ON THE EXCEPTIONAL CENTRAL SIMPLE NON-LIE MALCEV ALGEBRAS

BY

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Abstract. Malcev algebras belong to the class of binary Lie algebras. Any Lie algebra is a Malcev algebra. In this paper we show that for each seven-dimensional central simple non-Lie Malcev algebra any finite dimensional Malcev module is completely reducible also for positive characteristics. This contrasts with each modular semisimple Lie algebra. As a consequence we get that the classical structure theory for characteristic zero is valid also in the modular case if semisimplicity is replaced by $G_1$-separability.

The Wedderburn principal theorem is proved for Malcev algebras.

1. Introduction. Structures in algebra and other fields connected with an alternative Cayley algebra show exceptional features. If $C$ is an alternative algebra one recalls that the commutator algebra $C^-$ with the product defined by $a \circ b := a \cdot b - b \cdot a$ is a Malcev algebra. Let $D$ denote a Cayley algebra over a field $k$ with $\text{char}(k) \neq 2, 3$, and $e$ the unit of $D$. Then any algebra $A$ isomorphic to $D^-/k \cdot e$ is a central simple and non-Lie Malcev algebra and vice versa [4]. $A$ is called an exceptional Malcev algebra of type $G_1$, or of type $G_1$. $A$ is said to be of type $C^-M$ if $A$, or equivalently $D$ is split.

Any Lie algebra is a Malcev algebra. Malcev modules are a generalization of Lie modules over Lie algebras. E. J. Taft conjectured that any finite dimensional Malcev module over a Malcev algebra of type $G_1$ is completely reducible also for positive characteristics. In the following we prove the conjecture for $\text{char}(k) \neq 2, 3$ (Theorem 1). As is well known the analogous statement is false for any simple Lie algebra [3]. If $\text{char}(k) = 0$ the complete reducibility is shown for semisimple Malcev algebras [4]. Our proof applies the classification of irreducible Malcev modules in [1].

The Wedderburn principal theorem was recently extended by E. L. Stitzinger to Malcev algebras if $\text{char}(k) = 0$, and if the radical is $G_1$-potent [7]. We prove the theorem for an arbitrary radical $R$ if $\text{char}(k) = 0$, and for the modular case if $A/R$ is $G_1$-separable.
In the following we denote by $k$ a field with $\text{char}(k) \neq 2, 3$. $A$ and $M$ are presumed finite dimensional $k$-vector spaces.

2. Definitions. Let $A$ be a binary algebra over $k$. $J: A \times A \times A \to A$ denotes the Jacobi map with $(x, y, z) \mapsto J(x, y, z)$ where $J(x, y, z) = xy \cdot z + yz \cdot x + zx \cdot y$. $J$ alternates if $x^2 := x \cdot x = 0$ for all $x \in A$. Let $x, y, z, t \in A$. We recall that a Malcev algebra $A$ is defined by

$$x^2 = 0,$$

and

$$J(x, y, xz) = J(x, y, z)x.$$  

Then we have

$$(xy \cdot z)t + (yz \cdot t)x + (zt \cdot x)y + (tx \cdot y)z = ty \cdot xz,$$

$$2tJ(x, y, z) = J(t, x, yz) + J(t, y, zx) + J(t, z, xy).$$

[5]. An $A$-bimodule is called a Malcev module over a Malcev algebra $A$ if the semidirect sum or trivial extension $E := A \oplus M$ together with the product $(x + m) \cdot (y + m') := xy + xm' + my$ for $m, m' \in M$ is a Malcev algebra. $M$ is called a Lie module over $A$ if $xm = -mx$ and $J(x, y, m) = 0$. For a Malcev module $M$ the module nucleus $N_M$ defined by $N_M := \{m \in M | \forall x, y \in A: J(x, y, m) = 0\}$ is the maximal Lie submodule. Subsequently, $A$ always denotes a Malcev algebra, and $M$ a Malcev module over $A$. We define an $A$-module homomorphism $f$ of $M$ into a second $A$-Malcev module $M'$ by $f(xm) = xf(m)$; if moreover $f$ is injective, $f$ is a monomorphism of the $A$-modules etc.

$M$ is irreducible over $A$ if $M \neq \{0\}$, and $\{0\}$ and $M$ are the only submodules over $A$. If $M = \bigoplus M_i$, $1 \leq i \leq s$, $s \in \mathbb{N}$, each $M_i$ an irreducible submodule, then $M$ is called completely reducible over $A$. Then equivalently for any submodule $P$ there is a submodule $N$ so that $M = P \oplus N$ [3]. If $A$ is canonically considered as an $A$-bimodule, and $M$ isomorphic to $A$ then $M$ is called regular. For $A$ of type $C_\infty$ the irreducible Malcev modules are up to isomorphism the regular module and the one-dimensional zero module [1].

Let $\rho: A \to \text{End}_k(M)$ denote the canonical representation with $\rho(x): m \mapsto mx$. $k[Y]$ denotes the ring of polynomials in the indeterminate $Y$. A map $\varphi$ of $A$ into the subset of irreducible polynomials with $x \mapsto \varphi_x$ is called a primary function, a map of $A$ in $k$ a root. Then

$$M_\varphi := \{m \in M | \forall x \in A \exists r \in \mathbb{N}: (\varphi_x(\rho(x)))^r(m) = 0\}.$$

If $\varphi_x = Y - \gamma(x)$ for any $x$ then $M_\varphi$ is designated by $M_\varphi$ or $M_\varphi(A)$. Set $M_r(A) := \{m \in M | \forall x \in A: \rho(x)(m) = \gamma(x) \cdot m\}$. $M_\varphi$ is called a primary component, and $M_r$ a root space. If $M_\varphi \neq \{0\}$ then $\varphi$ is called characteristic or essential for $M$, and similarly for roots. $A$ splits over $M$ if for any $\rho(x)$ the
roots of its minimum polynomial $m_x$ in $k[Y]$ are in $k$. $M$ is smooth if moreover those roots are distinct that is if any $m_x$ is separable over $k$. A splitting subalgebra is defined in the obvious way.

