composition*

$$K_{q-1} \xrightarrow{i} Y_{q-1} \xrightarrow{\gamma_q} K_q$$

where i is the inclusion of the fibre.

(iii) *Suppose N is a smooth, compact, closed, n-manifold, $v \in H^p(T(\nu_N))$ and $n - p < 2k + 2$. Then any lifting of v: $T(\nu_N) \to H = Y_0$ to Y_{q-1} lifts to Y_q. Furthermore, if (N, v) is adapted to M_k and $\tilde{v}$: $T(\nu_N) \to Y_{q-1}$ is such a lifting, then γ_q is the unique map such that $(\gamma_q i)^* = d_q$ and $\gamma_q \tilde{v} = 0$.*

PROOF. The properties of χ and [3, (5.1)] yield (i) and (ii).

Let ch( ) = Hom(, R/Z). For any CW-complex X,

$$(Y_q)_p(X) = \mathrm{ch}\big(\chi(Y_q)^p(X)\big) = \mathrm{ch}\big((E_q)^p(X)\big).$$

To prove the first part of (iii) we wish to show that

$$(Y_q)^p(T(\nu)) \to (Y_{q-1})^p(T(\nu))$$

is an epimorphism for all $q > 0$ and $n - p < 2k + 2$. By S-duality this is equivalent to

$$(Y_q)_p(N) \to (Y_{q-1})_p(N)$$

being an epimorphism for $p < 2k + 2$, which in turn, is equivalent to

$$(E_{q-1})^p(N) \to (E_q)^p(N)$$

being a monomorphism for $p < 2k + 2$. By [3, (5.1)(ii)], $L_q \xrightarrow{e_q} E_{q-1} \to E_q$ is a fibration and by (5.2)(iv), $L_{q,2k} \to E_{q-1,2k+1}$ is zero. (L_q and E_q are Ω-spectra.) The desired result now follows since

$$L_q^{p-1}(N) \xrightarrow{e_{q*}} E_{q-1}^p(N) \to E_q^p(N)$$

is exact and $e_{q*} = 0$ for $p < 2k + 2$.

Suppose N and $\tilde{v}$ are as above and (N, v) is adapted to M_k. Then $\gamma_q \tilde{v} = 0$ by the above. Since $\to Y_q \to Y_{q-1} \to$ is constructed from an acyclic resolution of M_k, the image of $H^*(Y_{q-1})$ in $H^*(Y_q)$ is M_k and thus

$$0 \to M_k \to H^*(Y_{q-1}) \xrightarrow{i^*} H^*(K_{q-1})$$

is exact. The map $\tilde{v}$: $H^*(Y_{q-1}) \to H^*(T(\nu))$ factors through M_k and hence splits the above exact sequence. Therefore γ_q^*: $H^*(K_q) \to H^*(Y_{q-1})$, and hence γ_q, is uniquely determined by the conditions that $\gamma_q \tilde{v} = 0$ and $(\gamma_q i)^* = d_q$.

We define $B(k) = \text{proj lim } Y_q$.

In §5 we construct (N, v) adapted to M_k, but in fact, it is easy to see that they exist from results in [4].

COROLLARY 3.2. *Suppose Y is a spectrum which is trivial at odd primes, $H^*(Y) \approx M_k$ and 1: $Y \to H$ represents $1 \in M_k$. If for some (N, v) adapted to M_k there is a map $\tilde{v}$: $T(\nu_N) \to Y$ such that $1\tilde{v} = v$, then Y and $B(k)$ are homotopy equivalent.*

PROOF. We lift $Y \to H = Y_0$ to Y_q by induction on q. Consider the commutative diagram:

$$\begin{tikzcd} & Y \arrow[r, "f"] \arrow[d] & Y_{q-1} \arrow[r, "\gamma_q"] \arrow[d] & K_q \\ T(\nu) \arrow[ur, "\tilde{v}"] \arrow[r, "v"] & H & = Y_0 & \end{tikzcd}$$

By (3.1)(iii), $\gamma_q f\tilde{v} = 0$. Furthermore, $\tilde{v}^*$: $H^*(Y) \to H^*(T(\nu))$ is a monomorphism. Hence $\gamma_q f = 0$ and therefore, f lifts to Y_q. We may therefore find a map F: $Y \to B(k)$ which induces an isomorphism in cohomology and is thus a homotopy equivalence.

