4-MANIFOLDS, 3-FOLD COVERING SPACES AND RIBBONS

BY

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ABSTRACT. It is proved that a PL, orientable 4-manifold with a handle presentation composed by 0-, 1-, and 2-handles is an irregular 3-fold covering space of the 4-ball, branched over a 2-manifold of ribbon type. A representation of closed, orientable 4-manifolds, in terms of these 2-manifolds, is given. The structure of 2-fold cyclic, and 3-fold irregular covering spaces branched over ribbon discs is studied and new exotic involutions on $S^4$ are obtained. Closed, orientable 4-manifolds with the 2-handles attached along a strongly invertible link are shown to be 2-fold cyclic branched covering spaces of $S^4$. The conjecture that each closed, orientable 4-manifold is a 4-fold irregular covering space of $S^4$ branched over a 2-manifold is reduced to studying $\gamma \neq S^1 \times S^2$ as a nonstandard 4-fold irregular branched covering of $S^3$.

1. Introduction. We first remark that the foundational paper [8] might be useful as an excellent account of definitions, results and historical notes.

Let $F$ be a closed 2-manifold (not necessarily connected nor orientable) locally flat embedded in $S^4$. To each transitive representation $\omega: \pi_1(S^4 - F) \to S_n$ into the symmetric group of n letters there is associated a closed, orientable, PL 4-manifold $W^4(F, \omega)$ which is a $n$-fold covering space of $S^4$ branched over $F$. This paper deals with the problem of representing each closed, orientable PL 4-manifold $W^4$ as a $n$-fold covering space of $S^4$ branched over a closed 2-manifold.

I. Berstein and A. L. Edmonds proved [9] that in some cases (for instance $S^1 \times S^1 \times S^1 \times S^1$) $n$ has to be at least 4. They also pointed out to the author that, for $CP^2$, $F$ must be nonorientable (using the Euler characteristic number). More generally, S. Cappell and J. Shaneson pointed out that, if $F$ is orientable, the signature of $W^4$ must be zero.

We conjecture that each such $W^4$ is an irregular simple 4-fold covering space of $S^4$ branched over a closed surface $F$ (simple means that the representation $\omega$, where $W^4 \cong W^4(F, \omega)$, sends meridians into transpositions).

The manifold $W^4$ admits a handle representation $W^4 = H^0 \cup \lambda H^1 \cup \mu H^2$
Thus, by duality, $W^4$ is obtained by pasting together two manifolds $V^4 = H^0 \cup \lambda H^1 \cup \mu H^2$, $U^4 = H^0 \cup \gamma H^1$ along their common boundary, which is $\gamma \neq S^1 \times S^2$. Our idea is to represent $V^4$ and $U^4$ as coverings of $D^4$ branched over a 2-manifold with boundary in $S^3$, and then match the two coverings.

In this paper we prove that a manifold with presentation $H^0 \cup \lambda H^1 \cup \mu H^2$ is, in fact, a dihedral 3-fold covering of $D^4$ branched over a 2-manifold of a special type (which we call a ribbon manifold because it is a natural generalization of ribbon discs).

In the case that $\mu H^2$ is attached along a strongly invertible link in $\lambda \neq S^1 \times S^2 = \partial (H^0 \cup \lambda H^1)$, then we show that the closed 4-manifold $W^4 = H^0 \cup \lambda H^1 \cup \mu H^2 \cup \gamma H^3 \cup H^4$ is actually a 2-fold cyclic branched covering of $S^4$. For the case of 4-fold irregular branched coverings of $S^4$, we show our conjecture reduces to studying $\gamma \neq S^1 \times S^2$ as a nonstandard 4-fold irregular branched covering of $S^3$.

It is shown in [7] that each manifold $W^4 = H^0 \cup \lambda H^1 \cup \mu H^2 \cup \gamma H^3 \cup H^4$ is uniquely determined by $H^0 \cup \lambda H^1 \cup \mu H^2$. From this point of view, our presentation of $H^0 \cup \lambda H^1 \cup \mu H^2$ as an irregular 3-fold covering space also provides a representation for closed, orientable 4-manifolds.

We study also the structure of the 2- and dihedral 3-fold covering spaces of ribbon discs. We obtain in this way some contractible 4-manifolds of Mazur, and this allows us to find many 2-knots in $S^4$ with the same 2-fold cyclic covering space, even $S^4$ itself, thus obtaining new examples of exotic involutions on $S^4$.

Lastly, we note some possible applications of these results to the study of 3-manifolds and classical knots.

I am indebted to Robert Edwards, Charles Giffen and Cameron Gordon for helpful conversations.

2. A simple case. We begin with the simple case of a manifold which is presented by one 0-handle, one 1-handle, and one 2-handle of a special type, both in order to obtain special results for this case and to illustrate the method.

Consider $S^1 \times B^3$ with presentation $H^0 \cup H^1$, i.e. one 0-handle plus one 1-handle. Its boundary, $S^1 \times S^2$, is illustrated in Figure 1.

