ON THE EXISTENCE OF UNIFORMLY DISTRIBUTED
SEQUENCES IN COMPACT TOPOLOGICAL SPACES. I

BY

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ABSTRACT. We prove the existence of uniformly distributed sequences for an
arbitrary probability measure on a separable dyadic space, e.g. on a
separable compact topological group. Some counterexamples for the non-
existence of u.d. sequences in certain dense subsets are given.

Let X be a compact topological space, μ a probability measure on X. A
sequence (x_n) in X is called μ-uniformly distributed (μ-u.d.), if

$$\lim_{N\to\infty} \sum_{n=1}^{N} f(x_n) = \int f \, d\mu$$

holds for any continuous, complex valued function f on X (see [7, p. 171 ff.]
for basic properties and examples of u.d. sequences).

It follows immediately from the definition that the closure of a u.d.
sequence contains the support of the measure. One can therefore restrict
oneself to the case of separable spaces. It can be shown that for metrizable X
u.d. sequences always exist and behave in much the same way as in the
classical case of the torus group. For compact separable topological groups
and Haar measure it follows from results in [12] (see also [1], [11]) that u.d.
sequences exist. On the other hand it was shown in [9] that on the Stone-Čech
compactification of the integers u.d. sequences exist only if the measure is
concentrated on a countable set. One of the aims of this paper is to give
further conditons for the existence of u.d. sequences.

First we treat the following 'lifting problem': if $\pi\colon X \to Y$ is surjective and
continuous, X metrizable and if a sequence (y_n) is u.d. with respect to the
image measure $\pi(\mu)$ on Y, does there always exist a μ-u.d. sequence (x_n) on
X, for which $\pi(x_n) = y_n$? The special case where $X = Y \times Z$, $\pi(y, z) = y$,
$\mu = \nu \otimes \rho$ has been considered in [4]. In the general case it turns out that a
lifting is possible for any $\pi(\mu)$-u.d. sequence in Y, iff the map π is open in
μ-almost all points of X, in particular if π is an open map.

Using this result, we can show that for dyadic spaces with an open basis of

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cardinality $\aleph_1$ u.d. sequences exist for any measure μ (a topological space X is called dyadic, if it is a continuous image of $\{0, 1\}^I$ for some index set I, see [2], [3]). In particular, if one assumes the continuum hypothesis, u.d. sequences exist for arbitrary measures on compact separable groups. Finally we give an example which shows that the dyadic spaces form a proper subclass of those spaces for which u.d. sequences exist for any measure.

In the last section we give three counterexamples which show that some properties of u.d. sequences, valid on metrizable spaces, do not carry over to arbitrary separable spaces: it is in general not possible to choose the u.d. sequence in the support of the measure μ. On a topological group G the u.d. sequence cannot be chosen in an arbitrary dense subgroup of G. If μ is the Haar measure on G, it has been shown in [11] that it is possible to construct a u.d. sequence in a given countable dense subgroup, but we show that it is not possible to choose it in an arbitrary countable dense *subset*. (On metrizable groups any dense sequence can be rearranged to a u.d. sequence, compare [7, p. 185], and [6].)

PROPOSITION 1. *Let X, Y be compact, metrizable topological spaces, π: $X \times Y \to X$ the projection onto the first coordinate, μ a probability measure on $X \times Y$, $\nu = \pi(\mu)$. If (x_n) is μ-u.d. in X, there exists a sequence (y_n) in Y, such that (x_n, y_n) is μ-u.d. in $X \times Y$.*

PROOF. Let P_n be a sequence of partitions of X with the following properties:

(i) P_{n+1} refines P_n;

(ii) each $A \in P_n$ satisfies $\nu(\partial A) = 0$ and its diameter is less than n^{-1}. (∂A denotes the boundary of A.)

In a similar fashion we can choose a sequence of partitions Q_n of Y. By [7, pp. 172, 174] a sequence (x_n, y_n) is μ-u.d. iff

$$\lim N^{-1} \sum_{n=1}^{N} c_A(x_n) c_B(y_n) = \mu(A \times B)$$

for $A \in \cup P_n$, $B \in \cup Q_n$.

