ON THE STABLE DECOMPOSITION OF $\Omega^2 S^{r+2}$

BY

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Abstract. In this paper we show that $\Omega^2 S^{r+2}$ is stably homotopy equivalent to a wedge of suspensions of other spaces $C^1_k$, and that $C^1_k$ is homotopy 2-equivalent to the Brown-Gitler spectrum.

1. Introduction. In this paper we show that $\Omega^2 S^{r+2}$ is stably homotopy equivalent to a wedge of suspensions of other spaces $C^1_k$, that $C^1_k$ cannot be further decomposed into a wedge, and that $C^1_k$ is homotopy 2-equivalent to the Brown-Gitler spectrum $B([k/2])$ [3].

Let

$$C'_k = C_{2,k} \vee \Sigma^k \left( \bigwedge S^r \right),$$

where $C_{n,k}$ is the space of $k$ distinct points in $\mathbb{R}^n$. Snaith [12] showed that $\vee_{k=1}^{\infty} C'_k$ is stably homotopy equivalent to $\Omega^2 S^{r+2}$, if $r > 0$. F. Cohen, Mahowald, and Milgram [6] showed that

$$C^1_k = \begin{cases} S^{k(r-1)}C^1_k & \text{if } r \text{ odd,} \\ S^{kr}C^0_k & \text{if } r \text{ even.} \end{cases}$$

Our two main results are the following.

**Theorem A.** $C^0_k$ is stably homotopy equivalent to $(\vee_{i=1}^{[k/2]} C^1_i) \vee S^0$.

**Theorem B.** $C^1_k$ is homotopy 2-equivalent to $S^{kr}B([k/2])$.

Thus, $\Omega^2 S^{r+2}$ is stably homotopy equivalent to a wedge of suspensions of $C^1_i$. More precisely, we have the following corollary.

**Corollary C.** $\Omega^2 S^{r+2}$ is stably homotopy equivalent to $\vee_{k=1}^{\infty} S^{k(r-1)}C^1_k$ if $r$ is odd and to

$$\vee_{k=1}^{\infty} \left( S^{kr} \vee \vee_{i=1}^{[k/2]} S^{kr}C^1_i \right)$$
for $r$ even and positive. For completeness, we note that a component of $\Omega^2 S^2$ is homotopy equivalent to $\Omega^2 S^3$. Furthermore, each piece of $\Omega^2 S^{r+2}$ is homotopy 2-equivalent to a Brown-Gitler spectrum.

We note that $C^0_k = C_{2,k}/\Sigma_k \cup$ (base point) and that $C_{2,k}/\Sigma_k = K(B_k, 1)$, where $B_k$ is the $k$th braid group [2]. Hence Theorems A and B describe the stable homotopy type of $K(B_k, 1)$.

Finally, we note that $C^1_k$ cannot be further decomposed into a wedge. However, our low-dimensional computations lead us to believe that $C_{3,k}$ can be decomposed into much smaller pieces.

2. Results about the Brown-Gitler spectrum. Let $A$ be the mod two Steenrod algebra, $\chi: A \to A$ the canonical antiautomorphism, and define $M_k$ to be the $A$-module:

$$M_k = A/A\{\chi(Sq^i)| i > k\}.$$ 

One of the properties of the spectrum $B(k)$ is that

$$\chi^*(B(k); Z_2) \cong M_k.$$ 

In the course of proving that $h_1h_i$ represents a homotopy element [11], Mahowald proves that $H^*(C^0_k; Z_2) \cong S^kM_{[k/2]}$. If it had been known that $C^1_k$ and $S^k B([k/2])$ were homotopy 2-equivalent, Mahowald’s proof could have been simplified. (Unfortunately, our proof of Theorem B does not simplify Mahowald’s proof, since we use his technique to prove B.) J. F. Adams noted that property (a) does not characterize $B(k)$. Other properties of $B(k)$ are the following (see [3]):

(b) If $H = K(Z_2)$ is the Eilenberg-Mac Lane spectra and $\alpha: B(k) \to H$ corresponds to $1 \in H^0(B(k); Z_2)$, then $\alpha_*: B(k)_q(X) \to H_q(X; Z_2)$ is an epimorphism for $q < 2k + 2$ and $X$ a CW-complex.

c) If $M^n$ is a smooth $n$-manifold, $\nu$ the normal bundle, and $T(\nu)$ the Thom spectrum of $\nu$, then $\alpha_*: B(k)^q(T(\nu)) \to H_q(T(\nu); Z_2)$ is an epimorphism if $n - q < 2k + 2$.

(d) $\pi_i(B(k)) \cong (\Lambda^k)^i$, for $i < 2k$, where $\Lambda^k$ is the graded vector space with basis the symbols $\lambda_i, I = (i_1, \ldots, i_l), 2i_j > i_{j+1}, i_j > k, \dim \lambda_I = \Sigma i_j$.

