ON COMPLETE HYPERSURFACES OF NONNEGATIVE
SECTIONAL CURVATURES
AND CONSTANT \( m \)TH MEAN CURVATURE\(^1 \)

BY

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Abstract. The main result is that if \( M = M^n \) is a complete Riemann
manifold of nonnegative sectional curvature and \( X: M \to R^{n+1} \) is an
isometric immersion such that \( X(M) \) has a positive constant \( m \)th mean
curvature, then \( X(M) \) is the product of a Euclidean space \( R^n - d \) and a
d-dimensional sphere, \( m < d < n \).

1. Introduction. Let \( n > 2 \), \( M = M^n \) a Riemann manifold, and \( X: M \to \)
\( R^{n+1} \) an isometric immersion into Euclidean space. Let \( \lambda_1(x) < \cdots < \lambda_n(x) \)
be ordered principal curvatures of \( X(M) \). In analogy with the case \( n = 2 \), we
call \( X(M) \) a Weingarten hypersurface (or a \( W \)-hypersurface) if
\( W(\lambda_1(x), \ldots, \lambda_n(x)) \equiv \text{const} \) for some nontrivial function \( W(\lambda) = \)
\( W(\lambda_1, \ldots, \lambda_n) \).

Let \( M \) be complete and have nonnegative sectional curvatures. Without
any assumption that \( X(M) \) is a Weingarten hypersurface, a theorem of
Sacksteder [15] implies that \( M \) and \( X \) have factorizations,
\[
M = R^{n-d} \times M_0^d \quad \text{and} \quad X = X_1 \times X_0
\]
(1.1)
such that \( M_0^d \) is a complete Riemann manifold, the first map \( X_1 \) in
\[
X_1: R^{n-d} \to R^{n-d} \quad \text{and} \quad X_0: M_0^d \to R^{d+1}
\]
(1.2)
is the identity and the second is an isometric immersion; \( X_0(M_0^d) \) does not
contain any (complete) lines and is the boundary of a convex body; cf. [8] for
an analogue when \( X: M^n \to R^{n+p} \). Of course, either factor \( R^{n-d} \) or \( M_0^d \) can
be missing (i.e., reduce to a point). If, in addition, it is supposed that \( X(M) \) is
a \( W \)-Weingarten hypersurface, then it is of interest to find conditions on \( W \)
which assure that, in the Sacksteder decomposition (1.1) of \( X(M) \),
(a) \( X_0(M_0^d) \) is compact, or even
(b) \( X_0(M_0^d) \) is a sphere.

For the most part, this paper deals with the situation where \( W(\lambda) = \sigma_m(\lambda) \)
is the \( m \)th elementary symmetric function of \((\lambda_1, \ldots, \lambda_n)\), \( 1 < m < n \).

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\[ \sigma_m(\lambda) = \sum \cdots \sum \lambda_{(1)} \lambda_{(2)} \cdots \lambda_{(m)}, \quad \text{where } i(1) < \cdots < i(m). \quad (1.3) \]

\( H(x) = \sigma_1(\lambda(x))/n \) is the mean curvature, \( R(x) = 2!(n-2)! \sigma_2(\lambda(x))/n! \) the scalar curvature, \( K(x) = \sigma_n(\lambda(x)) \) the Gauss-Kronecker curvature; and more generally, \( H_m(x) = m!(n-m)! \sigma_m(\lambda(x))/n! \) the \( m \)th mean curvature \( (1 < m < n) \). Let \( X(M) \) be a \( \sigma_m \)-Weingarten hypersurface, with \( \sigma_m(\lambda(x)) \equiv C_0 > 0 \).

It is known that (b) holds if \( m = 1 \) ([5]; cf. [12] for \( n = 2 \) and [7] for a related result) or if \( m = 2 \) ([6]). The object of this note is to prove:

**Theorem (\(*\)).** Let \( M = M^n \) be a complete connected Riemann manifold of class \( C^2 \) with nonnegative sectional curvature. Let \( X: M \to \mathbb{R}^{n+1} \) be an isometric immersion of class \( C^2 \) such that \( X(M) \) has a positive constant \( m \)th mean curvature \( H_m(x) = C_0 > 0 \), \( x \in M \), for some \( m, 1 < m < n \). Then, in the Sacksteder decomposition (1.1)-(1.2), \( X_0(M_0^d) \) is a sphere (of dimension \( d, m < d < n \)).