Put $m_n = |P_n| |Q_n|$ ($|P_n|$ denotes the number of elements of P_n). We may assume that m_n tends to infinity. Assume that $A \in P_n$ and that M and N are natural numbers with $N > M$. Then

$$\left| (N - M)^{-1} \sum_{n=M+1}^{N} c_A(x_n) - \nu(A) \right|$$

$$\leqslant 2N(N - M)^{-1} \sup_{k > M} \left| K^{-1} \sum_{n=1}^{K} c_A(x_n) - \nu(A) \right|$$

and this expression becomes arbitrarily small for large M, provided that the

quotient $(N - M)/N$ does not decrease too fast. Using this observation, one can construct a monotone increasing sequence of natural numbers N_k ($N_1 = 0$), and a nondecreasing map $\varphi: \mathbf{N} \to \mathbf{N}$ with the following properties:

(i) $|(N_{k+1} - N_k)^{-1} \Sigma_{n=N_k+1}^{N_{k+1}} c_A(x_n) - \nu(A)| \leqslant m_{\varphi(k)}^{-2}$ holds for all $A \in P_{\varphi(k)}$.

(ii) $N_{k+1} - N_k \geqslant m_{\varphi(k)}^2$.

(iii) $\lim_{k \to \infty} N_{k+1}/N_k = 1$.

For $A \in P_{\varphi(k)}$ put $I_A = \{n: N_k < n \leqslant N_{k+1}, x_n \in A\}$. For each $B \in Q_{\varphi(k)}$ we can find a natural number $|I_{AB}|$ such that $||I_{AB}| - \mu(A \times B)\nu(A)^{-1}|I_A||< 1$ and $\Sigma_{B \in Q_{\varphi(k)}}|I_{AB}| = |I_A|$.

Now choose $y_n \in Y$ such that $|I_{AB}| = |\{n: N_k < n \leqslant N_{k+1}, (x_n, y_n) \in A \times B\}$. It follows that

$$\left|(N_{k+1} - N_k)^{-1} \sum_{n=N_k+1}^{N_{k+1}} c_{A \times B}(x_n, y_n) - \mu(A \times B)\right|$$

$$= \left|(N_{k+1} - N_k)^{-1}|I_{AB}| - \mu(A \times B)\right|$$

$$\leqslant (N_{k+1} - N_k)^{-1} + \mu(A \times B)|(N_{k+1} - N_k)^{-1}\nu(A)^{-1}|I_A| - 1|$$

$$\leqslant m_{\varphi(k)}^{-2} + \mu(A \times B)\nu(A)^{-1}m_{\varphi(k)}^{-2}$$

$$\leqslant 2m_{\varphi(k)}^{-2}.$$

Since for $l > k$, $P_{\varphi(l)}$ (resp. $Q_{\varphi(l)}$) is a refinement of $P_{\varphi(k)}$ (resp. $Q_{\varphi(k)}$) we get:

$$\left|(N_{l+1} - N_l)^{-1} \sum_{n=N_l}^{N_{l+1}} c_{A \times B}(x_n, y_n) - \mu(A \times B)\right| \leqslant 2m_{\varphi(l)}^{-1}.$$

Since N_{m+1}/N_m tends to zero we get for $N \geqslant N_0(l)$:

$$\left|N^{-1} \sum_{n=1}^{N} c_{A \times B}(x_n, y_n) - \mu(A \times B)\right| \leqslant 3m_{\varphi(l)}^{-1}.$$

REMARKS. (1) An example in [10] shows that Proposition 1 is not true in general for nonmetrizable, compact X, even if $\mu = \nu \otimes \rho$ for some measure ρ.

(2) If $\mu = \nu \otimes \rho$, X, Y metrizable, it has been shown in [4] that (x_n, y_n) is μ-u.d. in X for almost all sequences in Y (with respect to the product measure $\rho^{\mathbf{N}}$ on $Y^{\mathbf{N}}$). If μ is not of the form $\nu \otimes \rho$, it follows that the set of sequences (y_n) for which (x_n, y_n) is μ-u.d. has measure zero in $Y^{\mathbf{N}}$ with respect to any probability measure ρ on Y. For example if $X = Y$ and μ is the diagonal measure of ν, i.e.

$$\int_{X \times X} f(x, y) \, d\mu(x, y) = \int_X f(x, x) \, d\nu(x),$$

then (x_n, y_n) is μ-u.d. iff for any $\varepsilon > 0$ the set $\{n: d(x_n, y_n) \geqslant \varepsilon\}$ has density zero in $\mathbf{N}$.

