

4-MANIFOLDS, 3-FOLD COVERING SPACES AND RIBBONS

BY

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ABSTRACT. It is proved that a PL, orientable 4-manifold with a handle presentation composed by 0-, 1-, and 2-handles is an irregular 3-fold covering space of the 4-ball, branched over a 2-manifold of ribbon type. A representation of closed, orientable 4-manifolds, in terms of these 2-manifolds, is given. The structure of 2-fold cyclic, and 3-fold irregular covering spaces branched over ribbon discs is studied and new exotic involutions on S^4 are obtained. Closed, orientable 4-manifolds with the 2-handles attached along a strongly invertible link are shown to be 2-fold cyclic branched covering spaces of S^4 . The conjecture that each closed, orientable 4-manifold is a 4-fold irregular covering space of S^4 branched over a 2-manifold is reduced to studying $\gamma \# S^1 \times S^2$ as a nonstandard 4-fold irregular branched covering of S^3 .

1. Introduction. We first remark that the foundational paper [8] might be useful as an excellent account of definitions, results and historical notes.

Let F be a closed 2-manifold (not necessarily connected nor orientable) locally flat embedded in S^4 . To each transitive representation $\omega: \pi_1(S^4 - F) \rightarrow \mathfrak{S}_n$ into the symmetric group of n letters there is associated a closed, orientable, PL 4-manifold $W^4(F, \omega)$ which is a n -fold covering space of S^4 branched over F . This paper deals with the problem of representing each closed, orientable PL 4-manifold W^4 as a n -fold covering space of S^4 branched over a closed 2-manifold.

I. Bernstein and A. L. Edmonds proved [9] that in some cases (for instance $S^1 \times S^1 \times S^1 \times S^1$) n has to be at least 4. They also pointed out to the author that, for CP^2 , F must be nonorientable (using the Euler characteristic number). More generally, S. Cappell and J. Shaneson pointed out that, if F is orientable, the signature of W^4 must be zero.

We conjecture that each such W^4 is an irregular simple 4-fold covering space of S^4 branched over a closed surface F (simple means that the representation ω , where $W^4 \cong W^4(F, \omega)$, sends meridians into transpositions).

The manifold W^4 admits a handle representation $W^4 = H^0 \cup \lambda H^1 \cup \mu H^2$

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$\cup \gamma H^3 \cup H^4$. Thus, by duality, W^4 is obtained by pasting together two manifolds $V^4 = H^0 \cup \lambda H^1 \cup \mu H^2$, $U^4 = H^0 \cup \gamma H^1$ along their common boundary, which is $\gamma \# S^1 \times S^2$.

Our idea is to represent V^4 and U^4 as coverings of D^4 branched over a 2-manifold with boundary in S^3 , and then match the two coverings.

In this paper we prove that a manifold with presentation $H^0 \cup \lambda H^1 \cup \mu H^2$ is, in fact, a *dihedral 3-fold* covering of D^4 branched over a 2-manifold of a special type (which we call a ribbon manifold because it is a natural generalization of ribbon discs).

In the case that μH^2 is attached along a strongly invertible link in $\lambda \# S^1 \times S^2 = \partial(H^0 \cup \lambda H^1)$, then we show that the closed 4-manifold $W^4 = H^0 \cup \lambda H^1 \cup \mu H^2 \cup \gamma H^3 \cup H^4$ is actually a *2-fold cyclic* branched covering of S^4 . For the case of 4-fold irregular branched coverings of S^4 , we show our conjecture reduces to studying $\gamma \# S^1 \times S^2$ as a nonstandard 4-fold irregular branched covering of S^3 .

It is shown in [7] that each manifold $W^4 = H^0 \cup \lambda H^1 \cup \mu H^2 \cup \gamma H^3 \cup H^4$ is uniquely determined by $H^0 \cup \lambda H^1 \cup \mu H^2$. From this point of view, our presentation of $H^0 \cup \lambda H^1 \cup \mu H^2$ as an irregular 3-fold covering space also provides a representation for closed, orientable 4-manifolds.

We study also the structure of the 2- and dihedral 3-fold covering spaces of ribbon discs. We obtain in this way some contractible 4-manifolds of Mazur, and this allows us to find many 2-knots in S^4 with the same 2-fold cyclic covering space, even S^4 itself, thus obtaining new examples of exotic involutions on S^4 .

Lastly, we note some possible applications of these results to the study of 3-manifolds and classical knots.

I am indebted to Robert Edwards, Charles Giffen and Cameron Gordon for helpful conversations.

2. A simple case. We begin with the simple case of a manifold which is presented by one 0-handle, one 1-handle, and one 2-handle of a special type, both in order to obtain special results for this case and to illustrate the method.

Consider $S^1 \times B^3$ with presentation $H^0 \cup H^1$, i.e. one 0-handle plus one 1-handle. Its boundary, $S^1 \times S^2$, is illustrated in Figure 1.

We consider a knot K , contained in $S^1 \times S^2$, such as the one shown in Figure 1, which is *strongly invertible*, i.e. reflection u in the E -axis induces on K an involution with two fixed points. Now we add a 2-handle to $S^1 \times B^3$ so that K is the attaching sphere. More precisely, we have an embedding $h: \dot{B}^2 \times B^2 \rightarrow \partial(S^1 \times B^3)$ so that $h(\dot{B}^2 \times 0)$ equals K , and let $W^4 = S^1 \times B^3 \cup_h \dot{B}^2 \times B^2$.