We consider a knot $K$, contained in $S^1 \times S^2$, such as the one shown in Figure 1, which is strongly invertible, i.e. reflection $u$ in the $E$-axis induces on $K$ an involution with two fixed points. Now we add a 2-handle to $S^1 \times B^3$ so that $K$ is the attaching sphere. More precisely, we have an embedding $h: \hat B^2 \times B^2 \to \partial (S^1 \times B^3)$ so that $h(\hat B^2 \times 0)$ equals $K$, and let $W^4 = S^1 \times B^3 \cup_h B^2 \times B^2$.

Let $U: S^1 \times B^3 \to S^1 \times B^3$ be the involution, with two discs $D_1, D_2$ as
fixed-point set, which canonically extends the reflection $u$ above.

Let $V: B^2 \times B^2 \to B^2 \times B^2$ be the reflection in $D = B^1 \times B^1$. We can represent $S^3 = \partial (B^2 \times B^2)$ by stereographic projection onto $R^3 + \infty$ in such a way that $V$ induces on $S^3$ the reflection $v$ in the $y$-axis. In this representation $B^2 \times B^2$ is a regular neighborhood, $X$, of the unit circle $C$ in the $(x, y)$-plane; the belt-sphere is the $z$-axis, and the belt-tube is $Y = S^3 - \text{int} X$. Finally, let $(m, l)$ be a meridian-longitude system on $\partial X$ (see Figure 2).

Up to isotopy, we may suppose that $uh = hv$, so that $(U, V)$ is an involution on $W^4 = S^1 \times B^3 \cup_h B^2 \times B^2$. The fixed-point set is the disc $(D_1 \cup D_2) \cup_h D$, where $h$ pastes $\partial D$ to $\partial (D_1 \cup D_2)$ along $\alpha \cup \beta$ of Figure 2.

The orbit-space, which is $D^4$, can be described as follows. First, $p: S^1 \times B^3 \to S^1 \times B^3 / U$ is the 2-fold branched covering space of $D^4$.
branched over two disjoint discs $D_1', D_2'$ (see Figure 3(a)) such that $p|S^1 \times S^2: S^1 \times S^2 \to S^3$ is the standard covering over $\partial (D_1' \cup D_2')$. In this covering $p(K)$ is an arc with its endpoints in $\partial D_1'$ and $\partial D_2'$. Second, $q: B^2 \times B^2 \to B^2 \times B^2/V$ is the 2-fold branched covering space of $D^4$ branched over a disc $D'$ such that $p|\partial (B^2 \times B^2): \partial (B^2 \times B^2) \to S^3$ is the standard covering over the trivial knot $\partial D'$. In this covering, $q(C)$ is the arc shown in Figure 3(c). We must paste along regular neighborhoods $q(X)$ and $p(U(K))$ of $q(C)$ and $p(K)$, respectively, by the map $phq^{-1}$, obtaining $D^4 = S^1 \times B^3/U \cup phq^{-1} B^2 \times B^2/V$.

**Figure 3**

The branching set is $(D_1' \cup D_2') \cup phq^{-1} D'$, which can be visualized as follows. Deform $D_1'$ and $D_2'$ by isotopy as illustrated in Figure 3(b), thus obtaining the “ribbon” $D_1' \cup D_2'$ (and, in fact, if we pull $D_1' \cup D_2'$ back into $S^3$ in the way suggested by the shaded part of Figure 3(b), then we obtain a ribbon immersion of $D_1' \cup D_2'$). Pasting $B^2 \times B^2/V$ to $D^4 = S^1 \times B^3/U$ along the balls $q(X)$ and $p(U(K))$ and then “absorbing” the bulge $B^2 \times B^2/V$ on $D^4$ back into $D^4$, we obtain the branch set in the aspect of Figure 3(b) $\cup$ 3(c) joined by the arrow. Of course, the number of twists in the boundary of the ribbon depends on the number of times that $h(l)$ goes around $\partial (U(K))$. The ribbon of Figure 3(b) corresponds to the choice $h(l) \sim L + 5M$ (on $\partial U(K)$). Note that the number of components of the branch-set is one if and only if $K$ connects the two components of $D_1 \cup D_2$, in which case the branch set is a ribbon disc.

We collect together these results in the following theorem.

**Theorem 1.** The manifold $W^4 = H^0 \cup H^1 \cup H^2$, where $H^2$ is attached
along a strongly invertible knot of $S^1 \times S^2$, is a 2-fold covering space of $D^4$, branched over a ribbon disc or over the union of a disc and either an annulus or a Möbius band.

3. Ribbons, Mazur manifolds and exotic involutions. We have immediately an amusing result. Note that the manifold $W^4$ corresponding to the knot $K$, in Figure 1, is the one discovered by B. Mazur [6], which has the property that $W^4 \times I \approx B^5$. So, the double $2W^4 \cong S^4$ and $W^4$ is contractible. Then, the ribbon 2-knot $R$ corresponding to the ribbon of Figure 3(b) (i.e., the 2-knot obtained by pushing a copy of the ribbon disc into each of the two sides of $S^3$ in $S^4$) has $2W^4 = S^4$ as 2-fold covering space. This is another example of exotic involution in $S^4$, first discovered by C. Gordon [1]. (It is easily checked that $\pi_1(S^4 - R) \neq \mathbb{Z}$, showing that $R$ is not the trivial knot in $S^4$.)