Along the way to proving Theorem B, we prove the following characterization of $B(k)$.

**Theorem 2.1.** If $Y$ is a spectrum which is trivial at odd primes and $Y$ satisfies properties (a) and (b), then $Y$ is homotopy equivalent to $B(k)$.

One may easily verify that the following sequence of $A$-modules is exact:

$$0 \to M_{[k/2]} \xrightarrow{\alpha} M_k \xrightarrow{\beta} M_{k-1} \to 0,$$

where $\alpha(1) = \chi(Sq^k)$ and $\beta(1) = 1$. 

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Theorem 2.2. The maps $\alpha$ and $\beta$ may be realized by a cofibration $B(k - 1) \to B(k) \to S^k B([k/2])$ and hence there is a map $h: S^{k-1} B([k/2]) \to B(k - 1)$ such that

$$B(k) = B(k - 1) \cup_h C(S^{k-1} B([k/2])).$$

In §3 we recall some results of [3], prove a lemma characterizing the $k$-invariants of $B(k)$ and prove Theorems 2.1 and 2.2. In §4 we make some calculations in the Adams spectral sequence of $C_k \wedge K(Z_2, 1)$. In making these calculations we utilize the following results of Mahowald [11].

Let $f: \Omega S^3 \to \Omega S^5$ be the first James-Hopf invariant map. Then $\Omega f: \Omega^2 S^3 \to \Omega^2 S^5$ defines a stable map $g: C_{2k} \to C_k^3 = S^2 C^1_k$.

Theorem 2.3. $C_{2k}$ and $C_{2k+1}$ satisfy property (a). Furthermore, there is a commutative diagram

$$\begin{array}{ccc}
H^*(S^{2k} C_k^1) & \xrightarrow{\delta^*} & H^*(C_{2k}) \\
\downarrow & & \downarrow \\
M_{[k/2]} & \xrightarrow{\alpha} & M_k
\end{array}$$

In §5 we prove Theorem B and in §6 and §7 we prove Theorem A.

3. The $k$-invariants of $B(k)$. Throughout this section, $k$ is a fixed integer. In [3, (5.1)], a collection of spectra $E_q$ and $L_q$ and maps $e_q: L_q \to E_{q-1}$ were constructed. Also a functor $\chi$ on spectra was defined. Let $Y_q = \chi(E_q)$, $K_q = \chi(L_q)$ and $\gamma_q = \chi(e_q)$. Suppose $N$ is a smooth, closed, compact, $n$-manifold, $\nu$ is its normal bundle, $T(\nu)$ is the Thom spectrum of $\nu$ (the Thom class is in $H^0(T(\nu))$) and $\nu \in H^p(T(\nu))$. We will say that $(N, \nu)$ is adapted to $M_k$ if $n - p < 2k + 2$ and

$$0 \to A \{ \chi(S^q)[i > k] \} \to A \xrightarrow{\nu^*} H^*(T(\nu))$$

is exact, where $\nu^*(a) = a\nu$. In §4 we describe an $A$-free acyclic resolution of $M_k$,

$$C_q \xrightarrow{d_q} C_{q-1} \to \cdots \to C_0 \to M_k \to 0.$$

Proposition 3.1. (i) $Y_0 = K_0$ and $K_0, K_1, K_2, \ldots$ are generalized Eilenberg-Mac Lane spectra with $\pi_*(K_q)$ a graded $\mathbb{Z}_2$ vector space. Also $K_0 = H$.

(ii) $Y_q$ may be taken as a fibration over $Y_{q-1}$ with fibre $K_q$ and $k$-invariant $\gamma_q$ ($\gamma_q$ has degree + 1). $H^*(K_q) = C_q$ and $d_q: C_q \to C_{q-1}$ is realized by the composition

$$K_{q-1} \xrightarrow{i} Y_{q-1} \xrightarrow{\gamma_q} K_q$$

where $i$ is the inclusion of the fibre.
(iii) Suppose $N$ is a smooth, compact, closed, $n$-manifold, $v \in H^p(T(v_N))$ and $n - p < 2k + 2$. Then any lifting of $v$: $T(v_N) \to H = Y_0$ to $Y_q$ lifts to $Y_{q_x}$. Furthermore, if $(N, v)$ is adapted to $M_k$ and $\tilde{v}: T(v_N) \to Y_{q-1}$ is such a lifting, then $\gamma_q$ is the unique map such that $(\gamma_q)i^* = d_q$ and $\gamma_q\tilde{v} = 0$.