Let $U: S^1 \times B^3 \rightarrow S^1 \times B^3$ be the involution, with two discs D_1, D_2 as

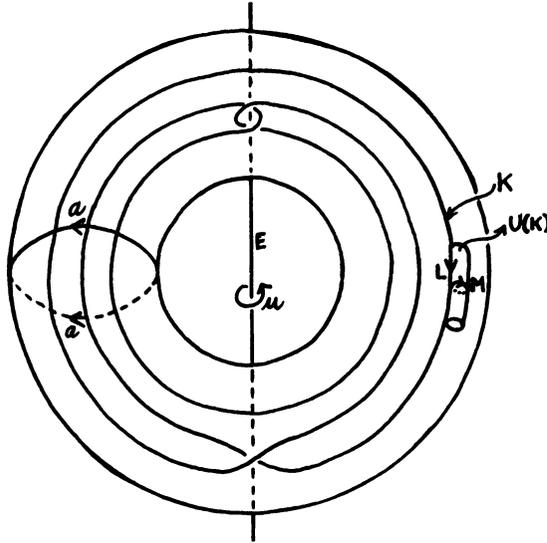


FIGURE 1

fixed-point set, which canonically extends the reflection u above.

Let $V: B^2 \times B^2 \rightarrow B^2 \times B^2$ be the reflection in $D = B^1 \times B^1$. We can represent $S^3 = \partial(B^2 \times B^2)$ by stereographic projection onto $R^3 + \infty$ in such a way that V induces on S^3 the reflection v in the y -axis. In this representation $B^2 \times B^2$ is a regular neighborhood, X , of the unit circle C in the (x, y) -plane; the belt-sphere is the z -axis, and the belt-tube is $Y = S^3 - \text{int } X$. Finally, let (m, l) be a meridian-longitude system on ∂X (see Figure 2).

Up to isotopy, we may suppose that $uh = hw$, so that (U, V) is an involution on $W^4 = S^1 \times B^3 \cup_h B^2 \times B^2$. The fixed-point set is the disc $(D_1 \cup D_2) \cup_h D$, where h pastes ∂D to $\partial(D_1 \cup D_2)$ along $\alpha \cup \beta$ of Figure 2.

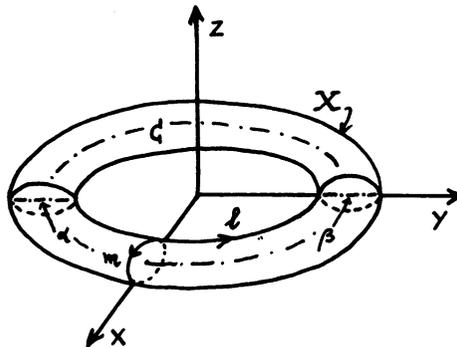


FIGURE 2

The orbit-space, which is D^4 , can be described as follows. First, $p: S^1 \times B^3 \rightarrow S^1 \times B^3/U$ is the 2-fold branched covering space of D^4

branched over two disjoint discs D'_1, D'_2 (see Figure 3(a)) such that $p|_{S^1 \times S^2}: S^1 \times S^2 \rightarrow S^3$ is the standard covering over $\partial(D'_1 \cup D'_2)$. In this covering $p(K)$ is an arc with its endpoints in $\partial D'_1$ and $\partial D'_2$. Second, $q: B^2 \times B^2 \rightarrow B^2 \times B^2/V$ is the 2-fold branched covering space of D^4 branched over a disc D' such that $p|\partial(B^2 \times B^2): \partial(B^2 \times B^2) \rightarrow S^3$ is the standard covering over the trivial knot $\partial D'$. In this covering, $q(C)$ is the arc shown in Figure 3(c). We must paste along regular neighborhoods $q(X)$ and $p(U(K))$ of $q(C)$ and $p(K)$, respectively, by the map phq^{-1} , obtaining $D^4 = S^1 \times B^3/U \cup_{phq^{-1}} B^2 \times B^2/V$.

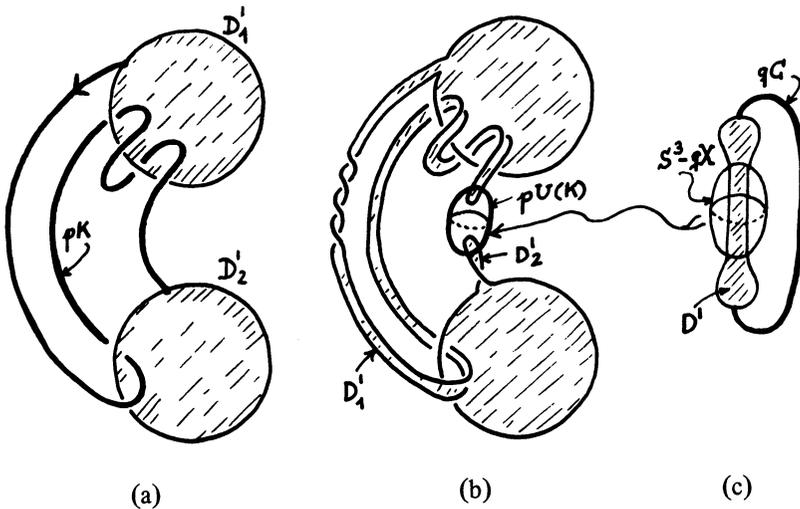


FIGURE 3

The branching set is $(D'_1 \cup D'_2) \cup_{phq^{-1}} D'$, which can be visualized as follows. Deform D'_1 and D'_2 by isotopy as illustrated in Figure 3(b), thus obtaining the “ribbon” $D'_1 \cup D'_2$ (and; in fact, if we pull $D'_1 \cup D'_2$ back into S^3 in the way suggested by the shaded part of Figure 3(b), then we obtain a ribbon immersion of $D'_1 \cup D'_2$). Pasting $B^2 \times B^2/V$ to $D^4 = S^1 \times B^3/U$ along the balls $q(X)$ and $p(U(K))$ and then “absorbing” the bulge $B^2 \times B^2/V$ on D^4 back into D^4 , we obtain the branch set in the aspect of Figure 3(b) \cup 3(c) joined by the arrow. Of course, the number of twists in the boundary of the ribbon depends on the number of times that $h(l)$ goes around $\partial(U(K))$. The ribbon of Figure 3(b) corresponds to the choice $h(l) \sim L + 5M$ (on $\partial U(K)$). Note that the number of components of the branch-set is one if and only if K connects the two components of $D_1 \cup D_2$, in which case the branch set is a ribbon disc.

