INTERTWINING DIFFERENTIAL OPERATORS FOR

Mp(n, R) AND SU(n, n)

BY

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ABSTRACT. For each of the two series of groups, three series of representations \( U_n, D_n, \) and \( H_n (n \in \mathbb{Z}) \) are considered. For each series of representations there is a differential operator with the property, that raised to the \( n \)th power \( (n > 0) \), it intertwines the representations indexed by \(-n\) and \(n\). The operators are generalizations of the d'Alembertian, the Dirac-operator and a combination of the two. Unitarity of subquotients of representations indexed by negative integers is derived from the intertwining relations.

0. Introduction. Motivated by the aspects of the conformal group as a physical symmetry group, as suggested by I. E. Segal [11], we recently studied, jointly with Michele Vergne, some representation theoretical aspects of \( SU(2, 2) \) [5]. One result, that was obtained, was that powers of the d'Alembertian

\[
\Box = \frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x_1^2} - \frac{\partial^2}{\partial x_2^2} - \frac{\partial^2}{\partial x_3^2},
\]

as well as powers of the Dirac operator \( \mathcal{D} \) (a \(4 \times 4\) matrix for which \( \mathcal{D}^2 = \Box \)) are intertwining between two series of representations of \( SU(2, 2) \). We shall see that a similar phenomenon takes place for \( Mp(n, R) \) and \( SU(n, n) \).

Specifically, we consider an \( n \times n \) matrix \( D = \partial / \partial x_{a\beta} \) with first order differential operators as entries, corresponding to a parametrization of the space of \( n \times n \) symmetric (hermitian) matrices. We prove that \( \det D \) as well as

\[
\mathcal{D} = \begin{pmatrix} 0 & D' \\ c(D) & 0 \end{pmatrix},
\]

where \( c(D)D' = D'c(D) = (\det D)I_n \), are intertwining between two series \( U_i \).
and $D_l$, $l \in \mathbb{Z}$, of representations of $Mp(n, \mathbb{R})$ ($SU(n, n)$) in the sense that for $l > 0$

$$(\det D)^l U_{-l} = U_l (\det D)^l, \quad (\nabla)^{2l+1} D_{-l} = D_l (\nabla)^{2l+1}. \quad (0.1)$$

We shall refer to $\nabla$ as the Dirac-type operator.

For $D'$ and $\det D$ the equations (0.1) were expected from [5, V.5.1 and V.6.1]. However, we shall take a somewhat different approach, which has the equations involving $c(D)$ above as a straightforward consequence.

We remark that as a special case of algebraic results, B. Kostant [7] obtained quasi-invariance properties of the wave-operator $\Box$. The operators $\det D'$, $(D')'$, and $c(D)'$, for $l \in \mathbb{N}$, were studied by L. Gårding [3], and the Cauchy problem was solved. It was also noted that the principal formulas involving $(\det D)'$ were invariant under the transformation $x \to axa^*$ of the space of symmetric (hermitian) matrices, for $a$ in $SL(n, \mathbb{R})$ ($SL(n, \mathbb{C})$).

A basic observation in the proof of (0.1) is, that if $s$ is the operator acting on functions $f$ from the space of symmetric $n \times n$ matrices (space of hermitian $n \times n$ matrices) to $\mathbb{C}^n$ by $(sf)(s) = sf(s)$, then $c(D) = [(\det D), s]$. We use (0.1) to prove that $U_{-l}$ and $D_{-l}$, for $l > 0$, act unitarily on quotient spaces of functions modulo solutions to

$$\phi \in 0$$

and $\nabla^{2l-1}\phi = 0$, respectively. The representation $D_l$ can be written as $D_l^+ \oplus D_l^-$ where, by (0.1), $D_l^+$ and $D_l^-$ satisfy

$$\det D' c(D) D_{-l-1}^+ = D_l^- \det D' c(D), \quad \det D' D_{-l-1}^+ = D_l^+ \det D' D'. \quad (0.2)$$

We use this to study a series $H_l$ of representations obtained by induction from reducible, noncomplemented representations of the maximal parabolic subgroup. It is proved that these representations are related to the preceding ones by

$$H_l = \begin{bmatrix} \frac{l}{D'} & 0 \\ -l & \end{bmatrix} H_0 = \begin{bmatrix} D_{l-1}^- & 0 \\ 0 & D_l^+ \end{bmatrix} \begin{bmatrix} \frac{l}{D'} & 0 \\ -l & \end{bmatrix}. \quad (0.3)$$

The study of the representations $H_l$ was motivated in part by the work of A. Salam and G. Mack [10, p. 178].

The present article falls in five parts. (1) is the scalar case for $Mp(n, \mathbb{R})$, corresponding to $(\det D)'$, and (2) is the scalar case for $SU(n, n)$. In (3) the Dirac-type operator for $Mp(n, \mathbb{R})$ is related to representations obtained by induction from reducible, complemented representations of the maximal parabolic $P_-$ in $2n$-dimensional complex vector spaces, and (4) is the corresponding for $SU(n, n)$. Finally, in (5) representations obtained by induction
from reducible, noncomplemented representations of $P_-$ in $2n$ dimensional complex vector spaces, are proved in many cases to be unitary (and reducible). The details are carried out for $SU(n, n)$.

The first two chapters of the present paper are essentially contained in the author's Ph.D. Thesis [6], written under the direction of I. E. Segal. The author is indebted to Professor Segal for many helpful discussions. He is also thankful to Michele Vergne for friendly help and conversations.

1. The scalar case for $Mp(n, R)$. We shall begin with some generalities about $Sp(n, R)$ which is covered twice by $Mp(n, R)$.

$Sp(n, R)$ is the subgroup of $Gl(2n, R)$ consisting of those matrices $g$ that satisfy

$$g = \begin{bmatrix} 0 & 1_n \\ -1_n & 0 \end{bmatrix}.$$  \hfill (1.1)

If we write $g$ in terms of $n \times n$ blocks; $g = [a \ b]$, then (1.1) is equivalent to

$$ad' - bc' = 1; \quad ab' = ba'; \quad cd' = dc',$$  \hfill (1.2)

and to

$$a'd - c'b = 1; \quad a'c = c'a; \quad b'd = d'b,$$  \hfill (1.3)

where (1.3) is obtained by replacing $g$ by $g^{-1}$ in (1.1). The Lie algebra of $Sp(n, R)$ is thus [4, p. 341]

$$sp(n, R) = \left\{ \begin{bmatrix} x_1 & x_2 \\ x_3 & -x_1' \end{bmatrix} \right| x_1 \text{ arbitrary}; \quad x_2 = x_2'; \quad x_3 = x_3' \right\}.$$  \hfill (1.4)

We let $\mathcal{D} = \{ z = x + iy | x, y \text{ real } n \times n \text{ matrices}, x = x'; \quad y = y'; \quad y > 0 \}$. $\mathcal{D}$ is then a complex domain, and $Sp(n, R)$ acts on $\mathcal{D}$ by

$$g \cdot z = (az + b)(cz + d)^{-1}.$$  \hfill (1.5)

We recall from [5] that if $G$ is a group of holomorphic transformations on $\mathcal{D}$, $V$ a finite dimensional complex vector space, $J(g, z)$ a continuous function $G \times \mathcal{D} \to GL(V)$ which, for each fixed $g$ in $G$ is holomorphic in $z$ and satisfies

$$J\left( g_1 g_2, z \right) = J\left( g_1, g_2 z \right) J\left( g_2, z \right); \quad J\left( 1, z \right) = 1,$$  \hfill (1.6)

then a function $K(z, w): \mathcal{D} \times \mathcal{D} \to \text{End } V$, holomorphic in $z$, anti-holomorphic in $w$, is the reproducing kernel for the representation

$$(T_g f)(z) = J\left( g^{-1}, z \right)^{-1} f\left( g^{-1} z \right)$$  \hfill (1.7)

on a space of holomorphic functions $f: \mathcal{D} \to V$ if and only if

$$K(z, w) = K(w, z)^*,$$  \hfill (1.8)
\[ K(gz, gw) = J(g, z)K(z, w)J(g, w)^*, \text{ and} \]
\[ \sum_{i,j=1}^{n} \langle K(z_j, z_i) v_i, v_j \rangle > 0 \]

for all \( z_i \) in \( G \), \( v_i \) in \( V \), \( i = 1, 2, \ldots, n \), and \( n \) in \( \mathbb{N} \). We let * denote the complex adjoint of an operator. Thus, \( (x + iy)^* = x' - iy' \). If \( g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \) is in \( \text{Sp}(n, \mathbb{R}) \) and \( z, w \) are in \( G \), we then define

\[ J_1(g, z) = cz + d; \quad J_2(g, z) = (zc^t + dt)^{-1}, \]
\[ K_1(z, w) = ((z - w^*)/2i)^{-1}; \quad K_2(z, w) = ((z - w^*)/2i). \]

Then it follows easily from (1.2) and (1.3) that \( J_1 \) and \( J_2 \) both satisfy (1.4). By the same relations, it also follows that for \( i = 1, 2 \)

\[ K_i(gz, gw) = J_i(g, z)K_i(z, w)J_i(g, w)^*. \]

We shall in this section consider the one-parameter family of actions of \( \text{Sp}(n, \mathbb{R}) \) on the space \( \mathcal{O}_1 \) of holomorphic functions on \( G \) defined by, for \( \lambda \in \mathbb{R} \),

\[ (U_\lambda(g)f)(z) = (\det J_1(g^{-1}, z))^{-\lambda-((n+1)/2)} f(g^{-1}z). \]

It is easy to see that \( U_\lambda \) is holomorphically induced from the one-dimensional representation \( \tau_\lambda(k) = \det(J, (k, i))^{\lambda+(n+1)/2} \) of the maximal compact subgroup \( K \) (cf. [5, p. 61]).