In order to state these results with more generality, let us quote now the description of ribbon knots given by T. Yajima [10]. Let $C_0, C_1, \ldots , C_\lambda$ be unlinked trivial circles in $R^3$. Take disjoint small arcs $\alpha_j, \ldots, \alpha_\lambda$ on $C_0$, and a small arc $\gamma_i$ on $C_i$ ($i = 1, \ldots, \lambda$). For every $i$, connect $\alpha_i$ with $\gamma_i$ by a nontwisting narrow band $B_i$ which may run through $C_j$ (for some $j$) or may get tangled with itself or with other bands. Then, each ribbon knot is of this type for some $\lambda$. We shall say that a presentation of this form has type $(C_0, C_1, \ldots , C_\lambda)$.

Consider a ribbon $R$ of type $(C_0, C_1)$ and let $\Delta_0, \Delta_1$ be disjoint discs with boundary $C_0, C_1$ respectively. We may assume that the center line path $\beta$ of the band $B_j$ from $C_0$ to $C_1$ cuts $\text{Int} \Delta_0 \cup \Delta_1$ transversally, thus partitioning $\beta$ into a composition of (nontrivial) paths which we write as $\beta = \beta_1 * \cdots * \beta_k$ (some $k$). Let $\#R = \Sigma(-1)^j$ where $j$ runs over those indices from 1 to $k$ such that the subpath $\beta_j$ connects $C_0$ and $C_1$. Refer to Figure 3(a), where $\#R = 1$. Note that $\#R$ is always odd. We see that the 2-fold covering space branched over the disc $R$ is a manifold $W^4 = H^0 \cup H^1 \cup H^2$, where $H^2$ is added along a strongly-invertible knot, which is homologous to $\#R$ times a generator of $H_1(S^1 \times B^3)$.

We now use the trick of Mazur [6] to describe $2W^4$. The manifold $W^4 \times I$ is obtained by adding a 2-handle to $S^1 \times B^4$ along a curve $w$ in $S^1 \times S^3$ which is homologous to $\#R$ times a generator of $H_1(S^1 \times S^3)$. This defines the handle addition uniquely up to PL-homeomorphism since 1-knot theory of $S^1 \times S^3$ is essentially trivial.1 Thus, $2W^4 = \partial(W^4 \times I)$ is obtained by a spherical modification of $S^1 \times S^3$ along a curve which runs $\#R$ times the generator of $H_1(S^1 \times S^3)$. Hence, we have

1A "Dehn-twist" along a belt-sphere of $S^1 \times B^1$ changes the framing of $w$ by a map $w \to SO(3)$ which represents the nontrivial element of $\pi_1(SO(3))$ if and only if $\#R$ is odd. Thus, we cannot worry about framings here.
Theorem 2. All the ribbon 2-knots, corresponding to ribbon knots of type 
\((C_0, C_1)\) with the same number \(\# R\), have the same 2-fold covering space.

In particular, if \(\# R = \pm 1\), then \(2W^4 = S^4\), and so we have

Corollary 1. All the ribbon 2-knots of type \((C_0, C_1)\), with \(\# R = \pm 1\), have 
\(S^4\) as 2-fold covering space.

We see that, in contrast with the 3-dimensional analogue, the family of
2-knots in \(S^4\) with the same 2-fold covering space is very large indeed.

More generally, if \(\# R = 2m + 1\) we have \(\pi_1(2W^4) \cong \mathbb{Z}_{2m+1}\) and the
universal covering space of \(2W^4\) is \(2m \# S^2 \times S^2\). This composite \(2m \# S^2 \times S \to 2W^4 \to S^4\) is a regular dihedral branched cover over the ribbon 
2-knot.

4. 2-fold coverings. We generalize these results to 4-manifolds with several
1-handles, being somewhat less explicit than before in the description of
geometrical constructions.

The manifold \(\lambda \# S^1 \times B^3\) has the presentation \(H^0 \cup \lambda H^1\) (if \(\lambda = 0, \lambda \# S^1 \times B^3 = B^4\)). Its boundary \(\lambda \# S^1 \times S^2\) is represented by the hand-
lebody of Figure 4, with points on the boundary identified by reflection in the
\((x, y)\)-plane.

The reflection \(u\) in the \(x\)-axis is the restriction of an involution \(U\) in
\(\lambda \# S^1 \times B^3\) which has \(\lambda + 1\) disjoint discs as fixed-point set. The orbit
space of \(U\) is \(D^4\), and we show the branch set, \(C = C_0 \cup \cdots \cup C_\lambda\), in \(\partial D^4\)
in Figure 4.

Let \(p\) be the projection, and consider now a system \(A = A_1 \cup \cdots \cup A_\mu\).
of disjoint, simple arcs in $S^3$, meeting $C$ only in their endpoints. It is clear that $p^{-1}A$ is a strongly invertible link in $\lambda \# S^1 \times S^2$ (which means each $p^{-1}A_i$ is strongly-invertible with respect to $u$). We have the following theorem.