We collect together these results in the following theorem.

THEOREM 1. *The manifold $W^4 = H^0 \cup H^1 \cup H^2$, where H^2 is attached*

along a strongly invertible knot of $S^1 \times S^2$, is a 2-fold covering space of D^4 , branched over a ribbon disc or over the union of a disc and either an annulus or a Möbius band.

3. Ribbons, Mazur manifolds and exotic involutions. We have immediately an amusing result. Note that the manifold W^4 corresponding to the knot K , in Figure 1, is the one discovered by B. Mazur [6], which has the property that $W^4 \times I \approx B^5$. So, the double $2W^4 \cong S^4$ and W^4 is contractible. Then, the ribbon 2-knot R corresponding to the ribbon of Figure 3(b) (i.e., the 2-knot obtained by pushing a copy of the ribbon disc into each of the two sides of S^3 in S^4) has $2W^4 = S^4$ as 2-fold covering space. This is another example of exotic involution in S^4 , first discovered by C. Gordon [1]. (It is easily checked that $\pi_1(S^4 - R) \neq Z$, showing that R is not the trivial knot in S^4 .)

In order to state these results with more generality, let us quote now the description of ribbon knots given by T. Yajima [10]. Let $C_0, C_1, \dots, C_\lambda$ be unlinked trivial circles in R^3 . Take disjoint small arcs $\alpha_1, \dots, \alpha_\lambda$ on C_0 , and a small arc γ_i on C_i ($i = 1, \dots, \lambda$). For every i , connect α_i with γ_i by a nontwisting narrow band B_i which may run through C_j ($j = 0, \dots, \lambda$) or may get tangled with itself or with other bands. Then, each ribbon knot is of this type for some λ . We shall say that a presentation of this form has type $(C_0, C_1, \dots, C_\lambda)$.

Consider a ribbon R of type (C_0, C_1) and let Δ_0, Δ_1 be disjoint discs with boundary C_0, C_1 respectively. We may assume that the center line path β of the band B_1 from C_0 to C_1 cuts $\text{Int } \Delta_0 \cup \Delta_1$ transversally, thus partitioning β into a composition of (nontrivial) paths which we write as $\beta = \beta_1 * \dots * \beta_k$ (some k). Let $\#R = \sum (-1)^j$ where j runs over those indices from 1 to k such that the subpath β_j connects C_0 and C_1 . Refer to Figure 3(a), where $\#R = 1$. Note that $\#R$ is always odd. We see that the 2-fold covering space branched over the disc R is a manifold $W^4 = H^0 \cup H^1 \cup H^2$, where H^2 is added along a strongly-invertible knot, which is homologous to $\#R$ times a generator of $H_1(S^1 \times B^3)$.

We now use the trick of Mazur [6] to describe $2W^4$. The manifold $W^4 \times I$ is obtained by adding a 2-handle to $S^1 \times B^4$ along a curve w in $S^1 \times S^3$ which is homologous to $\#R$ times a generator of $H_1(S^1 \times S^3)$. This defines the handle addition uniquely up to PL -homeomorphism since 1-knot theory of $S^1 \times S^3$ is essentially trivial.¹ Thus, $2W^4 = \partial(W^4 \times I)$ is obtained by a spherical modification of $S^1 \times S^3$ along a curve which runs $\#R$ times the generator of $H_1(S^1 \times S^3)$. Hence, we have

¹A "Dehn-twist" along a belt-sphere of $S^1 \times B^1$ changes the framing of w by a map $w \rightarrow SO(3)$ which represents the nontrivial element of $\pi_1(SO(3))$ if and only if $\#R$ is odd. Thus, we cannot worry about framings here.

THEOREM 2. *All the ribbon 2-knots, corresponding to ribbon knots of type (C_0, C_1) with the same number $|\#R|$, have the same 2-fold covering space.*

In particular, if $\#R = \pm 1$, then $2W^4 = S^4$, and so we have

COROLLARY 1. *All the ribbon 2-knots of type (C_0, C_1) , with $\#R = \pm 1$, have S^4 as 2-fold covering space.*

We see that, in contrast with the 3-dimensional analogue, the family of 2-knots in S^4 with the same 2-fold covering space is very large indeed.

More generally, if $\#R = 2m + 1$ we have $\pi_1(2W^4) \cong Z_{2m+1}$ and the universal covering space of $2W^4$ is $2m\#S^2 \times S^2$. This composite $2m\#S^2 \times S \rightarrow 2W^4 \rightarrow S^4$ is a regular dihedral branched cover over the ribbon 2-knot.

4. 2-fold coverings. We generalize these results to 4-manifolds with several 1-handles, being somewhat less explicit than before in the description of geometrical constructions.

The manifold $\lambda\#S^1 \times B^3$ has the presentation $H^0 \cup \lambda H^1$ (if $\lambda = 0$, $\lambda\#S^1 \times B^3 = B^4$). Its boundary $\lambda\#S^1 \times S^2$ is represented by the handlebody of Figure 4, with points on the boundary identified by reflection in the (x, y) -plane.

The reflection u in the x -axis is the restriction of an involution U in $\lambda\#S^1 \times B^3$ which has $\lambda + 1$ disjoint discs as fixed-point set. The orbit space of U is D^4 , and we show the branch set, $C = C_0 \cup \dots \cup C_\lambda$, in ∂D^4 in Figure 4.