Remark. For noninteger \( \lambda \)'s, we only get a projective representation; by passing to the universal covering group, we get a proper representation. In the sequel we shall mostly be interested in the cases where \( \lambda \) is integer. In these cases, we need only pass to the double covering group of \( \text{Sp}(n, \mathbb{R}) \); the metaplectic group \( \text{Mp}(n, \mathbb{R}) \). We shall maintain the notation \( U_\lambda \) irrespective of the groups.

It follows easily from (1.2) and (1.3) that

\[ g \cdot (z^*) = (g \cdot z)^* \]

for all \( g \) in \( \text{Sp}(n, \mathbb{R}) \). (\( \text{Sp}(n, \mathbb{R}) \) also acts on the "lower half-plane".) In particular, \( g \) leaves the Shilov boundary \( \partial G \) of \( G \); \( \partial G = S = \{x + iy | y = 0\} \), invariant, and even though the action is not globally defined, we still get an action on measurable functions. We shall also maintain the notation \( U_\lambda \) for this action. It can be seen that there exists a subspace \( V_\lambda \) of \( C^\infty \)-functions on \( S \), which is invariant under this latter representation, and such that the restriction of \( U_\lambda \) to \( V_\lambda \) can be imbedded into an invariant subspace of a degenerate principal series representation (cf. [5, pp. 82–83], and below).

The space \( S \) of \( n \times n \) symmetric real matrices is a real vector space of dimension \( \frac{1}{2} n(n + 1) \). We write elements of \( S \) as \( x = [x_{\alpha \beta}]_{\alpha, \beta = 1}^{n} \) (\( x_{\alpha \beta} = x_{\beta \alpha} \)) or just \( x = [x_{\alpha \beta}] \), and let \( dx \) denote Lebesgue measure on \( S \) corresponding to
this parametrization. On functions from $S$ to $\mathbb{C}$, we define first order
differential operators $a_{\alpha \beta}$ by

$$a_{\alpha \beta} f = \frac{1}{2} \frac{\partial}{\partial x_{\alpha \beta}} f. \quad \text{(Thus, } a_{\alpha \beta} = a_{\alpha \alpha}).$$

For $\alpha = \beta$: $a_{\alpha \alpha} f = \frac{\partial}{\partial x_{\alpha \alpha}} f.$

We let $D$ be the differential operator, whose $(\alpha, \beta)$th entry is

$$(D)_{\alpha, \beta} = a_{\alpha \beta}. \quad (1.14)$$

We shall in this section study the $n$th order differential operator $\det D$. The
Fourier transform is defined by

$$\hat{f}(k) = \gamma_1 \int_S e^{-it^r x^k f(x)} dx,$$

and the inverse Fourier transform by

$$\hat{g}(x) = \gamma_2 \int_S e^{it^r x^k g(k)} dk.$$

For suitable pairs $(\gamma_1, \gamma_2)$, and nice functions $f, f = \hat{f} = \hat{f}$. Since for any (real
or complex) matrix $z$,

$$(\det D) e^{t^r x^k z} = \det z e^{t^r x^k}, \quad (1.15)$$

det $D$ is in particular proportional to the Fourier transform of the multiplication
operator $(i)^n \det k$. This could of course also be taken to be the
definition of $D$.

We want to analyze whether powers of $\det D$ can be intertwining operators,
and if so, for which pairs of $(U_{\lambda}, U_{\lambda'})$s.

We remark that if we take

$$u(x) = \begin{bmatrix} 1 & x \\ 0 & 1 \end{bmatrix} \in \text{Sp}(n, \mathbb{R}) \quad \text{for } x \in X,$$

then $(\det D)'U_{\lambda}(u(x)) = U_{\lambda}(u(x))(\det D)'$ for all $(\lambda, \lambda')$ and $r \in \mathbb{N}$, since
$\det D$ is a constant coefficient differential operator, and $\{u(x) \mid x \in S\}$ is the
translation subgroup.

Let us return to the Lie algebra. This is generated by the subalgebras

$$\left\{ \begin{pmatrix} x & \alpha \\ 0 & 0 \end{pmatrix} \right\}_{x \in S} \quad \text{and} \quad \left\{ \begin{pmatrix} 0 & \alpha \\ y & 0 \end{pmatrix} \right\}_{y \in S}.$$

In fact, we could replace the latter by the single element $(0_{n \times n}, 0)$, but we shall
find it convenient to study a more general $(0_{n \times n}, 0)$.

Specifically, fix $p$ and $q$ and let $y$ be the matrix in $S$ given by

$$\{y\}_{kl} = \delta_{pq} \delta_{ql} + \delta_{pl} \delta_{kq}. \quad (1.16)$$
For \( x \in S \) and real \( t \), we have
\[
x(tyx + 1)^{-1} = x - txyx
\]
and
\[
\det(tyx + 1) = 1 + t \tr yx
\]
to the first order in \( t \). Since furthermore
\[
\{ xyx \}_{\alpha \beta} = x_{\alpha p} x_{q \beta} + x_{\alpha q} x_{p \beta}
\]
and
\[
\tr yx = x_{qp} + x_{pq} \quad (= 2x_{qp})
\]
we see, that if we let
\[
Y_0 = \sum_{\alpha, \beta = 1}^{n} (x_{\alpha p} x_{q \beta} + x_{\alpha q} x_{p \beta}) a_{\alpha \beta}
\]
with \( a_{\alpha \beta} \) as in (1.13), and if we let \( Y_m = x_{qp} + x_{pq} \), then
\[
dU_\lambda \left( \begin{pmatrix} 0 & 0 \\ y & 0 \end{pmatrix} \right) = \frac{d}{dt} U_\lambda \left( \begin{pmatrix} 1 & 0 \\ ty & 1 \end{pmatrix} \right) \bigg|_{t=0}
\]
\[
= Y_0 + (\lambda + (n + 1)/2) Y_m. \tag{1.17}
\]

We now want to compute \([\det D, x_{i_0 j_0}]\), and to do this, we recall some basic facts from linear algebra.

Let \( \{ A \}_{ij} \) be an \( n \times n \) matrix. Let \( C_{ij} \) be the determinant of the matrix \( M_{ij} \), obtained from \( A \) by replacing the entries in the \( i \)th row and \( j \)th column by zeros, except for a one in the \((i,j)\)th place. Specifically,
\[
\{ M_{ij} \}_{rs} = (a_{rs} (1 - \delta_{ir}) (1 - \delta_{js}) + \delta_{ir} \delta_{js}). \tag{1.18}
\]
We call \( M_{ij} \) the \((i, j)\)-minor, and denote by \( C_{ij}^{rs} \) the determinant of the \((r, s)\)-minor of \( M_{ij} \). Then,
\[
\det A = \sum_{j=1}^{n} a_{ij} C_{ij}, \tag{1.19}
\]
or, more generally,
\[
\sum_{i=1}^{n} a_{ri} C_{si} = \sum_{i=1}^{n} a_{is} C_{ir} = \delta_{rs} \det A, \quad \text{and}
\]
\[
C_{ij} = \sum_{s=1}^{n} a_{rs} C_{ij}^{rs} - \sum_{s=1}^{n} a_{rs} \delta_{ir} C_{ij}^{rs} - a_{ij} C_{ij}^{ij} + a_{ij} \delta_{ir} C_{ij}^{ij} + \delta_{ir} C_{ij}^{ij}. \tag{1.20}
\]
We use the above relations on the differential operator $\det D = \det[a_{ij}]$. We put $i = i_0, r = j_0$. Then

\[
\det A = \sum_{j,s=1}^{n} a_{ij} a_{is} C_{ij}^{j*} (1 - \delta_{ij}) \]

\[
+ \sum_{j=1}^{n} a_{ij} a_{ij} C_{ij}^{j*} (-1 + \delta_{ij}) + \delta_{ij} \sum_{j=1}^{n} a_{ij} C_{ij}^{j*}.
\]

From (1.13) we get

\[
[a_{ij}, x_{ij}] = \frac{1}{2} (\delta_{ij} + \delta_{ij} \delta_{ij}),
\]

\[
[a_{ij}, x_{ij}] = \frac{1}{2} \left( \delta_{ij} + \delta_{ij} \delta_{ij} \right).
\]

Finally, since by construction, $[C_{ij}^{j*}, x_{ij}] = 0$, we get

\[
[\det D, x_{ij}] = \sum_{j,s=1}^{n} a_{ij} \frac{1}{2} (\delta_{ij} + \delta_{ij} \delta_{ij}) C_{ij}^{j*} (1 - \delta_{ij}) \]

\[
+ \sum_{j,s=1}^{n} \frac{1}{2} (\delta_{ij} + \delta_{ij} \delta_{ij}) a_{ij} C_{ij}^{j*} (1 - \delta_{ij}) \]

\[
+ \sum_{j=1}^{n} \frac{1}{2} a_{ij} (\delta_{ij} + \delta_{ij} \delta_{ij}) C_{ij}^{j*} (-1 + \delta_{ij}) \]

\[
+ \sum_{j=1}^{n} \frac{1}{2} \delta_{ij} (\delta_{ij} + \delta_{ij} \delta_{ij}) C_{ij}^{j*} \]

\[
= \frac{1}{2} \sum_{j=1}^{n} a_{ij} C_{ij}^{j*} (1 - \delta_{ij}) + \frac{1}{2} \sum_{s=1}^{n} a_{is} C_{is}^{s*} (1 - \delta_{is}) \]

\[
+ \frac{1}{2} a_{ij} C_{ij}^{j*} (-1 + \delta_{ij}) + \frac{1}{2} a_{ij} C_{ij}^{j*} (-1 + \delta_{ij}) \]

\[
+ \frac{1}{2} \delta_{ij} (C_{ij}^{j*} + C_{ij}^{j*}).
\]