**Proof.** The properties of $\chi$ and [3, (5.1)] yield (i) and (ii).

Let $\text{ch}(\cdot) = \text{Hom}(\cdot, R/Z)$. For any CW-complex $X$,

$$(Y_q)_p(X) = \text{ch}(\chi(Y_q)_p^*(X)) = \text{ch}(E_q)_p^*(X)).$$

To prove the first part of (iii) we wish to show that

$$(Y_q)_p^*(T(v)) \to (Y_{q-1})_p^*(T(v))$$

is an epimorphism for all $q > 0$ and $n - p < 2k + 2$. By S-duality this is equivalent to

$$(Y_q)_p(N) \to (Y_{q-1})_p(N)$$

being an epimorphism for $p < 2k + 2$, which in turn, is equivalent to

$$(E_{q-1})_p^*(N) \to (E_q)_p^*(N)$$

being a monomorphism for $p < 2k + 2$. By [3, (5.1)(ii)], $L_q \to E_{q-1} \to E_q$ is a fibration and by (5.2)(iv), $L_{q-2k} \to E_{q-1,2k+1}$ is zero. ($L_q$ and $E_q$ are $\Omega$-spectra.) The desired result now follows since

$$L_{q-1}^p(N) \to E_{q-1}^p(N) \to E_q^p(N)$$

is exact and $e_q^* = 0$ for $p < 2k + 2$.

Suppose $N$ and $\tilde{v}$ are as above and $(N, v)$ is adapted to $M_k$. Then $\gamma_q\tilde{v} = 0$ by the above. Since $\to Y_{q-1} \to$ is constructed from an acyclic resolution of $M_k$, the image of $H^*(Y_{q-1})$ in $H^*(Y_q)$ is $M_k$ and thus

$$0 \to M_k \to H^*(Y_{q-1}) \to H^*(K_{q-1})$$

is exact. The map $\tilde{v}: H^*(Y_{q-1}) \to H^*(T(v))$ factors through $M_k$ and hence splits the above exact sequence. Therefore $\gamma_q^*: H^*(K_q) \to H^*(Y_{q-1})$, and hence $\gamma_q^*$ is uniquely determined by the conditions that $\gamma_q\tilde{v} = 0$ and $(\gamma_qi)^* = d_q$.

We define $B(k) = \text{proj lim} Y_q$.

In §5 we construct $(N, v)$ adapted to $M_k$, but in fact, it is easy to see that they exist from results in [4].

**Corollary 3.2.** Suppose $Y$ is a spectrum which is trivial at odd primes, $H^*(Y) \simeq M_k$ and 1: $Y \to H$ represents 1 $\in M_k$. If for some $(N, v)$ adapted to $M_k$ there is a map $\tilde{v}: T(v_N) \to Y$ such that $1\tilde{v} = v$, then $Y$ and $B(k)$ are homotopy equivalent.

**Proof.** We lift $Y \to H = Y_0$ to $Y_q$ by induction on $q$. Consider the
commutative diagram:

\[
\begin{array}{ccc}
  Y & \xrightarrow{f} & Y_{q-1} \\
  \uparrow{\sim} & & \downarrow{\gamma_q} \\
  T(v) & \xrightarrow{v} & H = Y_0
\end{array}
\]

By (3.1)(iii), \( \gamma_q f \bar{u} = 0 \). Furthermore, \( \bar{\delta}^* : H^*(Y) \to H^*(T(v)) \) is a monomorphism. Hence \( \gamma_q f = 0 \) and therefore, \( f \) lifts to \( Y_q \). We may therefore find a map \( F : Y \to B(k) \) which induces an isomorphism in cohomology and is thus a homotopy equivalence.

**Proof of 2.1.** Suppose \( Y \) is a spectrum satisfying (a) and (b) of §2 and \( (N, v) \) is adapted to \( M_k \). By S-duality we have a commutative diagram

\[
\begin{array}{ccc}
  (Y)_{n-p}(N) & \xrightarrow{\bar{\delta}^*} & H_{n-p}(N) \\
  \downarrow{\bar{\delta}} & & \downarrow{\bar{\delta}} \\
  Y^p(T(v)) & \xrightarrow{\bar{\delta}^*} & H^p(T(v))
\end{array}
\]

Since \( n - p < 2k + 2 \), (b) implies that the horizontal maps are epimorphisms. Therefore there is a map \( \bar{\delta} : T(v) \to Y \) such that \( \bar{\delta} = v \) and, by 3.2, \( Y \) and \( B(k) \) are homotopy equivalent.

**Proof of 2.2.** Let \( \to Y_q \to Y_{q-1} \to \) be the tower used to construct \( B(k) \) and suppose \( (N, v) \) is adapted to \( M_{k-1} \). One can lift \( 1 : B(k - 1) \to H = Y_0 \) to \( B(k) \) just as in the proof of 3.2 to obtain a map \( f : B(k - 1) \to B(k) \) realizing \( \beta : M_k \to M_{k-1} \).