PROPOSITION 2. *Let X, Y be compact, metrizable spaces, $\pi\colon X \to Y$ continuous, surjective, d a metric for the topology of Y, μ a probability measure on X, $\nu = \pi(\mu)$. If (y_n) is a ν-u.d. sequence in Y, there exists a μ-u.d. sequence (x_n) in X, such that $d(\pi(x_n), y_n)$ converges to zero $(n \to \infty)$.*

PROOF. We consider the space $C(X)$ of continuous, real valued functions on X, equipped with supremum norm. If X is metrizable, this space is separable. Let $(f_n)_{n=1}^{\infty}$ be a countable dense subset in the unit ball of $C(X)$. Put $Z = Y \times [-1, 1]^N$. $\pi_1\colon X \to Z$ is defined by

$$\pi_1(x) = (\pi(x), f_1(x), f_2(x), \ldots).$$

Since (f_n) is dense in the unit ball of $C(X)$, π_1 is injective and consequently a homeomorphism onto $\pi_1(X)$. Let p be the projection of Z onto Y, $\mu_1 = \pi_1(\mu)$. Since $p \circ \pi_1 = \pi$, we have clearly $p(\mu_1) = \nu$.

By Proposition 1 we can lift our sequence (y_n) to a μ_1-u.d. sequence (z_n) in Z. Since the support of the measure μ_1 is contained in $\pi_1(X)$ and Z is metrizable, we can find a sequence (x_n) in X such that $d(\pi_1(x_n), z_n)$ tends to zero (d denotes again a metric for the topology of Z). It follows easily that $\pi_1(x_n)$ (and therefore x_n) is u.d. with respect to μ_1 (resp. μ).

DEFINITION. We call $\pi\colon X \to Y$ open in $x \in X$ if $\pi(U)$ is a neighbourhood of $\pi(x)$ for any neighbourhood U of x.

THEOREM 1. *Let X, Y be compact, metrizable spaces, $\pi\colon X \to Y$ continuous, surjective, μ a probability measure on X, $\nu = \pi(\mu)$. The following statements are equivalent:*

(i) *if (y_n) is ν-u.d. in Y, then there exists a μ-u.d. sequence (x_n) in X, such that $\pi(x_n) = y_n$.*

(ii) *the set of points $x \in X$, where π is not open has μ-measure zero.*

PROOF. (ii) $\Rightarrow$ (i): We fix metrics on X and Y and denote them by the same letter d. Let $K_\varepsilon(x)$ be the open ball with center x and radius ε. If π is not open in $x \in X$, there exists $\varepsilon > 0$ such that $\pi(x) \notin \pi(K_\varepsilon(x))^\circ$ ($^\circ$ denotes the interior). Put $M_\varepsilon = \{x \in X\colon x \notin \pi^{-1}(\pi(K_\varepsilon(x))^\circ)\}$. By (ii) $\mu(M_\varepsilon) = 0$ for all $\varepsilon > 0$.

We claim that $M_\varepsilon^- \subseteq M_{\varepsilon/2}$: Assume that $u_n \in M_\varepsilon$ converges to u. For each u_n there exists a sequence v_{nm} in Y, which converges to $\pi(u_n)$ and does not belong to $\pi(K_\varepsilon(u_n))$. Selecting an appropriate diagonal sequence, we find elements w_n, which converge to $\pi(u)$ and such that $w_n \notin \pi(K_\varepsilon(u_n))$. If $w_n = \pi(w_n')$ with $w_n' \in K_{\varepsilon/2}(u)$, then $w_n' \notin K_\varepsilon(u_n)$ and consequently

$$d(u, u_n) \geqslant d(w_n', u_n) - d(u, w_n') \geqslant \varepsilon/2.$$

This means that $w_n \notin \pi(K_{\varepsilon/2}(u))$ for large n.

Since M_ε^- is compact and satisfies $\mu(M_\varepsilon^-) = 0$, there exists an open set $U_\varepsilon \supseteq M_\varepsilon^-$ with $\mu(\partial U_\varepsilon) = 0$ and $\mu(U_\varepsilon) < \varepsilon$. There exists a number $\delta(\varepsilon) > 0$

such that $K_\delta(\pi(x)) \subseteq \pi(K_{2\varepsilon}(x))$ for all $x \in U'_\varepsilon$: otherwise we can find a sequence $(u_n) \subseteq U'_\varepsilon$, which converges to some element u and for which $K_{1/n}(\pi(u_n)) \not\subseteq \pi(K_{2\varepsilon}(u_n))$. This means that we can choose a sequence (v_n) in Y, which converges to $\pi(u)$ and satisfies $v_n \notin \pi(K_{2\varepsilon}(u_n))$. As above it follows that $v_n \notin \pi(K_\varepsilon(u))$ for large n, contradicting our construction of U_ε.