We recall that for any nilpotent splitting subalgebra $H$ there is a root space decomposition $A = \bigoplus A_\gamma$ with $\gamma \in \Delta$ [4, Lemma 5]; then

$$A_{\beta}A_{\gamma} \subset A_{\beta + \gamma} \quad \text{if } \beta \neq \gamma,$$

$$A_{\beta}^2 \subset A_{2\beta} + A_{-\beta},$$

$$J(A_0, A_\beta, A_\beta) \subset A_{-\beta},$$

$$J(A_0, A_\beta, A_\gamma) = \{0\} \quad \text{if } \beta \neq \gamma,$$

$$J(A_\beta, A_\gamma, A_\delta) = \{0\} \quad \text{if } \beta \neq \gamma \neq \delta \neq \beta,$$

$$J(A_\beta, A_\gamma, A_\gamma) = \{0\} \quad \text{if } \beta \neq 0, \gamma, -\gamma,$$

for $\beta, \gamma, \delta \in \Delta$. A nilpotent subalgebra $H$ of $A$ is called a Cartan subalgebra if $H = A_0$ [4]. $A$ is split if it has a splitting Cartan subalgebra.

Let $Z_3$ denote the integers modulo 3, and let the elements of $Z_3$ be represented by 1, 2, 3. Choose $\nu \in Z_3$. If $A$ is of type $C^\nu$ and $e \in k \setminus \{0\}$, then $A$ has a basis $T_\nu = \{h, x_\nu, x'_\nu \mid \nu \in Z_3\}$ with $x_\nu h = e x_\nu, x'_\nu h = -e x'_\nu, x_\nu x_{\nu+1} = 2x'_{\nu+2}, x'_\nu x'_{\nu+1} = e x_{\nu+2}, x_\nu x'_{\nu+1} = x'_\nu x_{\nu+1} = 0$ [4], [5]. Hence for any $\nu$, $(h, x_\nu, x'_\nu)$ is a basis of a split simple three-dimensional Lie (Malcev) algebra $B$ of type $A_1$. Then $(x_{\nu+1}, x'_{\nu+2})$ is the basis of a non-Lie Malcev module of type $M_2$ over $B$. If $H$ is a splitting Cartan subalgebra of $A$, and $A_\alpha \oplus H \oplus A_{-\alpha}$ the corresponding root space decomposition, then we may choose $T_\nu$ with $H = \langle h \rangle$ and $x_\nu \in A_\alpha$ [4]. The module of type $M_2$ is up to isomorphism the only non-Lie Malcev module over the Lie algebra of type $A_1$ [1].

For two algebras $B, C$ over $k$, $B \oplus C$ denotes their direct product. Similarly we designate the direct product of two $A$-submodules $M_1$ and $M_2$ by $M_1 \oplus M_2$. If $X$ is a vector space over $k$, $x_i \in X$ with $1 < i < r$, $r \in \mathbb{N}$, let $\langle x_1, \ldots, x_r \rangle$ denote the subspace generated by the $x_i$. For a map $f \colon X \rightarrow Y$, $Y$ a set, let $X' := f(X)$. For further definitions see [1], [2], [4].

3. The exceptional decomposition of a module. Theorem 1 is preceded by four lemmas.

**Lemma 1.** Let $A$ be a Malcev algebra, $H$ a nilpotent subalgebra, and $M$ a Malcev module over $A$. If $A = J(A, A, A)$, and $A = \bigoplus A_\gamma$ with $\gamma \in \Delta$ the primary decomposition over $H$ then for $M$ over $H$ we have

$$M = \bigoplus M_\gamma \quad \text{for } \gamma \in \Delta.$$

**Proof.** By base field extension we may consider roots instead of characteristic primary functions. Thus let $A = \bigoplus A_\gamma$, $\gamma \in \Delta$, be a $H$-root
space decomposition. Assume \( M \neq \bigoplus M_\gamma, \gamma \in \Delta \). Then there exists \( M_\beta \neq \{0\} \) with \( \beta \in \Delta \). From (8)-(10) then \( J(M_\beta, A, A) = \{0\} \). By (4)

\[ M_\beta \subset M_\beta A = M_\beta J(A, A, A) \subset J(M_\beta, A, A) = \{0\}. \]

Thus \( M_\beta = \{0\} \), proving the lemma. □

Let \( h \in A, h \neq 0 \). If \( H = \langle h \rangle \) and \( \alpha : H \to k \) a \( k \)-linear map we may identify \( \alpha \) with \( \alpha(h) \). We have

**Lemma 2.** Let \( A \) be a Malcev algebra, \( h \in A \) with \( h \neq 0 \), and \( H = \langle h \rangle \), \( M \) a Malcev module over \( A \). Suppose that \( A \) and \( M \) are smooth over \( H \). The root spaces are taken over \( H \). Let \( A = A_\alpha \oplus H \oplus A_{-\alpha} \) with \( \alpha \neq 0 \). For \( \beta \in \{\alpha, -\alpha\} \) let \( M_{2\beta} = \{0\} \). Then for \( m \in M_\beta, n \in M_0 \) and \( x, y \in A_\beta, y' \in A_{-\beta} \) with \( xy' = \delta h, \delta \in k \) we get

\[
\begin{align*}
mx \cdot y' &= -2my' \cdot x - 2\beta \delta m, \\
mx \cdot y &= -m \cdot xy, \\
nx \cdot y &= -n \cdot xy, \\
nx \cdot y' + \beta \delta n &\in N_M.
\end{align*}
\]

**Proof.** By (6) \( M_\gamma A_{\gamma} \subset M_{-\gamma} \). For (11) observe

\[
\beta mx \cdot x = (mh \cdot y')x \quad \text{and by (3)}
\]

\[
= -(hy' \cdot x)m - (y' \cdot m)h - (xm \cdot h)y' + xh \cdot my'
\]

\[
= \beta \delta hm + \delta hm \cdot h + \beta xm \cdot y' + \beta x \cdot my'
\]

\[
= -2\beta^2 \delta m - \beta mx \cdot y' - \beta my' \cdot x.
\]

Thus \( mx \cdot y' = -2my' \cdot x - 2\beta \delta m \).