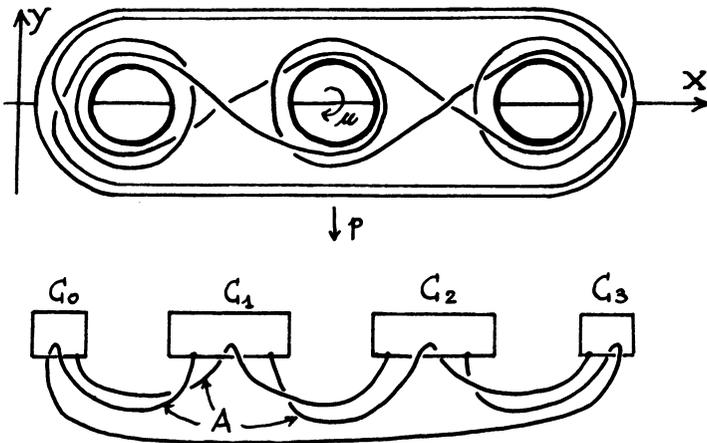


FIGURE 4

Let p be the projection, and consider now a system $A = A_1 \cup \dots \cup A_\mu$

of disjoint, simple arcs in S^3 , meeting C only in their endpoints. It is clear that $p^{-1}A$ is a *strongly invertible link* in $\lambda \# S^1 \times S^2$ (which means each $p^{-1}A_i$ is strongly-invertible with respect to u). We have the following theorem.

THEOREM 3. *The manifold $W^4 = H^0 \cup \lambda H^1 \cup \mu H^2$, where μH^2 is attached along a strongly invertible link in $\lambda \# S^1 \times S^2$, is a 2-fold cyclic covering space of D^4 branched over a 2-manifold.*

REMARK. The branching set is easily obtained in a way similar to that in §2.

Utilizing the ribbon presentation explained in §3 we have immediately the following theorem:

THEOREM 4. *If R is a ribbon knot of type $(C_0, C_1, \dots, C_\lambda)$, the 2-fold covering space branched over the corresponding ribbon disc in D^4 is the manifold $W^4 = H^0 \cup \lambda H^1 \cup \lambda H^2$, where λH^2 is attached along a strongly invertible link in $\lambda \# S^1 \times S^2$.*

REMARK. The manifold W^4 in the statement of the theorem has a spine composed by a bouquet of λ 1-cells $\{a_1, \dots, a_\lambda\} = A$ and λ 2-cells $\{w_1, \dots, w_\lambda\}$ so that the boundary of w_i , attached to A , is the word $T_i a_i T_i'$, where T_i is a word in the alphabet $A \cup A^{-1}$ and T_i' is the same word read backwards. This follows from the Yajima representation of a ribbon. Of course, from this property of the spine we see immediately that $H_*(W^4; Z/2) = 0$.

REMARK. If we want to know the structure of the 2-fold covering space of a 2-ribbon knot we have to look to the manifold $2W^4 = \partial(W^4 \times I)$. The triviality of $\pi_1 W^4$ implies that $W^4 = H^0 \cup \lambda H^1 \cup \lambda H^2$ is contractible, but in order to assure that the homotopy 4-sphere $2W^4$ is S^4 it is necessary and sufficient that $W^4 \times I$ be B^5 or, alternatively, that the Heegaard diagram provided by Lemma 1 in [7] goes to $(S^3; \emptyset)$ by Heegaard moves.

Another consequence of Theorem 3 is the following result.

THEOREM 5. *The closed 4-manifold $W^4 = H^0 \cup \lambda H^1 \cup \mu H^2 \cup \gamma H^3 \cup H^4$, where μH^2 is attached along a strongly invertible link in $\lambda \# S^1 \times S^2$, is a 2-fold cyclic covering space of S^4 branched over a closed 2-manifold.*

PROOF. Theorem 3 says that there is a 2-fold cyclic covering $p: H^0 \cup \lambda H^1 \cup \mu H^2 \rightarrow D^4$ branched over a 2-manifold F with boundary $\partial F \subset S^3 = \partial D^4$. But the cover $p|\partial(H^0 \cup \lambda H^1 \cup \mu H^2) = \gamma \# S^1 \times S^2 \rightarrow S^3$, branched over ∂F , must be standard (see [4]). Thus, ∂F is a system of $\gamma + 1$ unknotted and unlinked curves in S^3 . Put D^4 in S^4 and fill up the curves ∂F with discs in $S^4 - D^4$. We get a closed 2-manifold $F' \subset S^4$ and the corresponding 2-fold cyclic branched covering space is $H^0 \cup \lambda H^1 \cup \mu H^2 \cup \gamma \# S^1 \times B^3$. But this manifold is W^4 because of the results in [7]. \square

EXAMPLES. (a) Take $CP^2 = H^0 \cup H^2 \cup H^4$. Here H^2 is attached along a

trivial knot in $\partial H^0 = S^3$ with framing +1. Then $H^0 \cup H^2$ is a 2-fold cyclic covering of D^4 branched over a Möbius band F with its boundary in S^3 . The surface F' of the theorem is a projective plane.

(b) Take $S^2 \times S^2 = H^0 \cup 2H^2 \cup H^4$. Here $2H^2$ is attached along a link of two components, simply linked, each with framing 0. Then $H^0 \cup 2H^2$ is a 2-fold cyclic covering of D^4 branched over a (torus-open disc). The surface F' is now a torus in S^3 (!).

(c) Take $CP^2 \# -CP^2 = S^2 \times S^2 = H^0 \cup 2H^2 \cup H^4$. Here $2H^2$ is attached along a link of two components, simply linked, with framings 0 and 1, respectively. Then $H^0 \cup 2H^2$ is a 2-fold cyclic covering of D^4 branched over a (Klein-bottle-open disc). The surface F' is now a Klein bottle.