PROOF OF 2.1. Suppose Y is a spectrum satisfying (a) and (b) of §2 and (N, v) is adapted to M_k. By S-duality we have a commutative diagram

$$\begin{array}{ccc} (Y)_{n-p}(N) & \to & H_{n-p}(N) \\ \wr\wr & & \wr\wr \\ Y^p(T(\nu)) & \to & H^p(T(\nu)) \end{array}$$

Since $n - p < 2k + 2$, (b) implies that the horizontal maps are epimorphisms. Therefore there is a map $\tilde{v}$: $T(\nu) \to Y$ such that $1\tilde{v} = v$ and, by 3.2, Y and $B(k)$ are homotopy equivalent.

PROOF OF 2.2. Let $\to Y_q \to Y_{q-1} \to$ be the tower used to construct $B(k)$ and suppose (N, v) is adapted to M_{k-1}. One can lift 1: $B(k-1) \to H = Y_0$ to $B(k)$ just as in the proof of 3.2 to obtain a map f: $B(k-1) \to B(k)$ realizing β: $M_k \to M_{k-1}$.

Define a spectrum Z by $S^kZ = B(k) \cup_f B(k-1)$ and let g: $B(k) \to Z$ be the map of degree k corresponding to the inclusion map of $B(k)$ in S^kZ. Then $H^*(Z) = M_{[k/2]}$. Suppose (N, v) is adapted to M_k. Then $(N, \chi(\mathrm{Sq}^k)v)$ is adapted to $M_{[k/2]}$ and

$$\begin{tikzcd} T(\nu) \arrow[r, "\tilde{v}"] \arrow[dr, "v"'] & B(k) \arrow[r, "g"] \arrow[d, "1"] & Z \arrow[d, "1"] \\ & H \arrow[r, "\chi(\mathrm{Sq}^k)"'] & H \end{tikzcd}$$

commutes. Therefore by 3.2, Z and $B([k/2])$ are homotopy equivalent and the proof of 2.2 is complete.

4. A lemma. Throughout this section if k is an integer, $\bar{k} = [k/2]$. Let ξ_l be the l-plane bundle $C_{2,l} \times_{\Sigma_l} R^l$ over $\bar{C}_{2,l} = C_{2,l}/\Sigma_l$, where Σ_l acts on R^l by permuting the coordinates. Let $t(\xi_0)$ and $T(\xi_l)$ be the Thom space and Thom spectrum of ξ_l, respectively. It is immediate that $t(\xi_l) = C_l^1$. This section is devoted to proving

LEMMA 4.1. *For each $i \geqslant 0$ there is a smooth, closed, compact 2^i-manifold N_i, with normal bundle ν_i, and a map f_i: $\nu_i \to \xi_{2^i}$ such that the Stiefel-Whitney class $w_{2^i-1}(\nu_i) \neq 0$.*

PROOF. Let $K = K(Z_2, 1)$ and let $\iota \in H^1(K)$ be the generator. We first note that it is sufficient to prove there is an $[h] \in \pi_{2^i}(T(\xi_{2^i}) \wedge K)$ which is nonzero on $\mathrm{Sq}^{2^i-1}u \otimes \iota$, where u is the Thom class. For suppose h is such a map and p: $\overline{C}_{2,l} \times K \to \overline{C}_{2,l}$ is the projection. Then $T(p^*\xi_l) = T(\xi_l) \wedge K^+ \supset T(\xi_l) \wedge K$. Making h transverse to the zero section of $p^*\xi_{2^i}$, we obtain a 2^i-manifold N_i and maps f: $\nu_i \to \xi_{2^i}$ and s: $N_i \to K$ such that

$$(f_{N_i} \times s)^*(w_{2^i-1}(\xi_{2^i}) \otimes \iota) = w_{2^i-1}(\nu_i) \cup s \neq 0.$$