To obtain (12), consider

\[
\beta mx \cdot x = (mh \cdot x)x \quad \text{and again by (3)}
\]

\[
= -(xm \cdot h)x + xh \cdot mx = -2\beta mx \cdot x.
\]

Hence \( 3 \beta mx \cdot x = 0 \), therefore, by \( \text{char}(k) \neq 3 \) and \( \beta \neq 0 \), \( mx \cdot x = 0 \).

Linearization gives \( mx \cdot y = -my \cdot x \). Again applying (3)

\[
\beta mx \cdot y = (mh \cdot x)y = -(hx \cdot y)m - (xy \cdot m)h - (ym \cdot h)x + yh \cdot mx
\]

\[
= -\beta m \cdot xy - \beta my \cdot x - \beta mx \cdot y,
\]

and hence \( mx \cdot y = -m \cdot xy \), which is (12).

To establish (13), from (3)

\[
(nx \cdot x)h = -(hx \cdot n)x + hx \cdot nx = -\beta xn \cdot x - \beta x \cdot nx = 2\beta nx \cdot x.
\]

Since \( M_{2\beta} = \{0\} \) then \( nx \cdot x = 0 \). By means of linearization \( nx \cdot y = -ny \cdot x \).

Through further application of (3)

\[
\begin{align*}
ny \cdot x &= \beta^{-1} nx \cdot yh = \beta^{-1} [(yx \cdot h)n + (xh \cdot n)y + (hn \cdot y)x + (ny \cdot x)h]
\end{align*}
\]

\[
= -n \cdot xy - nx \cdot y - ny \cdot x = -n \cdot xy.
\]
Let \( w \in A_\beta \), and \( w' \in A_{-\beta} \). For (14) we obtain by (3) and (13)
\[
(nx \cdot y')w = -(xy' \cdot w)n - (y'w \cdot n)x - (wn \cdot x)y' + wx \cdot ny'
\]
\[
= -\beta \delta nw - (n \cdot wx)y' - ny' \cdot wx = -\beta \delta mw.
\]
Noting (8) then \((nx \cdot y')w' = (ny' \cdot x)w' = -\beta \delta mw'\). Therefore \((nx \cdot y' + \beta \delta n) \cdot A_\gamma = \{0\}\) for \( \gamma = 0, \alpha, -\alpha \) which proves (14).

**Corollary 1.** Let \( A \) be split of type \( A_1 \), \( M \) a Malcev module over \( A \), \( H \) a splitting Cartan subalgebra of \( A \), \( H = \langle h \rangle \), and \( M \) smooth for \( H \). For any root \( \beta \) of \( H \) with \( \beta \neq 0 \) and \( A_\beta \neq \{0\} \) let \( M_{2\beta} = \{0\} \).

Then
\[
M = NM \oplus J(M, A, A).
\]

**Proof.** Let \( M \neq N_M \). Take a basis \( \{x_\alpha, x_{-\alpha}, h\} \) for \( A \) with \( \alpha \in k \setminus \{0\} \), 
\( x_\alpha x_{-\alpha} = h \) and \( x_\beta h = \beta x_\beta \) for \( \beta \in \{\alpha, -\alpha\} \). By smoothness, \( M \) is split over \( H \). Since \( J \) alternates and (7)-(9) then \( M = N_M + (M_\alpha \oplus M_{-\alpha}) \). Let \( m \in M_\beta \) with \( mx_\beta \neq 0 \). By \( M_{2\beta} = \{0\} \) from (11) and (12)
\[
(mx_\beta \cdot x_{-\beta})x_\beta = -2\beta \delta mx_\beta
\]
with \( x_\beta x_{-\beta} = \delta h \neq 0 \). Hence \( \langle mx_\beta \rangle \oplus \langle mx_\beta \cdot x_{-\beta} \rangle \) is an irreducible non-Lie submodule of type \( M_2[1] \). Thus \( P = M_\alpha \cdot x_\alpha \oplus M_{-\alpha} \cdot x_{-\alpha} \) is a sum of submodules of type \( M_2 \). From (11) we have \( M = P + N_M \). Since \( J(mx_\beta, x_\beta, x_{-\beta}) = 3\beta \delta mx_\beta \) this sum is direct. Therefore \( M = N_M \oplus J(M, A, A) \). The complete reducibility of \( J(M, A, A) \) is trivial.

**Lemma 3.** Let \( A \) be of type \( C_\infty \), \( H \) a splitting Cartan subalgebra, and \( M \) an \( A \)-Malcev module.

Then \( M \) is smooth over \( H \).

**Proof.** Since \( N_M \cdot A = \{0\} \) the assertion is trivial for \( M = N_M \). Let \( M \neq N_M \), \( E \) the semidirect sum of \( A \) and \( M \), and \( H = \langle h \rangle \). We consider the root spaces over \( H \). By Lemma 1, \( M_{-\beta} \neq \{0\} \) implies \( A_{-\beta} \neq \{0\} \). Now \( E_0 = H \oplus M_0 \). Since \( J(H, M_0, E) = \{0\} \), from [5, Lemma 5.12] then \( HM_0 \subset N_M \).

Thus \( HM_0 \cdot A = \{0\} \). By this and (8) it follows that \( M_0 A_{\gamma} \subset 1(M_{\gamma}) \). Observing \( A_{-\beta} = A_{-\beta}A_{-\beta} \) for \( \beta \neq 0 \) and (3) one gets \( 1(M_\beta)A_\beta \subset 1(M_{-\beta}) \). Hence the sum of \( H \)-eigen spaces of \( M \) is a submodule.