These examples explain beautifully why $CP^2 \# S^2 \times S^2 = CP^2 \# S^2 \times S^2$, because forming connected sum of the real projective plane with torus or Klein bottle produces the same result.

6. 3-fold coverings. There now remains the case of attaching μH^2 to $H^0 \cup \lambda H^1$ along a general system of curves. In this case W^4 need no longer be a 2-fold branched covering space, but we show in the following that it is an irregular 3-fold covering of D^4 .

Firstly, we have to define $\lambda \# S^1 \times B^3 = H^0 \cup \lambda H^1$ as a *standard* irregular 3-fold covering space of D^4 branched along $\lambda + 2$ unlinked and unknotted copies of D^2 . In general, we represent $\lambda \# S^1 \times B^n$ in the following way. Let $s: \mathbf{R}^n \rightarrow \mathbf{R}^n$ defined by $s(x_1, \dots, x_n) = (x_1, \dots, -x_n)$, and consider $A = \{(x_1, \dots, x_n) \in [-1, 1]^n \mid x_1 = 1 \text{ and } x_n \in \pm[(2i-1)/3\lambda, 2i/3\lambda], \text{ for some } 0 < i \leq \lambda\}$. Then $\lambda \# S^1 \times B^n$ is $[-1, 2] \times [-1, 1]^n$ with the following identifications \mathcal{R} among elements $(y, x) \in [-1, 2] \times [-1, 1]^n$: $(y, x) \mathcal{R} (y, sx)$ if and only if $y = -1, 2$ or $x \in A$ and $y \in [-\frac{1}{2}, \frac{1}{2}] \cup [\frac{3}{2}, 2]$. We suggest that the reader draw a picture to illustrate and understand the case $n = 2$.

We can represent D^{n+1} by $[0, 1] \times [-1, 1]^n$ with the following identifications: $(y, x) \mathcal{R}' (y, sx)$ if and only if $y = 0, 1$ or $x \in A$ and $y \in [0, \frac{1}{2}]$.

We define the following map $\hat{p}_1: [-1, 2] \rightarrow [0, 1]$ by the following rule:

$$\hat{p}_1(t) = \begin{cases} -t & \text{if } -1 \leq t \leq 0, \\ t & \text{if } 0 \leq t \leq 1, \\ -t + 2 & \text{if } 1 \leq t \leq 2, \end{cases}$$

and the folding map $\hat{p}_{n+1}: [-1, 2] \times [-1, 1]^n \rightarrow [0, 1] \times [-1, 1]^n$ by $\hat{p}_{n+1} = \hat{p}_1 \times s$. Because \hat{p}_{n+1} is compatible with \mathcal{R} and \mathcal{R}' , it defines $p_{n+1}: \lambda \# S^1 \times B^n \rightarrow D^{n+1}$. This is an irregular 3-fold covering space branched over $\{0, 1\} \times [-1, 1]^{n-1} \cup \{0\} \times A/\mathcal{R}'$. Clearly, p_{n+1} corresponds to the (simple) representation $\omega: \pi_1(D^{n+1}\text{-branching set}) \rightarrow \mathfrak{S}_3$ such that $\omega(x) = (01)$

if x is a meridian around $\{0\} \times ([-1, 1]^{n-1} \cup A)/\mathcal{R}'$ or $\omega(x) = (02)$ if x is a meridian around $\{1\} \times [-1, 1]^{n-1}/\mathcal{R}'$.

In case $n = 3$, $p_4|\lambda \# S^1 \times S^2: \lambda \# S^1 \times S^2 \rightarrow S^3$ can be visualized by means of Figure 5, where the boundary of the handlebody X and the ball D^3 are identified by reflection in the (x, y) -plane, and p_4 identifies points by reflection through the axes R_{02} and P_{01} .

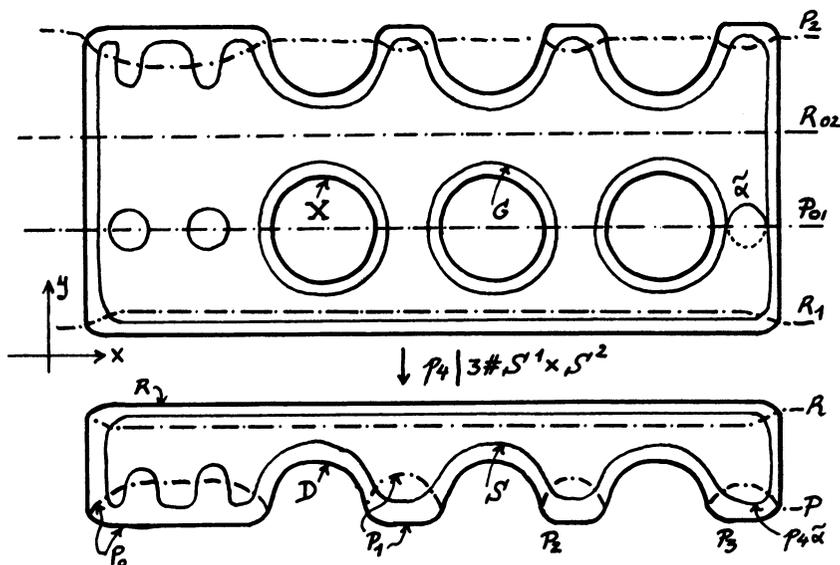


FIGURE 5

Here the boundary of the branching set is the union of $\partial(\{0\} \times ([-1, 1]^2 \cup A)/\mathcal{R}') = P$ and $\partial(\{1\} \times [-1, 1]^2/\mathcal{R}') = R$.

LEMMA 1. Let L be a link of n components in $\lambda \# S^1 \times S^2$. Then, after an isotopy of L there exists a system $A = A_1 \cup \dots \cup A_n$ of disjoint simple arcs in S^3 with the following properties:

- (a) Each arc A_i does not meet R and meets P only in its endpoints.
- (b) $p_4^{-1}A$ consists of L and a system A' of simple arcs.
- (c) $p_4|L$ is a 2-fold branched covering over A .
- (d) $p_4|A': A' \rightarrow A$ is a homeomorphism.

