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THICKENINGS OF CW COMPLEXES OF THE FORM $S^m \cup_\alpha e^n$

BY

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ABSTRACT. Necessary conditions are given for the existence of a thickening of $S^m \cup_\alpha e^n$ in codimension k. I give examples of such complexes requiring arbitrarily large codimension in order to thicken. Sufficient conditions are given for the existence of a tractable thickening in codimension $k + 1$. The methods used include the study of the reduced product space of a pair of CW complexes.

1. Let K be a simply-connected finite CW complex. A Poincaré thickening of K, or more briefly a thickening of K, as defined by Levitt in [12], consists of a Poincaré duality pair $(W, \partial W)$ together with a homotopy equivalence f: $K \to W$. (Note that, in contrast to Levitt, I do not assume that $\pi_1 \partial W$ is trivial.) Throughout this paper I will refer to a thickening by the map f: $K \to W$ if the boundary of W is clearly understood. If K is of dimension n, and the fundamental class $[W] \in H_*(W, \partial W)$ is of degree N, then the codimension of the thickening is $N - n$.

Thickenings arise, for example, in embedding theory. An embedding of the CW complex K^n up to homotopy type, in a smooth or PL manifold M^N, consists of a subcomplex L of M^N together with a homotopy equivalence h: $K \to L$. If W denotes a regular neighborhood of L in M^N, then the pair $(W, \partial W)$ together with the composition

$$K \xrightarrow{h} L \subseteq W$$

is a thickening of K in codimension $N - n$.

In this paper I study thickenings of complexes of the form $S^m \cup_\alpha e^n$ by analyzing possible cell structures for the pair $(W, \partial W)$. (I am indebted to Ethan Akin for pointing out that the methods of [5] might be applied to this more general problem.) The most simply stated result obtained is as follows: Let f: $S^m \cup_\alpha e^n \to W$ be a thickening in codimension k. Assume $m \geqslant 2$,

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$k \geqslant 2$. The quotient space $W/\partial W$ plays a role analogous to the Thom space of the normal bundle of an embedded smooth manifold, so it will be termed the Thom space of the given thickening. In the case at hand $W/\partial W$ has the homotopy type of a cell complex of the form $S^k \cup_\beta e^{k+n-m} \cup e^{n+k}$. If we are careful about orientations, the homotopy class of the attaching map $\beta \in \pi_{k+n-m-1}(S^k)$ is uniquely determined.

THEOREM 1.1. *Under the above conditions, there exist elements* $\psi_0 \in \pi_{k+n-m-1}(S^{k+m-1})$ *and* $\gamma_0 \in \pi_{n+k-2}(S^{k+n-m-1})$ *such that*

$$S^{m-1}\beta = (-1)^{mk}S^{k-1}\alpha + \psi_0 \circ \gamma_0. \tag{1}$$

Here S^i denotes suspension iterated i times. Condition (1) should be compared to the condition of Corollary 2 of [**5**]. It says that the Thom space $W/\partial W$, minus its top cell, differs from a Spanier-Whitehead dual of $S^m \cup_\alpha e^n$ by the presence of the term $\psi_0 \circ \gamma_0$. Note that if $S^m \cup_\alpha e^n$ is a stable space, i.e. that $n < 2m$, then the element ψ_0 lives in a zero group and so (1) reduces to $S^{m-1}\beta = (-1)^{mk}S^{k-1}\alpha$.

There are further conditions on α, β, γ_0, ψ_0, but (1) suffices for some interesting applications.

EXAMPLE. Suppose $S^m \cup_\alpha e^n$ thickens in codimension k, where $k < m$. Then condition (1) says that if we suspend α $k-1$ times and add the decomposable term $\psi_0 \circ \gamma_0$, the result can be desuspended more than $k-1$ times ($m-1 > k-1$). I say that α is "stably desuspendable modulo decomposables", amplifying on a phrase of [**9**]. This suggests choosing an indecomposable element in the stable homotopy groups of spheres, and desuspending as far as possible. Let p be an odd prime, $q = 2p(p-1)-2$. Then the stable group π_q^S has a summand Z_p, the first nontrivial p-primary component of the cokernel of the J-homomorphism. Let α^S be a generator of this Z_p-summand. Composition products of all lower stems are zero, by [**15**, Proposition 4.17], so α^S is indecomposable. According to Toda's calculation of unstable groups [**14**, pp. 132–133]

(a) The element α^S can be desuspended to $\pi_{2p-1+q}(S^{2p-1})$.

(b) Any element in the p-primary component of $\pi_{2p-2+q}S^{2p-2}$ is stably trivial–i.e., suspends to zero in the stable group π_q^S.

Let $\alpha \in \pi_{2p-1+q}S^{2p-1}$ suspend to $\alpha^S \in \pi_q^S$. Then I claim that the complex $K = S^{2p-1} \cup_\alpha e^{2p+q}$ thickens in codimension $2p-1$, but not in codimension $2p-2$.

Nonexistence of codimension $2p-2$ *thickening*. If there were a thickening of K of codimension $k = 2p-2$, then by Theorem 1 there would exist $\beta \in \pi_{2p-2+q}S^{2p-2}$, $\psi_0 \in \pi_{2p+q-2}S^{4p-4}$ and $\gamma_0 \in \pi_{4p+q-4}S^{2p+q-2}$ such that

$$S^{2p-2}\beta = S^{2p-3}\alpha + \psi_0 \circ \gamma_0. \tag{2}$$

Now suspend both sides to the stable group π_q^S. By (b) above, β suspends to zero. Thus (2) implies that α^S is decomposable, but α^S was chosen to be indecomposable. Thus no thickening of K in codimension $(2p-2)$ exists.

Existence of a codimension $(2p-1)$ *thickening*. For a moment consider the general question: Given a complex $S^m \cup_\alpha e^n$, can we find a thickening in codimension m? One way is to construct a Poincaré duality space of the form

$$S^m \cup_\alpha e^n \cup e^{m+n}, \tag{3}$$

remove a small ball of dimension $m+n$ from the $(m+n)$-cell, and utilize the complement. According to James **[10]**, there exists a complex of the form (3) satisfying Poincaré duality if and only if there exists $\alpha' \in \pi_{m+n-2}(S^{n-1})$ such that $[\alpha, \iota_m] = \alpha \circ \alpha'$. Here $[\ ,\ ]$ denotes the Whitehead product defined by J. H. C. Whitehead in **[18]**, and ι_m denotes the identity map $S^m \to S^m$. Thus

PROPOSITION 1.2. *The complex $S^m \cup_\alpha e^n$ thickens in codimension m if there exists $\alpha' \in \pi_{m+n-2}S^{n-1}$ satisfying $[\alpha, \iota_m] = \alpha \circ \alpha'$.*

In the case at hand, m is odd, so $[\alpha, \iota_m]$ is of order 2, and α is an element of order p, so $[\alpha, \iota_m] = 0$, and so by Proposition 1.2 there is a thickening of the required codimension.