Let \( n \in M_0, x \in A_\beta, x' \in A_{-\beta}, \beta \neq 0, \) and \( xx' = h \). With (14) and observing \( nh = nx \cdot x' - nx' \cdot x \) together with \( nx' \cdot x \) in \( 1(M_0) \) then
\[
\beta nx \cdot x' = -(nx \cdot x')x \cdot x' = -(nx' \cdot x)x' \cdot x = \beta nx' \cdot x.
\]
Thus \( nh = 0 \), therefore \( M_0 = 1(M_0) \). Consider now \( m \in M_\beta \). We show \( m \in 1(M_\beta) \). Assume that \( mh \neq \beta m \), and set \( \hat{m} := mh - \beta m \). By (8) we then have \( \hat{m}x' = mh \cdot x' - \beta mx' = mx' \cdot h = 0 \), hence \( \hat{m} \cdot A_{-\beta} = \{0\} \). Without
restriction let \( m \in \mathbb{Z} \), and \( h, x^\prime_r, x^\prime_s \) with \( \nu \in \mathbb{Z}_3 \) constitute a basis \( T_\beta \). For 
\( \mu, \nu \in \mathbb{Z}_3 \) with \( \mu \neq \nu \) set 
\( x^\prime_r := x^{\prime \mu}, x^\prime_s := x^{\prime \nu} \). From (11) together with 
\( m \cdot A_{-\beta} = \{0\} \), and (3), (12) one derives 
\( 2\beta \hat{m}y = -(\hat{m}x \cdot x^\prime)y = \beta \hat{m}y \). Thus \( \langle \hat{m} \rangle \) is irreducible over \( A \), implying \( \beta = 0 \) in contradiction to \( \beta \neq 0 \). Therefore 
\( M_\beta = \mathbb{C}(M_\beta) \). \( \square \)

**Lemma 4.** Let \( A \) be split of type \( G_1 \), \( H \) and \( M \) as in Lemma 3, \( H = \langle h \rangle \), and the root spaces taken over \( H \).

If \( \beta \neq 0 \), \( m \in M_\beta \), and \( T_\beta = \{x^\prime_r, x^\prime_s, h|\nu \in \mathbb{Z}_3\} \) then

\[
\sum mx^\prime_r \cdot x^\prime_s = -\beta m \quad \text{for } \nu \in \mathbb{Z}_3. \tag{15}
\]

**Proof.** Set 
\( x := x^\prime_r, y := x^\prime_s, z := x^\prime_t \), and similarly for \( y^\prime, z^\prime \). We get

\[
mx^\prime \cdot x = -\frac{1}{2}(yz \cdot m)x \quad \text{and by (3)}
\]

\[
= \frac{1}{2}\{(zm \cdot x)y + (mx \cdot y)z + (xy \cdot z)m - xz \cdot ym\}
\]

\[
= my^\prime \cdot y - mz^\prime \cdot z + \beta m + my \cdot y' \quad \text{with (12),}
\]

\[
= -my^\prime \cdot y - mz^\prime \cdot z - \beta m \quad \text{by (11)}.
\]

Hence \( mx^\prime \cdot x + my^\prime \cdot y + mz^\prime \cdot z = -\beta m \) which is (15). \( \square \)

For a \( k \)-vector space and an extension field \( K \) of \( k \) we define \( X_K := X \otimes_k K \). We get

**Theorem 1.** Let \( A \) be a Malcev algebra of type \( G_1 \) and \( M \) a Malcev module 
over \( A \).

Then \( M \) is completely reducible over \( A \).

**Proof.** For \( M = N_M \) the theorem is trivial. Suppose \( M \neq N_M \).

(1) Let \( A \) be of type \( \mathbb{C}_M \), and \( H \) a splitting Cartan subalgebra, \( M = M_a \oplus M_0 \oplus M_{-\alpha} \) the root space decomposition of \( M \) over \( H \) with \( \alpha \neq 0 \) according to Lemma 1. If \( M = M_0 \) then \( M = N_M \) by Lemma 3. Thus let \( M_a \neq \{0\} \), and \( m \in M_a \), \( m \neq 0 \). From (15) there exists \( z^\prime \in A_{-\alpha} \) with \( n := mz^\prime \neq 0 \). We show that \( n \) generates a regular submodule by an argument similar to that in [1]. Take the \( k \)-linear map \( f: A \rightarrow M \) defined by

\[
f(h) := n, \quad f(x) := \beta^{-1} xn \quad \text{if } x \in A_\beta
\]

when \( \beta \neq 0 \). We claim that \( f \) is a module homomorphism over \( A \). For 
\( x, y \in A_\beta, \beta \neq 0 \), we obviously have \( xf(h) = f(xh), hf(x) = f(hx) \), and by (13), \( xf(y) = f(xy) \). It remains to show for \( y^\prime \in A_{-\beta} \) that \( y^\prime f(x) = f(y^\prime x) \), equivalently

\[
x \cdot y^\prime = -\beta \eta n \quad \tag{16}
\]

where \( xy^\prime = \delta h, \delta \in k \). We may restrict ourselves to a basis \( T_\beta \) of \( A \) corresponding to \( H \) with \( x^\prime \in A_\beta \). Let \( x^\prime x^\prime_\mu = \delta h, \) and \( x^\prime x^\prime_\eta = \eta h \) with \( \delta, \eta \in k \), where \( \lambda, \mu, \nu \in \mathbb{Z}_3 \). Then by (11) and (12)
\[(mx'_\mu \cdot x'_\mu)x'_\mu = -\frac{1}{2} (mx'_\mu \cdot x'_\mu)x'_\mu - \beta \eta mx'_\lambda\]
\[= -\frac{1}{2} m(x'_\mu \cdot x'_\mu)x'_\mu - \beta \eta mx'_\lambda = -\beta \delta mx'_\lambda.\]

We derive the last equality from the multiplication relations for \(T_\beta\). Hence we have (16). Since \(f(h) = n \neq 0\), \(f\) is an \(A\)-module monomorphism of \(A\) in \(M\).