**Theorem 3.** The manifold $W^4 = H^0 \cup \lambda H^1 \cup \mu H^2$, where $\mu H^2$ is attached along a strongly invertible link in $\lambda \# S^1 \times S^2$, is a 2-fold cyclic covering space of $D^4$ branched over a 2-manifold.

**Remark.** The branching set is easily obtained in a way similar to that in §2. Utilizing the ribbon presentation explained in §3 we have immediately the following theorem:

**Theorem 4.** If $R$ is a ribbon knot of type $(C_0, C_1, \ldots, C_\lambda)$, the 2-fold covering space branched over the corresponding ribbon disc in $D^4$ is the manifold $W^4 = H^0 \cup \lambda H^1 \cup \lambda H^2$, where $\lambda H^2$ is attached along a strongly invertible link in $\lambda \# S^1 \times S^2$.

**Remark.** The manifold $W^4$ in the statement of the theorem has a spine composed by a bouquet of $\lambda$ 1-cells $(a_1, \ldots, a_\lambda) = A$ and $\lambda$ 2-cells $(w_1, \ldots, w_\lambda)$ so that the boundary of $w_i$, attached to $A$, is the word $T_i a_i T'_i$, where $T_i$ is a word in the alphabet $A \cup A^{-1}$ and $T'_i$ is the same word read backwards. This follows from the Yajima representation of a ribbon. Of course, from this property of the spine we see immediately that $H_*(W^4; \mathbb{Z}/2) = 0$.

**Remark.** If we want to know the structure of the 2-fold covering space of a 2-ribbon knot we have to look to the manifold $2W^4 = \partial (W^4 \times I)$. The triviality of $\pi_1 W^4$ implies that $W^4 = H^0 \cup \lambda H^1 \cup \lambda H^2$ is contractible, but in order to assure that the homotopy 4-sphere $2W^4$ is $S^4$ it is necessary and sufficient that $W^4 \times I$ be $B^5$ or, alternatively, that the Heegaard diagram provided by Lemma 1 in [7] goes to $(S^3; \partial)$ by Heegaard moves.

Another consequence of Theorem 3 is the following result.

**Theorem 5.** The closed 4-manifold $W^4 = H^0 \cup \lambda H^1 \cup \mu H^2 \cup \gamma H^3 \cup H^4$, where $\mu H^2$ is attached along a strongly invertible link in $\lambda \# S^1 \times S^2$, is a 2-fold cyclic covering space of $S^4$ branched over a closed 2-manifold.

**Proof.** Theorem 3 says that there is a 2-fold cyclic covering $p: H^0 \cup \lambda H^1 \cup \mu H^2 \to D^4$ branched over a 2-manifold $F$ with boundary $\partial F \subset S^3 = \partial D^4$. But the cover $p|\partial (H^0 \cup \lambda H^1 \cup \mu H^2) = \gamma \# S^1 \times S^2 \to S^3$, branched over $\partial F$, must be standard (see [4]). Thus, $\partial F$ is a system of $\gamma + 1$ unknotted and unlinked curves in $S^3$. Put $D^4$ in $S^4$ and fill up the curves $\partial F$ with discs in $S^4 - D^4$. We get a closed 2-manifold $F' \subset S^4$ and the corresponding 2-fold cyclic branched covering space is $H^0 \cup \lambda H^1 \cup \mu H^2 \cup \gamma \# S^1 \times B^3$. But this manifold is $W^4$ because of the results in [7].

**Examples.** (a) Take $CP^2 = H^0 \cup H^2 \cup H^4$. Here $H^2$ is attached along a
trivial knot in \( \partial H^0 = S^3 \) with framing +1. Then \( H^0 \cup H^2 \) is a 2-fold cyclic covering of \( D^4 \) branched over a Möbius band \( F \) with its boundary in \( S^3 \). The surface \( F' \) of the theorem is a projective plane.

(b) Take \( S^2 \times S^2 = H^0 \cup 2H^2 \cup H^4 \). Here \( 2H^2 \) is attached along a link of two components, simply linked, each with framing 0. Then \( H^0 \cup 2H^2 \) is a 2-fold cyclic covering of \( D^4 \) branched over a (torus-open disc). The surface \( F' \) is now a torus in \( S^3(7) \).

(c) Take \( CP^2 \# - CP^2 = S^2 \times S^2 = H^0 \cup 2H^2 \cup H^4 \). Here \( 2H^2 \) is attached along a link of two components, simply linked, with framings 0 and 1, respectively. Then \( H^0 \cup 2H^2 \) is a 2-fold cyclic covering of \( D^4 \) branched over a (Klein-bottle-open disc). The surface \( F' \) is now a Klein bottle.

These examples explain beautifully why \( CP^2 \# S^2 \times S^2 = CP^2 \# S^2 \times S^2 \), because forming connected sum of the real projective plane with torus or Klein bottle produces the same result.