REMARK. Proposition 1.2 does not provide a necessary condition for the existence of a thickening. Note that the boundary of the thickening given by Proposition 1.2 is a sphere.

Following Wall **[17]**, I define the suspension of a thickening $f\colon K \to W$ to be the pair $(W \times I, \partial(W \times I))$ together with the composition

$$K \xrightarrow{f} W \to W \times I.$$

Suspending a thickening increases the codimension by 1. Thus if a complex thickens in codimension k, it thickens in all higher codimensions. The nonexistence part of the example given above proves

THEOREM 1.3. *Let N be a given integer. There exists a* CW *complex K which fails to thicken in codimension N.*

Because of the relationship of thickenings to embeddings of complexes in manifolds there results

COROLLARY 1.4. *Let N be a given integer. There exists a* CW *complex K which fails to embed up to homotopy type in any* PL-*manifold in codimension N.*

Theorem 1.3 suggests the definition of the following number theoretic function. Let n be an integer. Define $T(n)$ by the statement: every finite CW complex of dimension n thickens in codimension $T(n)$; some finite CW

complex of dimension n fails to thicken in codimension $T(n) - 1$. Since every n-complex can be embedded in Euclidean space R^{2n+1}, we have

$$T(n) \leqslant n + 1 \quad \text{for all } n. \tag{4}$$

It is easy to improve (4) to $T(n) \leqslant n$. On the other hand, the example above shows that

$$T(2p^2 - 2) \geqslant 2p - 1 \quad \text{for } p \text{ an odd prime.} \tag{5}$$

Thus as $n \to \infty$, $T(n) \to \infty$. It is interesting to consider the ratio $T(n)/n$, which is bounded above by 1. Examples involving more than 2 cells show that as $n \to \infty$, $T(n)/n \geqslant \frac{1}{3}$, which is better than (5) for $p > 3$.

The paper is organized as follows. In §2 I make some general remarks about thickenings, state the principal theorem on existence of thickenings of a complex $S^m \cup_\alpha e^n$, and derive related results including Theorem 1.1. A proof of the main theorem is presented in §3. An auxiliary result concerning the relative homotopy groups of a space with a cell attached is required for the proof in §3; this result is obtained in §4.

2. Suspensions of thickenings. Tractable thickenings. The case $K = S^m \cup_\alpha e^n$. Let K be a simply-connected CW complex. Let $(W, \partial W)$, $f\colon K \to W$ be a thickening of K. The suspension of the given thickening consists of the pair

$$(W \times I, \partial(W \times I)) = (W \times I, W \times \{0, 1\} \cup \partial W \times I) \tag{6}$$

and the composition

$$K \xrightarrow{f} W \to W \times I, \quad \text{denoted } Sf\colon K \to W \times I. \tag{7}$$

Suspending a thickening increases the codimension by 1.

DEFINITION 2.1. A thickening $(W, \partial W)$, $f\colon K \to W$ is called *tractable* if $\pi_1(\partial W)$ is trivial and there exists a map $r\colon W \to \partial W$ such that the composition

$$W \xrightarrow{r} \partial W \subseteq W$$

is homotopic to the identity.

PROPOSITION 2.2. *The suspension of a thickening is a tractable thickening.*

PROOF. The map r defined by the composition

$$W \times I \to W \times 0 \subseteq W \times \{0, 1\} \cup \partial W \times I = \partial(W \times I)$$

has the required property. The van Kampen theorem implies that $\pi_1(\partial(W \times I))$ is trivial, since W is simply-connected. Thus the pair (6) is a tractable thickening of K. $\square$

REMARK. The mapping cone of the map

$$r\colon W \times I \to \partial(W \times I)$$

so defined is homotopy equivalent to the Thom space $W/\partial W$ of the original thickening.

REMARK. To relate these matters to Levitt's terminology in **[12]**, a tractable thickening is a thickening whose associated normal fibration admits a cross-section. Theorem 2.5 of **[12]** asserts that such a thickening desuspends, i.e. is equivalent to the suspension of a thickening, provided the codimension is metastable, i.e. larger than $n/2 + 2$, where n is the dimension of the given CW complex.

Now specialize to the case $K = S^m \cup_\alpha e^n$. The main theorem of this section gives necessary and sufficient conditions for the existence of a tractable thickening of K in codimension $k + 1$. Given such a thickening f: $K \to W$, with map r: $W \to \partial W$, I will show that the mapping cone of r is homotopy equivalent to a CW complex of the form

$$S^k \cup_\beta e^{k+n-m} \cup e^{n+k}.$$

If we are careful about orientations, the homotopy class of the attaching map $\beta \in \pi_{k+n-m-1}(S^k)$ is well defined.

We need some notation. Given elements ζ_1, ζ_2 in $\pi_*(X)$, where X is an arbitrary space, define iterated Whitehead products as follows:

$$\begin{aligned} W_0(\zeta_1, \zeta_2) &= \zeta_1, \\ W_1(\zeta_1, \zeta_2) &= [\zeta_1, \zeta_2], \\ W_{j+1}(\zeta_1, \zeta_2) &= [W_j(\zeta_1, \zeta_2), \zeta_2]. \end{aligned}$$

Let $S^m \vee S^k$ denote the one point union of S^m and S^k. Let i_1: $S^m \subseteq S^m \vee S^k$ and i_2: $S^k \subseteq S^m \vee S^k$ denote the inclusions. Finally, the cross term is the subgroup of $\pi_*(S^m \vee S^k)$ consisting of those elements which map to zero in the product $S^m \times S^k$.