We might remark that since \( M \) and \( X \) are of class \( C^2 \), they are real analytic, for \( \sigma_m(\lambda(x)) \equiv C_0 > 0 \) is a nonlinear elliptic analytic partial differential equation.

See [13] for the situation when \( C_m = 0 \).

In §§2 and 3, we adapt the arguments used by Cheng and Yau [6] in the case \( m = 2 \) to show that \( X_0(M_0^d) \) is compact for arbitrary \( m, 1 < m < n \). Our proof depends on a generalization of the elliptic operator used in [21] and [6]. It then follows from results of Alexandrov [2] that \( X_0(M_0^d) \) is a sphere; cf. Proposition 1.1 and remarks below. A different proof of the result in the compact case has been given by Nakagawa and Yokote [13, p. 479]. It should be noted that it has also been shown by Süss [17], Hsiung [11], U. Simon [16] and Yano [19] that when \( M_0^d \) is compact with nonnegative sectional curvature and \( X_0(M_0^d) \) is a \( \sigma_m \)-Weingarten hypersurface with \( \sigma_m(\lambda(x)) \equiv C_0 > 0 \) on \( M \), and certain additional convexity properties are satisfied, then \( X_0(M_0^d) \) is a sphere. By virtue of Sacksteder [15], Proposition 3.5 and its proof below, the additional convexity properties in Hsiung [11] and in U. Simon [16] are redundant and, in fact, Nakagawa and Yokote deduce their result from Hsiung's.

Actually, the fact that \( X_0(M_0^d) \) is a sphere can be deduced from the following consequence (applied to \( W = \sigma_m(\lambda) \) and \( c > 0 \)) of results of A. D. Alexandrov [2]:

**Proposition 1.1.** Let \( W(\lambda) = W(\lambda_1, \ldots, \lambda_n) \) be of class \( C^1 \) for \( 0 < \lambda_1 < \cdots < \lambda_n \) such that \( \partial W/\partial \lambda_k > 0 \) for \( 1 < k < n \). Let \( M = M^n \) be a compact connected Riemann manifold of class \( C^2 \) with nonnegative sectional curvature. Let \( X: M \to \mathbb{R}^{n+1} \) be an isometric immersion of class \( C^2 \) such that \( X(M) \) is a \( W \)-Weingarten hypersurface, say, with \( W(\lambda(x)) \equiv c \) on \( M \) and that
\[ \frac{\partial W(\lambda)}{\partial \lambda_k} > 0 \quad \text{for} \quad 1 < k < n \quad (1.4) \]

whenever \( \lambda \equiv \lambda(x), \ x \in M \) (for example, let \( (1.4) \) hold on the set \( \{ \lambda: W(\lambda) = c \} \)). Then \( X(M) \) is a sphere.

Self-intersections are not excluded a priori in this assertion, but it follows from Sacksteder's results [15] that, in fact, \( X(M) \) is the boundary of a convex body, so that \( X \) is an embedding when \( M \) and \( X \) are sufficiently smooth.

Because A. D. Alexandrov strived for extreme generality in the statements of his theorems, some do not seem to be clearly worded; cf. the comments in [13, pp. 479–480] and in a footnote of the translator of [2, III, p. 391]. Nevertheless, it is not difficult to see that the proof of Proposition 1.1 is contained in Alexandrov's arguments in the proof of the result of [2, V], with due reference to [2, III, pp. 390–391], and of course to [2, II, pp. 361–375]. (For an argument similar to [2, II, p. 371], in the special case \( W = \sigma_m(\lambda) \), see [1, p. 827].)

Proposition 1.1 applies not only to \( W(\lambda) = \sigma_m(\lambda) \) and \( c > 0 \) (cf. Proposition 3.5 below) but, for example, also to \( W(\lambda) = S^m_1 - S_m \) and \( c > 0 \), where \( S_m = \lambda_1^m + \cdots + \lambda_n^m \) and \( m = 1, 2, \ldots \) (and \( W \) reduces to \( S^2_1 - S_2 = 2\sigma_2 \) for \( m = 2 \)). Also, Proposition 1.1 and the proof of Theorem (*) have the following consequence:

**Corollary (*).** Let \( a_1, \ldots, a_n \) be nonnegative constants and

\[ W(\lambda) = \sum_{\mu=1}^{n} a_\mu \sigma_\mu(\lambda). \]

Let \( M = M^n \) be a complete Riemann manifold of class \( C^2 \) with nonnegative sectional curvatures. Let \( X: M \to \mathbb{R}^{n+1} \) be an isometric immersion of class \( C^2 \) such that \( X(M) \) is a \( W \)-Weingarten hypersurface with \( W(\lambda(x)) \equiv c > 0 \). Then, in the Sacksteder decomposition (1.1), \( X_0(M_0^d) \) is a sphere (of dimension \( d \), \( 0 < d < n \)).