Using the relations (1.20), this is readily seen to give

\[
[\det D, x_{ij}] = \frac{1}{2} (C_{ij} + C_{ij}) = C_{ij},
\]

(1.21)
We are now able to compute

\[
[\det D, Y_0] = \sum_{\alpha, \beta=1}^n \left[ \det D, x_{ap} x_{q\beta} + x_{aq} x_{p\beta} \right] a_{\alpha\beta}
\]

\[
= \sum_{\alpha, \beta=1}^n \left( x_{ap} C_{q\beta} + C_{ap} x_{q\beta} \right) a_{\alpha\beta} + \sum_{\alpha, \beta=1}^n \left( x_{aq} C_{p\beta} + C_{aq} x_{p\beta} \right) a_{\alpha\beta}
\]

\[
= \sum_{\alpha=1}^n x_{ap} \delta_{qa} \det D + \sum_{\beta=1}^n \delta_{p\beta} \det D x_{q\beta}
\]

\[
- \frac{1}{2} \sum_{\alpha, \beta=1}^n C_{ap} (\delta_{qa} + \delta_{q\beta} \delta_{a\beta}) + \sum_{\alpha=1}^n x_{aq} \delta_{p\alpha} \det D
\]

\[
+ \sum_{\beta=1}^n \delta_{p\beta} \det D x_{p\beta} - \frac{1}{2} \sum_{\alpha, \beta=1}^n C_{aq} (\delta_{p\alpha} + \delta_{p\beta} \delta_{a\beta})
\]

\[
= 2 Y_m \det D - (n - 1)C_{pq}.
\]

By (1.21) and the above, we thus have the relations

\[
[\det D, Y_0] = 2 Y_m \det D - (n - 1)C_{pq},
\]

\[
[\det D, Y_m] = 2C_{pq},
\]

\[
[\det D, C_{pq}] = 0. (1.22)
\]

It follows that

\[
\det D (Y_0 + (\lambda + (n + 1)/2) Y_m) = (Y_0 + (\lambda + 2 + (n + 1)/2) Y_m) \det D
\]

\[
+ (2(\lambda + (n + 1)/2) - (n - 1)) C_{pq} \det D_{r-1}.
\]

or, by induction, for \( r \in \mathbb{N} \),

\[
(\det D)^r (Y_0 + (\lambda + (n + 1)/2) Y_m)
\]

\[
= (Y_0 + (\lambda + (n + 1)/2 + 2r) Y_m)(\det D)^r
\]

\[
+ (2r(\lambda + (n + 1)/2) + 2r(r - 1) - r(n - 1)) C_{pq} (\det D)^{r-1}. \quad (1.23)
\]

We see, that \((\det D)^r\) is an intertwining operator exactly when \( \lambda = - r \). In this case, \( \lambda' = r \). We shall from now on consider \( U_\lambda, \lambda \in \mathbb{Z}, \) as a representation of \( \text{Mp}(n, \mathbb{R}) \), even though this only is strictly necessary for \( n \) even. The above analysis may thus be summarized as

**PROPOSITION 1.1.** Let \( r \in \mathbb{N} \). Then \( \forall X \in \text{sp}(n, \mathbb{R}) \):

\[
(\det D)^r dU_{-r}(X) = dU_r(X)(\det D)^r. \quad (1.24)
\]

The point now is that the class of functions on which the equation (1.24) can be integrated is sufficiently big to be of interest. Specifically, the vector space \( V_{-r} \), spanned by \( \{(U_{-r}(g) \det(K_1(\cdot, w))\theta)(x) | g \in \text{Mp}(n, \mathbb{R}), w \in \Omega\} \),
\( \alpha - (n + 1)/2 \in \mathbb{Z} \), and \( \alpha > (n + 1)/2 - r \) is invariant under \( U_{-r} \), and each function \( f \) in \( V_{-r} \) is the boundary value of a holomorphic function \( F_f \), which can be extended holomorphically across the Shilov boundary.

We remark that since the operators \((\det D)^r\) are boundary values of operators acting on the space \( \Theta_1 \) of holomorphic functions on \( \partial \), and since these, by exactly the same arguments as before, satisfy exactly the same intertwining relations, we could just integrate the relation (1.24) on holomorphic functions on \( \partial \). In fact, by the above properties of \( V_{-r} \), this would be exactly the same as integrating it on \( V_{-r} \). On holomorphic functions on \( \partial \), however, it is rather obvious, that the relation can be integrated. We shall therefore only give a sketch.

We first observe that the Lie algebra \( \mathfrak{sp}(n, \mathbb{R}) \) is generated by the subalgebras

\[
\begin{bmatrix}
0 & y \\
0 & 0
\end{bmatrix} \quad \text{and} \quad \begin{bmatrix}
0 & \theta \\
-\theta & 0
\end{bmatrix}
\]

For the first algebra, the relation can easily be integrated. Thus, since \( \text{Mp}(n, \mathbb{R}) \) is connected, we need only consider

\[
c(\theta) = \exp \left( \theta \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \right) = \begin{bmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{bmatrix}.
\]

Clearly, the map \((\theta, z) \rightarrow c(\theta)z\) is holomorphic and hence \((\theta, z) \rightarrow f(c(\theta)z)\) is holomorphic, if \( f \) is a holomorphic function on \( \partial \). Since the same can be said about the multipliers

\[
\det(-\sin \theta z + \cos \theta)^{-(n+1)/2-r}
\]

and

\[
\det(-\sin \theta z + \cos \theta)^{-(n+1)/2+r},
\]

it follows by power series expansions that

\[
\left( (\det D U_{-r}, c(\theta) f) \right)(z) = \left( U_r(c(\theta)) \det D f \right)(z). \tag{1.25}
\]

**Proposition 1.2 [5].** For \( r \in \mathbb{N} \) and \( f \) either a holomorphic function on \( \partial \) or an element of \( V_{-r} \), we have that

\[
\left( (\det D') U_{-r}, g f \right) = U_r(g)(\det D') f, \quad \forall g \in \text{Mp}(n, \mathbb{R}). \tag{1.26}
\]

From here, we proceed as in the case of \( SU(2, 2) \) [5]. We denote the forward light cone in \( \mathbb{S} \) by \( \mathbb{C}^+ \), i.e.

\[
\mathbb{C}^+ = \{ x \in \mathbb{S} | x > 0 \}.
\]

Then we know from [3] or [9] that there exists a constant \( k_1 \) such that for all
\[ \alpha > -1 \text{ and } z \in \mathbb{D} \]
\[ \int_{C^+} e^{+i \text{tr} z y} \det y^\alpha \, dy = k_1 (\det z/i)^{-\alpha-((n+1)/2)} \prod_{i=1}^{n} \Gamma(\alpha + (i + 1)/2). \]

It is also known [9, §§4.5 and 4.6] that if we let

\[ \Theta_0 = \{0\}, \quad \Theta_1 = \{x \in S \mid x > 0, \text{rank } x = 1\}, \ldots, \]
\[ \Theta_{n-1} = \{x \in S \mid x > 0, \text{rank } x = n - 1\}, \]

then there are semi-invariant measures \( \mu_j \) on \( \Theta_j \) and constants \( d_b(j) \), such that

\[ \int_{\Theta_j} e^{i \text{tr} z y} \, d\mu_j(y) = d_b(j)(\det z/i)^{-j/2}. \quad (1.28) \]

The formula (1.27), together with (1.7), (1.8), (1.9), (1.10), and (1.11), immediately gives that \( U_\alpha \) is unitary for \( \alpha > -1 \), and (1.28) shows that \( U_{(n-1)/2+j/2} \) for \( j = 0, 1, \ldots, n - 1 \) has an invariant subspace, on which it acts unitarily, namely the space of Fourier transforms of holomorphic functions in \( L^2(\Theta_j, \mu_j) \). From (1.14) and (1.27) it also follows that there are constants \( C_{n,r} \) such that

\[ (\det D)^r \det(x + z)^{-(n+1)/2} = C_{n,r} \det(x + z)^{-r-(n+1)/2}, \quad (1.29) \]

for \( z \in \mathbb{D} \) and \( x \in S \). As in the case of \( SU(2, 2) \) [5, pp. 91–96], this is exactly what is needed to conclude: If we define an equivalence relation on \( V_r \) by

\[ f \sim_r g \iff (\det D)^r(f - g) = 0, \]

and denote the equivalence classes by \([·]_r\), then, using the unitarity of \( U_r \) for \( r > 0 \) and Proposition 1.2, we get

**Proposition 1.3.** For integers \( r > 0 \), there exists a subspace of equivalence classes \([·]_r\), which can be given a Hilbert space structure, in which \( U_r \) acts unitarily.