THEOREM 2.3. *Suppose that $m, k \geqslant 2$. Then there exists a tractable thickening*

$$f\colon S^m \cup_\alpha e^n \to W, \qquad r\colon W \to \partial W,$$

in codimension $k + 1$ with the mapping cone of r homotopy equivalent to

$$S^k \cup_\beta e^{k+n-m} \cup e^{n+k}, \qquad \beta \in \pi_{k+n-m-1} S^k,$$

if and only if there exist elements

$$\begin{aligned} &\psi \in \pi_{k+n-m-1}(S^m \vee S^k) \quad \textit{in the cross term}, \\ &\gamma_j \in \pi_{n+k-2}(S^{k+n-m-1+j(k-1)}), \qquad j = 0, 1, \ldots, \end{aligned}$$

such that, in $\pi_{n+k-2}(S^m \vee S^k)$, the following relation holds:

$$[i_1 \circ \alpha, i_2] = [i_2 \circ \beta + \psi, i_1] + \sum_{j=0}^{\infty} (-1)^{j+1} W_j(i_2 \circ \beta + \psi, i_2) \circ \gamma_j. \quad (8)$$

REMARK. The sum occurring on the right-hand side of (8) is finite because γ_j must be zero for $j > (m-1)/(k-1)$.

The proof of Theorem 2.3 will be given in §3.

In order to interpret the theorem I recall certain facts about the homotopy groups of $S^m \vee S^k$. The reader is referred to [8] for details. One chooses certain iterated Whitehead products of the generators i_1 and i_2, called basic Whitehead products. The pth basic product is denoted by $\omega_p \in \pi_{n_p}(S^m \vee S^k)$, $p \geqslant -2$. For example,

$$\omega_{-2} = i_1, \quad \omega_{-1} = i_2, \quad \omega_0 = [i_1, i_2],$$
$$\omega_1 = [i_1, [i_1, i_2]], \qquad \omega_2 = [i_2, [i_1, i_2]].$$

Then each basic product induces a homomorphism

$$(\omega_p)_*: \pi_*(S^{n_p}) \to \pi_*(S^m \vee S^k).$$

For every p, $(\omega_p)_*$ is a monomorphism, and $\pi_*(S^m \vee S^k)$ is the direct sum of the images of $(\omega_p)_*$, $p \geqslant -2$. Thus, given any $\zeta \in \pi_i(S^m \vee S^k)$, ζ has a unique representation in the form

$$\zeta = \sum_{p \geqslant -2} \omega_p \circ \zeta_p, \qquad \zeta_p \in \pi_i(S^{n_p}). \tag{9}$$

The element ζ_p will be called the pth coordinate of ζ. The condition of the theorem asserts that there exist

$$\psi = \sum_{p \geqslant 0} \omega_p \circ \psi_p, \qquad \psi_p \in \pi_{n+k-m-1}(S^{n_p}),$$

and elements β and γ_j, $j \geqslant 0$, such that

$$[i_1 \circ \alpha, i_2] = [i_2 \circ \beta + \psi, i_1] + \sum_{j=0}^{\infty} (-1)^{j+1} W_j(i_2 \circ \beta + \psi, i_2) \circ \gamma_j. \tag{10}$$

We expand both sides of (10), using standard formulas for rearranging Whitehead products and compositions. The basic rearrangement formula is given in Barcus and Barratt [1, p. 69]. (The reader should note that Barcus and Barratt use a definition of the Whitehead product that differs in sign from Whitehead's definition.) Each side of (10) has a unique representation in the form (9), and so we obtain a sequence of relations equivalent to (10) by setting corresponding coordinates equal. These relations will involve the elements α, β, ψ, γ_j, and their higher Hopf invariants. The calculations are omitted; however, for the reader's convenience I state the first four relations, deriving from the coordinates of the basic products ω_{-1}, ω_0, ω_1, and ω_2. The following is then a direct consequence of Theorem 2.3.

THEOREM 2.4. *Suppose that $m, k \geqslant 2$. If there exists a tractable thickening of $S^m \cup_\alpha e^n$ in codimension $k+1$ with the mapping cone of r: $W \to \partial W$ having*

the homotopy type of

$$S^k \cup_\beta e^{k+n-m} \cup e^{n+k}, \qquad \beta \in \pi_{k+n-m-1}S^k,$$

then there exist elements ψ *and* γ_j *as in Theorem* 2.3 *such that*

$$\beta \circ \gamma_0 - [\beta, \iota_k] \circ \gamma_1 + (-1)^k[\iota_k, [\iota_k, \iota_k]] \circ S^{2k-2}\beta \circ \gamma_2 = 0, \tag{11}$$

where ι_k *denotes the identity map in* $\pi_k S^k$,

$$S^{k-1}\alpha = (-1)^{mk}S^{m-1}\beta + (-1)^{m+n}\psi_0 \circ \gamma_0, \tag{12}$$

$$S^{k-1}H\alpha = (-1)^{mk+m}S^{m-1}\psi_0 + (-1)^{m+n+1}\psi_1 \circ \gamma_0 \tag{13}$$

$$\begin{aligned} S^{m-1}H\beta = {} & (-1)^{(m+n)(n+k)}S^{m+k-2}\beta \circ S^{k+n-m-2}\psi_0 \circ H\gamma_0 \\ & + (-1)^{m+n+mk+1}\psi_2 \circ \gamma_0 + (-1)^m S^{k-1}\psi_0 \circ \gamma_1. \end{aligned} \tag{14}$$

Theorem 1.1 now follows as a corollary.

PROOF OF THEOREM 1.1. Let f: $K \to W$ be a thickening in codimension $k \geqslant 2$ with Thom complex $W/\partial W$ homotopy equivalent to

$$S^k \cup_\beta e^{k+n-m} \cup e^{k+n}. \tag{15}$$

Then the suspension of f, Sf: $K \to W \times I$, is a tractable thickening of K, by Proposition 2.2. Let r: $W \times I \to \partial(W \times I)$ be the map given in the proof of 2.2. By the remark following the proof of 2.2, the mapping cone of r has the form (15). Apply Theorem 2.4 to the suspended thickening; the codimension is indeed at least 3. Equation (12) of Theorem 2.4 is the desired consequence. (Note that in equation (1) I have replaced γ_0 by $(-1)^{m+n}\gamma_0$, to simplify the statement of 1.1.)