Now assume that U_ε and $\delta(\varepsilon)$ have been chosen for $\varepsilon = 1/n$ ($n = 1, 2, \dots$). If (y_n) is ν-u.d. in Y, there exists by Proposition 2 a μ-u.d. sequence (z_n) in X, such that $d(\pi(z_n), y_n)$ converges to zero. Since $\mu(\partial U_\varepsilon) = 0$, we have

$$\lim_{N \to \infty} N^{-1} \sum_{n=1}^{N} c_{U_\varepsilon}(z_n) = \mu(U_\varepsilon) < \varepsilon$$

for all $\varepsilon > 0$. We choose a sequence of indices N_k with the following properties:

(1) $d(\pi(z_n), y_n) < \delta(1/k)$ for $n \geqslant N_k$;
(2) $(N - N_k)^{-1} \sum_{n=N_k+1}^{N} c_{U_{1/k}}(z_n) < 1/k$ for $N \geqslant (1 + 1/k)N_k$;
(3) $N_{k+1} \geqslant (1 + 1/k)N_k$.

If $N_k \leqslant n < N_{k+1}$ and $z_n \in U'_{1/k}$, we can find $x_n \in X$ with $d(x_n, z_n) < 2/k$, and $\pi(x_n) = y_n$. Otherwise choose an arbitrary $x_n \in X$ with $\pi(x_n) = y_n$. By (2) the set $\bigcup_{k=1}^{\infty}\{n: N_k \leqslant n < N_{k+1}, z_n \in U_{1/k}\}$ has density zero in $\mathbb{N}$. On the complement of this set the distance from x_n to z_n tends to zero. It follows that (x_n) is μ-u.d.

(i) $\Rightarrow$ (ii): Assume that $\bigcup_{\varepsilon>0} M_\varepsilon$, i.e. the set of points where π is not open has positive outer measure. Then there exists $\varepsilon > 0$ such that $M_{2\varepsilon}$ has positive outer measure. If $x \in M_{2\varepsilon}$ and $d(x, y) < \varepsilon$, then $K_\varepsilon(y) \subseteq K_{2\varepsilon}(x)$. Consequently $\pi^{-1}(\pi(K_\varepsilon(y))^\circ) \subseteq \pi^{-1}(\pi(K_{2\varepsilon}(x))^\circ)$ and furthermore $x \in K_\varepsilon(y) \setminus \pi^{-1}(\pi(K_\varepsilon(y))^\circ)$. It follows that $M_{2\varepsilon} \subseteq \bigcup\{K_\varepsilon(y) \setminus \pi^{-1}(\pi(K_\varepsilon(y))^\circ)\}$ the union being taken over a countable dense subset of X. This shows the existence of an element $y \in X$, for which $\mu(K_\varepsilon(y) \setminus \pi^{-1}(\pi(K_\varepsilon(y))^\circ)) > 0$ for uncountably many $\varepsilon > 0$, i.e. we may in addition assume that $\mu(\partial K_\varepsilon(y)) = 0$. In this way we have found an open subset U of X, with $\mu(\partial U) = 0$ and $\mu(U \setminus \pi^{-1}(\pi(U)^\circ)) > 0$.

Now let ν_1 be the restriction of ν to $Y \setminus \pi(U)^\circ$ and $\nu_2 = \nu - \nu_1$. Since Y is metrizable, we can find a ν-u.d. sequence $(y_n)_{n \in \mathbb{N}}$, which is composed of a ν_1-u.d. subsequence $(y_n)_{n \in \mathbb{N}_1}$ and a ν_2-u.d. sequence $(y_n)_{n \in \mathbb{N}_2}$ such that $y_n \in Y \setminus \pi(U)$ for $n \in \mathbb{N}_1$ and $y_n \notin \partial\pi(U)$ for $n \in \mathbb{N}_2$.