We shall end this section with a look at noninteger \( r \)'s. We do this by Fourier transform. We consider \( C^\infty \) functions with compact support in the interior of the forward light cone \( C^+ \). We denote the Fourier transforms of the operators \( \det D, Y_0, Y_m, \) etc. by \( \hat{D}, \hat{Y}_0, \hat{Y}_m, \) etc. In particular

\[ \hat{\det D} = \det ik, \quad \hat{\zeta}_{\alpha\beta} = i\alpha_{\alpha\beta}, \quad \hat{\alpha}_{\alpha\beta} = i\zeta_{\alpha\beta}. \]
We let $\mu \in \mathbb{C}$. Then it follows from (1.20), (1.21), and (1.22) that
\[\det k^\mu, \hat{Y}_0 = 2\hat{D} \hat{C}_{pq} \det(-i)^2 \mu(\mu - 1) \det k^{\mu - 2}\]
\[+ \det(-i)[\hat{D}, \hat{Y}_0] \mu \det k^{\mu - 1}\]
\[= 2\det(-i)\hat{C}_{pq} \mu(\mu - 1) \det k^{\mu - 1} + 2\hat{Y}_m \mu \det k^\mu\]
\[- \det(-i)(n - 1) \mu \hat{C}_{pq} \det k^{\mu - 1}.\]
Also, $[\det k^\mu, \hat{Y}_m] = 2\hat{C}_{pq} \mu \det k^{\mu - 1} \det(-i)$. Hence,
\[\det k^\mu(\hat{Y}_0 + (\lambda + (n + 1)/2) \hat{Y}_m)\]
\[= (\hat{Y}_0 + (\lambda + (n + 1)/2 + 2\mu) \hat{Y}_m) \det k^\mu\]
\[+ \det k^{\mu - 1} \hat{C}_{pq} \det(-i)(2(\lambda + (n + 1)/2) \mu + 2\mu(\mu - 1) - (n - 1) \mu).\]
In particular, the last term vanishes if and only if $\lambda = -\mu$. In this case, $\lambda' = \mu$.

**Remark.** We let $d\hat{U}_{-\mu}$ and $d\hat{U}_\mu$ be the two representations of the Lie algebra obtained by Fourier transformation. They of course extend to be representations on all $C^\infty$ functions in the interior of the forward light cone, and since $\det k^\mu$ is a bijection of $C^\infty(C^+)$ onto itself, these modules are infinitesimally equivalent. However, since when we took the Fourier transforms, we completely neglected boundary behavior, if these in any manner can be integrated, the result, transformed back again by the inverse Fourier transform, will in general differ from the original $U_\lambda$'s by boundary terms. In this connection we observe that if we define two $C^\infty$ functions $f$ and $g$ in $C^+$ to be equivalent if for each point $p$ on the boundary $b(C^+)$ there is an open neighborhood $N_p$ such that $\det k^\mu(f - g)$ in $N_p \cap C^+$ is the restriction of a $C^\infty$ function in $N_p$, then $d\hat{U}_{-\mu}$ preserves equivalence classes.

We finally mention that for $r$ real and positive, the closure of the operators $d\hat{U}_r(x), x \in \text{sp}(n, \mathbb{R})$, are the differentials of a unitary representation $\hat{U}_r$ in $L^2(C^+, \det k^{-r}dk)$. Then, since the map
\[T_r: L^2(C^+, \det k^{-r}dk) \to L^2(C^+, \det k^{r}dk); (T_rf)(k) = \frac{1}{\det k^{r}} f(r)\]
is unitary, we can define $T_r \hat{U}_r T_r^{-1}$, which then has generators $d\hat{U}_{-r}(x), x \in \text{sp}(n, \mathbb{R})$.

2. **The scalar case for $SU(n, n)$**. In view of the analysis of $Mp(n, \mathbb{R})$, it is not surprising that similar generalizations hold for $SU(n, n)$. Due to the strong analogy, we shall only give a sketch.

$SU(n, n)$ is the subgroup of $SL(2n, \mathbb{C})$ consisting of those matrices $g$ that satisfy
\[g \begin{bmatrix} 0 & i \\ -i & 0 \end{bmatrix} g^* = \begin{bmatrix} 0 & i \\ -i & 0 \end{bmatrix};\]
writing \(g\) in \(n \times n\) blocks, this is equivalent to

\[
\begin{align*}
ad^* - bc^* &= 1; \quad ab^* = ba^*; \quad cd^* = dc^*, \\
\end{align*}
\]

and to

\[
\begin{align*}
a^*d - c^*b &= 1; \quad a^*c = c^*a; \quad b^*d = d^*b,
\end{align*}
\]

where (2.2) is (2.1) for \(g^{-1}\). The Lie algebra is given by

\[
su(n, n) = \left\{ \begin{bmatrix} x_1 & x_2 \\ x_3 & -x_1^* \end{bmatrix} | \text{Im tr } x_1 = 0; \ x_2 = x_2^*; \ x_3 = x_3^* \right\}.
\]

We let \(\mathfrak{g} = \{x + iy|x = x^*; y = y^*; y > 0\}\). \(SU(n, n)\) acts on \(\mathfrak{g}\) by \(g \cdot z = (az + b)(cz + d)^{-1}\). Similarly to the case of \(Mp(n, \mathbb{R})\) there are functions, defined for \(g = (a^ b)\) and \(z, w \in \mathfrak{g}\) by

\[
\begin{align*}
J_1(g, z) &= cz + d, \quad J_2(g, z) = (zc^* + d^*)^{-1}, \\
K_1(z, w) &= ((z - w^*)/2i)^{-1}, \quad \text{and} \quad K_2(z, w) = ((z - w^*)/2i),
\end{align*}
\]

which satisfy (1.4) and (1.11):

\[
\begin{align*}
J_i(g_1g_2, z) &= J_i(g_1, g_2z)J_i(g_2, z) \quad \text{and} \\
K_i(gz, gw) &= J_i(g, z)K_i(z, w)J_i(g, w^*), \quad i = 1, 2.
\end{align*}
\]

We shall consider the representations

\[
(U_{\lambda}(g)f)(z) = \left(\det J_1(g^{-1}, z)\right)^{-\lambda-n} f(g^{-1}z),
\]

for \(\lambda \in \mathbb{Z}\).

It follows from (2.1) and (2.2) that \((gz^*)^* = g \cdot z\), and we can thus restrict \(U_{\lambda}\) to act on measurable functions on the Shilov boundary \(\partial \mathfrak{g} = H\) of \(\mathfrak{g}\); \(H = \{x|x = x^*\}\).

On functions on \(H\), we define first order differential operators \(a_{\alpha\beta}\) to be dual to the variables \(x_{\alpha\beta}\) in the parametrization of elements in \(H\). Thus,

\[
a_{\alpha\beta}(x_{\alpha\beta}) = \delta_{\alpha\alpha}\delta_{\beta\beta}.
\]

We let \(D\) be the differential operator whose \((\alpha, \beta)\)th entry is

\[
(D)_{\alpha, \beta} = a_{\alpha\beta},
\]

and we shall here consider \(\det D\). We define the Fourier transform by

\[
f(k) = \gamma_3 \int e^{-i \text{tr } xk} f(x) \ dx,
\]

and the inverse by

\[
g(x) = \gamma_4 \int e^{i \text{tr } xk} g(k) \ dk.
\]

Since

\[
(\det D)e^{i \text{tr } xk} = \det ze^{i \text{tr } xk},
\]

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we observe that $\det D$ is proportional to the Fourier transform of $(i)^n \det k$.

As in the case of $\text{Mp}(n, \mathbb{R})$ we shall analyze for which pairs of $(\lambda, \lambda')$'s powers of $\det D$ are intertwining operators, and exactly as in that case we are reduced to studying the action of the subalgebra

$$\left\{ \begin{bmatrix} 0 & 0 \\ y & 0 \end{bmatrix} \middle| y = y^* \right\}.$$ 

Specifically, we take $\{ y \}_{kl} = \rho \delta_{pk} \delta_{ql} + \rho^* \delta_{pl} \delta_{kq}$, for a fixed $\rho \in \mathbb{C}$. For $x$ in $H$ and $t$ in $\mathbb{R}$, we then have

$$x(tyx + 1)^{-1} \approx x - txyx; \quad \det(tyx + 1) \approx 1 + t \text{tr } xy.$$

Since $(xyx)_{a\beta} = \rho x_{ap} x_{q\beta} + \rho^* x_{aq} x_{p\beta}$, and $\text{tr } xy = \rho x_{qp} + \rho^* x_{pq}$, we see, that if we put

$$Y_0 = \sum_{a, \beta = 1}^n (\rho x_{ap} x_{q\beta} + \rho^* x_{aq} x_{p\beta})a_{a\beta},$$

and

$$Y_m = \rho x_{qp} + \rho^* x_{pq},$$

then

$$\frac{d}{dt} U_\lambda \left( \begin{bmatrix} 0 & 0 \\ y & 1 \end{bmatrix} \right) \bigg|_{t=0} = Y_0 + (\lambda + n)Y_m. \quad (2.9)$$

We compute $[\det D, x_{i_{0}j_{0}}]$, maintaining the notation from $\text{Mp}(n, \mathbb{R})$:

$$\det D = \sum_{j=1}^n a_{i_{0}j} C_{i_{0}j} \Rightarrow [\det D, x_{i_{0}j_{0}}] = C_{i_{0}j_{0}}. \quad (2.10)$$

Hence,

$$[\det D, Y_0] = \sum_{a, \beta = 1}^n [\det D, \rho x_{ap} x_{q\beta} + \rho^* x_{aq} x_{p\beta}]a_{a\beta}$$

$$= \sum_{a, \beta = 1}^n \rho x_{ap} C_{p\beta} a_{a\beta} + \sum_{a, \beta = 1}^n \rho C_{ap} x_{q\beta} a_{a\beta}$$

$$+ \sum_{a, \beta = 1}^n \rho^* x_{aq} C_{p\beta} a_{a\beta} + \sum_{a, \beta = 1}^n \rho^* C_{aq} x_{p\beta} a_{a\beta}$$

$$= \sum_{a = 1}^n \rho x_{ap} (\det D) \delta_{a\alpha} + \sum_{a, \beta = 1}^n \rho C_{ap} (a_{a\beta} x_{q\beta} - \delta_{a\alpha})$$

$$+ \sum_{a = 1}^n \rho^* x_{aq} (\det D) \delta_{pa} + \sum_{a, \beta = 1}^n \rho^* C_{aq} (a_{a\beta} x_{p\beta} - \delta_{pa})$$

$$= 2 Y_m (\det D) - (n - 1)[\rho C_{qp} + \rho^* C_{pq}].$$
We put $C = \rho C_{qp} + \rho^* C_{pq}$. Then,

\[
[\det D, Y_0] = 2Y_m (\det D) - (n - 1)C;
[\det D, Y_m] = C; [\det D, C] = 0.
\]  

(2.11)

Hence,

\[
\det D(Y_0 + (\lambda + n)Y_m)
= (Y_0 + (\lambda + n + 2)Y_m)\det D + (\lambda + n - (n - 1))C,
\]

or, more generally,

\[
(\det D)^r(\lambda Y_0 + (\lambda + n)Y_m) = (\lambda Y_0 + (\lambda + n + 2r)Y_m)(\det D)^r
\]

+ \[r(\lambda + n) + (r - 1)r - r(n - 1)\]C(\det D)^{r-1}.  