We recall some results from [3]. Let Λ be the free associative algebra with unit over Z_2 generated by λ_i, $i = 0, 1, 2, \ldots$, modulo the relations: If $2i < j$,

$$\lambda_i\lambda_j = \sum \binom{s-1}{2s-(j-2i)} \lambda_{i+s}\lambda_{j-s}.$$

Grade Λ by $\dim \lambda_i = i$. Define $\lambda_{-1}\lambda_i$ by the above formula. If $I = (i_1, i_2, \ldots, i_l)$ and $2i_j \geqslant i_{j+1}$, define $\lambda_I = \lambda_{i_1}\lambda_{i_2} \cdots \lambda_{i_l}$ and $l(\lambda_I) = l$. A Z_2-basis for Λ is given by $\{\lambda_I\}$, $\lambda_{(\,)} = 1$. Let $\Lambda^* = \mathrm{Hom}(\Lambda, Z_2)$ and let $\{\lambda^I\}$ be the basis of Λ^* dual to $\{\lambda_I\}$ (λ^I is denoted by λ_I in [3]). Let $\Lambda^k \subset \Lambda^*$ be the subspace generated by $\{\lambda^I | l(\lambda_I) = l,\ i_l \geqslant k\}$. Let C^k be the free left A module generated by Λ_l^k. In [3] it is shown that the following is an A-free acyclic resolution of M_k.

$$\to C_l^k \xrightarrow{d} C_{l-1}^k \to \cdots \to C_0^k \in M_k$$

where

$$d\lambda^I = \sum \lambda^I(\lambda_i\lambda_J)\chi(\mathrm{Sq}^{i+1})\lambda^J$$

where the sum ranges over all J (admissible) and $i = -1, 0, 1, \ldots$; $\varepsilon(1) = 1$. The following two lemmas are easily proved.

LEMMA 4.2. *The map*

$$\mu\colon C^{\bar{k}} \to C^{k-1}/C^k$$

defined by $\mu(\lambda^I) = \lambda^{(I,k-1)}$ is an isomorphism of chain complexes.

Let $J_t = \{\lambda_I |$ the last entry of $I < t\} \subset \Lambda$.

LEMMA 4.3. *J_t is an ideal in Λ and $J_t\lambda_I \subset J_s$ where $s = t + [|\lambda_I|/2]$.*

PROOF. Induction on $l(I)$.

Let γ: $C^{\bar{k}} \to C^{k-1}$ be the A-linear map defined by $\gamma(\lambda^I) = \lambda^{(I,k-1)}$. Then 4.2 shows that $d\gamma + \gamma d = 0 \bmod C^k$ and hence we may define a map α: $C^{\bar{k}} \to C^k$ by $\alpha = d\gamma + \gamma d$. Then 4.2 and 4.3 yield the following lemma.

LEMMA 4.4. *The map α is A-linear, $d\alpha = \alpha d$ and $\alpha(\lambda^{(\,)}) = \chi(\mathrm{Sq}^k)\lambda^{(\,)}$ and hence α: $C^{\bar{k}} \to C^k$ is a map of resolutions over the map α: $M_{\bar{k}} \to M_k$ of* 2.2.

We next construct a resolution of $\overline{M}_k = M_k \otimes H^*(K)$. Let $\overline{C}_l^k = C_l^k \otimes H^*(K)$ with the diagonal A-module structure, that is

$$a(x \otimes y) = \sum a_i'(x) \otimes a_i''(y).$$

Let

$$\bar{d} = d \otimes \mathrm{id}\colon \overline{C}_l^k \to \overline{C}_{l-1}^k, \quad \bar{\varepsilon} = \varepsilon \otimes \mathrm{id}\colon \overline{C}_0^k \to \overline{M}_k, \quad \bar{\alpha} = \alpha \otimes \mathrm{id}\colon \overline{C}^{\bar{k}} \to \overline{C}^k.$$

LEMMA 4.5.

$$\cdots \to \overline{C}_l^k \xrightarrow{\bar{d}} \overline{C}_{l-1}^k \to \cdots \to \overline{C}_0^k \to \overline{M}_k \to 0$$

is an A-free, acyclic resolution of $\overline{M}_k$, $\bar{\alpha}$ is a chain map, and $\bar{\varepsilon}\bar{\alpha} = (\alpha \otimes \mathrm{id})\bar{\varepsilon}$. Furthermore,

$$\bar{\alpha}^*\colon \mathrm{Hom}_A\left(\overline{C}_l^k, Z_2\right) \to \mathrm{Hom}_A\left(\overline{C}_l^{\bar{k}}, Z_2\right)$$

is zero in dimensions $< 2k$.