Thus \(P = M_\mu \cdot x'_1 \oplus M_\mu \oplus M_{-\mu}\) obviously is a direct sum of regular submodules by (15). For \(n' \in M_0\) from (14) then \(p := n'x_1 \cdot x'_1 + \beta n' \in N_M\). Hence \(M_0 = N_M + P_0\). By \(N_M A = \{0\}\) and (16) the sum is direct, and we have \(M = N_M \oplus P. N_M\) and \(P\) are completely reducible over \(A\), hence \(M\) too.

(2) Suppose that \(A\) is not split. By [4] \(A\) has a Cartan subalgebra \(H\) with \(H = \langle h \rangle\). Now \(A = H \oplus A_\mu\) over \(H\) with \(\pi_h = Y^2 - c(h)\) and \(c(h) \in k \setminus k^2\). Then \(M = M_0 \oplus M_\mu\) by Lemma 1. Let \(K\) denote a splitting field of \(\pi_h\).

Set \(P := J(M, A, A)\). Then \(M = N_M \oplus P\) by the corresponding decomposition for \(M_K\). For \(P_0\) we choose a basis \(\{n_i | 1 \leq i < s, s \in \mathbb{N}\}\). Let \(P(i) := k \cdot n_i \oplus A \cdot n_i\). Then \(P(i)_K = K \cdot n_i \oplus A_K \cdot n_i\). \(P(i)_K\) is regular over \(A_K\) by (1). If \(f\) denotes the \(A_K\)-module monomorphism of \(A_K\) in \(M_K\) with \(f(h) = n_i\) then \(f(A)\) is regular over \(A\). Further

\[f(A) = f(k \cdot h) \oplus f(A \cdot h) = k \cdot f(h) \oplus A \cdot f(h) = P(i).\]

\(P(i)\) is irreducible, and \(P = \bigoplus P(i)\) for \(1 \leq i < s\). Hence \(M\) is completely reducible over \(A\).

The theorem is proved. \(\square\)

The following propositions are well known if \(\text{char}(k) = 0\) for the semisimple case [2], [4]. They extend the classical structure theory to positive characteristics for the exceptional case.

**Proposition 2.** Let \(A = \bigoplus A_i\) with \(1 \leq i < r, r \in \mathbb{N}\), where each \(A_i\) is of type \(G_1\). Let \(M\) be a Malcev module over \(A\).

Then \(M\) is completely reducible over \(A\).

Moreover \(M = N_M \oplus (\bigoplus P_j)\) with \(1 \leq j < s, s \in N_0, N_M A = \{0\}\), where for any index \(j\) there is an index \(i\) so that \(P_j\) is regular over \(A_i\), and \(P_j A_i = \{0\}\) if \(i \neq i\) for \(1 \leq i < r\).

**Proof.** If \(r = 1\) the first part of the statement is Theorem 1, and the second is a corollary of the proof of Theorem 1. We proceed by induction on the number of simple direct factors of \(A\), and assume that the statement is valid for \(r \in \mathbb{N}\). Let now \(A = \bigoplus A_i\) with \(1 \leq i < r + 1\). Set \(A' = \bigoplus A_i, 2 \leq i < r + 1\). We choose a Cartan subalgebra \(H\) of \(A\). Then \(H = H_1 \oplus H_2, H_1\) a Cartan subalgebra of \(A_1\) and \(H_2\) a Cartan subalgebra of \(A'\).

If \(N(1)\) designates the nucleus of \(M\) over \(A_1\) then

\[M = N(1) \oplus J(M, A_1, A_1)\]
is a sum of completely reducible $A_i$-submodules of $M$. $J(M, A_1, A_i)$ decomposes into a direct sum of regular $A_i$-submodules. We show that $N(1)$ and $J(M, A_1, A_i)$ are submodules over $A$. For this we may assume that $H$ is splitting over $A$. The root spaces of $A$ for $H$ unequal to $H$ are just those of $A_1$ for $H_1$ and of $A'$ for $H_2$ unequal to $H_1$ and $H_2$. The corresponding characteristic roots $y$ are obvious. When $y(H_1) \neq \{0\}$ then $y(H_2) = \{0\}$ thus $A_y \subset A_1$, and vice versa. Applying (4) and Lemma 3 together with (8)–(10) we get

$$J(M, A_1, A_i)A' \subset J(M, A_1, A') = \{0\}.$$  

For example if $\beta(H_1) \neq \{0\}$ then $\beta(H_2) = \{0\}$ hence $J(M_\beta, A_\beta, H_2) = \{0\}$ by smoothness of $A$ and $M$ over $H_2$. If $\Delta_i$ is the set of the characteristic roots of $H_1$ in $A_1$, let $(A_1)^{\delta} := \oplus (A_1)_\delta(H_1)$ with $\delta \in \Delta_i \setminus \{0\}$. From (8) for $H_1$ one has $N(1)A' \cdot (A_1)^{\delta} = \{0\}$. Noting Lemma 3 then $N(1)A' \cdot A_1 = \{0\}$. Thus $N(1)A' \subset N(1)$.

Hence the above yields a direct sum of $A$-modules. By the induction hypothesis $N(1)$ decomposes as asserted over $A'$. The proposition is evident.

The radical $R$ of $A$ is by definition the unique maximal solvable ideal. $A$ is called semisimple if $R = \{0\}$. Separability is defined as usual. In case of char($k$) = 0 any semisimple Malcev algebra is separable by the nondegeneracy of the Killing form. A is called $G_1$-separable if there is a base field extension $K$ of $k$ so that the base field extension $A_K$ decomposes into a direct sum of algebras of type $C_\infty$.