(2.12)

We see that in order for $(\det D)^r$ to be intertwining, the last term must vanish, i.e. $\lambda = -r$. In this case, $\lambda' = r$. Hence we have proved

**PROPOSITION 2.1.** For integers $r \geq 0$,

\[
(\det D)^r dU_{-r}(X) = dU_r(X)(\det D)^r \forall X \in su(n, n).  
\]

(2.13)

If we let $V_{-r}$ be the space consisting of those real analytic functions $f$ on $H$, that are boundary values of holomorphic functions on $\mathcal{D}$, and for which $U_{-r}([0, I_{n, 0}])f$ again is such a function, then it follows easily (cf. Chapter 1) that we have

**PROPOSITION 2.2** [5]. For $r$ in $\mathbb{N}$, $f$ in $V_{-r}$, and $g$ in $SU(n, n)$:

\[
(\det D)^r U_{-r}(g)f = U_r(g)(\det D)^r f.  
\]

(2.14)

In this section we let $C^+ = \{ x \in H | x > 0 \}$. Then it is again known from [3] or [9] that there exists a constant $k_2$ such that for $\alpha > -1$ and $z$ in $\mathcal{D}$

\[
\int_{C^+} e^{i tr z y} \det y^\alpha dy = k_2 (\det z / i)^{-\alpha - n} \prod_{i=1}^{n} \Gamma(i + \alpha).  
\]

(2.15)

Likewise, it is known [9] that on the orbits $\mathcal{O}_0 = \{ 0 \}$, $\mathcal{O}_1 = \{ x \in H | x > 0$, rank $x = 1 \}$, \ldots, $\mathcal{O}_{n-1} = \{ x \in H | x > 0$, rank $x = n - 1 \}$ there are semi-invariant measures $\mu_j$ such that for suitable constants $c_0(j)$

\[
\int_{\mathcal{O}_j} e^{i tr y^\alpha} d\mu_j(y) = c_0(j)(\det z)^{-\gamma}.  
\]

(2.16)

These formulas imply that $U_{\alpha}$ is unitary for $\alpha > -1$, and that $U_{-j}$ for $j = 1, 2, \ldots, n$ has an invariant subspace, on which it acts unitarily.

Finally, again parallel to the case of $Mp(n, R)$ we conclude:

**PROPOSITION 2.3.** On a space of equivalence classes of functions whose Fourier transforms are supported by $\overline{C^+}$; equivalence being defined by $f \sim g \iff$
(det $D')'(f - g) = 0$, one can construct a pre-Hilbert space structure, which is preserved by $U_r$, for all $r$ in $\mathbb{N}$.

**Remark.** By taking Fourier-transforms, we can get the relation

$$(\det k)^\mu d\hat{U}_{-\mu} = d\hat{U}_\mu (\det k)^\mu,$$

for any $\mu \in \mathbb{C}$, as an equation between Lie algebra representations on the space of $C^\infty$-functions whose support is compact and contained in $C^+$; just as in the case of $Mp(n, \mathbb{R})$.

3. The Dirac-type operator. The case of $Mp(n, \mathbb{R})$. There exists a distinguished operator $\mathcal{V}$; a $2n \times 2n$ matrix with differential operators as entries, for which $(\mathcal{V})^2 = \det D$. Recall from [5] that the Dirac operator associated to

$$\Box = \frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x_1^2} - \frac{\partial^2}{\partial x_2^2} - \frac{\partial^2}{\partial x_3^2}$$

can be defined as $\mathcal{V} = (c(\sigma), 0)$, where

$$\sigma = \begin{pmatrix}
\frac{\partial}{\partial t} + \frac{\partial}{\partial x_3} & \frac{\partial}{\partial x_1} - i \frac{\partial}{\partial x_2} \\
\frac{\partial}{\partial x_1} + i \frac{\partial}{\partial x_2} & \frac{\partial}{\partial t} - \frac{\partial}{\partial x_3}
\end{pmatrix},$$

and $c(\sigma)$ is the co-factor of $\sigma$;

$$c(\sigma) = \begin{pmatrix}
\frac{\partial}{\partial t} - \frac{\partial}{\partial x_3} & -\frac{\partial}{\partial x_1} + i \frac{\partial}{\partial x_2} \\
-\frac{\partial}{\partial x_1} - i \frac{\partial}{\partial x_2} & \frac{\partial}{\partial t} + \frac{\partial}{\partial x_3}
\end{pmatrix}.$$

We observe that, corresponding to (1.15), $\sigma'$ is proportional to the Fourier transform of the multiplication operator $(kf)(k) = k \cdot f(k)$ for $k \in H(2)$. Likewise, $c(\sigma)'$ is proportional to the Fourier transform of the multiplication operator $c(k)$, where $c(k)$ is the co-factor of $k$; $c(k) = (\det k)k^{-1}$. These observations clearly lead to a natural candidate for $\mathcal{V}$. Finally, we make the (key) observation, that if we promote $\Box$ to be the operator $\Box f = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}(f')$ on functions $f : H(2) \rightarrow \mathbb{C}^2$, and let $h$ and $c(h)$ be the above defined multiplication operators, then [5]

$$[\Box, h] = 2c(\sigma); \quad [\Box, c(h)] = 2\sigma.$$

We shall in this chapter be concerned with two series of representations of $Mp(n, \mathbb{R})$. To avoid a repetition of the technical arguments following Proposition 1.1, we shall here consider the space of holomorphic functions from $\mathcal{O}$.
to \( C' \). We define, for \( g^{-1} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \lambda \in \mathbb{Z}, \) and \( z \in \mathfrak{g} \)

\[
(D_{\lambda}^+ (g) f)(z) = \frac{(cz + d)^{-1}}{\det(cz + d)^{\lambda + (n+1)/2}} f\left( \frac{az + b}{cz + d} \right), \tag{3.1}
\]

\[
(D_{\lambda}^- (g) f)(z) = \frac{(zc^t + d^t)^{\lambda + (n+3)/2}}{\det(zc^t + d^t)} f\left( \frac{az + b}{cz + d} \right), \text{ and}
\]

\[D_{\lambda} (g) = D_{\lambda}^+ (g) \oplus D_{\lambda}^- (g).\]

We let \( D = D' \) be the operator from (1.13), extended in an obvious way to functions from \( \mathfrak{g} \) to \( C' \), corresponding to a parametrization of \( \mathfrak{g} \) by matrices \( z \) for which the \((i, j)\)th entry \( z_{ij} = z_{ji} \in C \). Then all the formulas from Chapter 1 remain unchanged. We let \( c(D) \) be the \( n \times n \) matrix whose \((i, j)\)th entry is \( C_{ij} = \text{the determinant of } M_{ij} \), as defined in (1.18). Thus, \( C(D)' = c(D) \) and \( D \cdot c(D) = c(D) \cdot D = (\det D)I_n \). Finally, we let \( \mathcal{V}' = (\begin{smallmatrix} 0 & P \\ c(D) & 0 \end{smallmatrix}) \). Then \((\mathcal{V}')^{2r} = (\det D)'^r \) for \( r \in \mathbb{N} \).

We want to analyze the relation between odd powers of \( \mathcal{V} \) and the representations \( D_{\lambda} \). We begin by investigating for what \((\lambda, \lambda')\), for a given integer \( l > 0 \),

\[
(\det D)'^l D_{\lambda} (g) = (D_{\lambda}^+ (g))(\det D)'^l D.
\tag{3.2}
\]

As in the scalar case, we are immediately reduced to studying the elements \((\begin{smallmatrix} 0 & \theta \\ 0 & 0 \end{smallmatrix})\) of \( \text{sp}(n, \mathbb{R}) \). We maintain the notation of Chapter 1. Then,

\[
dD_{\lambda}^+ \begin{pmatrix} 0 & 0 \\ y & 0 \end{pmatrix} = \frac{d}{dt} D_{\lambda}^+ \begin{pmatrix} 1 & 0 \\ ty & 1 \end{pmatrix} \bigg|_{t=0} = yz + (Y_0 + (\lambda' + (n + 1)/2) Y_m)I_n, \text{ and}
\]

\[
dD_{\lambda}^- \begin{pmatrix} 0 & 0 \\ y & 0 \end{pmatrix} = \frac{d}{dt} D_{\lambda}^- \begin{pmatrix} 1 & 0 \\ ty & 1 \end{pmatrix} \bigg|_{t=0} = -zy + (Y_0 + (\lambda + (n + 3)/2) Y_m)I_n. \tag{3.3}
\]