REMARK. Equation (8) of Theorem 2.3 concerns elements of degree $n + k - 2$. The coordinates of both sides will be zero for all p such that $n_p > n + k - 2$. For example, the first few basic products occur in the following degrees:

$$\begin{aligned} \omega_0 &= [i_1, i_2] \in \pi_{m+k-1}(S^m \vee S^k), \\ \omega_1 &= [i_1, [i_1, i_2]] \in \pi_{2m+k-2}(S^m \vee S^k), \\ \omega_2 &= [i_2, [i_1, i_2]] \in \pi_{m+2k-2}(S^m \vee S^k). \end{aligned}$$

For $p \geqslant 3$ the basic products ω_p occur in degrees at least equal to

$$\min(3m + k - 3, m + 3k - 3). \tag{16}$$

It follows, therefore, by Theorem 2.3, that (11) and (12) of Theorem 2.4 are sufficient conditions for the existence of a tractable thickening if $n + k - 2 < \min(2m + k - 2, m + 2k - 2)$ or $n < \min(2m, m + k)$. (I have already remarked that if $n < 2m$, the element ψ_0 is in a zero group and (12) is simplified.) Similarly, conditions (11)–(14) together are sufficient if $n + k - 2$

is less than the quantity (16) above, or

$$n < \min(3m - 1, m + 2k - 1). \tag{17}$$

EXAMPLE. Let $\sigma \in \pi_{15}S^8$ denote the homotopy class of the Hopf map $S^{15} \to S^8$ as defined in [**13**, p. 109]. The complex $K = S^8 \cup_\sigma e^{16}$ is homotopy equivalent to a manifold, the Cayley projective plane, and so it has thickenings in every codimension. In fact if ξ: $E \to K$ is a fibration with fibre S^{k-1}, and $\bar{E} \to K$ denotes the associated mapping cylinder fibration, the pair $(\bar{E}, E)$ together with a section $K \to \bar{E}$ define a thickening of K in codimension k. Every thickening of K is equivalent to one obtained in this way. The Thom space of the thickening has the form

$$S^k \cup_\beta e^{k+8} \cup e^{k+16}, \qquad \beta \in \pi_{k+7}(S^k), \tag{18}$$

and I ask, which β's arise as attaching maps in these Thom spaces?

According to the discussion following the above proof of Theorem 1.1, equations (11)–(14) give necessary and sufficient conditions for the existence of a tractable thickening provided (17) holds. In the present case $m = 8$, $n = 16$, and so (17) says $16 < \min(23, 2k + 7)$ or $k \geqslant 5$. So I assume $k \geqslant 5$ and study equations (11)–(14). Let $\iota_p \in \pi_p S^p$ denote the homotopy class of the identity map. Then $H(\alpha) = H(\sigma) = \iota_{15}$. The elements ψ_1, ψ_2, and γ_2 live in zero groups. The element γ_0 lives in $\pi_{k+14}S^{k+7}$, and so, since $k \geqslant 5$, γ_0 is a suspension, implying $H(\gamma_0) = 0$. Thus (11)–(14) reduce to

$$\beta \circ \gamma_0 - [\beta, \iota_k] \circ \gamma_1 = 0, \tag{11'}$$

$$S^{k-1}\sigma = S^7\beta + \psi_0 \circ \gamma_0, \tag{12'}$$

$$\iota_{k+14} = S^7\psi_0, \tag{13'}$$

$$S^7H(\beta) = S^{k-1}\psi_0 \circ \gamma_1. \tag{14'}$$

Equation (13′) may be desuspended to state: $\psi_0 = \iota_{k+7}$. Then (12′) and (14′) may be solved for γ_0 and γ_1:

$$\gamma_0 = S^{k-1}\sigma - S^7\beta, \qquad \gamma_1 = S^7(H\beta).$$

Substituting these values in (11′) our necessary and sufficient conditions reduce to

$$\beta \circ S^{k-1}\sigma - \beta \circ S^7\beta - [\beta, \iota_k] \circ S^7H(\beta) = 0. \tag{19}$$

PROPOSITION 2.5. *Suppose $K = S^8 \cup_\sigma e^{16}$, and that $k \geqslant 5$. There exists a tractable thickening $(W, \partial W)$ of K in codimension $k + 1$ with the mapping cone of r: $W \to \partial W$ homotopy equivalent to $S^k \cup_\beta e^{k+8} \cup e^{k+16}$ if and only if β satisfies equation* (19).

Note that (19) is automatically satisfied if $\beta = 0$. This corresponds to the thickening $K \times D^{k+1}$. Other values of β allowed by (19) can be determined

by referring to Toda's computations [16] of the homotopy groups of spheres. For example, if $k \geqslant 9$, then $\pi_{k+7}S^k$ is the stable group of order 240 generated by $S^{k-8}\sigma$, so

$$\beta = dS^{k-8}\sigma \quad \text{for some } d \in Z/240Z,$$

$H(\beta) = 0$, and (19) reduces to

$$(d^2 - d)(S^{k-8}\sigma \circ S^{k-1}\sigma) = 0. \tag{20}$$

The order of $S^{k-8}\sigma \circ S^{k-1}\sigma$ is a power of 2 decreasing from 16 for $k = 9$ to 2 for $k \geqslant 16$. So for $k \geqslant 16$, (20) is true for any d, and so all possible values of β occur.

3. Proof of Theorem 2.3. I begin with some general remarks. A relative CW complex is a pair (X, A) such that X is obtained from A by attaching cells in successive stages of increasing dimension. The n-skeleton X^n is the union of A and all of the cells of X of dimension at most n. The cells are to be regarded as part of the structure of the CW complex. The attaching maps are not to be so regarded. Suppose, however, that A is simply-connected, and that $X^1 = A$. If e^n is an n-cell of (X, A), there are essentially two choices of attaching maps in $\pi_{n-1}(X^{n-1})$ for the cell e^n. An orientation of the relative CW complex (X, A) is then a choice of attaching maps for the cells. An orientation determines, for each cell e^n of (X, A),

(a) an attaching map $S^{n-1} \to X^{n-1}$,

(b) a characteristic map $(D^n, S^{n-1}) \to (X^n, X^{n-1})$,

(c) the corresponding generator of the free abelian group $H_n(X^n, X^{n-1})$.

The homology groups $\{H_n(X^n, X^{n-1})\}$ together with the boundary maps for the various triples form the cellular chain complex for the relative CW complex (X, A), denoted $C_*(X, A)$. An orientation of (X, A) chooses a basis for this free abelian chain complex.

Suppose that the attaching maps of the cells of (X, A) are all homologically trivial. Then the boundary operators in $C_*(X, A)$ are trivial, and

$$C_*(X, A) \approx H_*(X, A). \tag{21}$$

Given such an oriented relative CW complex (X, A), I will say that a homology class $u \in H_n(X, A)$ is carried by the cell e^n, in symbols $u = [e^n]$, if under the isomorphism (21), the generator of $C_n(X, A)$ assigned by the orientation to e^n in (c) above maps to u.