PROOF. The first part of 4.5 is immediate from 4.4. Suppose $v \in \mathrm{Hom}_A(\overline{C}_l^k, Z_2)$, $|v| < 2k$, and $\lambda^I \otimes \iota^l \in \overline{C}_l^{\bar{k}}$, $|\lambda^I \otimes \iota^l| = |\alpha^* v| = |v| - k$. In $\overline{C}^k$,

$$\begin{aligned} \chi(\mathrm{Sq}^j)\lambda^I \otimes \iota^l &= \sum \chi(\mathrm{Sq}^t)(\lambda^I \otimes \mathrm{Sq}^{j-t}\iota^l) \\ &= \sum \chi(\mathrm{Sq}^t)\binom{l}{j-t}(\lambda^I \otimes \iota^{l+j-t}). \end{aligned}$$

By 4.3, if $\lambda^I \in C^m$, $\lambda^I(\lambda_j\lambda_J) = 0$ for $j + 1 + [|\lambda_J|/2] < m$. Consider

$$\alpha^* v(\lambda^I \otimes \iota^l) = v\big((d\gamma + \gamma d)(\lambda^I) \otimes \iota^l\big).$$

Since

$$\begin{aligned} v(\gamma d\lambda^I \otimes \iota^l) &= \sum v\big(\lambda^I(\lambda_j\lambda_J)\chi(\mathrm{Sq}^{j+1})\lambda^{(J,k-1)} \otimes \iota^l\big) \\ &= \sum \binom{l}{j+1}\lambda^I(\lambda_j\lambda_J)v(\lambda^{(J,k-1)} \otimes \iota^{l+j+1}), \end{aligned}$$

this is zero as $\binom{l}{j+1} = 0$ for $j + 1 > l$, and for $j + 1 \leq l$,

$$j + 1 + \left[|\lambda_J|/2\right] = j + 1 + \left[(|\lambda_I| - j)/2\right] = \left[(j + 1 + |\lambda_I|)/2\right] < \bar{k}.$$

The same argument shows that $v(d\gamma(\lambda^I \otimes \iota^l)) = 0$ and the proof of 4.5 is complete.

Let $1 \otimes (\iota^l)^*$ denote the element of $\mathrm{Hom}_A(\overline{C}_0^k, Z_2)$ which is one on $1 \otimes \iota^l$.

LEMMA 4.6. *On $1 \otimes (\iota^{2^{l+1}})^* \in \mathrm{Hom}_A(\overline{C}_0^{2^l}, Z_2)$,*

$$\bar{\alpha}^*\left(1 \otimes (\iota^{2^{l+1}})^*\right) = 1 \otimes (\iota^{2^l})^*.$$

PROOF.

$$\bar{\alpha}(1 \otimes \iota^{2^i}) = \chi(\mathrm{Sq}^{2^i})(1) \otimes \iota^{2^i} = \sum \chi(\mathrm{Sq}^t)(1 \otimes \mathrm{Sq}^{2^i - t}\iota^{2^i}) = 1 \otimes \iota^{2^{i+1}}.$$

In [11], the following is proved:

LEMMA 4.7. *$H^*(T(\xi_k)) \approx M_{\bar{k}}$ and there is a map g: $T(\xi_{2^{i+1}}) \to T(\xi_{2^i})$ such that $g^* = \alpha$: $M_{2^i - 1} \to M_{2^i}$.*