**Added in Proof (6/20/78).** Using different methods in a forthcoming paper, we shall show that Theorem (*) remains valid if "\( H_m(x) \equiv C_0 \)" is replaced by "\( W(\lambda(x)) \equiv c \)" where \( W \) satisfies the conditions of Proposition 1.1.

2. **Preliminaries for (a).** As mentioned above, Cheng and Yau [6] show that a complete hypersurface in \( \mathbb{R}^{n+1} \) with nonnegative sectional curvatures and positive constant scalar curvature satisfies (b). From their arguments, we shall extract a general result (Lemma 2.1 below) dealing with the question of the validity of (a).

Unless otherwise indicated, all sums are over the range \( 1, \ldots, n \).

If \( M \) is a Riemann manifold of class \( C^3 \), let \( \omega_1, \ldots, \omega_n \) be a local \( C^2 \) orthonormal field of 1-forms on \( M \) satisfying the usual structure equations
\[ d\omega_i = -\sum_m \omega_{im}\omega_m, \quad \omega_{im} + \omega_{mi} = 0, \]
\[ d\omega_{ij} = -\sum_m \omega_{im}\omega_{mj} + \Omega_{ij}, \]
\[ \Omega_{ij} = \frac{1}{2} \sum_k \sum_m R_{ijkm}\omega_k\omega_m, \quad R_{ijkm} + R_{ijkm} = 0. \]

Here and below, products of exterior forms are understood to be exterior products. If a function \( f \) is of class \( C^2(M) \), its gradient or covariant derivative is defined by
\[ df = \sum_m f_m\omega_m, \]
and its second covariant derivative by
\[ \sum_m f_m\omega_m = df_i - \sum_m f_m\omega_{mi}. \]

Similarly, if \( \Sigma \Sigma h_{ij}\omega_i \otimes \omega_j \) is a \( C^2 \) symmetric tensor on \( M \), its first and second covariant derivatives are defined by
\[ \sum_m h_{ij,m}\omega_m = dh_{ij} - \sum_m h_{mj}\omega_{mi} - \sum_m h_{im}\omega_{mj}, \]
\[ \sum_m h_{ij,mk}\omega_m = dh_{ij,k} - \sum_m h_{ij,m}\omega_{mk} - \sum_m h_{im,k}\omega_{mj} - \sum_m h_{mj,k}\omega_{mi}. \]

Correspondingly, the Laplacian of the function \( f \) and of the tensor \( \Sigma \Sigma h_{ij}\omega_i \otimes \omega_j \) are defined by
\[ \Delta f = \sum_k f_{kk} \quad \text{and} \quad \Delta h_{ij} = \sum_k h_{ij, kk}. \quad (2.1) \]

For any continuous symmetric tensor \( \Sigma \Sigma \phi_{ij}\omega_i \otimes \omega_j \), define the differential operator \( L_\phi \) of second order by
\[ L_\phi f = \sum_1 \phi_{ij} f_{ij}. \quad (2.2) \]

This operator is called elliptic if \( (\phi_{ij}) \) is positive definite, and degenerate elliptic if \( (\phi_{ij}) \) is nonnegative definite.

**Proposition 2.1** [6]. Let \( M \in C^3 \) and \( \phi \in C^1 \). Then \( L_\phi \) is formally selfadjoint if and only if
\[ \sum_j \phi_{ij,j} \equiv 0 \quad \text{for } 1 < i < n \text{ on } M. \quad (2.3) \]

A form of the maximum principle gives

**Proposition 2.2** [5], [6]. Let \( L_\phi \) be formally selfadjoint and elliptic (possibly degenerate). Let \( E \subset M \) be open with compact closure \( \overline{E} \), \( 0 < f \in C^2(\overline{E}), \)
0 < g \in C^2(\bar{E}), g \equiv 0, \text{ and } g = 0 \text{ on } \partial E. \text{ Then}

\[ -\int_E g(L_\phi g) \, dx / \int_E g^2 \, dx > \inf_E \left[ -(L_\phi f) / f \right]. \tag{2.4} \]

It will be convenient to list here, for ready reference, some assumptions which we will use from time to time in this and the following section.