Theorem A of [5] asserts the existence of cell structures on a space compatible with its homology. The proof given in [5] is easily adapted to yield a relative version. The following is then a useful special case.

PROPOSITION 3.1. *Let $B \xrightarrow{g} Y$ be a map of simply-connected spaces such that the homology groups $H_k(f)$ are free abelian on elements $\{y_{k,i}\}$. Suppose that f:*

$A \to B$ is a homotopy equivalence. Then one can attach cells $\{e_i^k\}$ to A, obtaining an oriented relative CW *complex (X, A) and a map*

$$h\colon \begin{array}{ccc} A & \to & B \\ \cap\!\shortmid & & \downarrow g \\ X & \to & Y \end{array}$$

which is the given homotopy equivalence f on A and induces relative homology isomorphisms. The attaching maps for the cells e_i^k of (X, A) are homologically trivial and for each e_i^k, the homology class $[e_i^k]$ carried by e_i^k satisfies

$$h_*[e_i^k] = y_{k,i}.$$

I now proceed with the proof of Theorem 2.3. Given $K = S^m \cup_\alpha e^n$ and a tractable thickening of K in codimension $k + 1$:

$$(W, \partial W), \quad f\colon K \to W, \quad r\colon W \to \partial W.$$

Consider the diagram

$$\begin{tikzcd} \partial W \arrow[r, "i"] & W \arrow[l, bend right, "r"'] \end{tikzcd} \tag{21'}$$

where $i\colon \partial W \to W$ denotes the inclusion and $i \circ r \simeq 1_W$. The idea is to construct a cell structure on the diagram (21′) with as few cells as possible, and then to analyze the possible attaching maps.

The simplest way to proceed is as follows. The equivalence $f\colon K \to W$ defines a cell structure on W. I will use the map $r\colon W \to \partial W$ and Proposition 3.1 to extend this cell structure to one on ∂W. Thus I am regarding ∂W as obtained from W by attaching cells, rather than the other way around. This somewhat confusing state of affairs reduces the numbers of cells involved and makes the computations possible.

The case $n = m + 1$ involves torsion in the homology of K; it is best to treat this case separately, so for the time being, assume $n \geqslant m + 2$.

The complex $K = S^m \cup_\alpha e^n$ is oriented as it is given to us by the selection of the attaching map α. Since $n \geqslant m + 2$ the boundary homomorphism in the cellular chain complex is zero. Therefore the cells of K carry homology classes determined by the given orientation; there are corresponding cohomology classes $x_m \in H^m(S^m \cup_\alpha e^n)$, $x_n \in H^n(S^m \cup_\alpha e^n)$ which will prove more useful. Since $f\colon K \to W$ is an equivalence there are corresponding generators $x'_m \in H^m(W)$, $x'_n \in H^n(W)$. Let $[W] \in H_{n+k+1}(W, \partial W)$ denote the fundamental class of the Poincaré pair $(W, \partial W)$.

Now we can obtain generators for the homology groups of the map r: $W \to \partial W$. The homology sequence of the pair $(W, \partial W)$ is split by r_*:

$$0 \to H_{j+1}(W, \partial W) \xrightarrow{\partial} H_j(\partial W) \underset{r_*}{\overset{i_*}{\rightleftarrows}} H_j(W) \to 0,$$

since $i \circ r \simeq 1_W$. It follows that ∂ induces an isomorphism

$$\partial' \colon H_{j+1}(W, \partial W) \to H_j(r).$$

The cap product

$$[W] \cap \colon H^{n+k-j}(W) \to H_{j+1}(W, \partial W)$$

is an isomorphism. Composing with ∂', it follows that $H_j(r)$ is infinite cyclic generated by

$$\begin{aligned} y'_k &= -\partial'([W] \cap x'_n) \in H_k(r) \quad \text{if } j = k, \\ y'_{k+n-m} &= \partial'([W] \cap x'_m) \in H_{k+n-m}(r) \quad \text{if } j = k + n - m, \\ y'_{n+k} &= \partial'[W] \in H_{k+n}(r) \quad \text{if } j = k + n, \end{aligned}$$

and zero for other values of j. The signs used are chosen to simplify the signs in the final result. Now apply Proposition 3.1 to the map $r\colon W \to \partial W$, the homotopy equivalence $f\colon K \to W$, and the generators y'_k, y'_{k+n-m}, y'_{n+k} of $H_*(r)$. We obtain a complex $K' = K \cup e^k \cup e^{k+n-m} \cup e^{n+k}$ and a homotopy equivalence of maps:

$$\begin{array}{ccc} K & \xrightarrow{f} & W \\ \cap\!\!\mid & & \downarrow r \\ K' & \xrightarrow{f'} & \partial W \end{array} \tag{22}$$

The homology groups $H_*(K', K)$ are free abelian on classes y_k, y_{k+n-m}, y_{n+k} carried by the cells e^k, e^{k+n-m}, e^{n+k} respectively. We have

$$(f', f)_* y_i = y'_i \quad \text{for } i = k, k + n - m, n + k,$$

according to the proposition.

There are cohomology classes $y_k^*, y_{k+n-m}^*, y_{n+k}^* \in H^*(K', K)$ defined by

$$y_i \cap y_i^* = 1 \quad \text{for } i = k, k + n - m, n + k.$$

Since (f', f) is a homotopy equivalence of maps (22), corresponding to the inclusion $i\colon \partial W \subseteq W$ is a map $i'\colon K' \to K$. Since $i \circ r \simeq 1_W$, the map i', when restricted to K, may be assumed to be the identity. This means first that the cohomology sequence of the pair (K', K) is split:

$$0 \to H^*(K', K) \to H^*(K') \underset{\curvearrowleft}{\xrightarrow{(i')^*}} H^*(K) \to 0.$$

We have generators x_m, x_n for $H^*(K)$. The above splitting yields a basis for $H^*(K')$:

$$\begin{aligned} z_j &= y_j^* \text{ restricted to } K' \quad \text{for } j = k, k + n - m, n + k, \\ z_j &= (i')^* x_j \quad \text{for } j = m, n. \end{aligned}$$

These cohomology classes are carried by the various cells of K'. The complex

K', homotopy equivalent to ∂W, satisfies Poincaré duality. In fact it is an easy exercise to check that

$$z_{k+n-m} \cup z_m = z_{n+k}, \qquad z_n \cup z_k = -z_{n+k}. \tag{23}$$

This cup product structure is of use in determining coefficients of certain relative Whitehead products.