6. 3-fold coverings. There now remains the case of attaching \( \mu H^2 \) to \( H^0 \cup \lambda H^1 \) along a general system of curves. In this case \( W^4 \) need no longer be a 2-fold branched covering space, but we show in the following that it is an irregular 3-fold covering of \( D^4 \).

Firstly, we have to define \( \lambda \# S^1 \times B^3 = H^0 \cup \lambda H^1 \) as a standard irregular 3-fold covering space of \( D^4 \) branched along \( \lambda + 2 \) unlinked and unknotted copies of \( D^2 \). In general, we represent \( \lambda \# S^1 \times B^n \) in the following way. Let \( s: \mathbb{R}^n \to \mathbb{R}^n \) defined by \( s(x_1, \ldots, x_n) = (x_1, \ldots, -x_n) \), and consider \( A = \{(x_1, \ldots, x_n) \in [-1, 1]^n | x_1 = 1 \text{ and } x_n \in \pm[(2i - 1)/3\lambda, 2i/3\lambda], \text{ for some } 0 < i < \lambda \} \). Then \( \lambda \# S^1 \times B^n \) is \([0, 1] \times [-1, 1]^n \) with the following identifications \( \mathcal{R} \) among elements \((y, x) \in [-1, 1] \times [-1, 1]^n \): \((y, x) \mathcal{R} (y, sx)\) if and only if \( y = -1, 2 \text{ or } x \in A \text{ and } y \in [-\frac{1}{2}, \frac{1}{2}] \cup [\frac{3}{2}, 2] \). We suggest that the reader draw a picture to illustrate and understand the case \( n = 2 \).

We can represent \( D^{n+1} \) by \([0, 1] \times [-1, 1]^n \) with the following identifications: \((y, x) \mathcal{R}' (y, sx)\) if and only if \( y = 0, 1 \text{ or } x \in A \text{ and } y \in [0, \frac{1}{2}] \).

We define the following map \( \hat{p}_1: [-1, 2] \to [0, 1] \) by the following rule:

\[
\hat{p}_1(t) = \begin{cases} 
-t & \text{if } -1 < t < 0, \\
t & \text{if } 0 < t < 1, \\
-t + 2 & \text{if } 1 < t < 2,
\end{cases}
\]

and the folding map \( \hat{p}_{n+1}: [-1, 2] \times [-1, 1]^n \to [0, 1] \times [-1, 1]^n \) by \( \hat{p}_{n+1} = \hat{p}_1 \times s \). Because \( \hat{p}_{n+1} \) is compatible with \( \mathcal{R} \) and \( \mathcal{R}' \), it defines \( p_{n+1}: \lambda \# S^1 \times B^n \to D^{n+1} \). This is an irregular 3-fold covering space branched over \( \{0, 1\} \times [-1, 1]^{n-1} \cup \{0\} \times A / \mathcal{R}' \). Clearly, \( p_{n+1} \) corresponds to the (simple) representation \( \omega: \pi_1(D^{n+1}-\text{branching set}) \to \mathbb{S}_3 \) such that \( \omega(x) = (01) \).
if $x$ is a meridian around $\{0\} \times ([-1, 1]^n \cup A')/\mathcal{R}'$ or $\omega(x) = (02)$ if $x$ is a meridian around $\{1\} \times [-1, 1]^n \cup A'/\mathcal{R}'$.

In case $n = 3$, $p_4|\lambda \# S^1 \times S^2$: $\lambda \# S^1 \times S^2 \to S^3$ can be visualized by means of Figure 5, where the boundary of the handlebody $X$ and the ball $D_3$ are identified by reflection in the $(x, y)$-plane, and $p_4$ identifies points by reflection through the axes $R_{02}$ and $P_{01}$.

![Figure 5](image-url)

Here the boundary of the branching set is the union of $\partial((\{0\} \times ([-1, 1]^2 \cup A')/\mathcal{R}')) = P$ and $\partial((\{1\} \times [-1, 1]^2)/\mathcal{R}') = R$.

**Lemma 1.** Let $L$ be a link of $n$ components in $\lambda \# S^1 \times S^2$. Then, after an isotopy of $L$ there exists a system $A = A_1 \cup \cdots \cup A_n$ of disjoint simple arcs in $S^3$ with the following properties:

(a) Each arc $A_i$ does not meet $R$ and meets $P$ only in its endpoints.

(b) $p_4^{-1}A$ consists of $L$ and a system $A'$ of simple arcs.

(c) $p_4|L$ is a 2-fold branched covering over $A$.

(d) $p_4|A'$: $A' \to A$ is a homeomorphism.

**Proof.** It is easily checked that if $A$ satisfies (a) and also if $p_4^{-1}A$ contains $n$ closed components, then $A$ satisfies (c) and (d). In order to find such a system $A$ we divide the proof into several steps (we refer to Figure 5).

**Step 1. Putting $L$ in the interior of $X$.**

Isotope $L$ so that $\partial X \cap L$ is a system of points symmetric with respect to reflection in the $(x, y)$-plane (see Figure 5). Connect each pair of symmetric points of this system with a selfsymmetric arc lying in $\partial X$, and use regular
neighborhoods of these arcs to isotope $L$ into the interior of $X$.