Since the hypothesis of characteristic 0 in the proof of [2, Theorem 2] is only used to establish that $M$ is reducible that proof actually gives the following slightly stronger result

**Proposition 3.** Let $A$ be a Malcev algebra and $M$ a Malcev module over $A$. If $A$ is $G_1$-separable, then any derivation of $A$ in $M$ is inner. □

**Corollary 2.** Let $A$ be a Malcev algebra, and $C$ a $G_1$-separable subalgebra. Then any derivation of $C$ in $A$ can be extended to an inner derivation of $A$. □

For an ideal $I$ of $A$ let $\mathfrak{C}(I) := I$ and $\mathfrak{C}(I):= \mathfrak{C}^{-1}(I) \cdot I + (\mathfrak{C}^{-1}(I) \cdot I)$ · $A$ if $r \in N$. $I$ is called $\mathfrak{C}$-nilpotent if $\mathfrak{C}(I) = \{0\}$ for some $r \in N$ [2]. The index $n_{\mathfrak{C}}$ of $\mathfrak{C}$-nilpotency is the minimal $n_{\mathfrak{C}} \in N$ with $\mathfrak{C}^{n_{\mathfrak{C}}}(I) = \{0\}$. Nilpotency and $\mathfrak{C}$-nilpotency of $I$ are equivalent. The nilradical $N$ of $A$ is by definition the maximal nilpotent ideal, hence $N \subset R$. We recall, if $B$ is a subalgebra of $A$, and $A = B \oplus R$ then this decomposition is called a Wedderburn or Levi decomposition, and $B$ a Wedderburn or Levi factor of $A$.

Similarly as in [2, Theorem 3] we get as a further consequence of Theorem 1.
Proposition 4. Let $A$ be a Malcev algebra with radical $R$. $n_X$ denotes the index of $X$-nilpotency of the nilradical of $A$. Suppose that $\text{char}(k) > 2n_X - 1$.

Let $B$ be a Levi factor, and $C$ a $G_1$-separable subalgebra of $A$.

Then there is an inner automorphism $a$ of $A$ with

$$C^a \subset B.$$ □

Corollary 3. Let $A$ be as in Proposition 4, and $A/R$ $G_1$-separable. Then any two Levi factors are conjugate by an inner automorphism of $A$. □

4. The Wedderburn splitting. Let $S$ be an ideal of $A$, and $S^2 = \{0\}$. Let $\varphi: A \to A/S$ with $x \mapsto x^S := x + S$ denote the canonical map. If $H$ is a nilpotent subalgebra of $A$ and $\gamma$ a linear root of $H$ we define $\gamma: H^\varphi \to k$ by $\gamma(h) := \gamma(h)$ if $h \in H$. Then obviously

$$(A_\gamma(H))^\varphi = (A^\varphi)_\gamma(H^{\varphi}) \quad \text{and} \quad (A_\gamma(H))^\varphi = A_\gamma(H)/S_\gamma(H). \quad (17)$$

$S$ is a Malcev module over $A^\varphi$ in the canonical way. If $C \subset A^\varphi$ denote $C^\varphi := \varphi^{-1}(C)$. Let $M_\gamma(h) := M_\gamma(\langle h \rangle)$.

Lemma 5. Let $S$ be an ideal of $A$ with $S^2 = \{0\}$, and $L$ an abelian subalgebra of $A/S$. Furthermore, let $S$ be smooth over $L$. Then $A$ contains a subalgebra $H$ with $H^\varphi = L$ and $H^3 = \{0\}$.

Proof. If $\dim(L) = 0$ the assertion is trivial. We use induction on the dimension of $L$ and assume the statement of the lemma for some $n \in \mathbb{N}_0$. Suppose $\dim(L) = n + 1$ and $c \in L$, $c \neq 0$. By the hypothesis of the induction there exists a subalgebra $T$ of $A$ with $T^3 = \{0\}$, $T^\varphi \subset L$, $\dim(T^\varphi) = n$, and $c \in T^\varphi$. Then $T \subset A_\varphi(T)$, and $L \subset A_\varphi(T)^\varphi$ by (17).

We choose $h \in A_\varphi(T)$ with $h = c$. Further $S = \bigoplus S_\gamma(h)$ for $\gamma \in \Delta$ denotes the root space decomposition over $h$. Let $h_i \in T$ for $i = 1, \ldots, n$, the $h_i$ linearly independent. Then $h_i h = \sum \langle r_\gamma \rangle$ with $\gamma \in \Delta$ and $\langle r_\gamma \rangle \in S_\gamma(h) \cap A_\varphi(T)$. Set $h_i^* := h_i - \sum \beta^{-1} r_\beta$ for $\beta \in \Delta \setminus \{0\}$. Note $h_i^* \in A_\varphi(T) \cap A_\varphi(h)$. Let $H$ be the subalgebra of $A$ generated by $h$ and the $h_i^*$ for $i = 1, \ldots, n$. Hence $H \subset A_\varphi(T) \cap A_\varphi(h)$, and $H^2 \subset S_\varphi(L)$. Thus $H^3 = \{0\}$. □

We prove

Theorem 5. Let $A$ be a Malcev algebra over $k$, $R$ the radical of $A$, and $\text{char}(k) = 0$, or $\text{char}(k) > 3$. If $\text{char}(k) > 3$ let $A/R$ be $G_1$-separable.

Then $A$ decomposes

$$A = B \oplus R$$

where $B$ is a semisimple subalgebra of $A$ with $B \cong A/R$.

Proof. If $A/R = \{0\}$ or $R = \{0\}$ then the theorem is trivial. Assume that $A/R \neq \{0\}$ and $R \neq \{0\}$. By standard reduction we may assume $R^2 = \{0\}$, and $R$ an irreducible $A$-Malcev module. Further we may suppose that $k$ is
algebraically closed. So \( A/R = \bigoplus C_i \) with \( 1 \leq i \leq n \), \( n \in \mathbb{N} \), any \( C_i \) a simple split subalgebra. In the course of proof we will distinguish different cases. Let \( A^i = A/R, x^i = x + R \), and \( \varphi: x \mapsto x \).