Thus $g \wedge \mathrm{id}$: $T(\xi_{2^{i+1}}) \wedge K \to T(\xi_{2^i}) \wedge K$ realizes $\bar{\alpha}$: $\overline{M}_{2^i-1} \to \overline{M}_{2^i}$. Therefore $g \wedge \mathrm{id}$ induces a map of the corresponding Adams spectral sequences, which on the E_1 level is,

$$\alpha^*: \mathrm{Hom}_A\left(\bar{C}^{2^i}, Z_2\right) \to \mathrm{Hom}_A\left(\bar{C}^{2^{i-1}}, Z_2\right).$$

We show that $1 \otimes (\iota^{2^i})^*$ lives to E_∞ for all i. Suppose $d_s(1 \otimes (\iota^{2^i})) = 0$ for all i and all $s < r$. Then

$$d_r\left(1 \otimes (\iota^{2^i})^*\right) = d_r\left(\alpha^*\left(1 \otimes (\iota^{2^{i+1}})^*\right)\right) = \alpha^*\left(d_r\left(1 \otimes (\iota^{2^{i+1}})^*\right)\right) = 0,$$

since α^* is zero on $\mathrm{Hom}_A(\bar{C}^{2^i}, Z_2)$ in dimensions $< 2^{i+1}$.

Let $[h] \in \pi_{2^i}(T(\xi_{2^i}) \wedge K)$ represent $1 \otimes (\iota^{2^i})^*$. Since $\chi(\mathrm{Sq}^{2^i-1})\iota = \iota^{2^i}$ and

$$\bar{\varepsilon}\left(\sum \mathrm{Sq}^j 1 \otimes \chi(\mathrm{Sq}^{2^i-1-j})\iota\right) = (\varepsilon \otimes \mathrm{id})\left(\mathrm{Sq}^{2^i-1}1 \otimes \iota\right) = \mathrm{Sq}^{2^i-1}1 \otimes \iota,$$

h is nonzero on $\mathrm{Sq}^{2^i-1}u \otimes \iota$ and the proof of 4.1 is complete.

5. Proof of Theorem B. Note the diagonal map of A induces a map

$$\mu: M_{k+l} \to M_k \otimes M_l.$$

LEMMA 5.1. *μ is an injection if $k < l = 2^i$ and when $k = l = 2^i$, the kernel is $\{0, \mathrm{Sq}^{2^{i+2}-1}\}$.*

PROOF. In [3] it is shown that $M_k = \{\chi(\mathrm{Sq}^I) | I = (i_1, \ldots, i_l)$ is admissible and $i_1 \leqslant k\}$.

One may easily verify 5.1 directly for $l = 1$. We prove 5.1 by induction on k and i. Suppose 5.1 is true for $i - 1$. Then induction on k and the following diagram give 5.1 for i.

$$\begin{array}{ccccc}
M_{\bar{k}+2^{i-1}} & \to & M_{\bar{k}} & \otimes & M_{2^{i-1}} \\
\downarrow \alpha & & & \downarrow & \\
M_{k+2^i} & \to & M_k & \otimes & M_{2^i} \\
\downarrow & & & \downarrow & \\
M_{k-1+2^i} & \to & M_{k-1} & \otimes & M_{2^i}
\end{array}$$

Let F: $\xi_k \times \xi_l \to \xi_{k+l}$ be the bundle map defined as follows: Let p: $R^2 \to R$ be the first coordinate. If $x = \{x_1, \ldots, x_k\} \in C_{2,k}, y = \{y_1, \ldots, y_l\} \in C_{2,l}$, $z = (z_1, \ldots, z_k) \in R^k$ and $w = (w_1, \ldots, w_l) \in R^l$, let $F(\{x, z\}, \{y, w\}) = \{u, (z, w)\}$ where

$$\begin{aligned} u &= x_i - \left(\max\{p(x_j)\} + 1, 0\right), \qquad i \leqslant k, \\ &= y_{i-k} + \left(\min\{p(y_j)\} + 1, 0\right), \quad k < i \leqslant k + l. \end{aligned}$$

Since $T(F)$: $T(\xi_k) \wedge T(\xi_l) \to T(\xi_{k+l})$ carries the Thom class to the tensor product of Thom classes, $T(F)$ realizes μ: $M_{\bar{k}+\bar{l}} \to M_{\bar{k}} \otimes M_{\bar{l}}$ if l is even.