**Step 2. Putting $L$ onto a symmetric surface.**

Put $L$ in normal projection with respect to the $(x, y)$-plane (by small isotopy), and consider the “checkerboard surface” $F$ spanned by the link $L$. That is, we color one set of regions into which the $(x, y)$-plane is divided black, and the complementary set white, in such a way that any two regions with a common boundary are colored differently. Suppose that the region which contains $\infty$ is colored white, then join each two black regions, with a common double point, by a ribbon with a half-twist in the natural way. By a (gross) isotopy of $L$ if necessary, we may assume that the normal projection of $L$ is connected and separates the “holes” of $X$ from one another and from $\infty$. In addition, we put a small “kink” in each component of $L$, which impedes into a black region of $F$, and reconstruct $F$ from $L$ in this new form. If a black region intersects $R^3 - \text{int } X$ delete the interior of a small regular neighborhood of this intersection (which is a disc) so that the new surface is now contained in the interior of $X$. Call the deleted surface $F$ again.

We construct an orientable surface $G$, containing $L$ as follows. Consider a (relative) regular neighborhood $V$ of $F$ rel $\partial F$ in $X$. Then $G = \partial V$ is an orientable surface containing $\partial F$, and hence $L \subset \partial F \subset G$. Because of the kinks, no component of $L$ separates $G$. Because of the hole separating condition on $L$, $G$ is parallel to $\partial X$, except for a number of extra-holes. By isotopy, position $G$ such that these extra-holes are over the $P_{01}$ axis and so that the new surface, still called $G$, is equal to $p_4^{-1}S$, where $S$ is a 2-sphere contained in the interior of the ball $D$ (as the one shown in Figure 5). The surface $G$ contains each component of $L$ as a nonseparating curve.

**Step 3. Putting each component of $L$ onto a symmetric surface.**

Let us call $L_1, L_2, \ldots, L_n$ the components of $L$ and consider $n$ 2-spheres $S_j$, parallel to $S$. Call $G_i = p_4^{-1}S_j$. We can isotope each $L_i$ onto $G_i$. Hence, in each surface $G_i$ we have a nonseparating knot $L_i$.

**Step 4. Symmetrizing $L_i$.**

There is an orientation preserving homeomorphism of $G_i$ sending the nonseparating curve $\tilde{\alpha}_i$ corresponding to $\tilde{\alpha}$ in Figure 5 onto $L_i$ (the proof can be done using W. B. R. Lickorish’s methods [5]). This homeomorphism is isotopic in $G_i$ to $\tilde{f}_i$, which is a lifting of a homeomorphism $f_i: (S_j, P \cap S_j, R \cap S_j) \rightarrow (H. M. Hilden [2])$. So, after an isotopy in $G_i$ we can suppose that $\tilde{f}_i\tilde{\alpha}_i = L_i$. This isotopy can now be extended to $X$, using a regular neighborhood of $G_i$, which does not meet the other surfaces. Then $f_i(p_4\tilde{\alpha}_i)$ is a simple arc which meets $P$ exactly in its endpoints, does not meet $R$ and $p_4^{-1}f_i(p_4\tilde{\alpha}_i)$ contains $\tilde{f}_i\tilde{\alpha}_i = L_i$.

The conditions of the lemma are fulfilled by $A = \bigcup_i f_i(p_4\tilde{\alpha}_i)$. □

Let $W^4 = H^0 \cup \lambda H^1 \cup \mu H^2$, where $\mu H^2 = H_1^2 \cup \cdots \cup H_\mu^2$, and let $h_i$:
$\hat{B}^2 \times B^2 \to \partial (H^0 \cup \lambda H^1) = \lambda \# S^1 \times S^2$ be the attaching map of the 2-handle $H^2_i$. Put the link $L = \bigcup_i h_i(\hat{B}^2 \times (0,0))$ in $\lambda \# S^1 \times S^2$ as in Lemma 1, so that $P_4(\bigcup_i h_i(\hat{B}^2 \times B^2))$ is a regular neighborhood $U(A)$ of $A = p_4 L$. As in §2, suppose that the involution $h_i Vh_i^{-1}$ preserves fibers of $p_4; \lambda \# S^1 \times S^2 \to S^3$. Now, add $H^2_i$ to $H^0 \cup \lambda H^1$ using $h_i$. Calling $g_i$ the natural projection $H^2_i \to H^2_i|\nu$, add $H^2_i|\nu$ to $D^4$ using the composition $p_4 h_i g_i^{-1}$, and add $H^2_i|\nu$ to $p_4^{-1}(U(A_i))-U(L_i)$ using $p_4 h_i g_i^{-1}$ followed by the inverse of the homeomorphism $p_4|p_4^{-1}U(A_i)-U(L_i) \to U(A_i)$. Thus we obtain $W^4$ as $H^0 \cup \lambda H^1 \cup \mu H^2 \cup (\bigcup_i H^2_i/\nu)$, and $p = p_4 \cup \bigcup_i g_i \cup \bigcup_i (\text{id}: H^2_i/\nu \to H^2_i/\nu)$ is a 3-fold covering over $D^4 \cup \bigcup_i H^2_i/\nu \approx B^4$.