We consider \( D_{zy} \). Since \((\begin{smallmatrix} zy \\ \end{smallmatrix})_{rs} = \delta_{qs}z_{rp} + \delta_{ps}z_{rq}, \) we get that

\[
(\begin{pmatrix} D_{zy} \end{pmatrix})_{rs} = \sum_i \left( \delta_{qs}a_{ri}z_{ip} + \delta_{ps}a_{ri}z_{iq} \right)
\]

\[
= \sum_i \left( \delta_{qs}z_{ip}a_{ri} + \delta_{ps}z_{iq}a_{ri} \right) + \frac{n + 1}{2} \left( \delta_{qs}\delta_{rp} + \delta_{ps}\delta_{rq} \right).
\]
Since moreover
\[
\{ [D, Y_0] \}_{rs} = [a_{rs}, Y_0] = \sum_{\beta} (\delta_{sp} z_{q\beta} a_{\beta} + \delta_{r\beta} z_{q\beta} a_{\beta}) \\
+ \sum_{\alpha} (z_{\alpha p} \delta_{rq} a_{as} + z_{\alpha p} \delta_{sq} a_{ar}),
\]
\[
\{ DY_m \}_{rs} = \{ Y_mD \}_{rs} + (\delta_{qr} \delta_{ps} + \delta_{qs} \delta_{pr}),
\]
we get, using \( a_{\alpha \beta} = a_{\beta \alpha} \) and \( z_{\alpha \beta} = z_{\beta \alpha} \), that
\[
\{ (\text{det } D)D(-zy + (Y_0 + (\lambda + (n + 3)/2) Y_m) I_n) \}_{rs}
\]
\[
= \{ (\text{det } D)yz D \}_{rs} + \{ (\text{det } D)'(Y_0 + (\lambda + (n + 3)/2) Y_m) D \}_{rs}
\]
\[
+ (\text{det } D)'(\lambda + (n + 3)/2 - (n + 1)/2)(\delta_{qr} \delta_{ps} + \delta_{qs} \delta_{pr}).
\]
To compute \( \{ (\text{det } D)'yz D \}_{rs} \), observe that \( \{ [\text{det } D, yz] \}_{rs} = \delta_{qr} C_{sp} + \delta_{pr} C_{sq} \).
It then follows easily that
\[
\{ (\text{det } D)'yz D \}_{rs} = \{ yz(\text{det } D)'D \}_{rs} + l(\delta_{qr} \delta_{ps} + \delta_{pr} \delta_{qs})(\text{det } D)'.
\]
Using (1.23), we thus get
\[
\{ (\text{det } D)'D(-zy + (Y_0 + (\lambda + (n + 3)/2) Y_m) I_n) \}_{rs}
\]
\[
= \{ yz(\text{det } D)'D \}_{rs} + \{ (Y_0 + (\lambda + 2l + (n + 3)/2) Y_m)(\text{det } D)'D \}_{rs}
\]
\[
+ \{ (2l(\lambda + (n + 3)/2) + 2l(l - 1) - l(n - 1))C_{pq}(\text{det } D)'^{-1}D \}_{rs}
\]
\[
+ (\lambda + 1 + l)(\text{det } D)'(\delta_{qr} \delta_{ps} + \delta_{qs} \delta_{pr}).
\]
In particular, to have (3.2) satisfied, we must have \( \lambda = -1 - l \) and \( \lambda' = l \).
To deal with the other half of the problem, observe that if \( \text{det } D = (\text{det } D)I_n \), then

**Lemma 3.1.**
\[
C(D) = [\text{det } D, z],
\]
and in fact, for integers \( l > 0 \),
\[
(l + 1)(\text{det } D)'C(D) = [((\text{det } D)'^{-1}, z].
\]

**Proof.** (3.6) clearly follows from (3.5) which again follows from (1.21).
To complete the investigation, we need therefore only check whether, for
\[ g \in \text{Mp}(n, \mathbb{R}) \]

\[
\left[ (\det D)^{l+1}, z \right] D_{-l-1}^{-1} (g) = D_{-l}^{-} (g) \left[ (\det D)^{l+1}, z \right].
\]

(3.7)

This relation is trivially satisfied for \( g = (l_0 \, x), \, x \in S \), since \( (\det D)^{l+1}, z \) is a constant coefficient differential operator. But these elements, together with those that project onto \( [0, l_0] \in \text{Sp}(n, \mathbb{R}) \) generate \( \text{Sp}(n, \mathbb{R}) \). This means that if we put \( (C_0 f)(z) = f(-z^{-1}) \), then we must have

\[
\left[ (\det D)^{l+1}, z \right] \frac{z^{-1}}{(\det z)^{-1-l+(n+1)/2}} C_0
\]

\[ = \frac{z}{(\det z)^{l+(n+3)/2}} C_0 \left[ (\det D)^{l+1}, z \right].
\]

That this equation is valid follows from Proposition 1.2 since

\[
\left( (\det D)^{l+1} z - z (\det D)^{l+1} \right) \frac{z^{-1}}{(\det z)^{-1-l+(n+1)/2}} C_0
\]

\[ = (\det D)^{l+1} \frac{1}{(\det z)^{-1-l+(n+1)/2}} C_0
\]

\[ + z (\det D)^{l+1} \frac{1}{(\det z)^{-1-l+(n+1)/2}} C_0 z
\]

\[ = \frac{1}{(\det z)^{l+(n+3)/2}} C_0 (\det D)^{l+1}
\]

\[ + z \frac{1}{(\det z)^{l+(n+3)/2}} C_0 (\det D)^{l+1} z
\]

\[ = -\frac{z}{(\det z)^{l+(n+3)/2}} C_0 z (\det D)^{l+1}
\]

\[ + \frac{z}{(\det z)^{l+(n+3)/2}} C_0 (\det D)^{l+1} z
\]

\[ = \frac{z}{(\det z)^{l+(n+3)/2}} C_0 \left[ (\det D)^{l+1}, z \right].
\]

We have thus brought the analysis to an end. Similarly to the scalar case we can then state

**Proposition 3.2.** Let \( l > 0 \) be an integer and let \( f \) be a holomorphic function from \( \mathcal{D} \) to \( \mathbb{C}^n \). Then, for all \( g \) in \( \text{Mp}(n, \mathbb{R}) \)

\[ \Psi^{2l+1} D_{-l-1}^{-1} (g) f = D_{-l}^{-} (g) \Psi^{2l+1} f.
\]

To settle the question of unitarity, observe that it follows quite readily from [1], [2], and [8] that there exists a constant \( k_1 \) such that for \( \alpha > -1 \) and
\[ z \in \mathbb{D} : \]
\[ \int_{C^+} e^{i \text{$_{tr}$} y c(y)(\det y)^{\alpha} \, dy} = k_1(z/i)(\det z/i)^{-\alpha - ((n+3)/2)} \prod_{i=2}^{n} \Gamma(\alpha + (i + 3)/2), \quad (3.8) \]
and
\[ \int_{C^+} e^{i \text{$_{tr}$} y y (\det y)^{\alpha} \, dy} \]
\[ = k_1(z/i)^{(-1)}(\det z/i)^{-\alpha - ((n+1)/2)} \prod_{i=1}^{n-1} \Gamma(\alpha + (i + 1)/2). \]

By (1.8), (1.9), (1.10), and (1.11) it then follows that \( D_{-1} \) is unitary for \( \lambda > -1 \).

We observe that since, for \( \nu \in \mathbb{C}^n \), \( D e^{i \text{$_{tr}$} y \nu} = iye^{i \text{$_{tr}$} y \nu} \), and \( c(D)e^{i \text{$_{tr}$} y \nu} = (i)^{n-1}c(y)e^{i \text{$_{tr}$} y \nu} \), it follows from (1.27) and (3.8) by analytic continuation in \( \alpha \), that for all \( \alpha \) in \( \mathbb{R} \), \( z, w \) in \( \mathbb{D} \) and \( \nu \in \mathbb{C}^n \):

\[ \text{det} \, D \, \text{det} \left( \frac{z - w^*}{i} \right)^{-\alpha - ((n+1)/2)} \nu \]
\[ = (i)^n \prod_{i=1}^{n} \left( \alpha + \frac{i + 1}{2} \right) \text{det} \left( \frac{z - w^*}{i} \right)^{-\alpha - 1 - ((n+1)/2)} \nu, \]
\[ C(D) \text{det} \left( \frac{z - w^*}{i} \right)^{-\alpha - ((n+1)/2)} \nu \]
\[ = (i)^{-1} \prod_{i=2}^{n} \left( \alpha + \frac{i + 1}{2} \right) \text{det} \left( \frac{z - w^*}{i} \right)^{-\alpha - ((n+3)/2)} \nu, \]
\[ \text{D det} \left( \frac{z - w^*}{i} \right)^{-\alpha - ((n+1)/2)} \nu \]
\[ = (i)^{\alpha + \frac{n+1}{2}} \text{det} \left( \frac{z - w^*}{i} \right)^{-1} \text{det} \left( \frac{z - w^*}{i} \right)^{-\alpha - ((n+1)/2)} \nu. \quad (3.9) \]

By comparison with (1.28), it then follows that \( D_{-1} \) has an invariant subspace on which it is unitary, namely the completion of the pre-Hilbert space of all finite linear combinations of the functions (see (1.10))

\[ z \rightarrow K_1(z, w) \det K_1(z, w)^{-1 + ((n+1)/2)} \nu_1 \bigoplus K_2(z, w) \det K_2(z, w)^{-((n+1)/2)} \nu_2, \]
with \( w \) in \( \mathbb{D} \), and \( \nu_i \) in \( \mathbb{C}^n \) for \( i = 1, 2 \), in the metric given by (1.8).