Define a spectrum \( Z \) by \( S^k Z = B(k) \cup_f B(k - 1) \) and let \( g : B(k) \to Z \) be the map of degree \( k \) corresponding to the inclusion map of \( B(k) \) in \( S^k Z \). Then \( H^*(Z) = M_{[k/2]} \). Suppose \( (N, v) \) is adapted to \( M_k \). Then \( (N, \chi(S^k) v) \) is adapted to \( M_{[k/2]} \) and

\[
\begin{array}{ccc}
  T(v) & \xrightarrow{\bar{\delta}} & B(k) \xrightarrow{g} Z \\
  \uparrow{\sim} & \downarrow{1} & \downarrow{1} \\
  H & \xrightarrow{\chi(S^k)} H
\end{array}
\]

commutes. Therefore by 3.2, \( Z \) and \( B([k/2]) \) are homotopy equivalent and the proof of 2.2 is complete.

**4. A lemma.** Throughout this section if \( k \) is an integer, \( \tilde{k} = [k/2] \). Let \( \xi_i \) be the \( l \)-plane bundle \( C_{2,\tilde{l}} \times_{\Sigma_i} R^l \) over \( C_{2,l} = C_{2,l}/\Sigma_l \) where \( \Sigma_l \) acts on \( R^l \) by permuting the coordinates. Let \( t(\xi_0) \) and \( T(\xi_i) \) be the Thom space and Thom spectrum of \( \xi_i \), respectively. It is immediate that \( t(\xi_i) = C_{l} \). This section is devoted to proving
Lemma 4.1. For each $i > 0$ there is a smooth, closed, compact $2^i$-manifold $N_i$, with normal bundle $v_i$, and a map $f_i: v_i \to \varepsilon_{2^i}$ such that the Stiefel-Whitney class $w_{2^i-1}(v_i) \neq 0$.

Proof. Let $K = K(Z_2, 1)$ and let $\iota \in H^1(K)$ be the generator. We first note that it is sufficient to prove there is an $[h] \in \pi_{2^i}(T(\varepsilon_{2^i}) \land K)$ which is nonzero on $S^2_{2^i-1}u \land \iota$, where $u$ is the Thom class. For suppose $h$ is such a map and $p: C_{2^i} \times K \to C_{2^i}$ is the projection. Then $T(p^*\xi) = T(\xi) \land K^+ \supset T(\xi) \land K$. Making $h$ transverse to the zero section of $p^*\varepsilon_{2^i}$, we obtain a $2^i$-manifold $N_i$ and maps $f: v_i \to \varepsilon_{2^i}$ and $s: N_i \to K$ such that

$$(f_{N_i} \times s)^*(w_{2^i-1}(\varepsilon_{2^i}) \land \iota) = w_{2^i-1}(v_i) \land s \neq 0.$$ 

We recall some results from [3]. Let $\Lambda$ be the free associative algebra with unit over $Z_2$ generated by $\lambda_i$, $i = 0, 1, 2, \ldots$, modulo the relations: If $2i < j$,

$$\lambda_i\lambda_j = \sum \left(2s - (j - 2i)\right)\lambda_i+s\lambda_j-s.$$

Grade $\Lambda$ by $\dim \lambda_i = i$. Define $\lambda_{i-1}\lambda_i$ by the above formula. If $I = (i_1, i_2, \ldots, i_l)$ and $2i_j > j_{j+1}$, define $\lambda_I = \lambda_{i_1}\lambda_{i_2}\cdots\lambda_{i_l}$ and $l(I) = I$. A $Z_2$-basis for $\Lambda$ is given by $\{\lambda_I\}$, $\lambda(I) = 1$. Let $\Lambda^* = \text{Hom}(\Lambda, Z_2)$ and let $\{\lambda^I\}$ be the basis of $\Lambda^*$ dual to $\{\lambda_I\}$ ($\lambda^I$ is denoted by $\lambda_I$ in [3]). Let $\Lambda^k \subset \Lambda^*$ be the subspace generated by $\{\lambda^I|l(I) = I, i_j > k\}$. Let $C^k$ be the free left $A$ module generated by $\Lambda^k$. In [3] it is shown that the following is an $A$-free acyclic resolution of $M_k$.

$$\to C^k \to C_{i-1} \to \cdots \to C_0 \in M_k$$

where

$$d\lambda^I = \sum \lambda^J(\lambda_I\lambda_J)(\text{Sq}^{i+1})\lambda^J$$

where the sum ranges over all $J$ (admissible) and $i = -1, 0, 1, \ldots$; $\varepsilon(1) = 1$. The following two lemmas are easily proved.

Lemma 4.2. The map

$$\mu: C^k \to C^{k-1}/C^k$$

defined by $\mu(\lambda^I) = \lambda^{(t(k-1))}$ is an isomorphism of chain complexes.

Let $J_t = \{\lambda_I|\text{the last entry of } I < t\} \subset \Lambda$.

Lemma 4.3. $J_t$ is an ideal in $\Lambda$ and $J_t\lambda_I \subset J_s$ where $s = t + [\lambda_I]/2$.

Proof. Induction on $l(I)$.