(1) Let \( \text{char}(k) = 0, A \) a Lie algebra, and \( R \) a Lie module over \( A \). Then \( A \) is a Lie algebra:

\[ J(A, A, A) = \{0\} \text{ there is nothing to show. Otherwise } J(A, A, A) = R. \]

By Lemma 5 there is obviously a Cartan subalgebra \( H \) of \( A \) so that \( H^\varphi \) is a Cartan subalgebra of \( A_\alpha \), and \( H^3 = \{0\} \). Decompose \( A \) into \( H \)-root spaces. Since \( H = A_\alpha \), then \( J(A_\alpha, A_\alpha, A_\alpha) = \{0\} \). From \( \dim((A_\beta)^\varphi) < 1 \) for \( \beta \neq 0 \) and \( R \) Lie we then have \( J(A, A, A) = \{0\} \). Hence \( A \) is a Lie algebra for which the theorem is known.

(2) It remains to treat the case that \( R \) is not a Lie module over \( A \), or \( A \) not a Lie algebra, or \( \text{char}(k) > 3 \) with \( A \) \( G_2 \)-separable. We proceed by induction on the number \( n \) of the simple ideals of \( A \).

Let \( n = 1 \). Suppose that \( A \) is a Lie algebra of type \( A_1 \). Let \( h \in A \) so that \( \langle h \rangle \) is a Cartan subalgebra of \( A \). Decompose \( A \) and \( R \) over \( h \). Let us consider three cases for \( R \). If \( R \) is the one-dimensional zero module then \( R = \langle r_0 \rangle \) and \( A = A_a \oplus A_0 \oplus A_{-a}, \alpha \neq 0 \), with \( R \subset A_0 \). We choose \( h' \in A_a A_{-a} \) with \( h' = h \). Then obviously \( A_a \oplus \langle h' \rangle \oplus A_{-a} \) is a Levi factor of \( A \).

If \( R \) is non-Lie then \( R \) is necessarily of type \( M_2 \) over \( A_1 \), and \( R = R_a \oplus R_{-a} \) with \( R_\beta = \langle r_\beta \rangle \) where \( R_\beta \subset A_\beta \) for \( \beta \in \{\alpha, -\alpha\} \). Let \( x_\beta \in A_\beta \) with \( x_\beta \neq 0 \). Then \( J(x_\beta, x_{-\beta}, A) = 0 \). Any Lie triple of elements generates a Lie subalgebra. Hence \( \langle h, x_\beta, x_{-\beta} \rangle \) is a Levi factor of \( A \).

Assume third that \( R \) is regular over \( A \). Hence \( A = A_a \oplus A_0 \oplus A_{-a} \)

\[ A_0 = \langle h, r_0 \rangle, A_\beta = \langle x_\beta, r_\beta \rangle \text{ with } \beta \in \{\alpha, -\alpha\}, \text{ and } r_\beta = \beta^{-1} x_\beta r_0. \]

Note that a canonically \( A \)-module isomorphism is induced by \( h \mapsto r_0, \) and \( x_\beta \mapsto r_\beta \).

Suppose that \( \{x_\alpha, x_{-\alpha}, h\} \) is a standard basis for \( A \). After eventual substitutions \( h - \gamma r_0/\alpha, \) or \( x_{-\alpha} - \delta r_{-\alpha} \) with \( \gamma, \delta \in k, \) for \( h \) or \( x_{-\alpha} \) if necessary then

\[ \langle x_\alpha, x_{-\alpha}, h \rangle \text{ is a Levi factor of } A. \]

Now let \( A \) be of type \( C_2^r \) and \( R \) regular over \( A \). Take a basis \( T_a \) of \( A \), \( T_a = \{y_i, y_i', u_i | v \in \mathbb{Z}_3 \} \) and set \( C := \langle u, y_1, y_1' \rangle \). \( C \) is a subalgebra of type \( A_1 \). \( R \) has a \( C \)-decomposition

\[ R = B_1 R \oplus B_2 R, \]

with \( B_1 R \) regular and \( N_1 R, N_2 R \) of type \( M_2 \) over \( C \). In view of the minimal solvable ideals of \( C^\varphi \), and its completely reducible radical, \( C^\varphi^{-1} \) contains a Levi factor \( B_1 \). Let \( x, x', h \in B_1 \), with \( x = y_1, x' = y_1', h = u_1, \) and \( H := \langle h \rangle \).

We decompose \( A \) over \( H \) into root spaces, \( A = A_a \oplus A_0 \oplus A_{-a} \).

We claim \( A_\gamma = \langle A_\gamma \rangle \). For \( \nu \in \{2, 3\} \) choose \( x_\nu \in A_\nu, x'_\nu \in A_{-\nu} \) with \( x_\nu = y_\nu, x'_\nu = y'_\nu \).

Let \( r_0 \in R_\gamma, r_0 \neq 0 \). If \( \beta \neq 0 \) and \( z \in A_\beta \) set \( r_z := \beta^{-1} r_\beta \).