A second consequence of the fact that $i'|K \simeq 1_K$ is as follows. I have

$$i' \colon K' = K \cup e^k \cup e^{k+n-m} \cup e^{n+k} \to K$$

and so the attaching map for the k-cell is homotopically trivial; I can write

$$K' = (K \vee S^k) \cup e^{k+n-m} \cup e^{n+k}. \tag{24}$$

I have now obtained a cell structure on the diagram (21); in other words, I have replaced (21) by the homotopy equivalent diagram

$$K' \overset{i'}{\supseteq} K, \tag{25}$$

where K' has the form (24) and $i'|K$ is the identity. Now let C_r denote the mapping cone of r. The equivalence (f', f) mapping diagram (25) to (21′) induces a homotopy equivalence

$$K'/K \xrightarrow{f''} C_r.$$

Since K' has the form (24), the equivalence f'' gives a cell structure for C_r of the form

$$S^k \cup_\beta e^{k+n-m} \cup e^{n+k} = K'/K. \tag{26}$$

We have chosen generators y_k, y_{k+n-m}, y_{n+k} for $H_*(K', K) \approx \tilde{H}_*(K'/K)$. These determine an orientation for the CW complex K'/K and hence a choice for the attaching map $\beta \in \pi_{k+n-m-1}S^k$. The theorem I am proving puts conditions on the homotopy class β, given the tractable thickening I started with. The following lemma is an important step in the proof; it asserts that I can refine the cell structure (25) slightly.

LEMMA 3.2. *Suppose that $K = S^m \cup_\alpha e^n$ tractably thickens in codimension $k+1$. Suppose that $m \geqslant 2$, $k \geqslant 2$, and that the mapping cone of r: $W \to \partial W$ has the form* (26). *Then there exists a cell structure for the diagram*

$$\partial W \overset{r}{\subseteq} W$$

of the form (25) *where K' has the following structure*:

$$K' = (S^m \vee S^k) \cup_{i_1 \circ \alpha} e^n \cup_{i_2 \circ \beta + \psi} e^{k+n-m} \cup e^{n+k}, \qquad \psi \text{ in the cross-term.}$$

Furthermore, the map i': $K' \to K$ is a retraction, carries the subcomplex

$$(S^m \vee S^k) \cup_{i_2 \circ \beta + \psi} e^{k+n-m}$$

into the sphere $S^m \subseteq K$, and pinches S^k to a point.

The proof of Lemma 3.2 will be given later in this section. For the time being I assume the lemma and continue the proof of Theorem 2.3.

First assume that there exists a tractable thickening of $K = S^m \cup_\alpha e^n$ in codimension $k + 1$. Let K' be the complex given by the lemma above. Label some subcomplexes of K' as follows:

$$V = (S^m \vee S^k) \cup_{i_1 \circ \alpha} e^n \cup_{i_2 \circ \beta + \psi} e^{k+n-m},$$

$$X = (S^m \vee S^k) \cup_{i_1 \circ \alpha} e^n,$$

$$Y = (S^m \vee S^k) \cup_{i_2 \circ \beta + \psi} e^{k+n-m}.$$

This describes a triad of subcomplexes of K':

$$\begin{array}{ccc} S^m \vee S^k & \subseteq & X \\ \cap\!| & & \cap\!| \\ Y & \subseteq & V \end{array}$$

The attaching map of the top cell of K' is an element ζ of $\pi_{n+k-1}(V)$. Let j: $V \subseteq (V, S^m \vee S^k)$ denote the inclusion. I focus attention upon the image $j_*(\zeta) \in \pi_{n+k-1}(V, S^m \vee S^k)$.

Consider the diagram of inclusion maps:

$$\begin{tikzcd} (X, S^m \vee S^k) \arrow[d] \arrow[dr, "i_x"] & & (Y, S^m \vee S^k) \arrow[dl, "i_y"'] \arrow[d] \\ (V, Y) & (V, S^m \vee S^k) \arrow[l, "j_y"] \arrow[r, "j_x"'] & (V, X) \end{tikzcd} \tag{27}$$

The vertical maps are excisions and so induce isomorphisms of homology groups. By the Blakers-Massey theorem [2] they induce isomorphisms of relative homotopy groups in degrees less than $2n + k - m - 2$. Since $n \geqslant m + 2$, $2n + k - m - 2 > n + k - 1$, the degree of $j_*(\zeta)$. The diagonal sequences are two of the inclusions for the exact sequence of homotopy groups of a triple. The butterfly lemma [6, p. 32], applied to homotopy groups in degree $n + k - 1$, states that

$$\pi_{n+k-1}(V, S^m \vee S^k) \approx \pi_{n+k-1}(X, S^m \vee S^k) \oplus \pi_{n+k-1}(Y, S^m \vee S^k). \tag{28}$$

This isomorphism is induced by the inclusions i_X and i_Y of diagram (27). Using (28), express $j_*(\zeta)$ as a sum

$$j_*\zeta = (i_X)_*\zeta_1 + (i_Y)_*\zeta_2. \tag{29}$$

For the following lemma, let α': $(D^n, S^{n-1}) \to (X, S^m \vee S^k)$ denote a

characteristic map for the n-cell of X, and let i_2: $S^k \subseteq S^m \vee S^k$ denote the inclusion.

LEMMA 3.3. *The element* $\zeta_1 \in \pi_{n+k-1}(X, S^m \vee S^k)$ *vanishes under the restriction of* i': $K' \to K$ *to the pair* $(X, S^m \vee S^k)$ *and therefore satisfies* $\zeta_1 = r[\alpha', i_2]$ *for some* r *in* Z. *Here* $[\alpha', i_2]$ *denotes the relative Whitehead product defined in* [4].