PROOF. It is easily checked that if A satisfies (a) and also if $p_4^{-1}A$ contains n closed components, then A satisfies (c) and (d). In order to find such a system A we divide the proof into several steps (we refer to Figure 5).

Step 1. Putting L in the interior of X .

Isotope L so that $\partial X \cap L$ is a system of points symmetric with respect to reflection in the (x, y) -plane (see Figure 5). Connect each pair of symmetric points of this system with a selfsymmetric arc lying in ∂X , and use regular

neighborhoods of these arcs to isotope L into the interior of X .

Step 2. Putting L onto a symmetric surface.

Put L in normal projection with respect to the (x, y) -plane (by small isotopy), and consider the “checkerboard surface” F spanned by the link L . That is, we color one set of regions into which the (x, y) -plane is divided black, and the complementary set white, in such a way that any two regions with a common boundary are colored differently. Suppose that the region which contains ∞ is colored white, then join each two black regions, with a common double point, by a ribbon with a half-twist in the natural way. By a (gross) isotopy of L if necessary, we may assume that the normal projection of L is connected and separates the “holes” of X from one another and from ∞ . In addition, we put a small “kink” in each component of L , which impedes into a black region of F , and reconstruct F from L in this new form. If a black region intersects $R^3 - \text{int } X$ delete the interior of a small regular neighborhood of this intersection (which is a disc) so that the new surface is now contained in the interior of X . Call the deleted surface F again.

We construct an orientable surface G , containing L as follows. Consider a (relative) regular neighborhood V of F rel ∂F in X . Then $G = \partial V$ is an orientable surface containing ∂F , and hence $L \subset \partial F \subset G$. Because of the kinks, no component of L separates G . Because of the hole separating condition on L , G is parallel to ∂X , except for a number of extra-holes. By isotopy, position G such that these extra-holes are over the P_{01} axis and so that the new surface, still called G , is equal to $p_4^{-1}S$, where S is a 2-sphere contained in the interior of the ball D (as the one shown in Figure 5). The surface G contains each component of L as a nonseparating curve.

Step 3. Putting each component of L onto a symmetric surface.

Let us call L_1, L_2, \dots, L_n the components of L and consider n 2-spheres S_i , parallel to S . Call $G_i = p_4^{-1}S_i$. We can isotope each L_i onto G_i . Hence, in each surface G_i we have a nonseparating knot L_i .

Step 4. Symmetrizing L_i .

There is an orientation preserving homeomorphism of G_i sending the nonseparating curve $\tilde{\alpha}_i$ corresponding to $\tilde{\alpha}$ in Figure 5 onto L_i (the proof can be done using W. B. R. Lickorish’s methods [5]). This homeomorphism is isotopic in G_i to \tilde{f}_i , which is a lifting of a homeomorphism $f_i: (S_i, P \cap S_i, R \cap S_i) \rightarrow (S_i, P \cap S_i, R \cap S_i)$ (H. M. Hilden [2]). So, after an isotopy in G_i we can suppose that $\tilde{f}_i \tilde{\alpha}_i = L_i$. This isotopy can now be extended to X , using a regular neighborhood of G_i which does not meet the other surfaces. Then $f_i(p_4 \tilde{\alpha}_i)$ is a simple arc which meets P exactly in its endpoints, does not meet R and $p_4^{-1}f_i(p_4 \tilde{\alpha}_i)$ contains $\tilde{f}_i \tilde{\alpha}_i = L_i$.

The conditions of the lemma are fulfilled by $A = \cup_i f_i(p_4 \tilde{\alpha}_i)$. \square

Let $W^4 = H^0 \cup \lambda H^1 \cup \mu H^2$, where $\mu H^2 = H_1^2 \cup \dots \cup H_\mu^2$, and let $h_i:$

$\hat{B}^2 \times B^2 \rightarrow \partial(H^0 \cup \lambda H^1) = \lambda \# S^1 \times S^2$ be the attaching map of the 2-handle H_i^2 . Put the link $L = \cup_i h_i(\hat{B}^2 \times (0, 0))$ in $\lambda \# S^1 \times S^2$ as in Lemma 1, so that $P_4(\cup_i h_i(\hat{B}^2 \times B^2))$ is a regular neighborhood $U(A)$ of $A = p_4 L$. As in §2, suppose that the involution $h_i V h_i^{-1}$ preserves fibers of $p_4: \lambda \# S^1 \times S^2 \rightarrow S^3$. Now, add H_i^2 to $H^0 \cup \lambda H^1$ using h_i . Calling g_i the natural projection $H_i^2 \rightarrow H_i^2/V$, add H_i^2/V to D^4 using the composition $p_4 h_i g_i^{-1}$, and add H_i^2/V to $p_4^{-1}(U(A_i)) - U(L_i)$ using $p_4 h_i g_i^{-1}$ followed by the inverse of the homeomorphism $p_4|_{p_4^{-1}U(A_i) - U(L_i)} \rightarrow U(A_i)$. Thus we obtain W^4 as $H^0 \cup \lambda H^1 \cup \mu H^2 \cup (\cup_i H_i^2/V)$, and $p = p_4 \cup \cup_i g_i \cup \cup_i (\text{id}: H_i^2/V \rightarrow H_i^2/V)$ is a 3-fold covering over $D^4 \cup \cup_i H_i^2/V \approx B^4$.