Since the co-factor of a rank \( r \) matrix for \( r < n - 2 \) is zero, a similar result for the other orbits \( \mathbb{D}_j \) will only hold true for the representations \( D^+_{((n-j+1)/2)_2} \), \( j = 0, 1, \ldots, n - 2 \). For every integer \( r > 0 \), we define
equivalence classes of functions $f$ from $\mathcal{O}$ to $\mathbb{C}^n \oplus \mathbb{C}^n$ by

$$[f]_{2r+1} = [g]_{2r+1} \iff \nabla^{2r+1} (f - g) = 0.$$  

Then (cf. [5]) (3.9) together with Proposition 3.2 easily gives

**Proposition 3.3.** For each integer $r \geq 0$, there exists a subspace of the equivalence classes $[\cdot]_{2r+1}$ which can be given a Hilbert space structure in which $D_{2r+1}$ is unitary.

4. The Dirac-type operator. The case of $SU(n, n)$. We shall here describe the analogue of Chapter 3 for $SU(n, n)$. We consider functions defined on $H$; it is then straightforward to translate the results to holomorphic functions on $\mathcal{O}$.

Let $f$ be a measurable function from $H$ to $\mathbb{C}^n$, $\lambda \in \mathbb{Z}$, and $x \in H$. We define

$$\left( \begin{array}{c} \text{for} \quad g^{-1} = \left( \begin{array}{cc} a & b \\ c & d \end{array} \right) \in SU(n, n), \text{finally, we put} \\ D_{\lambda} (g) = D_{\lambda}^+ (g) \oplus D_{\lambda}^- (g). \end{array} \right.$$  

Let $D'$ be the transpose of the matrix $D$ in (2.5), and let $c(D)$ be the $n \times n$ matrix whose $(i, j)$th entry is $C_{ij} = C_{ij}(D)$ which is nothing more than the determinant of the matrix $M_{ij}$ obtained from $D$ as in (1.18). Then, since $C_{ij}(D) = C_{ji}(D')$,

$$D'c(D) = c(D)D' = \frac{\det D}{D_{11}}$$  

for $g^{-1} = \left( \begin{array}{cc} a & b \\ c & d \end{array} \right) \in SU(n, n)$. Finally, we put

$$D_{\lambda} (g) = D_{\lambda}^+ (g) \oplus D_{\lambda}^- (g).$$  

Let $D'$ be the transpose of the matrix $D$ in (2.5), and let $c(D)$ be the $n \times n$ matrix whose $(i, j)$th entry is $C_{ij} = C_{ij}(D)$ which is nothing more than the determinant of the matrix $M_{ij}$ obtained from $D$ as in (1.18). Then, since $C_{ij}(D) = C_{ji}(D')$,

$$D'c(D) = c(D)D' = \frac{\det D}{D_{11}}$$

i.e. if $D = \left( \begin{array}{cc} 0 & D' \\ c(D) & 0 \end{array} \right)$, then $(D')^2 = (\det D)'$ for $r \in \mathbb{N}$. Again we investigate for what $(\lambda, \lambda')$

$$\left( \begin{array}{c} \text{for a given integer} \quad l > 0, \text{and again we are reduced to studying the elements} \\ \lambda \text{ of } su(n, n); \gamma \in H. \text{We maintain the notation of Chapter 2 and get} \\ dD_{\lambda}^+ \left( \begin{array}{cc} 0 & 0 \\ y & 0 \end{array} \right) = \frac{d}{dt} \left. D_{\lambda}^+ \left( \begin{array}{cc} 1 & 0 \\ ty & 1 \end{array} \right) \right|_{t=0} = \gamma \chi + (Y_0 + (\lambda' + n) Y_m) I_n,$$  

$$dD_{\lambda}^- \left( \begin{array}{cc} 0 & 0 \\ y & 0 \end{array} \right) = \frac{d}{dt} \left. D_{\lambda}^- \left( \begin{array}{cc} 1 & 0 \\ ty & 1 \end{array} \right) \right|_{t=0} = -\chi \gamma + (Y_0 + (\lambda + n + 1) Y_m) I_n.$$  

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To proceed, we observe the formulas

\[
\begin{align*}
\{D'xv\}_r &= \sum_{i=1}^{n} a_{ir}(x_{ip}\rho \delta_{qs} + x_{iq}\rho \delta_{ps}), \\
\{yxD'\}_r &= \sum_{i=1}^{n} (\rho \delta_{pr}x_{qi} + \rho \delta_{q}x_{ps})a_{st}, \\
\{[D', Y_0]\}_r &= [a_{sr}, Y_0] = \sum_{\beta=1}^{n} \rho \delta_{pr}x_{q\beta}a_{s\beta} \\
&\quad + \sum_{\alpha=1}^{n} \rho x_{ap}\delta_{sq}a_{\alpha} + \sum_{\beta=1}^{n} \rho \delta_{q}x_{p\beta}a_{s\beta} + \sum_{\alpha=1}^{n} \rho x_{aq}\delta_{sp}a_{\alpha}, \\
\{[D', Y_m]\}_r &= (\rho \delta_{sq}\delta_{rp} + \rho \delta_{sp}\delta_{rq}), \\
\{[\text{det } D, yx]\}_r &= \rho \delta_{pr}C_{qs} + \rho \delta_{pq}C_{ps}, \\
\{[\text{det } D]', yx]\}_r &= l(\text{det } D)'^{-1}(\rho \delta_{pr}C_{qs} + \rho \delta_{pq}C_{ps}).
\end{align*}
\]

From these formulas, together with (2.12), it then follows that

\[
\begin{align*}
\{(\text{det } D)'D'(-xy + (Y_0 + (\lambda + 1 + n)Y_m)I_n)\}_r &= \{yx(\text{det } D)'D'\}_r + (\lambda + 1 + l)(\text{det } D)'(\rho \delta_{pr}\delta_{qs} + \rho \delta_{pq}\delta_{ps}) \\
&\quad + \{(Y_0 + (\lambda + 1 + n + 2l)Y_m)I_n(\text{det } D)'D'\}_r \\
&\quad + \{(l(\lambda + 1 + n) + l(l - 1) - l(n - 1))C(\text{det } D)'^{-1}D'\}_r.
\end{align*}
\]

In particular, if (4.2) is to be satisfied, then \(\lambda = -l - 1\) and \(\lambda' = l\).

The other half of the problem follows similarly to that of Chapter 3, once we have noted

**Lemma 4.1.** For integers \(l \geq 0\),

\[
[(\text{det } D)'^{l+1}, x] = (l + 1)(\text{det } D)'c(D).
\]

**Proof.** This follows by induction from (2.8).

We let \(V_{-l}\) be the space of those functions on \(H\) with values in \(\mathbb{C}^n \oplus \mathbb{C}^n\) that are boundary values of holomorphic functions on \(\mathcal{D}\), and for every \(x\) in \(H\) can be continued across the boundary as a holomorphic function in a neighborhood \(N_x\) of \(x\), and which maintain this property when acted upon by \(D_{-l-1}\). We can then state, similarly to the preceding cases:

**Proposition 4.2.** For integers \(l \geq 0\), \(g \in SU(n, n)\), and \(f \in V_{-l}\):

\[
\Phi^{2l+1}D_{-l-1}(g)f = D_l(g)\Phi^{2l+1}f.
\]

The question of unitarity is again settled by formulas obtainable from [1],
Specifically, there exists a constant $k_2$ such that for $\alpha > -1$ and $z \in \mathbb{C}$:

$$\int_{C^+} e^{i\tau yc(y)}(\det y)^{\alpha} \, dy = k_2 \left( \frac{z}{i} \right)^{\alpha-1} \left( \frac{1}{1 + \alpha} \right) \prod_{i=2}^{n} \Gamma(i + \alpha),$$

and

$$\int_{C^+} e^{i\tau y}(\det y)^{\alpha} \, dy = k_2 \left( \frac{z}{i} \right)^{\alpha-1} \left( \frac{1}{1 + \alpha} \right) \prod_{i=1}^{n-1} \Gamma(i + \alpha).$$

It follows again by (1.8), (1.9), (1.10), and (1.11) that $D_\lambda$ is unitary for $\lambda > -1$. It is easy to see, using $\det A' = \det A$, that for $v \in \mathbb{C}^n$:

$$D' e^{i\tau v} = iye^{i\tau v},$$

and

$$c(D)e^{i\tau yv} = (i)^{-1}c(y)e^{i\tau yv}.$$
DIFFERENTIAL OPERATORS FOR $\text{Mp}(n, \mathbb{R})$ AND $\text{SU}(n, n)$

$\mathbb{C}^n \oplus \mathbb{C}^n$ by

$$[f]_{2l+1} = [g]_{2l+1} \iff \nabla^{2l+1} (f - g) = 0.$$  

Again analogous to the proof for the case of $\text{SU}(2, 2)$ in [5], we get, using Proposition 4.2 and (4.9)

**Proposition 4.3.** For each integer $l > 0$, there exists a subspace of the equivalence classes $[\cdot]_{2l+1}$ which can be given a Hilbert space structure, in which $D_{-1-1}$ is unitary.

5. A combination. We conclude this article with a study of a series of representations obtained by induction from noncomplemented finite dimensional representations of the maximal parabolic subgroup $P_-$.

We shall give the details for the groups $\text{SU}(n, n)$, whereas the corresponding results for $\text{Mp}(n, \mathbb{R})$, due to the large similarity, are omitted.