(H1) \( M = M^n \) is a complete connected Riemann manifold of class \( C^4 \) of nonnegative sectional curvature.

(H2) \( X: M \to R^{n+1} \) is an isometric immersion of class \( C^4 \) such that \( X(M) \) contains no complete lines, and so is the boundary of a convex body (for otherwise, we can replace \( M, X \) by \( M'_0, X_0 \) in the Sacksteder decomposition).

(H3) \( X(M) \) is not compact.

(H4) \( W = W(\lambda) \) is a \( C^2 \) function \( \Phi \) of \( S_1, \ldots, S_m \) for some \( m > 1 \), where \( S_\mu = \lambda_1^\mu + \cdots + \lambda_m^\mu \), i.e., \( W(\lambda) = \Phi(S_1, \ldots, S_m) \).

Under the assumptions (H1)–(H2), there is a unit normal vector field \( N: M \to S^{n+1} \) such that the second fundamental form \( \sum h_{ij} \omega_i \otimes \omega_j \) is nonnegative definite. We, of course, have the Codazzi relations

\[ h_{ij,k} = h_{ji,k} = h_{ik,j} = h_{jk,i}. \tag{2.5} \]

When (H3) holds, the normal image \( N(M) \) is contained in a hemisphere, so that there exists a constant vector \( U \in R^{n+1} \) such that the Euclidean scalar product

\[ N(x) \cdot U > 0 \quad \text{on } M; \tag{2.6} \]

Sacksteder (cf., e.g., [18]). If \{\( e_1, \ldots, e_n \}\) is a local orthonormal frame field on \( X(M) \), then the Gauss and Weingarten formulas are

\[ X_{,ij} = h_{ij}N \quad \text{and} \quad N_{,j} = -\sum_i h_{ij} e_j, \]

so that

\[ N_{,ij} = -\sum_k h_{ik} h_{kj} N - \sum_k h_{ki,j} e_k; \]

and consequently

\[ L_\phi X = \sum_i \sum_j \phi_j h_{ji} N \quad \text{or} \quad L_\phi (X \cdot U) = \sum_i \sum_j \phi_j h_{ji} (N \cdot U), \tag{2.7} \]

\[ L_\phi (N \cdot U) = -\sum_i \sum_j \sum_k \phi_j h_{jk} h_{kl} (N \cdot U) - \sum_i \sum_j \sum_k \phi_j h_{ji,k} (e_k \cdot U); \tag{2.8} \]

cf. [7, p. 83].

**Proposition 2.3.** Assume (H1)–(H3) and (2.6). Let \( L_\phi \) be elliptic and satisfy

\[ \sum_i \sum_j \phi_j h_{ji,k} = 0 \quad \text{for } 1 < k < n \text{ on } M. \tag{2.9} \]
Then \( N(x) \cdot U > 0 \) on \( M \)

**Proof.** Even if \( L_\phi \) is degenerate elliptic, we have that

\[
\sum_i \sum_j \phi_{ij} h_{jk} h_{ki} > 0 \quad \text{for } 1 < k < n.
\]

Hence, by (2.6) and (2.9), \( L_\phi (N \cdot U) < 0 \) on \( M \). The strong minimum principle of E. Hopf [10] for elliptic operators implies that if \( N \cdot U = 0 \) at some point of \( M \), then \( N \cdot U \equiv 0 \) on \( M \). But this would show that, for every \( x^0 \in M \), the line \( X(x^0) + tU, -\infty < t < \infty \), is on \( X(M) \), which contradicts (H2).

**Proposition 2.4.** Assume (H1)–(H3). Let \( L_\phi \) be formally selfadjoint, elliptic, and let (2.9) hold. Let \( E = E(r) = \{ x \in M : X(x) \cdot U < r \} \) for \( r > 0 \). Then

\[
4 \int_{E(r)} \frac{1}{r} \sum_i \sum_j \phi_{ij} h_{ji} \, dx / r \, \text{vol} \, E(r/2) > \inf_{E(r)} \sum_i \sum_j \sum_k \phi_{ij} h_{jk} h_{ki}. \tag{2.10}
\]

**Proof.** If we let \( g = r - X(x) \cdot U \) and \( f = N(x) \cdot U > 0 \), then Proposition 2.2 gives

\[
\int_{E(r)} \frac{1}{r} \sum_i \sum_j \phi_{ij} h_{ji} (N \cdot U) \, dx / \int_{E(r)} (r - X(x) \cdot U)^2 \, dx
\]

\[
> \inf_{E(r)} \sum_i \sum_j \sum_k \phi_{ij} h_{jk} h_{ki}.
\]

Since the left side of this relation is at most the left side of (2.10), the assertion follows.