The branching set of p , lying over D^4 , is a system of disjoint discs $\hat{P}_0 \cup \dots \cup \hat{P}_\lambda \cup \hat{R}$ which intersect S^3 in a system $P_0 \cup \dots \cup P_\lambda \cup R$ of unlinked and unknotted curves (see Figures 5 and 6). The branching set of p lying over H_i^2/V is a disc which can be visualized as a band B_i , attached to $\hat{P}_0 \cup \dots \cup \hat{P}_\lambda$ along two different arcs in the boundary. Pushing B_i into S^3 , this band, with center line A_i , links $P_0 \cup \dots \cup P_\lambda \cup R$ as A_i does, producing ribbon singularities. This shows that the branching set is an obvious generalization of a ribbon disc if we allow, in Yajima's description of ribbon knots (see §3), an arbitrary number of bands. We call such surfaces *ribbon manifolds*. So we have

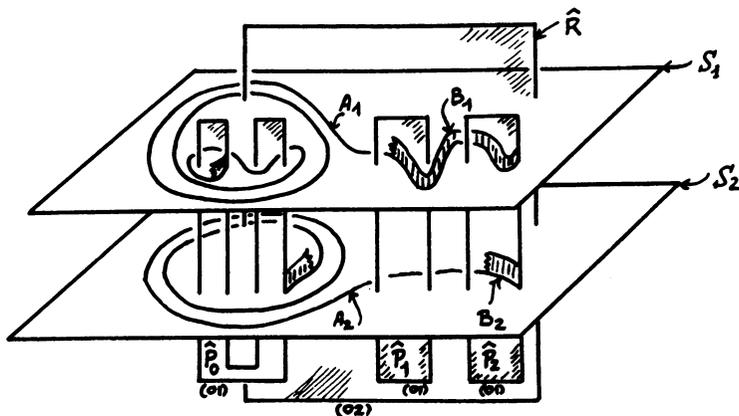


FIGURE 6

THEOREM 6. *Each $W^4 = H^0 \cup \lambda H^1 \cup \mu H^2$ is a 3-fold irregular covering space of D^4 , the branching set being a ribbon manifold.*

As an immediate consequence of Theorem 6 we have

COROLLARY 2. *The double $2V^4$ of an orientable 4-manifold V^4 with 2-spine is a 3-fold irregular covering space of S^4 branched over a closed 2-manifold. \square*

REMARKS. (1) The branching set which results from the proof of this theorem is a ribbon manifold of a special type as shown in Figure 6, because the arc A_i links \hat{R} in a special way as a result of the application of Hilden's Theorem in Step 4 of Lemma 2. The branched cover corresponds to a representation $\pi_1(D^4\text{-ribbon manifold}) \rightarrow \mathfrak{S}_3$ which sends Wirtinger generators linking \hat{P}_i to (01) and the ones linking \hat{R} to (02) (see Figure 6).

We call such a representation of a ribbon manifold a *colored ribbon manifold*. Note that if a colored ribbon disc is given it is very easy to exhibit a handle presentation for the corresponding 3-fold cover.

(2) A pseudo-handlebody structure on W^4 is a representation $W^4 = H^0 \cup \lambda H^1 \cup \mu H_s^2$, where H_s^2 means that $H_s^2 \cap (H^0 \cup \lambda H^1)$ is a *knotted* solid torus in $\partial H_s^2 = S^3$. Because W^4 has a 2-spine, W^4 has a handlebody representation with 0-, 1- and 2-“true” handles, and it is a 3-fold covering space of D^4 branched over a 2-manifold.

But we can obtain this 3-fold covering directly from the pseudo-handlebody structure of W^4 , and also an *explicit* true handlebody structure for W^4 . The reason is that we can define in H_s^2 a 3-fold covering space p over D^4 (instead of a 2-fold one), and use the lemma to modify the attaching knot in ∂H_s^2 to be symmetric with respect to p and with two fixed points. Now in an equivariant way we can match this projection with the one defined in $H^0 \cup \lambda H^1$ to get the result.

EXAMPLE. Let W^4 be the manifold $H^0 \cup H^1 \cup H_s^2$ where the attaching sphere of H_s^2 is the curve in Figure 1 and $(H^0 \cup H^1) \cap H_s^2$ is a regular neighborhood of the knot 8_{17} , in ∂H_s^2 . (We choose this knot because its invertibility is not known.) We symmetrize 8_{17} in Figure 7(a) and have only to paste the ball $p(U(K))$ in Figure 3(b) to a regular neighborhood of the arc α in Figure 7(b). The resulting ribbon manifold F is shown in Figure 7(c). Of course, the covering over the discs D_1^1 , $D_2^1 \# D_3^1$ and D_4^1 gives $S^1 \times B^3$ (compare Remark (1)) and we can lift the core of the ribbon band of F to get a handlebody representation $H^0 \cup H^1 \cup H^2$ for W^4 .

(3) (Application to 3-dimensional topology.) Each closed, orientable 3-manifold M^3 can be obtained by special Dehn surgery on a link $L = L_1 \cup \cdots \cup L_\mu$ in S^3 ; we mean by “special” that the new meridian of the surgery (in L_i , for instance) goes *one* time around a longitude on $\partial U(L_i)$. Consider $W^4 = H^0 \cup \mu H^2$ where μH^2 is attached to $\partial H^0 = S^3$ along L using the framing corresponding to the Dehn-surgery in L . Then there exists a 3-fold dihedral covering $p: W^4 \rightarrow D^4$ branched over a ribbon manifold R , and $p|(\partial W^4 = M^3): M^3 \rightarrow S^3$ is a 3-fold dihedral covering space branched over ∂R . Here, $\pi_1 W^4 = 1$. The ribbon manifold R consists of a disc D_1 and a disc with bands D_2 . The representation of $\pi_1(D^4 - R) \rightarrow \mathfrak{S}_3$, corresponding to the cover, sends meridians of D_1 (resp. D_2) on the transposition (01) (resp. (02)). (Compare this representation of 3-manifolds with the one in [3].)