Since a Lie triple \( x_\nu, x'_\nu, h \) generates a Lie subalgebra, \( x_\nu h = ax_\nu + \delta_r x_\nu \)

with \( \delta_r \in k \). We show \( \delta_r = 0 \). For
\[ ax_r x \cdot x' = xx_r \cdot x' h \]
\[ = (x' h \cdot x r) x + (hx \cdot x') x_r + (xx_r \cdot x)x h \text{ by (3)} \]
\[ = \alpha x_r x \cdot x' + \delta_x (r x_r \cdot x)x + \alpha x_r h = \alpha x_r h - \delta_x r x h \]
\[ = \alpha x_r x \cdot x' - 3\alpha \delta_x r x, \]

hence \( \delta_x = 0. \)

Therefore \( A_\alpha = A_\alpha (A_\alpha), \) and equally for \(-\alpha. \) Thus \( A \) is smooth for \( H. \)

By Corollary 1, \( A \) is completely reducible over \( B_1. \) Hence
\[ A = B_{1R} \oplus B_1 \oplus N_{1R} \oplus N_{2R} \oplus N_1 \oplus N_2 \]
with \( N_1, N_2 \) of type \( M_2 \) over \( B_1. \) We may assume \( x_2, x'_3 \in N_1 \) and \( x_3, x_2' \in N_2. \)
If \( x_2 x_2' = h + \eta x_0 \) with \( \eta \in k \setminus \{0\}, \) replace \( x_2 \) by \( x_2' := x_2 - \eta x_0. \) Hence we may suppose \( x_2 x_2' = h. \)

We assert that \( B := B_1 \oplus N_1 \oplus N_2 \) is an algebra of type \( C. \) We let \( y := x_2, \)
\( y' := x_2', z := x_3, z' := x_3'. \) Then
\[ yz' = (2a)^{-1} y h \cdot x y = (2a)^{-1} \{(x h \cdot y) y + (y x \cdot h) y\} \text{ by (3)} \]
\[ = xy \cdot y = -2y \cdot z'. \]
Thus \( yz' = 0. \) Similarly \( zy' = 0. \) Further
\[ zz' = (2a)^{-1} x y' \cdot x y, \text{ and with (3)} \]
\[ = \frac{1}{2} \{xx' + yy'\} = h. \]
From this with (3)
\[ yz = a^{-2} z' x' \cdot x' y' = 2x' \]
and similarly \( y' z' = a x. \) Therefore \( B^2 \subset B. \) Hence \( B \) is a Levi factor of \( A. \)

If \( R \) is the one-dimensional zero module, take \( B_1 \) as before. Similarly \( A \) has a \( B_1 \)-module decomposition
\[ A = B_1 \oplus R \oplus N_1 \oplus N_2. \]
By a similar argument one derives that \( B := B_1 \oplus N_1 \oplus N_2 \) is a Levi factor.
Thus the theorem is shown if \( A \) is of type \( A_1 \) or of type \( C_{\infty} \) when \( \text{char}(k) \neq 2, 3. \)
Let \( \text{char}(k) = 0. \) Then by \([1, \text{ Satz 11}]\) we know if \( A \) is a simple Lie algebra not of type \( A_1 \) then \( R \) is a Lie module over \( A, \) and the decomposition exists by (1). Hence we have shown the theorem for \( n = 1. \)

We assume as induction hypothesis that the theorem is valid if \( A \) has exactly \( n \) simple direct factors, \( n \in \mathbb{N}. \) Let \( A = \oplus \text{C}_i, 1 < i < n + 1. \) By (1) and \([1, \text{ Satz 11}]\) the remaining part of the proof is obviously reduced to the case that \( C_i \) is either of type \( C_{\infty}, \) or \( C_i \) is of type \( A_1 \) with \( R \) non-Lie over \( C_i. \)
In the latter case by the classification \( R \) is a module of type \( M_2 \) over \( C_i. \) Set \( G := \oplus \text{C}_i, 2 < i < n + 1. \) In view of \([2, \text{ Theorem 1}]\) or of Proposition 2 respectively, we have either \( RC_i = R \) and \( RG = \{0\}, \) or \( RC_i = \{0\}. \)
Let $B_1$ be a Levi factor of $G_1^{-1}$, existing by the preceding argument. $H_1$ denotes a Cartan subalgebra of $B_1$. Now $RB_1 = R$, or $RB_1 = \{0\}$. In the first case let $\hat{A}_0 := (G_1)^{-1}(H_1)$. Hence $(\hat{A}_0)^{\circ} = G$ by (17). By the induction hypothesis $\hat{A}_0$ contains a Levi factor $B_2$. Take a Cartan subalgebra $H_2$ of $B_2$, and set $H' := H_1 \oplus H_2$. Then $H^2 = \{0\}$ by smoothness. We decompose $A$ into $H$-root spaces $A = \bigoplus \gamma A_\gamma$ with $\gamma \in \Delta$, $\Delta$ the set of characteristic roots of $H$ in $A$.

If $\Delta_i$ is the set of the characteristic roots of $H_i$ in $B_i$ for $i \in \{1, 2\}$ and $\gamma \in \Delta$, let $\gamma^* : H \to k$ be the trivial linear extension with $\gamma^*(h) := \gamma(h)$ if $h \in H_i$, and $\gamma^*(h) = 0$ for $h \in H_j$ if $j \in \{1, 2\}$ and $j \neq i$. Set $\Delta_i^* := \{\gamma^* | \gamma \in \Delta_i\}$. Then $\Delta = \Delta_1^* \cup \Delta_2^*$. Hence $B_2 \subset H_2 \oplus (\bigoplus A_\beta)$, $\beta \in \Delta_2^* \setminus \{0\}$. Observing (17), $B_1 \subset (H_1 \oplus (\bigoplus A_\alpha)) + R$ with $\alpha \in \Delta_1^* \setminus \{0\}$. Because of $RB_2 = \{0\}$ and the composition of the root spaces with (5), $B_1B_2 = \{0\}$. Thus $B := B_1 \oplus B_2$ is a Levi factor of $A$.

Finally suppose $RB_1 = \{0\}$. Decompose $A$ as a $B_1$-module, $A = B_1 \oplus R \oplus V$ with $V^\circ = G$. Since $B_1V \subset R \cap V$ then $B_1V = \{0\}$. Further $V \oplus R = G_1^{-1}$ is a subalgebra. It contains a Levi factor $B_2$ by the hypothesis of the induction. Thus $B := B_1 \oplus B_2$ is a Levi factor of $A$.

This proves the theorem. \hfill \Box

**References**


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