Let $\gamma: C^k \to C^{k-1}$ be the $A$-linear map defined by $\gamma(\lambda^I) = \lambda^{(t(k-1))}$. Then 4.2 shows that $d\gamma + \gamma d = 0$ mod $C^k$ and hence we may define a map $\alpha: C^k \to C^k$ by $\alpha = d\gamma + \gamma d$. Then 4.2 and 4.3 yield the following lemma.
Lemma 4.4. The map $\alpha$ is $A$-linear, $d\alpha = ad$ and $\alpha(Sq^k) = \chi(Sq^k)\alpha$ and hence $\alpha: C^k \to C^k$ is a map of resolutions over the map $\alpha: M_k \to M_k$ of 2.2.

We next construct a resolution of $\overline{M}_k = M_k \otimes H^*(K)$. Let $\overline{C}^k = C_k \otimes H^*(K)$ with the diagonal $A$-module structure, that is

$$a(x \otimes y) = \sum a'_i(x) \otimes a''_i(y).$$

Let

$$d = d \otimes \text{id}: \overline{C}^k \to \overline{C}^k_{i-1}, \quad \varepsilon = \varepsilon \otimes \text{id}: \overline{C}^k \to M_k, \quad \overline{\alpha} = \alpha \otimes \text{id}: \overline{C}^k \to \overline{C}^k.$$

Lemma 4.5.

$$\cdots \to \overline{C}^k_1 \xrightarrow{d} \overline{C}^k_0 \to \cdots \to \overline{C}^k_0 \to M_k \to 0$$

is an $A$-free, acyclic resolution of $\overline{M}_k$, $\overline{\alpha}$ is a chain map, and $\overline{\varepsilon}\overline{\alpha} = (\alpha \otimes \text{id})\overline{\varepsilon}$.

Furthermore,

$$\overline{\alpha}^*: \text{Hom}_A(\overline{C}^k, Z_2) \to \text{Hom}_A(\overline{C}^k, Z_2)$$

is zero in dimensions $< 2k$.

Proof. The first part of 4.5 is immediate from 4.4. Suppose $v \in \text{Hom}_A(\overline{C}^k, Z_2)$, $|v| < 2k$, and $\lambda^l \otimes \iota^l \in \overline{C}^k_1$, $|\lambda^l \otimes \iota^l| = |\alpha^*v| = |v| - k$. In $C^k$,

$$\chi(Sq^j)\lambda^l \otimes \iota^l = \sum \chi(Sq^j)(\lambda^l \otimes Sq^{j-l} \iota^l)$$

$$= \sum \chi(Sq^j)(j+1-l)(\lambda^l \otimes \iota^{l+j-l}).$$

By 4.3, if $\lambda^l \in C^m$, $\lambda^l(\alpha_\lambda\lambda_j) = 0$ for $j + 1 + [\lambda_j]/2 < m$. Consider

$$\alpha^*v(\lambda^l \otimes \iota^l) = v((d\gamma + \gamma d)(\lambda^l \otimes \iota^l)).$$

Since

$$v(\gamma d\lambda^l \otimes \iota^l) = \sum v(\lambda^l(\lambda_\lambda\lambda_j)(\lambda(Sq^{j+1})(\lambda^{j+1}) \otimes \iota^l))$$

$$= \sum (j+1)(\lambda^l(\lambda_\lambda\lambda_j)v(\lambda^{j+1} \otimes \iota^{j+1}),$$

this is zero as $[\lambda_\lambda]/2 = 0$ for $j + 1 > l$, and for $j + 1 < l$,

$$j + 1 + [\lambda_j]/2 = j + 1 + [(\lambda_j - j)/2] = [(j + 1 + \lambda_j)/2] < \overline{k}.$$

The same argument shows that $v(d\gamma(\lambda^l \otimes \iota^l)) = 0$ and the proof of 4.5 is complete.

Let $1 \otimes (\iota^l)^* \in \text{Hom}_A(\overline{C}^k, Z_2)$ which is one on $1 \otimes \iota^l$.

Lemma 4.6. On $1 \otimes (\iota^{l+1})^* \in \text{Hom}_A(\overline{C}^k, Z_2)$,

$$\overline{\alpha}^*(1 \otimes (\iota^{l+1})^*) = 1 \otimes (\iota^l)^*.$$
Proof.

\[ \alpha((1 \otimes \iota')^r) = \chi(Sq^r)(1 \otimes \iota') = \sum \chi(Sq^r)(1 \otimes Sq^{2^r-1} \iota^r) = 1 \otimes \iota^{2^r+1}. \]

In [11], the following is proved:

Lemma 4.7. \( H^*(T(\xi_k)) \approx M_k \) and there is a map \( g: T(\xi_{2^r+1}) \to T(\xi_2) \) such that \( g^* = \alpha: M_{2^r-1} \to M_{2^r-2} \).