Proof of Theorem B. If Q_l is an l-manifold, let ν_l denote its normal bundle and $u_l \in H^0(T(\nu_l))$ the Thom class. We construct a Q_l and maps g_l: $\nu_l \to \xi_l$ by induction on l such that (Q_l, u_l) is adapted to $M_{\bar{l}}$. Since $M_0 = Z_2$ and $\xi_1 = R$, we may take $Q_1 = S^1$. Suppose $Q_{l'}$ has been defined for $l' < l$. Let k be the least positive integer such that $l = 2^i + k$. If $k < 2^i$, let $Q_l = Q_k \times Q_{2^i}$ and let g_l be the composition

$$\nu_k \times \nu_{2^i} \xrightarrow{g_k \times g_{2^i}} \xi_k \times \xi_{2^i} \xrightarrow{F} \xi_{k+2^i}.$$

By 5.1,

$$T(g_l)^*: M_{\bar{l}} = H^*(T(\xi_l)) \to H^*(T(\nu_l))$$

is an injection and hence (Q_l, u_l) is adapted to $M_{\bar{l}}$. Suppose $k = 2^i$. By 4.2 there is a 2^{i+1}-manifold N and a map f: $\nu_N \to \xi_{2^{i+1}}$ such that $T(f^*)(\mathrm{Sq}^{2^{i+1}-1}u) \neq 0$. Let $Q_{2^{i+1}} = Q_{2^i} \times Q_{2^i} \cup N$ and $g_{2^{i+1}} = F(g_{2^i} \times g_{2^i}) \cup f$. Again by 5.1, $(Q_{2^{i+1}}, u_{2^{i+1}})$ is adapted to M_{2^i}.

Theorem B now follows from 3.2 since $T(g_k)$: $T(\nu_k) \to T(\xi_k)$ is the required lifting.

6. $H_*(K(B_k, 1))$. The homological properties of $K(B_k, 1) = C_{2,k}/\Sigma_k$ have been studied by Fadell and Neuwirth **[8]**, Fox and Neuwirth **[9]**, Arnold **[1]**, Fuks **[10]**, Birman **[2]**, and F. Cohen et al. **[5]**. May **[5**, Theorem 5.11**]** constructs a map j_k: $C_{2,k}/\Sigma_k \to (\Omega^2 S^2)_0$ such that j_{k_*}: $H_*(C_{2,k}/\Sigma_k) \to H_*((\Omega^2 S^2)_0)$ is a monomorphism and such that $\cup j_k$: $\cup C_{2,k}/\Sigma_k \to (\Omega^2 S^2)_0$ is a homotopy equivalence. Using j_k, F. Cohen **[5]** computes $H_*(C_{2,k}/\Sigma_k; Z_p)$ as a module over A_p. His results concerning the A-action are incorrect; the following theorem is a corrected version.[2]

Theorem 6.1. *$H_*(C_{2,k}/\Sigma_k; Z_2) \subset P[e_j]$ is generated by monomials $(e_{j_1})^{r_1} \cdots (e_{j_t})^{r_t}$ such that $\sum_{i=1}^t r_i 2^{j_i} \leqslant k$, where $|e_j| = 2^j - 1$. If Sq^s_* is defined to be the dual of Sq^s, then Sq^s_* is determined by the formulae*:

$$\mathrm{Sq}^{2^s}_*(e_j) = 0 \quad \text{if } s > 0,$$

$$\mathrm{Sq}^1_*(e_{j+1}) = e_j^2 \quad \text{if } j \geqslant 1,$$

$$\mathrm{Sq}^1_*(e_1) = 0.$$

$H_*(C_{2,k}/\Sigma_k; Z_p) \subset E(\lambda) \otimes E(e_j) \otimes P[\beta e_j]$ is generated by monomials

[2]F. Cohen was aware of these corrections and agrees with them.