The branching set of $p$, lying over $D^4$, is a system of disjoint discs $\hat{P}_0 \cup \cdots \cup \hat{P}_\lambda \cup \hat{R}$ which intersect $S^3$ in a system $P_0 \cup \cdots \cup P_\lambda \cup R$ of unlinked and unknotted curves (see Figures 5 and 6). The branching set of $p$ lying over $H^2_i/\nu$ is a disc which can be visualized as a band $B_i$, attached to $\hat{P}_0 \cup \cdots \cup \hat{P}_\lambda$ along two different arcs in the boundary. Pushing $B_i$ into $S^3$, this band, with center line $A_i$, links $P_0 \cup \cdots \cup P_\lambda \cup R$ as $A_i$ does, producing ribbon singularities. This shows that the branching set is an obvious generalization of a ribbon disc if we allow, in Yajima's description of ribbon knots (see §3), an arbitrary number of bands. We call such surfaces ribbon manifolds. So we have

**Figure 6**

**Theorem 6.** Each $W^4 = H^0 \cup \lambda H^1 \cup \mu H^2$ is a 3-fold irregular covering space of $D^4$, the branching set being a ribbon manifold.

As an immediate consequence of Theorem 6 we have

**Corollary 2.** The double $2V^4$ of an orientable 4-manifold $V^4$ with 2-spine is a 3-fold irregular covering space of $S^4$ branched over a closed 2-manifold. □
Remarks. (1) The branching set which results from the proof of this theorem is a ribbon manifold of a special type as shown in Figure 6, because the arc \( A_i \) links \( R \) in a special way as a result of the application of Hilden's Theorem in Step 4 of Lemma 2. The branched cover corresponds to a representation \( \pi_1 (D^4\text{-ribbon manifold}) \to \mathbb{S}_3 \) which sends Wirtinger generators linking \( \hat{L}_1 \) to (01) and the ones linking \( \hat{R} \) to (02) (see Figure 6).

We call such a representation of a ribbon manifold a **colored ribbon manifold**. Note that if a colored ribbon disc is given it is very easy to exhibit a handle presentation for the corresponding 3-fold cover.

(2) A pseudo-handlebody structure on \( W^4 \) is a representation \( W^4 = H^0 \cup \lambda H^1 \cup \mu H^2 \), where \( H^2 \) means that \( H^2 \cap (H^0 \cup \lambda H^1) \) is a **knotted** solid torus in \( \partial H^2 = S^3 \). Because \( W^4 \) has a 2-spine, \( W^4 \) has a handlebody representation with 0-, 1- and 2-"true" handles, and it is a 3-fold covering space of \( D^4 \) branched over a 2-manifold.

But we can obtain this 3-fold covering directly from the pseudo-handlebody structure of \( W^4 \), and also an **explicit** true handlebody structure for \( W^4 \). The reason is that we can define in \( H^2 \) a 3-fold covering space \( p \) over \( D^4 \) (instead of a 2-fold one), and use the lemma to modify the attaching knot in \( \partial H^2 \) to be symmetric with respect to \( p \) and with two fixed points. Now in an equivariant way we can match this projection with the one defined in \( H^0 \cup \lambda H^1 \) to get the result.

**Example.** Let \( W^4 \) be the manifold \( H^0 \cup H^1 \cup H^2 \) where the attaching sphere of \( H^2 \) is the curve in Figure 1 and \( (H^0 \cup H^1) \cap H^2 \) is a regular neighborhood of the knot \( 8_{17} \), in \( \partial H^2 \). (We choose this knot because its invertibility is not known.) We symmetrize \( 8_{17} \) in Figure 7(a) and have only to paste the ball \( p(U(K)) \) in Figure 3(b) to a regular neighborhood of the arc \( \alpha \) in Figure 7(b). The resulting ribbon manifold \( F \) is shown in Figure 7(c). Of course, the covering over the discs \( D^1 \), \( D^2 \neq D^3 \) and \( D^4 \) gives \( S^1 \times B^3 \) (compare Remark (1)) and we can lift the core of the ribbon band of \( F \) to get a handlebody representation \( H^0 \cup H^1 \cup H^2 \) for \( W^4 \).

(3) (Application to 3-dimensional topology.) Each closed, orientable 3-manifold \( M^3 \) can be obtained by special Dehn surgery on a link \( L = L_1 \cup \cdots \cup L_\mu \) in \( S^3 \); we mean by "special" that the new meridian of the surgery (in \( L_\mu \), for instance) goes _one_ time around a longitude on \( \partial U(L_\mu) \).