Now assume that there exists a μ-u.d. sequence (x_n) in X, for which $\pi(x_n) = y_n$. By the regularity of ν, there exists an open subset V of $\pi(U)$ such that $\nu(\partial V) = 0$ and

$$\nu(\pi(U)^\circ \setminus V) < \mu(U \setminus \pi^{-1}(\pi(U)^\circ)).$$

Since $\partial(\pi^{-1}(V)) \subseteq \pi^{-1}(\partial V)$ we have $\mu(\partial(\pi^{-1}(V))) = 0$.

$\mu(\partial U) = 0$, consequently $\lim_{N \to \infty} N^{-1} \sum_{n=1}^{N} c_U(x_n) = \mu(U)$. Now we

decompose the left side of this equation, using that

$$U = \left(U \cap \pi^{-1}(V) \right) \cup \left(U \setminus \pi^{-1}(V) \right)$$
$$\subseteq \left(U \cap \pi^{-1}(V) \right) \cup \pi^{-1}(\pi(U) \setminus V).$$

If $x_n \in \pi^{-1}(\pi(U) \setminus V)$, then $y_n \in \pi(U)$. By our construction of y_n it follows that $y_n \in \pi(U)^\circ$ and $n \in N_2$. Since $\nu_2(\partial (\pi(U)^\circ)) = 0$ and clearly $\nu_2(\partial V) = 0$ the limit corresponding to $U \setminus \pi^{-1}(V)$ can be estimated by $\nu_2(\pi(U)^\circ \setminus V)$ and we would get:

$$\mu(U) \leqslant \mu\left(U \cap \pi^{-1}(V) \right) + \nu_2\left(\pi(U)^\circ \setminus V \right)$$
$$< \mu\left(U \cap \pi^{-1}(V) \right) + \mu\left(U \setminus \pi^{-1}(\pi(U)^\circ) \right)$$
$$\leqslant \mu(U) \quad \text{a contradiction.}$$

DEFINITION [2]. A topological space X is called *dyadic*, if there exists some index set I and a continuous surjective map $\pi: \{0, 1\}^I \to X$.

THEOREM 2. *Let X be a dyadic space with a base of cardinality $\aleph_1$, μ a probability measure on X. Then there exists a μ-u.d. sequence (x_n) in X.*

PROOF. By [2], there exists a set I of cardinality $\aleph_1$ and a continuous surjective map $\pi: \{0, 1\}^I \to X$. By the Hahn-Banach theorem there exists a probability measure ν on $\{0, 1\}^I$ such that $\pi(\nu) = \mu$. If (y_n) is ν-u.d. in $\{0, 1\}^I$, then $(\pi(y_n))$ is μ-u.d. in X. In this way we have reduced the problem to the case $X = \{0, 1\}^I$. We identify I with the first uncountable ordinal ω_1. For $\alpha < \omega_1$ put $X_\alpha = \{0, 1\}^\alpha$. We consider the projections $p_\alpha: X \to X_\alpha$ and $p_{\alpha\beta}: X_\alpha \to X_\beta$ ($\beta < \alpha$) and the measures $\nu_\alpha = p_\alpha(\nu)$. By induction we define ν_α-u.d. sequences $(x_{n\alpha})$ in X_α such that $p_{\alpha\beta}(x_{n\alpha}) = x_{n\beta}$: if $\alpha = \beta + 1$ for some β, then we may apply Theorem 1 since X is metrizable and $p_{\alpha\beta}$ open; if α is a limit ordinal, X_α is the projective limit of the spaces X_β ($\beta < \alpha$) and $(x_{n\beta})_{\beta<\alpha}$ defines a unique element $x_{n\alpha}$ in X_α. By the Stone-Weierstrass theorem $(x_{n\alpha})$ is ν_α-u.d. In the same way one gets from the sequences $(x_{n\alpha})_{\alpha<\omega_1}$ a ν-u.d. sequence (x_n). (Cf. the last remark in [10].)

COROLLARY. *Let G be a compact separable topological group, μ a probability measure. If we assume the continuum hypothesis, there exists a μ-u.d. sequence in G.*

PROOF. By [8] any compact topological group is a dyadic space, so the result follows from our theorem. One can also give a direct proof if one considers a well-ordered sequence of closed normal subgroups $(H_\alpha)_{\alpha<\omega_1}$, such that G/H_α is metrizable and $\cap H_\alpha = \{e\}$ (i.e. G is the projective limit of the groups G/H_α). (Cf. the proof of Lemma 2 in [9].)