**Corollary 2.1.** If, in addition to the assumptions of Proposition 2.4, we assume that

\[
\int_{E(r)} \frac{1}{r} \sum_i \sum_j \phi_{ij} h_{ji} \, dx / r \, \text{vol} \, E(r/2) \to 0 \quad \text{as } r \to \infty, \tag{2.11}
\]

e.g., that

\[
\sum_i \sum_j \phi_{ij} h_{ji} \quad \text{is bounded on } M, \tag{2.12}
\]

then

\[
\inf_M \sum_i \sum_j \sum_k \phi_{ij} h_{jk} h_{ki} = 0. \tag{2.13}
\]

Thus the arguments above lead to the following general result.

**Lemma 2.1.** Assume (H1)–(H2), and that (2.11) (say, (2.12)) holds but that (2.13) does not. Also, assume that \( L_\phi \) is elliptic, formally selfadjoint, and that
(2.9) holds. Then $X(M)$ is compact, i.e., (H3) cannot hold.

It will be seen, in the next section, that the assumption that "$L_\phi$ is formally selfadjoint" is quite restrictive.

3. Compactness. The object of this section is to prove the following:

**Lemma 3.1.** Let $m$ be fixed, $1 < m < n$. Assume conditions (H1)-(H2) of §2 and that $X(M)$ is a $\sigma_m$-Weingarten hypersurface with $\sigma_m(\lambda(x)) = C_0 > 0$ on $M$. Then $X(M)$ is compact.

This is a consequence of Lemma 2.1 and Propositions 3.1 and 3.4-3.7 below. In this section, we assume (H1)-(H2) and (H4). We associate with $W$, the symmetric tensor $\phi$ or $\sum \phi_j \omega_j \otimes \omega_j$ defined by

$$\phi = \sum_{\mu=1}^m \mu (\partial W/\partial S_\mu) h^{\mu-1},$$

where $h^\mu = (h_{ij})^\mu$ is the $\mu$th power of $h = (h_{ij})$, the second fundamental matrix. Thus $h^0 = I$ is the identity matrix and the $ij$th element of $h^\mu$ is

$$h_{ij}^\mu = \sum_{i(1)} \cdots \sum_{i(\mu-1)} h_{i(1)j(1)} h_{i(2)j(2)} \cdots h_{i(\mu-1)j}.$$

Hence (3.1) means that

$$\phi_{ij} = \sum_{\mu=1}^m \mu (\partial W/\partial S_\mu) h_{ij}^{\mu-1}, \text{ where } h_{ij}^0 = \delta_{ij}. \quad (3.3)$$

Correspondingly, the associated operator is

$$L_\phi f = \sum_i \sum_j \phi_{ij} f_{ij} = \sum_i \sum_j \sum_{\mu=1}^m \mu (\partial W/\partial S_\mu) h_{ij}^{\mu-1} f_{ij}. \quad (3.4)$$

**Proposition 3.1.** Under the assumptions (H1)-(H2), (H4) and $W(\lambda(x)) \equiv$ const on $M$, the relations (2.9) hold.

**Proof.** Let $s_\mu(x) = S_\mu(\lambda(x))$. Then

$$s_\mu(x) = \text{tr } h^\mu = \sum_i \sum_{i(1)} \cdots \sum_{i(\mu-1)} h_{i(1)i(2)} \cdots h_{i(\mu-1)i}.$$