$\lambda^l(\beta^{\varepsilon_1}e_{j_1})^{r_1}\cdots(\beta^{\varepsilon_t}e_{j_t})^{r_t}$ *such that* $2(l+\sum_{i=1}^t r_i p^{j_i}) \leqslant k$, *where* $|\lambda| = 1$, *and* $|\beta^\varepsilon e_j| = 2p^j - 1 - \varepsilon$. P_*^s *is determined by the formulae*:

$$P_*^{p^s}(e_j) = 0,$$

$$P_*^{p^s}(\beta e_j) = 0 \quad \text{if } s > 0,$$

$$P_*^1(\beta e_j) = -(\beta e_{j-1})^p \quad \text{if } j \geqslant 2,$$

$$P_*^1(\beta e_1) = 0.$$

We will also need the results of May [5] and F. Cohen [5] on $H_*(\Omega^2 S^3)$. Define $\xi_j = Q^{j-1}(\iota_1) \in H_*(\Omega^2 S^3; Z_2)$, $j \geqslant 1$ where $\iota_1 \in H_1(\Omega^2 S^3; Z_2)$ is the generator. Define $\mathrm{wt}(\xi_j) = 2^{j-1}$, $\mathrm{wt}(x\cdot y) = \mathrm{wt}(x) + \mathrm{wt}(y)$. For $p > 2$, define $\xi_j = Q_1^j(\iota_1) \in H_*(\Omega^2 S^3; Z_p)$, $j \geqslant 1$. Define $\mathrm{wt}(\xi_j) = p^j$, $\mathrm{wt}(\iota_1) = 1$.

THEOREM 6.2. $H_*(C_k^1; Z_2) \subset H_*(\Omega^2 S^3; Z_2) = Z_2[\xi_j]$ *is generated by all monomials of* wt k.

$H_*(C_k^1; Z_p) \subset H_*(\Omega^2 S^3; Z_p) = E(\iota_1) \otimes E(\xi_j) \otimes P[\beta\xi_j]$ *is generated by all monomials of* wt k.

The Nishida relations with lower indices read:

$$\mathrm{Sq}_*^{2s} Q_1(x) = Q_1 \mathrm{Sq}_*^s(x), \quad \mathrm{Sq}_*^{2s+1} Q_1(x) = |x|\big(\mathrm{Sq}_*^s(x)\big)^2,$$

$$P_*^{p^s} Q_1(x) = Q_1 P_*^s(x),$$

$$P_*^s Q_1(x) = 0 \quad \text{if } s \not\equiv O(p),$$

$$P_*^{p^s}\beta Q_1(x) = \beta Q_1 P_*^s(x), \qquad P_*^{p^{s+1}}\beta Q_1(x) = -Q_0 P_*^s \beta(x),$$

$$P_*^s \beta Q_1(x) = 0 \quad \text{if } r \not\equiv 0, 1(p).$$

COROLLARY 6.3. *The Steenrod operations on the elements in* 6.2 *are determined by the following formulae*:

$$\mathrm{Sq}_*^{2^s}(\xi_j) = 0 \quad \text{if } s > 0,$$

$$\mathrm{Sq}_*^1(\xi_j) = \xi_{j-1}^2 \quad \text{if } j > 1,$$

$$\mathrm{Sq}_*^1(\xi_1) = 0, \quad P_*^{p^s}(\xi_j) = 0,$$

$$P_*^{p^s}(\beta\xi_j) = 0 \quad \text{if } s > 0,$$

$$P_*^1(\beta\xi_j) = -(\beta\xi_{j-1})^p \quad \text{if } j \geqslant 2,$$

$$P_*^1(\beta\xi_1) = 0.$$

THEOREM 6.4. $H^*(C_{2,k}/\Sigma_k)$ *and* $H^*(\bigvee_{i=1}^{[k/2]} C_i^1)$ *are isomorphic as modules over the Steenrod algebra for* $p = 2$ *or* p *odd.*

PROOF. We define an isomorphism

$$\theta\colon H_*(C_{2,k}/\Sigma_k) \to H_*\left(\bigvee_{i=1}^{[k/2]} C_i^1\right) \subset H_*(\Omega^2 S^3)$$

as follows. If $p = 2$, define $\theta(e_j) = \xi_j$. If $p > 2$, define $\theta(\lambda) = \iota_1$, $\theta(e_j) = \xi_j$, $\theta(\beta e_j) = \beta\xi_j$, and extend θ multiplicatively. θ commutes with the action of the Steenrod algebra by 6.1 and 6.3. If a monomial in $H_*(C_{2,k}/\Sigma_k)$ is such that $\Sigma_{i=1}^{t} r_i 2^{j_i} \leq r$ or $2(l + \Sigma_{i=1}^{t} r_i p^{j_i}) \leq r$, then its image under θ has wt $\leq k/2$ and conversely. Since $H_*(\bigvee_{i=1}^{[k/2]} C_i^1) \subset H_*(\Omega^2 S^3)$ consists of all monomials of wt $\leq k/2$, this proves that θ is an isomorphism.