PROOF OF LEMMA 3.3. Subject (29) to the inclusion j_Y: $(V, S^m \vee S^k) \to (V, Y)$. Since $(j_Y)_*(i_Y)_* = 0$ the result is

$$(j_Y)_* j_* \zeta = (j_Y)_*(i_X)_* \zeta_1. \tag{30}$$

By Lemma 3.2, i': $K' \to K$ carries Y into $S^m \subseteq K$ and so defines a map of pairs

$$i'\colon (K', Y) \to (K, S^m).$$

I claim that

$$(i')_*(j_Y)_* j_* \zeta = 0. \tag{31}$$

To prove (31), consider the diagram:

$$\begin{array}{ccc} V & \xrightarrow{j} & (V, S^m \vee S^k) \\ i'\downarrow & & \downarrow i' j_Y j \\ K & \to & (K, S^m) \end{array} \tag{32}$$

Since i': $V \to K$ extends over $K' = V \cup_\zeta e^{n+k}$, $i'_*(\zeta) = 0$ in $\pi_{n+k-1}(K)$. Commutativity of (32) then implies (31). From (31) and (30) it follows that ζ_1 vanishes under the composition $i' \circ j_Y \circ i_X$: $(X, S^m \vee S^k) \to (K, S^m)$. This composition is a fancy way of writing $i'|(X, S^m \vee S^k)$. By definition, $X = K \vee S^m$, and by Lemma 3.2, i' maps K by the identity and pinches S^k to a point. Thus

$$i'\colon (X, S^m \vee S^k) \to (K, S^m)$$

induces isomorphisms of homology groups. A refined Blakers-Massey theorem [3] asserts that i'_* is an epimorphism in degree $n + k - 1$ with kernel generated by the relative Whitehead product $[\alpha', i_2]$. Here α': $(D^n, S^{n-1}) \to (X, S^m \vee S^k)$ denotes a characteristic map for the n-cell of X. Since ζ_1 vanishes under $(i'|(X, S^m \vee S^k))_*$, I have

$$\zeta_1 = r[\alpha', i_2] \quad \text{for some } r \in Z. \quad \square$$

Referring to (29), note that the element ζ_2 lies in the group $\pi_{n+k-1}(Y, S^m \vee S^k)$. This latter group can be explicitly computed provided $n \geqslant 2m$. Recall the cell structure of Y:

$$Y = (S^m \vee S^k) \cup_{i_2 \circ \beta + \psi} e^{k+n-m}.$$

Let $\delta = i_2 \circ \beta + \psi$ denote the attaching map for e^{k+n-m} and let δ': $(D^{k+n-m}, S^{k+n-m-1}) \to (Y, S^m \vee S^k)$ be the characteristic for the cell e^{k+n-m} of Y. Define iterated relative Whitehead products of δ' and i_2: $S^k \subseteq S^m \vee S^k$ by the formulas

$$W_0(\delta', i_2) = \delta', \qquad W_{j+1}(\delta', i_2) = [W_j(\delta', i_2), i_2].$$

PROPOSITION 3.4. *Suppose $n \geqslant 2m$. The group $\pi_{n+k-1}(Y, S^m \vee S^k)$ splits as a direct sum*

$$\pi_{n+k-1} \approx Z \oplus \bigoplus_{j=0}^{\infty} \pi_{n+k-1}(D^{k+n-m+j(k-1)}, S^{k+n-m+j(k-1)-1}).$$

Elements of $\pi_{n+k-1}(Y, S^m \vee S^k)$ have a unique representation in the form

$$s[\delta', i_1] + \sum_{j=0}^{\infty} W_j(\delta', i_2) \circ \gamma_j' \tag{33}$$

where $s \in Z$ and $\gamma_j' \in \pi_{n+k-1}(D^{k+n-m+j(k-1)}, S^{k+n-m+j(k-1)-1})$.

The proof of Proposition 3.4 is given in §4. The point is that the well-known operations of relative Whitehead product and composition suffice to compute the needed group in a certain range.

I can now complete the proof of Theorem 2.3. The element $j_*\zeta$ is expressed in (29) as a sum of two elements. The first of these is described in Lemma 3.3. Assume $n \geqslant 2m$. The second element ζ_2 is represented in the form (33) according to Proposition 3.4. Therefore

$$j_*\zeta = r[\alpha', i_2] + s[\delta', i_1] + \sum_{j=0}^{\infty} W_j(\delta', i_2) \circ \gamma_j', \tag{34}$$

where r, s are integers, and γ_j' is as in 3.4. The complex $K' = (S^m \vee S^k) \cup_{i_1 \circ \alpha} e^n \cup_\delta e^{k+n-m} \cup_\zeta e^{n+k}$ satisfies Poincaré duality. This implies that cohomology classes carried by the various cells of K' are dual under cup product. These cohomology classes have previously been denoted by z_m, z_n, z_k, z_{k+n-m}, and equations (23) express the duality. Just as in [5, p. 152], one can now determine the coefficients r and s, knowing the cup products are given by (23). In the present case I obtain $-r = s = 1$.

Now apply ∂: $\pi_{n+k-1}(V, S^m \vee S^k) \to \pi_{n+k-2}(S^m \vee S^k)$ to both sides of (34). The left side vanishes. The relative Whitehead product satisfies $\partial[x, y] = -[\partial x, y]$. The result is therefore

$$0 = [i_1 \circ \alpha, i_2] - [\delta, i_1] + \sum_{j=0}^{\infty} (-1)^j W_j(\delta, i_2) \circ \gamma_j, \tag{35}$$

since $\partial\delta' = \delta$, $\partial\alpha' = \alpha$; here $\partial\gamma_j' = \gamma_j$. This proves (8) of Theorem 2.3, assuming $n \geqslant 2m$.

Next assume $n < 2m$. Then $k + n - m < k + m$. So $\psi = 0$, and K' has

the form

$$(S^m \cup_\alpha e^n) \vee \left(S^k \cup_\beta e^{k+n-m}\right) \cup_\zeta e^{n+k}. \tag{36}$$

Write $L = S^k \cup_\beta e^{k+n-m}$, so that

$$K' = (K \vee L) \cup_\zeta e^{n+k}$$

where $\zeta \in \pi_{n+k-1}(K \vee L)$. Now

$$\pi_{n+k-1}(K \vee L) \approx \pi_{n+k-1}(K) \oplus \pi_{n+k-1}(L) \oplus \text{cross-term}$$

so $\zeta = \zeta_K + \zeta_L + \zeta'$ for some ζ' in the cross-term, where ζ_K and ζ_L are the images of ζ under the projections $K \vee L \to K$ and $K \vee L \to L$. Consider the complex

$$K'' = (K \vee L) \cup_{\zeta'} e^{n+k}.$$

It is easy to see that K'' satisfies Poincaré duality. Since ζ' projects to zero in K and L there are maps

$$\begin{array}{ccc} K'' & \xrightarrow{h_1} & L \\ {\scriptstyle h_2}\downarrow & & \\ K & & \end{array} \tag{37}$$

such that h_1 maps K to the basepoint in L and h_2 maps L to the basepoint in K.