Hence, we have, with $i = i(\mu)$,

$$s_{\mu,k} = \sum_{\kappa=1}^\mu \sum_i \sum_{i(1)} \cdots \sum_{i(\mu-1)} h_{i(1)i(2)} \cdots h_{i(\kappa-1)i(\kappa)} k \cdots h_{i(\mu-1)i},$$

so that

$$s_{\mu,k} = \mu \text{ tr}(h^{\mu-1} h_k) = \mu \sum_i \sum_j h_{ij}^{\mu-1} h_{ji,k}. \quad (3.6)$$
Note that if \( w(x) = W(\lambda(x)) \), then \( w(x) \equiv c_0 \) on \( M \) implies that

\[
0 = w_{,k} = \sum_{\mu=1}^{m} \left( \frac{\partial W}{\partial S_{\mu}} \right) s_{\mu,k} = \sum_i \sum_j \phi_{ij} h_{ij,k} \quad \text{for} \ 1 < k < n.
\]

**Proposition 3.2.** Assume (H1)–(H2), (H4) and (3.1). Then \( L_\phi \) is elliptic if and only if

\[
\partial W(\lambda)/\partial \lambda_i > 0 \quad \text{for} \ \lambda = \lambda(x), \ x \in M, \ 1 < i < n. \quad (3.7)
\]

**Proof.** In fact, the eigenvalues of \( \phi \) are

\[
\sum_{\mu=1}^{m} \mu \left( \frac{\partial W}{\partial S_{\mu}} \right) \lambda_{\mu,i}^{-1} = \partial W/\partial \lambda_i \quad \text{with} \ \lambda = \lambda(x), \ 1 < i < n. \quad (3.8)
\]

**Proposition 3.3.** Under the assumptions (H1)–(H2), (H4) and (3.1), a sufficient condition for \( L_\phi \) to be formally selfadjoint is that

\[
\sum_{\mu=1}^{m} \sum_{\kappa=1}^{\mu-1} \mu \left( \frac{\partial^2 W}{\partial S_{\mu} \partial S_{\kappa}} \right) \lambda_{\mu,i}^{-1} + \sum_{\mu=\kappa+1}^{m} \mu \left( \frac{\partial W}{\partial S_{\mu}} \right) \lambda_{\mu,-\kappa}^{-1} = 0 \quad (3.9)
\]

for \( 1 < i < n \). This is the case, e.g., if \( W \) satisfies

\[
\kappa \sum_{\mu=1}^{m} \mu \left( \frac{\partial^2 W}{\partial S_{\mu} \partial S_{\kappa}} \right) \lambda_{\mu,i}^{-1} + \sum_{\mu=\kappa+1}^{m} \mu \left( \frac{\partial W}{\partial S_{\mu}} \right) \lambda_{\mu,-\kappa}^{-1} = 0 \quad (3.10)
\]

for \( 1 < i < n, \ 1 < \kappa < m \).

**Proof.** From (3.3),

\[
\sum_{j} \phi_{ij,j} = \sum_{j} \left\{ \sum_{\mu=1}^{m} \sum_{k=1}^{\mu-1} \mu \left( \frac{\partial^2 W}{\partial S_{\mu} \partial S_{\kappa}} \right) s_{\kappa,j} h_{ij,j}^{-1} + \sum_{\mu=2}^{m} \mu \left( \frac{\partial W}{\partial S_{\mu}} \right) h_{ij,j}^{-1} \right\}.
\]

Note that (3.2) implies that

\[
h_{ij,j}^{-1} = \sum_{\kappa=1}^{\mu-1} \sum_{k} h_{\kappa,k}^{-1} h_{kp,j} h_{ij,j}^{\mu-k-1}. \quad (3.11)
\]

By the Codazzi relations (2.5), we have \( h_{kp,j} = h_{jp,k} \), so that, by (3.6),

\[
\sum_{j} h_{ij,j}^{-1} = \sum_{\kappa=1}^{\mu-1} (\mu - \kappa)^{-1} \sum_{k} h_{\kappa,k}^{-1} s_{\mu-k,k}
\]

\[
= \sum_{\kappa=1}^{\mu-1} \sum_{j} h_{ij,j}^{\mu-k-1} s_{\kappa,j}
\]

for \( \mu > 2 \). If, at a fixed arbitrary point \( x \) of \( M \), \( h = \text{diag}(\lambda_1(x), \ldots, \lambda_n(x)) \),
then, by (3.6) and the last relation,
\[ s_{k,j} = \kappa \sum_k \lambda_k^{\kappa-1} h_{k,k,j}, \]
\[ \sum_j h_{j,j}^{\mu} = \sum_{\kappa=1}^{\mu-1} \lambda_{\kappa-1}^{\mu-\kappa-1} s_{k,j} = \sum_k \sum_{\kappa=1}^{\mu-1} \lambda_k^{\mu-\kappa-1} \lambda_k^{\kappa-1} h_{k,k,j}. \]
Hence, \( \sum_j \phi_{j,j} \) becomes
\[ \sum_k \left( \sum_{\kappa=1}^{m} \sum_{\mu=1}^{m} \mu \kappa \left( \partial^2 W / \partial S_x \partial S_\mu \right) \lambda_{\mu-1}^{\kappa-1} \lambda_k^{\mu-1} \right) h_{k,k,i}. \]
Thus, the first part of the assertion follows, and the second is obvious (since \( \partial W / \partial S_{m+1} = 0 \)).