COROLLARY 6.5.

$$H^*(C_{2,k}/\Sigma_k;\, Z_2) = \bigoplus_{i=1}^{[k/2]} S^i M_{[i/2]},$$

$$H^*(C_{2,k}/\Sigma_k;\, Z_p) = \bigoplus_{\substack{i=1\\ i\equiv 0,\,1(p)}}^{[k/2]} S^{2[i/p](p-1)+\varepsilon} M_{[i/p]},$$

where $\varepsilon = 0$ if $i \equiv 0$ (p) and $\varepsilon = 1$ if $i \equiv 1$ (p), and $M_k = A/A\{\chi(\beta^\varepsilon \mathcal{P}^i) | 2pi + \varepsilon > 2k\}$.

PROOF. The case $p = 2$ follows by Mahowald's results and the case p odd follows from results of R. Cohen [7] on $H^*(C_i;\, Z_p)$.

COROLLARY 6.6. *$H^*(C_{2,k}/\Sigma_k;\, Z_2)$ is generated as a module over the Steenrod algebra by elements of dimension $\leq k/2$. $H^*(C_{2,k}/\Sigma_k;\, Z_p)$ is generated as a module over the Steenrod algebra by elements of dimension $\leq 2[[k/2]/p](p-1) + \varepsilon$, where $\varepsilon = 1$ if $[k/2] \equiv 1(p)$ and 0 otherwise.*

7. Proof of Theorem A. Let $Z \to \Omega S^2$ be the universal covering space. Let $h\colon \Omega S^3 \to Z$ be a lifting of $\Omega H\colon \Omega S^3 \to \Omega S^2$. Then h is a homotopy equivalence, and let $g\colon Z \to \Omega S^3$ be a homotopy inverse. Let $b\colon \Omega^2 S^3 \to \bigvee_{i=1}^{\infty} C_i^1$ be a stable homotopy inverse to the Snaith map. Let $p\colon \bigvee_{i=1}^{\infty} C_i^1 \to \bigvee_{i=1}^{[k/2]} C_i^1$ be the projection. Define $f\colon C_{2,k}/\Sigma_k \to \bigvee_{i=1}^{[k/2]} C_i^1$ by $f = pb(\Omega g) j_k$, where $j_k\colon C_{2,k}/\Sigma_k \to (\Omega^2 S^2)_0 = \Omega(Z)$. Theorem A is proved if we show that f_* is an isomorphism on $H_*(\ ;\, Z_p)$, all p. b_* and $(\Omega g)_*$ are isomorphisms as b is a homotopy equivalence and Ωg is a stable homotopy equivalence. If $p = 2$, p_* is an isomorphism in dimension $\leq k/2$ and so is j_{k*}, as the lowest dimensional element in $H_*(C_{2,k+1}/\Sigma_{k+1};\, Z_2)$ which is not in the image $H_*(C_{2,k}/\Sigma_k;\, Z_2) \to H_*(C_{2,k+1}/\Sigma_{k+1};\, Z_2)$ is $(e_1)^{(k+1)/2}$ or $(e_1)^{(k+2)/2}$. Hence f_* is an isomorphism in dimensions $\leq k/2$. By Corollary 6.6, f^* is onto the generators of $H^*(C_{2,k}/\Sigma_k;\, Z_2)$ over A and hence onto. By 6.4, both sides have the same rank and thus f^* is an isomorphism and so is f_*. The argument for p odd is similar with $k/2$ replaced by

$$2\left[\frac{[k/2]}{p}\right](p-1)+\varepsilon.$$

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