EXAMPLE. If one assumes the continuum hypothesis, there exists a separable compact space X which is not dyadic and such that any probability measure on X admits a u.d. sequence.

Let A be the algebra of almost periodic functions on $\mathbf{Z}$ and c_0 the algebra of complex sequences $(a_n)_{n=-\infty}^{\infty}$ for which

$$\lim_{n\to\infty} a_n = \lim_{n\to-\infty} a_n = 0.$$

We consider the subalgebra B of $l_\infty(\mathbf{Z})$ generated by A and c_0. B determines a compactification X of $\mathbf{Z}$. Any maximal ideal in B has either the form $I + c_0$, where I is a maximal ideal in A or $A + J$, where J is a maximal ideal in c_0. It follows that X can be written as a disjoint union of G and $\mathbf{Z}$, where G denotes the Bohr compactification of $\mathbf{Z}$. $\mathbf{Z}$ is open in X and X induces the group topology on G. If μ is a probability measure on X, it can be decomposed into measures on G and $\mathbf{Z}$ and by Theorem 2 u.d. sequences exist for both of them. But X is not dyadic, since it is a nonmetrizable compactification of $\mathbf{Z}$ [3, p. 61].

Some counterexamples. (1) There exists a probability measure μ on $X = [0, 1]^c$ which admits a u.d. sequence (under assumption of the continuum hypothesis), but for which no u.d. sequence exists, which is contained in the support of μ:

Let $\beta\mathbf{N}$ be the Stone-Čech compactification of the natural numbers and μ a probability measure on $\beta\mathbf{N}$, which is not concentrated on a countable subset. There exists an injective, continuous map $j : \beta\mathbf{N} \to X$. By [9, Proposition], there exists no μ-u.d. sequence in $\beta\mathbf{N}$, but by Theorem 2 μ admits a u.d. sequence in X.

(2) Let G be the Bohr compactification of the integers. There exist probability measures on G, for which no u.d. sequences exist, which are contained in the dense subgroup $\mathbf{Z}$:

Let μ be a point measure, concentrated in a point $x \in G \setminus \mathbf{Z}$. Assume that $(x_n) \subseteq \mathbf{Z}$ is μ-u.d. The dual group of G is the torus group T (with discrete topology). If χ is a continuous character on G, then $\lim N^{-1}\Sigma_1^N \chi(x_n) = \chi(x)$. By duality x_n and x can be identified with characters on T. It follows that $N^{-1}\Sigma_1^N x_n$ converges to x, pointwise on T. Since $x_n \in \mathbf{Z}$, they define continuous characters on T and so x would be Borel-measurable. But by [5, Corollary 22.19, p. 346] a Borel measurable character is already continuous, which contradicts $x \notin \mathbf{Z}$.

(3) There exists a dense subset M in $\{+1, -1\}^c$, such that there exists no sequence in M, which is u.d. with respect to the product measure:

If (x_n) is a dense sequence in $\{+1, -1\}^c$, it defines a system $(A_\alpha)_{\alpha\in c}$ of subsets of $\mathbf{N}$ by $A_\alpha = \{n : x_{n\alpha} = 1\}$ (where $x_n = (x_{n\alpha})_{\alpha\in c}$). If A is a subset of $\mathbf{N}$, we use the notation $A^1 = A$, $A^{-1} = \mathbf{N} \setminus A$. Now if $\alpha_1, \ldots, \alpha_k$ are different indices in c, and $i_1, \ldots, i_k \in \{+1, -1\}$, then $A_{\alpha_1}^{i_1} \cap \cdots \cap A_{\alpha_k}^{i_k} = \{n : x_{n\alpha_1} = i_1, \ldots, x_{n\alpha_k} = i_k\} \neq \varnothing$.