Consider \( W^4 = H^0 \cup \mu H^2 \) where \( \mu H^2 \) is attached to \( \partial H^0 = S^3 \) along \( L \) using the framing corresponding to the Dehn-surgery in \( L \). Then there exists a 3-fold dihedral covering \( p: W^4 \to D^4 \) branched over a ribbon manifold \( R \), and \( p|_{(\partial W^4 \cup M^3)}: M^3 \to S^3 \) is a 3-fold dihedral covering space branched over \( \partial R \). Here, \( \pi_1 W^4 = 1 \). The ribbon manifold \( R \) consists of a disc \( D_1 \) and a disc with bands \( D_2 \). The representation of \( \pi_1 (D^4 - R) \to \mathbb{S}_3 \), corresponding to the cover, sends meridians of \( D_1 \) (resp. \( D_2 \)) on the transposition (01) (resp. (02)). (Compare this representation of 3-manifolds with the one in [3].)
Figure 7

(4) Lemma 1 seems interesting in its own right because it gives a procedure for "symmetrizing" knots so that they can be represented by an arc (see Figure 7) with its endpoints in $P$ and linking $R$ a number of times. The minimum of this number is a measure of the strong noninvertibility of the knot.

7. Final remarks. (1) In [7] it is shown that each 4-manifold, represented by $W^4 = H^0 \cup \lambda H^1 \cup \mu H^2 \cup \gamma H^3 \cup H^4$ is completely determined by $H^0 \cup \lambda H^1 \cup \mu H^2$. This shows that the result of Theorem 6 is not as special as it might appear, inasmuch as it provides a surjection from a subset of colored ribbon manifolds (see Remark (1) in §6) to the set of all closed, orientable, PL 4-manifolds.

We call a colored ribbon manifold allowable if the boundary of the corresponding 3-fold covering space is $\gamma \not\# S^1 \times S^2$ for some $\gamma$. The enumeration of colored ribbon manifolds which are representatives of closed 4-manifolds corresponds to the following problem:
Problem 1. When is a colored ribbon manifold allowable?

Thus allowable colored ribbon manifolds provide a representation of PL, closed, orientable 4-manifolds. In order that this representation be more useful it would be convenient to translate, in terms of colored ribbon manifolds, the concept of homeomorphism between 4-manifolds. Hence we state

Problem 2. Given two colored ribbon manifolds which represent the same closed 4-manifold, find a combinatorial way of passing from one to the other.

(2) Let \( H^0 \cup \lambda H^4 \cup \mu H^2 \cup \gamma H^3 \cup H^4 \) be a handle presentation for a closed, orientable 4-manifold \( W^4 \). By duality \( W^4 \) is obtained by pasting together two manifolds \( V^4 = H^0 \cup \lambda H^1 \cup \mu H^2 \) and \( U^4 = \gamma \# S^1 \times B^3 \). It is important to remark that the manifold \( W^4 = V^4 \cup \gamma \# S^1 \times B^3 \) is independent of the way of pasting the boundaries together [7].

We have 3-fold irregular branched coverings \( q_1: \gamma \# S^1 \times B^3 \rightarrow D^4 \) and \( q_2: V^4 \rightarrow D^4 \), provided by Theorem 6, which are special in the sense of Remark (1) in \( \S 6 \). But in some cases, as Berstein and Edmonds pointed out, these 3-fold covering spaces cannot be pasted together (see \( \S 1 \)).

Now \( V^4 \) and \( \gamma \# S^1 \times B^3 \) "a fortiori" have irregular 4-fold covering presentations \( p_1: \gamma \# S^1 \times B^3 \rightarrow D^4 \) and \( p_2: V^4 \rightarrow D^4 \), which can be obtained by adding to the branching set of \( q_1 \) (resp. \( q_2 \)) a new properly embedded trivial disc, unlinked with the branching set, and by sending its meridian into the transposition \( (03) \in \Sigma_4 \).

The conjecture that each \( W^4 \) is a 4-fold irregular covering space of \( S^4 \) branched over a closed 2-manifold, follows from the next conjecture, where \( p'_1, p'_2 \) stand for the restriction to the boundary of \( p_1, p_2 \), respectively.

Conjecture. The coverings \( p'_1 \) and \( p'_2 \) are cobordant, i.e. there is a 4-fold irregular covering \( P: (\lambda \# S^1 \times S^2) \times I \rightarrow S^3 \times I \), which is equal to \( p'_i \) in \( (\lambda \# S^1 \times S^2) \times \{ i \}, i = 1, 2 \), and branched over a 2-manifold with boundary equal to the union of the branching sets of \( p'_1, p'_2 \).

In solving this conjecture the following criterion may be useful:

Lemma 2. Let \( p: \lambda \# S^1 \times S^2 \rightarrow S^3 \) be a special covering (in the sense of Remark (1) of \( \S 6 \)) such that \( \hat{R} \) bounds a disc which does not cut any other component of the branching set; then \( p \) is standard.

Proof. It is clear that, by the conditions of the lemma, \( \lambda \# S^1 \times S^2 \) is a 2-fold covering space branched over (branching set of \( p - \hat{R} \)). Because this cover is standard (see [4]), it consists of a system of \( \lambda + 1 \) unknotted and unlinked components. □

Note that the solution of the above conjecture is closely related to the solution of Problem 1.
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