**Proposition 3.4.** If \( W = s_m(\lambda) \), then (3.10) holds. Hence if (H1)–(H2) holds and \( W = s_m(\lambda) \) in (3.1), then \( L_{\phi} \) is selfadjoint.

It is not difficult to see that if \( W(\lambda) \) is a symmetric homogeneous polynomial of degree \( m \) in \( (\lambda_1, \ldots, \lambda_n) \), then (3.10) holds only if \( W \) is a constant multiple of \( s_m \).

**Proof.** Standard formulas show that
\[ \mu \partial s_m / \partial S_\mu = (-1)^{\mu-1} s_{m-\mu} \quad \text{for } 1 < \mu < m; \tag{3.12} \]
cf. [4, \$80 or 161], where \( p_\mu = (-1)^{\mu} s_\mu \). Hence
\[ \mu \kappa \partial s_m / \partial S_\kappa S_\mu = (-1)^{\mu+\kappa} s_{m-\mu-\kappa} \quad \text{for } 1 < \kappa, \mu < m-1, \text{and } \mu + \kappa < m. \tag{3.13} \]
Thus, if \( W = s_m \), then the left side of (3.10) is
\[ \sum_{\mu=1}^{m-\kappa} (-1)^{\mu+\kappa} s_{m-\mu-\kappa} \lambda_{\mu-1}^{\kappa-1} - \sum_{\mu=\kappa+1}^{m} (-1)^{\mu} s_{m-\mu} \lambda_{\mu-1}^{\kappa-1} = 0. \]

**Proposition 3.5.** If \( 0 < \lambda_1^0 < \cdots < \lambda_n^0 \) and \( s_m(\lambda^0) > 0 \), then \( \partial s_m(\lambda^0) / \partial \lambda_k > 0 \) for \( 1 < k < n \). Hence, if (H1)–(H2) holds, \( W = s_m(\lambda) \) in (3.1), and \( s_m(\lambda(\chi)) > 0 \) on \( M \), then \( L_{\phi} \) is elliptic.

**Proof.** Note that
\[ \partial s_m(\lambda) / \partial \lambda_i = s_m^{(i)}(\lambda), \tag{3.14} \]
where \( s_m^{(i)} \) denotes the \( (m - 1) \)st elementary symmetric function in \( \lambda_1, \ldots, \lambda_i, \lambda_{i+1}, \lambda_{i+1}, \ldots, \lambda_n \). We shall first verify the following:
CLAIM. Let $0 < \lambda_1^0 < \cdots < \lambda_n^0$. Then $\sigma_m(\lambda^0) = 0$ if and only if there exist $m - 1$ indices $(1 <) i(1) < \cdots < i(m - 1) (< n)$ such that $\lambda_k^0 = 0$ if $k \neq i(\mu)$, $1 < \mu < m - 1$.

This is clear if $m = 1$. Assume that $m > 1$ and that the claim holds if $m$ is replaced by $m - 1$. Suppose that $\lambda_j^0 \neq 0$ for some $j$. Since $m\sigma_m(\lambda) = \sum \lambda_i \sigma_m^{(i)}(\lambda)$, $\sigma_m(\lambda^0) = 0$ implies that $\sigma_m^{(i)}(\lambda^0) = 0$. Thus, by the induction hypothesis, there exist $m - 2$ indices $(1 <) i(1) < \cdots < i(m - 2) (< n)$, $i(\mu) \neq j$, such that $\lambda_k^0 = 0$ if $k \neq j$ or $k \neq i(\mu)$, $1 < \mu < m - 2$. This gives the claim.