For $\text{SU}(n, n)$,

$$P_- = \left\{ \begin{pmatrix} a & 0 \\ c & a^{-1} \end{pmatrix} \middle| \text{det } a \text{ is real and } ca^* = ac^* \right\},$$

and hence there are some very natural representations of $P_-$, namely

$$\mu_l \begin{pmatrix} a & 0 \\ c & a^{-1} \end{pmatrix} = \begin{pmatrix} a & 0 \\ c & a^{-1} \end{pmatrix} (\text{det } a)^{l+n}.$$

Similarly to the preceding cases, the induced representations obtained from $\mu_l$ yield actions on measurable functions on $H$ (see [5]) since $H$ can be identified with the subgroup $\{ (g^0) | x \in H \}$ of $G$ (which is mapped onto an open dense subset of $G/P_-$). The actions $H_l(g)$ thus obtained are

$$(H_l(g)f)(x) = \begin{pmatrix} (cx + d)^* & 0 \\ -c^* & (cx + d)^{-1} \end{pmatrix} \text{det}(cx + d)^{-(n+l)} f \left( \frac{ax + b}{cx + d} \right),$$

if $g^{-1} = (c, d)$, and if $f$ is any measurable function from $H$ to $\mathbb{C}^{2n}$.

We note that if $z \in \mathbb{R}$ and if $g = (c, d) \in G$, then

$$\text{det}(cz + d) = \text{det}(zc^* + d^*).$$

This fact is trivially true on the subgroup

$$P_+ = \left\{ \begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix} \middle| \text{det } a \text{ is real, and } ab^* = ba^* \right\}$$

and on $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \in G$. These elements generate the group, and hence, by (2.5), the assertion follows for all $g$ in $G$. 

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In terms of the representations studied in Chapter 2 and Chapter 4,
\[
H_i(g) = \begin{pmatrix}
D_{l-1}(g) & 0 \\
-c(g^{-1})*U_i(g) & D^+(g)
\end{pmatrix},
\]
(5.4)
and from (5.3) it follows, that if we let \(\mathfrak{g}(2n)\) be the space of holomorphic functions from \(\mathfrak{g}\) to \(\mathbb{C}^{2n}\), then we can think of \(H_i\) as an action on that space, with (5.2) as boundary value. We shall do that, and omit the technicalities of passing to the boundary. In this connection, we recall that it was observed in the preceding chapters, that the differential operators \(D', c(D),\) and \(\det D\) can be extended to operate on holomorphic functions on \(\mathfrak{g}\) in such a way that the intertwining relations remain valid.

**Lemma 5.5.** Let \(l\) be an integer and let \(f\) be a holomorphic function from \(\mathfrak{g}\) to \(\mathbb{C}^n\). Then
\[
D'(\frac{z}{\det z^{l+n}} f(-z^{-1})) = -l \frac{1}{\det z^{l+n}} f(-z^{-1}) + \frac{z^{-1}}{\det z^{l+n}} (Df)(-z^{-1}).
\]
(5.6)

**Proof.** Since \(D'\) is a matrix with first order differential operators as entries, and since \((\det z)^{-l}\) is a scalar valued function, the formula will follow from (4.7) (with \(l = 0\)), once we have proved that for any vector \(v \in \mathbb{C}^n\),
\[
D'(\det z^{-l}v) = -lz^{-1}\det z^{-l}v.
\]
This, however, follows from (4.9).

**Proposition 5.7.** Let \(l\) be an integer. Then for all \(f\) in \(\mathfrak{g}(2n)\) and all \(g\) in \(G\),
\[
\begin{bmatrix}
l & 0 \\
-D' & l
\end{bmatrix}
\begin{bmatrix}
D_{l-1}(g) & 0 \\
-c(g^{-1})*U_i(g) & D^+(g)
\end{bmatrix}f
= \begin{bmatrix}
D_{l-1}(g) & 0 \\
0 & D^+(g)
\end{bmatrix}
\begin{bmatrix}
l & 0 \\
-D' & l
\end{bmatrix}f.
\]
(5.8)

**Proof.** We need only check the relation for \(g^{-1} = (\begin{smallmatrix} 1 & -1 \\ 0 & 1 \end{smallmatrix})\). For this element, the only nontrivial equation in (5.8) is (5.6).

In (5.8) the operator \(\begin{bmatrix} - & 0 \\ D' & - \end{bmatrix}\) is clearly invertible on \(\mathfrak{g}(2n)\) when \(l \neq 0\). Hence, by Chapter 4, one can put a Hilbert space structure \(K_l\) on a subspace of \(\mathfrak{g}(2n)\) in which \(H_i\) is unitary for \(l > 1\).

When \(l = 0\), in addition to (5.8), we also have
\[
\begin{bmatrix}
0 & 0 \\
-D' & 0
\end{bmatrix}H_0 = H_0\begin{bmatrix}
0 & 0 \\
-D' & 0
\end{bmatrix}
\]
and
\[
\begin{bmatrix}
D^- & 0 \\
0 & D_0^+
\end{bmatrix}\begin{bmatrix}
0 & 0 \\
-D' & 0
\end{bmatrix} = \begin{bmatrix}
0 & 0 \\
-D' & 0
\end{bmatrix}\begin{bmatrix}
D^- & 0 \\
0 & D_0^+
\end{bmatrix}.\]
(5.9)
In this case, we see that \(K_0^+ = \{(\varphi') \in \Theta(2n)\} \) and \(K_0^- = \{(\varphi^+) \in \Theta(2n)| D\varphi^A = 0\}\) are invariant subspaces. The restriction of \(H_0\) to \(K_0^+\) is \(D_0^+\), which is unitary on a subspace. On \(K_0^-/K_0^+\), \(H_0\) is equivalent to the representation \(D_{-1}\) in the space of solutions to \(D\varphi = 0\). Again there is a subspace on which \(H_0\) is unitary. Finally, there is also a subspace of \(\Theta(2n)/K_0^- = (\Theta(2n)/K_0^+)/(K_0^-/K_0^+)\), where \(H_0\) is unitary, because \(H_0\) there is equivalent to \(D_0^+\).

As for the case \(-l < 0\), it follows from (4.7) that any operator of the form

\[
E_{l,\delta_1,\delta_2} = \begin{pmatrix}
0 & \delta_2 \det D', c(D) \\
\delta_1 \det D'D' & 0
\end{pmatrix}
\]  

(5.10)

has the property that

\[
E_{l,\delta_1,\delta_2} \begin{pmatrix}
D_{-1} & 0 \\
0 & D_{-1}^+
\end{pmatrix} = \begin{pmatrix}
D_{-1} & 0 \\
0 & D_{+}^+
\end{pmatrix} E_{l,\delta_1,\delta_2}.
\]  

(5.11)

Hence, if we define, for any complex number \(\alpha\),

\[
D_{l,\alpha} = \begin{pmatrix}
det D' & \left[ \det D', z \right] \\
\alpha \det D'D' & \det D'
\end{pmatrix}
\]

then we get from (5.8) and (5.9):

**Proposition 5.13.** For any nonnegative integer \(l\), and for all \(g\) in \(G\)

\[
D_{l,\alpha} H_{-l}(g) = H_l(g) D_{l,\alpha}.
\]

**Remark.** \(D_{l,0} = (D_{1,0})^l\).

The trivial fact, that \((0^0, 1)\) commutes with the representation

\[
\begin{pmatrix}
D_{-1} & 0 \\
0 & D_{+}^+
\end{pmatrix}
\]

for any pair of complex numbers \(\gamma_1\) and \(\gamma_2\), is, by (5.8), translated into

**Corollary 5.14.** Let \(M_{x,l} = \begin{pmatrix}
x^1 & 0 \\
0 & x^{-1}
\end{pmatrix}\), where \(x\) is an arbitrary complex number. Then

\[
H_l(g)M_{x,l} = M_{x,l}H_l(g)
\]

for any integer \(l\) and for all \(g\) in \(G\).

We observe that, either by using (5.8) to find the Hilbert space structure \(K_l\) that makes \(H_l\) unitary for \(l > 1\), or directly from (2.1) and (2.2), it follows
that if we define, for \( z \) and \( w \) in \( \mathbb{D} \) and \( \beta \) in \( \mathbb{C} \),

\[
F_{l, \beta}(z, w) = \left[ \begin{array}{cc} (z - w^*)/2i & 1/2i \\ -1/2i & \beta((z - w^*)/2i)^{-1} \end{array} \right] \det((z - w^*)/2i)^{-(l+n)},
\]

(5.15)

and if \( J_l(g, z) \) is the automorphic factor for which

\[
(H_l(g)f)(z) = J_l(g^{-1}, z)^{-1}f(g^{-1}z),
\]

then

**Proposition 5.16.** \( F_{l, \beta}(gz, gw) = J_l(g, z)F_{l, \beta}(z, w)J_l(g, w)^* \).

From (4.8) and (2.15) it finally follows that there are constants \( K_{l,n} \) such that, for \( l > 1 \)

\[
F_{l, \beta}(z, w) = K_{l,n} \int_{C^n} \left[ \begin{array}{cc} (l/2) \cdot k^{-1} & 1/2i \\ -1/2i & (2 \cdot \beta) \cdot k / (l + n) \end{array} \right] \det k \, e^{i \text{tr}(z-w^*) k} \, dk
\]

(5.17)

and hence, since

\[
\left[ \begin{array}{cc} (l/2)k^{-1} & 1/2i \\ -1/2i & (2 \cdot \beta) \cdot k / (l + n) \end{array} \right]
\]

is a positive operator (when \( k \in C^+ \)) if and only if \( \beta > (n + l)/4l \), we get, using (1.8),

**Proposition 5.18.** For \( l > 1 \) and \( \beta > (n + l)/4l \), \( F_{l, \beta} \) is the reproducing kernel for the representation \( H_l \) in \( K_l \).

**References**


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