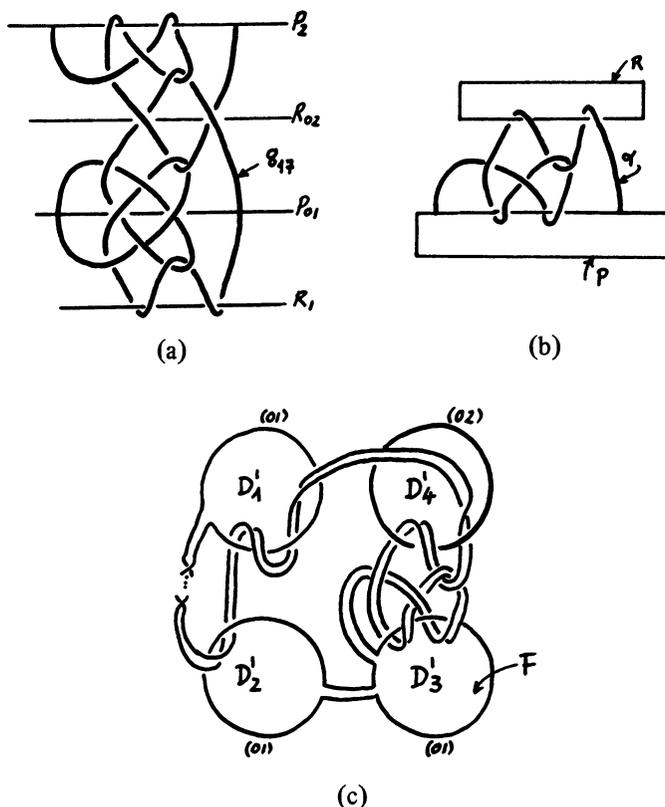


FIGURE 7

(4) Lemma 1 seems interesting in its own right because it gives a procedure for “symmetrizing” knots so that they can be represented by an arc (see Figure 7) with its endpoints in P and linking R a number of times. The minimum of this number is a measure of the strong noninvertibility of the knot.

7. Final remarks. (1) In [7] it is shown that each 4-manifold, represented by $W^4 = H^0 \cup \lambda H^1 \cup \mu H^2 \cup \gamma H^3 \cup H^4$ is completely determined by $H^0 \cup \lambda H^1 \cup \mu H^2$. This shows that the result of Theorem 6 is not as special as it might appear, inasmuch as it provides a surjection from a subset of colored ribbon manifolds (see Remark (1) in §6) to the set of all closed, orientable, PL 4-manifolds.

We call a colored ribbon manifold *allowable* if the boundary of the corresponding 3-fold covering space is $\gamma \# S^1 \times S^2$ for some γ . The enumeration of colored ribbon manifolds which are representatives of closed 4-manifolds corresponds to the following problem:

Problem 1. When is a colored ribbon manifold allowable?

Thus allowable colored ribbon manifolds provide a representation of PL , closed, orientable 4-manifolds. In order that this representation be more useful it would be convenient to translate, in terms of colored ribbon manifolds, the concept of homeomorphism between 4-manifolds. Hence we state

Problem 2. Given two colored ribbon manifolds which represent the same closed 4-manifold, find a combinatorial way of passing from one to the other.

(2) Let $H^0 \cup \lambda H^4 \cup \mu H^2 \cup \gamma H^3 \cup H^4$ be a handle presentation for a closed, orientable 4-manifold W^4 . By duality W^4 is obtained by pasting together two manifolds $V^4 = H^0 \cup \lambda H^1 \cup \mu H^2$ and $U^4 = \gamma \# S^1 \times B^3$. It is important to remark that the manifold $W^4 = V^4 \cup \gamma \# S^1 \times B^3$ is *independent* of the way of pasting the boundaries together [7].

We have 3-fold irregular branched coverings $q_1: \gamma \# S^1 \times B^3 \rightarrow D^4$ and $q_2: V^4 \rightarrow D^4$, provided by Theorem 6, which are special in the sense of Remark (1) in §6. But in some cases, as Bernstein and Edmonds pointed out, these 3-fold covering spaces cannot be pasted together (see §1).

Now V^4 and $\gamma \# S^1 \times B^3$ "a fortiori" have irregular 4-fold covering presentations $p_1: \gamma \# S^1 \times B^3 \rightarrow D^4$ and $p_2: V^4 \rightarrow D^4$, which can be obtained by adding to the branching set of q_1 (resp. q_2) a new properly embedded trivial disc, unlinked with the branching set, and by sending its meridian into the transposition $(03) \in \mathfrak{S}_4$.

The conjecture that each W^4 is a 4-fold irregular covering space of S^4 branched over a closed 2-manifold, follows from the next conjecture, where p'_1, p'_2 stand for the restriction to the boundary of p_1, p_2 , respectively.

CONJECTURE. *The coverings p'_1 and p'_2 are cobordant, i.e. there is a 4-fold irregular covering $P: (\lambda \# S^1 \times S^2) \times I \rightarrow S^3 \times I$, which is equal to p'_i in $(\lambda \# S^1 \times S^2) \times \{i\}$, $i = 1, 2$, and branched over a 2-manifold with boundary equal to the union of the branching sets of p'_1, p'_2 .*

In solving this conjecture the following criterion may be useful:

LEMMA 2. *Let $p: \lambda \# S^1 \times S^2 \rightarrow S^3$ be a special covering (in the sense of Remark (1) of §6) such that \hat{R} bounds a disc which does not cut any other component of the branching set; then p is standard.*

PROOF. It is clear that, by the conditions of the lemma, $\lambda \# S^1 \times S^2$ is a 2-fold covering space branched over (branching set of $p - \hat{R}$). Because this cover is standard (see [4]), it consists of a system of $\lambda + 1$ unknotted and unlinked components. \square

Note that the solution of the above conjecture is closely related to the solution of Problem 1.

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