Thus \( g \wedge id: T(\xi_{2^r+1}) \wedge K \to T(\xi_2) \wedge K \) realizes \( \alpha: M_{2^r-1} \to M_{2^r-2} \). Therefore \( g \wedge id \) induces a map of the corresponding Adams spectral sequences, which on the \( E_1 \) level is,

\[ \alpha^*: \text{Hom}_A(\overline{C}^{2^r}, Z_2) \to \text{Hom}_A(\overline{C}^{2^r-1}, Z_2). \]

We show that \( 1 \otimes (\iota^r)^* \) lives to \( E_\infty \) for all \( r \). Suppose \( d_r(1 \otimes (\iota^r)^*) = 0 \) for all \( r \) and all \( s < r \). Then

\[ d_r(1 \otimes (\iota^r)^*) = d_r(\alpha^*(1 \otimes (\iota^{2^r+1})^*))) = \alpha^*(d_r(1 \otimes (\iota^{2^r+1})^*)) = 0, \]

since \( \alpha^* \) is zero on \( \text{Hom}_A(\overline{C}^{2^r}, Z_2) \) in dimensions \( < 2^{r+1} \).

Let \( [h] \in \pi_2(T(\xi_2) \wedge K) \) represent \( 1 \otimes (\iota^r)^* \). Since \( \chi(Sq^{2^r-1}) = 2^r \) and

\[ \bar{\varepsilon} \left( \sum Sq^1 \otimes \chi(Sq^{2^r-1-1}) \iota \right) = (\varepsilon \otimes id)(Sq^{2^r-1-1} \otimes \iota) = Sq^{2^r-1} \otimes \iota, \]

\( h \) is nonzero on \( Sq^{2^r-1} \) and the proof of 4.1 is complete.

5. Proof of Theorem B. Note the diagonal map of \( A \) induces a map

\[ \mu: M_{k+1} \to M_k \otimes M_r. \]

Lemma 5.1. \( \mu \) is an injection if \( k < r = 2^l \) and when \( k = r = 2^l \), the kernel is \( \{0, Sq^{r+1} \} \).

Proof. In [3] it is shown that \( M_k = (\chi(Sq^l)|I = (i_1, \ldots, i_r) \) is admissible and \( i_1 < k \).

One may easily verify 5.1 directly for \( r = 1 \). We prove 5.1 by induction on \( k \) and \( i \). Suppose 5.1 is true for \( i = 1 \). Then induction on \( k \) and the following diagram give 5.1 for \( i \).

\[
\begin{array}{ccc}
M_{k+2^r-1} & \to & M_k \otimes M_{2^r-1} \\
\downarrow \alpha & & \downarrow \\
M_{k+2^r} & \to & M_k \otimes M_{2^r} \\
\downarrow & & \downarrow \\
M_{k-1+2^r} & \to & M_{k-1} \otimes M_{2^r}
\end{array}
\]

Let \( F: \xi_k \times \xi_l \to \xi_{k+l} \) be the bundle map defined as follows: Let \( p: R^2 \to R \) be the first coordinate. If \( x = \{x_1, \ldots, x_k\} \in C_{2,k}, y = \{y_1, \ldots, y_l\} \in C_{2,l}, z = \{z_1, \ldots, z_k\} \in R^k \) and \( w = \{w_1, \ldots, w_l\} \in R^l \), let \( F(\{x, z\}, \{y, w\}) = \{u, (z, w)\} \) where
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$$u = x_i - \left( \max \{ p(x_j) \} + 1, 0 \right), \quad i < k,$$

$$= y_{i-k} + \left( \min \{ p(y_j) \} + 1, 0 \right), \quad k < i < k + l.$$

Since $T(F): T(\xi_k) \land T(\xi_l) \to T(\xi_{k+l})$ carries the Thom class to the tensor product of Thom classes, $T(F)$ realizes $\mu: M_{k+l} \to M_k \otimes M_l$ if $l$ is even.

**Proof of Theorem B.** If $Q_l$ is an $l$-manifold, let $\nu_l$ denote its normal bundle and $u_l \in H^0(T(\nu_l))$ the Thom class. We construct a $Q_l$ and maps $g_i: V_i \to \xi_i$ by induction on $l$ such that $(Q_l, u_l)$ is adapted to $M_l$. Since $M_0 = Z_2$ and $\xi_1 = R$, we may take $Q_1 = S^1$. Suppose $Q_l$ has been defined for $l' < l$. Let $k$ be the least positive integer such that $l = 2^l + k$. If $k < 2^l$, let $Q_l = Q_k \times Q_{2^l}$ and let $g_l$ be the composition

$$g_l: \nu_k \times \nu_{2^l} \to \xi_k \times \xi_{2^l} \to \xi_{k+2^l}.$$

By 5.1,

$$T(g_l)^*: M_l = H^*(T(\xi_l)) \to H^*(T(\nu_l))$$

is an injection and hence $(Q_l, u_l)$ is adapted to $M_l$. Suppose $k = 2^l$. By 4.2 there is a $2^{l+1}$-manifold $N$ and a map $f: V_l \to \xi_{2^l}$ such that $T(f^*)(S^{2^{l+1}})$ is a homotopy equivalence. Using $j_k$, F. Cohen [5] computes $H_*(C_{2k}/\Sigma_k; Z_p)$ as a module over $A_p$. His results concerning the $A$-action are incorrect; the following theorem is a corrected version.\(^2\)