Conversely let $\mathfrak{A}$ be a family of subsets of $\mathbf{N}$, with cardinality c and which is maximal with respect to the property that $A_{\alpha_1}^{i_1} \cap \cdots \cap A_{\alpha_k}^{i_k} \neq \varnothing$ for any

distinct elements $A_{\alpha_1}, \ldots, A_{\alpha_k} \in \mathfrak{A}$ and $i_1, \ldots, i_k \in \{+1, -1\}$. (We index the elements of $\mathfrak{A}$ by the elements of c). $\mathfrak{A}$ defines a sequence (x_n) in $\{+1, -1\}^c$ by $x_{n\alpha} = 1$ if $n \in A_\alpha$ and $x_{n\alpha} = -1$ if $n \notin A$. If $\alpha_1, \ldots, \alpha_k \in c$ and $i_1, \ldots, i_k \in \{+1, -1\}$ then $\{n: x_{n\alpha_1} = i_1, \ldots, x_{n\alpha_k} = i_k\} = A_{\alpha_1}^{i_1} \cap \cdots \cap A_{\alpha_k}^{i_k} \neq \varnothing$. It follows that $M = \{x_n\}$ is dense in $\{+1, -1\}^c$.

Assume that (y_n) is a u.d. sequence in $\{+1, -1\}^c$, which is contained in M, i.e. we have a map $p: \mathbf{N} \to \mathbf{N}$ such that $y_n = x_{p(n)}$. Put $\beta_{in} = n^{-1}|\{k \leqslant n: p(k) = i\}|$. Then $\sum_{i=1}^\infty \beta_{in} = 1$ for all n and if n is fixed, $\beta_{in} \neq 0$ holds only for a finite number of $i \in \mathbf{N}$. We have

$$\lim_{n \to \infty} \sum_{i=1}^\infty \beta_{in} f(x_i) = \lim_{n \to \infty} n^{-1} \sum_{k=1}^n f(y_k) = \int f \, d\mu$$

for all continuous or Riemann integrable functions $f: \{+1, -1\}^c \to \mathbf{R}$. If one takes the characteristic function of x_i, it follows that $\lim_{n \to \infty} \beta_{in} = 0$.

Now we claim that there exists a subset B of $\mathbf{N}$ for which

$$\limsup_{n \to \infty} \sum_{i=1}^\infty \beta_{in} c_B(i) = 1 \quad \text{and} \quad \liminf_{n \to \infty} \sum_{i=1}^\infty \beta_{in} c_B(i) = 0.$$

The set B is constructed by induction: Assume that a subset B_n of the interval $[1, N]$, and $\varepsilon > 0$ are given. Since $\lim_{m \to \infty} \beta_{im} = 0$, there exists $m > n$, such that $\beta_{im} < \varepsilon N^{-1}$ for all $i \leqslant N$. Put $M = \max\{i: \beta_{im} > 0\} \cup \{N\}$. If we write B_m for B_n, regarded as a subset of $[1, M]$, then $\sum_{i=1}^\infty \beta_{im} c_{B_m}(i) < \varepsilon$. On the other hand, if we put $B_m = B_n \cup [N + 1, M]$, then

$$\sum_{i=1}^\infty \beta_{im} c_{B_m}(i) \geqslant \sum_{i=N+1}^N \beta_{im} = 1 - \sum_{i=1}^N \beta_{im} \geqslant 1 - \varepsilon.$$

Both sums do not change their value if we add a subset of $[M + 1, \infty[$ to B_m. If one applies alternately these two enlargement procedures and ε tends to zero, one gets the desired set B.

If $B = A_\alpha \in \mathfrak{A}$, then $\lim_{m \to \infty} \sum_{i=1}^\infty \beta_{im} c_B(i) = \lim_{m \to \infty} m^{-1}|\{k: y_{k\alpha} = 1\}| = 2^{-1}$ which is impossible. Since $\mathfrak{A}$ is maximal, there exist $\alpha_1, \ldots, \alpha_k \in c$, $i_0, i_1, \ldots, i_k \in \{+1, -1\}$ such that $B^{i_0} \cap A_{\alpha_1}^{i_1} \cap \cdots \cap A_{\alpha_k}^{i_k} = \varnothing$. Replacing B by $\mathbf{N} \setminus B$ if necessary, we may assume that $i_0 = -1$ i.e. $B \subseteq A_{\alpha_1}^{i_1} \cap \cdots \cap A_{\alpha_k}^{i_k}$. But then

$$\limsup_{m \to \infty} \sum_{i=1}^\infty \beta_{im} c_B(i) \leqslant \lim_{m \to \infty} m^{-1}|\{j: y_{j\alpha_1} = i_1, \ldots, y_{j\alpha_k} = i_k\}| = 2^{-k}$$

which is again a contradiction.

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