COMPLETION OF THE PROOF. Suppose that $0 < \lambda_1^0 < \cdots < \lambda_n^0$ and that $\partial \sigma_m(\lambda^0)/\partial \lambda_i = 0$ for some $i$, $1 < i < n$. Then $\sigma_m^{(i)}(\lambda^0) = 0$, and so there exist $m - 2$ indices $i(1) < \cdots < i(m - 2)$, $i(\mu) \neq i$, such that $\lambda_j^0 = 0$ if $j \neq i$ or $j \neq i(\mu)$ for $1 < \mu < m - 2$. Thus $\sigma_m(\lambda^0) = 0$. This proves the first part of the proposition, and the other part follows from Proposition 3.2.

REMARK. It is clear that this proof shows that if $c_0 > 0$ and $c_1 > 0$, then there exists a $\delta = \delta(c_0, c_1) > 0$ such that

$$\partial \sigma_m(\lambda)/\partial \lambda_k > \delta > 0 \quad \text{if} \quad \sigma_m(\lambda) > c_0 > 0, \quad 0 < \lambda_1 < \cdots < \lambda_n < c_1. \quad (3.15)$$

PROPOSITION 3.6. Assume (H1)-(H2), (H4) and that $W(\lambda)$ is a homogeneous function of degree $m$ in (3.1). Then

$$\text{tr} (\phi h) = \sum_i \sum_j \phi_{ij} h_{ji} = mW(\lambda(x)), \quad (3.16)$$

so that (2.12) holds if $W(\lambda(x)) \equiv \text{const on } M$.

PROOF. By (3.8), the eigenvalues of $\phi h$ are

$$\sum_{\mu = 1}^m \mu (\partial W/\partial S_{\mu}) \lambda_i^\mu = \lambda_i \partial W/\partial \lambda_i, \quad 1 < i < n,$$

so that (3.16) follows from Euler's theorem.

PROPOSITION 3.7. If (H1)-(H2) holds, $W = \sigma_m(\lambda)$ in (3.1), and $\sigma_m(\lambda(x)) > C_0 > 0$ on $M$, then

$$\inf_M \text{tr} (\phi h^2) = \inf_M \sum_i \sum_j \sum_k \phi_{ij} h_{jk} h_{ki} > 0. \quad (3.17)$$

PROOF. The eigenvalues of $\phi h^2$ are

$$\sum_{\mu = 1}^m \mu (\partial W/\partial S_{\mu}) \lambda_i^{\mu + 1} \quad \text{for} \quad 1 < i < n,$$

so that

$$\text{tr} (\phi h^2) = \sum_{\mu = 1}^m \mu (\partial W/\partial S_{\mu}) S_{\mu + 1}.$$
When \( W = \sigma_m \), (3.12) implies that
\[
\text{tr} (\phi h^2) = \sum_{\mu = 1}^{m} (-1)^{\mu-1} \sigma_{m-\mu} S_{\mu+1} = \sum_{\mu = 2}^{m+1} (-1)^{\mu} \sigma_{m+1-\mu} S_{\mu}.
\]
Thus, by Newton's formula (cf. [3, p. 244]),
\[
\text{tr} (\phi h^2) = \sigma_m \sigma_1 - (m + 1)\sigma_{m+1},
\]
where \( \sigma_{m+1} = 0 \) if \( m = n \). If \( m < n \), then each term \( \lambda_{i(1)} \cdots \lambda_{i(m+1)} \) of \( \sigma_{m+1} \) occurs \( m + 1 \) times in \( \sigma_m \sigma_1 \), for the factor \( \lambda_{i(j)} \) can be considered to be in \( \sigma_1 \) for \( j = 1, \ldots, m + 1 \). Thus
\[
\text{tr} (\phi h^2) > \sum_{\mu = 1}^{m} \left\{ \sum_{i(1)} \cdots \sum \lambda_{i(1)} \cdots \lambda_{i(\mu-1)} \lambda_{i(\mu)} \cdots \lambda_{i(m)} \right\},
\]
where the inner sum is over \( 1 < i(1) < \cdots < i(m) < m \). Hence
\[
\text{tr} (\phi h^2) > \lambda_n (\lambda_n \cdots \lambda_{n-m+1}).
\]
Since \( \sigma_m < (\lambda_n \cdots \lambda_{n-m+1})n!/(n-m)!m! < \lambda_n^m n!/(n-m)!m! \), we see that
\[
\text{tr} (\phi h^2) > [m!/(n-m)!\sigma_m/n!]^{1+1/m} > [m!(n-m)!C_o/n!]^{1+1/m} > 0.
\]

REFERENCES


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