**Theorem 6.1.** $H_*(C_{2k}/\Sigma_k; Z_p) \subset P[e_j]$ is generated by monomials $(e_j)^{t_1} \cdots (e_j)^{t_r}$ such that $\sum_{i=1}^r t_i 2^i < k$, where $|e_j| = 2^l - 1$. If $Sq^*$ is defined to be the dual of $Sq^*$, then $Sq^*$ is determined by the formulae:

$$Sq^r(e_j) = 0 \quad if \ s > 0,$$

$$Sq^1(e_{j+1}) = e_j^2 \quad if \ j > 1,$$

$$Sq^0(e_j) = 0.$$

$H_*(C_{2k}/\Sigma_k; Z_p) \subset E(\lambda) \otimes E(e_j) \otimes P[\beta e_j]$ is generated by monomials

\(^2\)F. Cohen was aware of these corrections and agrees with them.
\[ \lambda^i(\beta^*e_j)^{\lambda_1} \cdots (\beta^*e_j)^{\lambda_q} \text{ such that } 2(l + \sum_{j=1}^i p^j) < k, \text{ where } |\lambda| = 1, \text{ and } |eta^*e_j| = 2p^j - 1 - e. P^* is determined by the formulae:}

\[
P^\ast_{\ast}(e_j) = 0,
\]

\[
P^\ast_{\ast}(\beta e_j) = 0 \text{ if } s > 0,
\]

\[
P^\ast_1(\beta e_j) = -(\beta e_{j-1})^p \text{ if } j > 2,
\]

\[
P^\ast_1(\beta e_1) = 0.
\]

We will also need the results of May [5] and F. Cohen [5] on \( H_\ast(\Omega^2S^3) \).

Define \( \xi_j = Q^{j-1}(t_1) \in H_\ast(\Omega^2S^3; Z_2), j > 1 \) where \( t_1 \in H_1(\Omega^2S^3; Z_2) \) is the generator. Define \( \text{wt}(\xi_j) = 2^{j-1}, \text{wt}(x \cdot y) = \text{wt}(x) + \text{wt}(y) \). For \( p > 2 \), define \( \xi_j = Q^i(t_1) \in H_\ast(\Omega^2S^3; Z_p), j > 1 \). Define \( \text{wt}(\xi_j) = p^j, \text{wt}(t_1) = 1 \).

**Theorem 6.2.** \( H_\ast(C_k; Z_2) \subset H_\ast(\Omega^2S^3; Z_2) = Z_2[\xi] \) is generated by all monomials of \( \text{wt} k \).

\( H_\ast(C_k; Z_p) \subset H_\ast(\Omega^2S^3; Z_p) = E(t_1) \otimes E(\xi) \otimes P[\beta \xi] \) is generated by all monomials of \( \text{wt} k \).

The Nishida relations with lower indices read:

\[
\text{Sqr}_p(\beta, x) = \beta Q_1(x), \quad \text{Sqr}_p^{j+1}(x) = |x|(\text{Sqr}_p(x))^2,
\]

\[
P^\ast_{\ast}Q_1(x) = Q_1P^\ast_{\ast}(x),
\]

\[
P^\ast_1Q_1(x) = 0 \text{ if } s \not\equiv 0 (p),
\]

\[
P^\ast_1\beta Q_1(x) = \beta Q_1P^\ast_{\ast}(x), \quad P^\ast_{\ast+1}\beta Q_1(x) = -Q_0P^\ast_{\ast}\beta(x),
\]

\[
P^\ast_1\beta Q_1(x) = 0 \text{ if } r \not\equiv 0, 1(p).
\]

**Corollary 6.3.** The Steenrod operations on the elements in 6.2 are determined by the following formulae:

\[
\text{Sqr}_p^{\ast}(\xi_t) = 0 \text{ if } s > 0,
\]

\[
\text{Sqr}_p^{\ast}(\xi_t) = \xi_t^{s-1} \text{ if } j > 1,
\]

\[
\text{Sqr}_p^{\ast}(\xi_t) = 0, \quad P^\ast_{\ast}(\xi_t) = 0,
\]

\[
P^\ast_{\ast}(\beta \xi_t) = 0 \text{ if } s > 0,
\]

\[
P^\ast_1(\beta \xi_t) = -(\beta \xi_{t-1})^p \text{ if } j > 2,
\]

\[
P^\ast_1(\beta \xi_1) = 0.
\]

**Theorem 6.4.** \( H^\ast(C_2^k/\Sigma_k) \) and \( H^\ast(\sqrt{\Sigma_{k-1}}C_1) \) are isomorphic as modules over the Steenrod algebra for \( p = 2 \) or \( p \) odd.
Proof. We define an isomorphism
\[ \theta: H_*(C_{2,k}/\Sigma_k) \to H_* \left( \bigvee_{i=1}^{[k/2]} C_i \right) \subset H_*(\Omega^2S^3) \]
as follows. If \( p = 2 \), define \( \theta(e) = \xi \). If \( p > 2 \), define \( \theta(\lambda) = \iota_1 \), \( \theta(e_j) = \xi_j \), \( \theta(\beta_j) = \beta \xi_j \), and extend \( \theta \) multiplicatively. \( \theta \) commutes with the action of the Steenrod algebra by 6.1 and 6.3. If a monomial in \( H_*(C_{2,k}/\Sigma_k) \) is such that \( \Sigma_{i=1}^j r_{2^i} < r \) or \( 2(l + \Sigma_{i=1}^j r_{p^i}) < r \), then its image under \( \theta \) has wt < \( k/2 \) and conversely. Since \( H_*(\bigvee_{i=1}^{[k/2]} C_i) \subset H_*(\Omega^2S^3) \) consists of all monomials of wt < \( k/2 \), this proves that \( \theta \) is an isomorphism.

Corollary 6.5.
\[ H^*(C_{2,k}/\Sigma_k; Z_2) = \bigoplus_{i=1}^{[k/2]} S^i M_{(i/2)} \]
\[ H^*(C_{2,k}/\Sigma_k; Z_p) = \bigoplus_{i=0,1(p)} S^{2i/p(p-1)+e} M_{i/p} \]
where \( e = 0 \) if \( i \equiv 0 \) (p) and \( e = 1 \) if \( i \equiv 1 \) (p), and \( M_k = A/A(\chi(p^e)[2pi+e > 2k]) \).

Proof. The case \( p = 2 \) follows by Mahowald’s results and the case \( p \) odd follows from results of R. Cohen [7] on \( H^*(C_i; Z_p) \).

Corollary 6.6. \( H^*(C_{2,k}/\Sigma_k; Z_2) \) is generated as a module over the Steenrod algebra by elements of dimension < \( k/2 \). \( H^*(C_{2,k}/\Sigma_k; Z_p) \) is generated as a module over the Steenrod algebra by elements of dimension < \( 2[\lceil k/2 \rceil/p](p-1) + e \), where \( e = 1 \) if \( \lceil k/2 \rceil = l(p) \) and 0 otherwise.

7. Proof of Theorem A. Let \( Z \to \Omega S^2 \) be the universal covering space. Let \( h: \Omega S^3 \to Z \) be a lifting of \( \Omega H: \Omega S^3 \to \Omega S^2 \). Then \( h \) is a homotopy equivalence, and let \( g: Z \to \Omega S^3 \) be a homotopy inverse. Let \( b: \Omega^2 S^3 \to \bigvee_{i=1}^{\infty} C_i \) be a stable homotopy inverse to the Snaithe map. Let \( p: \bigvee_{i=1}^{\infty} C_i \to \bigvee_{i=1}^{[k/2]} C_i \) be the projection. Define \( f: C_{2,k}/\Sigma_k \to \bigvee_{i=1}^{[k/2]} C_i \) by \( f = pb(\Omega g)j_\text{g} \), where \( j_\text{g}: C_{2,k}/\Sigma_k \to (\Omega^2 S^3)_0 = \Omega(Z) \). Theorem A is proved if we show that \( f_* \) is an isomorphism on \( H_*(\ ; Z_p) \), all \( p \). \( b_* \) and \( (\Omega g)_* \) are isomorphisms as \( b \) is a homotopy equivalence and \( \Omega g \) is a stable homotopy equivalence. If \( p = 2, p_* \) is an isomorphism in dimension < \( k/2 \) and so is \( j_{k*} \), as the lowest dimensional element in \( H_*(C_{2,k+1}/\Sigma_{k+1}; Z_2) \) which is not in the image \( H_*(C_{2,k}/\Sigma_k; Z_2) \to H_*(C_{2,k+1}/\Sigma_{k+1}; Z_2) = (e_1)^{(k+1)/2} \) or \( (e_1)^{(k+2)/2} \). Hence \( f_* \) is an isomorphism in dimensions < \( k/2 \). By Corollary 6.6, \( f^* \) is onto the generators of \( H^*(C_{2,k}/\Sigma_k; Z_2) \) over \( A \) and hence onto. By 6.4, both sides have the same rank and thus \( f^* \) is an isomorphism and so is \( f_* \). The argument for \( p \) odd is similar with \( k/2 \) replaced by
2\left\lfloor \frac{k/2}{p} \right\rfloor (p - 1) + e.

BIBLIOGRAPHY


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