Now write

$$V = K \vee L, \quad X = K \vee S^k, \quad Y = S^m \vee L.$$

The resulting triad

$$\begin{array}{ccc} S^m \vee S^k & \subseteq & X \\ \cap\!\shortmid & & \cap\!\shortmid \\ Y & \subseteq & V \end{array}$$

is now studied as in the discussion following Lemma 3.2. The attaching map ζ' lives in $\pi_{n+k-1}(V)$. Letting $j\colon V \subseteq (V, S^m \vee S^k)$ denote the inclusion, I consider the element

$$j_*\zeta \in \pi_{n+k-1}(V, S^m \vee S^k).$$

This element splits as a sum (29). Because of the maps (37), the argument of Lemma 3.3 applies to ζ_1 and ζ_2. Therefore

$$j_*\zeta = r[\alpha', i_2] + s[\beta', i_1] \tag{38}$$

for some integers r and s. Duality implies that $-r = s = 1$ as before. So (38) leads to equation (8) of Theorem 2.3 with ψ and the γ_j's equal to zero. This completes the proof of half of Theorem 2.3.

On the other hand, suppose that $K = S^m \cup_\alpha e^n$ is given, that $\beta \in \pi_{k+n-m-1}(S^k)$ is given, and that there exist elements ψ and $\{\gamma_j\}$ as in

Theorem 2.3 such that (8) holds. Take $\delta = i_2 \circ \beta + \psi \in \pi_{k+n-m-1}(S^k \vee S^m)$ and form the complex

$$V = (S^m \vee S^k) \cup_{i_1 \circ \alpha} e^n \cup_\delta e^{k+n-m}.$$

Let δ': $(D^{k+n-m}, S^{k+n-m-1}) \to (V, S^m \vee S^k)$ denote the characteristic map of the $(k+n-m)$-cell of V. Consider the element in $\pi_{n+k-1}(V, S^m \vee S^k)$:

$$\theta = -[\alpha', i_2] + [\delta', i_1] + \sum_{j=0}^{\infty} W_j(\delta', i_2) \circ \gamma_j', \tag{39}$$

where γ_j' denotes the element in $\pi_{n+k-1}(D^{k+n-m+j(k-1)}, S^{k+n-m+j(k-1)-1})$ whose boundary is γ_j. The boundary of θ in (39) is zero, by my assumption that (8) holds. Therefore there exists $\zeta \in \pi_{n+k-1}(V)$ such that $j_*(\zeta) = \theta$. To construct a thickening of K of the desired form, start with the complex

$$K' = V \cup_\zeta e^{n+k}$$

Remember that

$$V = (S^m \vee S^k) \cup_{i_1 \circ \alpha} e^n \cup_{i_2 \circ \beta + \psi} e^{k+n-m}.$$

The projection $S^m \vee S^k \to S^m$ extends over e^{k+n-m} since i_2 vanishes and ψ is in the cross-term. Therefore there is a map $V \xrightarrow{g} K$ which is the identity on K, pinches S^k to a point and carries $S^m \vee S^k \cup e^{k+n-m}$ into S^m. I ask, does g extend over $K' = V \cup_\zeta e^{n+k}$? Or equivalently, is $g_*\zeta = 0$? Consider the diagram below:

$$\begin{array}{ccccc}
\pi_{n+k-1}(S^m \vee S^k) & \to & \pi_{n+k-1}(V) & \xrightarrow{j_*} & \pi_{n+k-1}(V, S^m \vee S^k) \\
\downarrow g_* & & \downarrow g_* & & \downarrow g_* \\
\pi_{n+k-1}(S^m) & \to & \pi_{n+k-1}(K) & \xrightarrow{(j')_*} & \pi_{n+k-1}(K, S^m)
\end{array}$$

We have $j_*\zeta = \theta$ (see (39)). The relative map g: $(V, S^m \vee S^k) \to (K, S^m)$ carries e^{k+n-m} into S^m and so $g_*\delta' = 0$. Since g pinches S^k to a point, $g_*[\alpha', i_2] = 0$. Therefore $g_*\theta = g_*j_*\zeta = 0$. By commutativity of the above diagram, $(j')_*g_*\zeta = 0$. It follows by diagram chase that I can adjust ζ by an element in the kernel of j_* obtaining a new attaching map ζ' which does vanish under g_*: $\pi_{n+k-1}(V) \to \pi_{n+k-1}(K)$. Let $K'' = V \cup_{\zeta'} e^{n+k}$. Then $j_*\zeta' = \theta$, the coefficients of $[\alpha', i_1]$ and $[\delta', i_2]$ are ± 1 and so K'' satisfies Poincaré duality. The map g: $V \to K$ extends to a map G: $K'' \to K$. Let Z_G denote the mapping cylinder of G. It is easy to check that

(a) (Z_G, K'') satisfies relative Poincaré duality with fundamental class of degree $n + k + 1$.

(b) if r: $Z_G \to K''$ denotes the restriction of Z_G onto K followed by

inclusion into K'', then, with r, (Z_G, K'') is a tractable thickening of K in codimension $k+1$, and

(c) the mapping cone of r is homotopy equivalent to $K''/K = S^k \cup_\beta e^{k+n-m} \cup e^{n+k}$. This completes the proof of Theorem 2.3 except for the case $n = m+1$ which was put aside at the start.

REMARK ON THE PROOF. The key idea in the proof is to study the element $j_*\zeta \in \pi_{n+k-1}(V, S^m \vee S^k)$. Equation (29) leads to uniquely determined elements $\zeta_1 \in \pi_{n+k-1}(X, S^m \vee S^k)$ and $\zeta_2 \in \pi_{n+k-1}(Y, S^m \vee S^k)$. The relative Whitehead products $[\alpha', i_2]$ and $[\delta', i_1]$ are situated in these groups, and one expects right away that, because of the cup product structure given in (23),

$$\zeta_1 = \pm[\alpha', i_2] + \text{error}, \qquad \zeta_2 = \pm[\delta', i_1] + \text{error},$$

where the errors are cohomologically trivial. Since $\partial j_*\zeta = 0$, this leads to something like (8):

$$\pm[i_1 \circ \alpha, i_2] = \pm[\delta, i_1] + \text{error}.$$

Presumably the error here involves "higher order terms" than $[i_1, i_2]$, so, expanding both sides and looking at the coordinates of $[i_1, i_2]$, one expects to derive equation (12) of Theorem 2.4 and therefore Theorem 1.1. The problem is to get one's hands on the error terms. Lemma 3.3 deals with ζ_1 in a rather simple fashion. The real difficulty lies in computing ζ_2. It should be possible to prove enough about the error for ζ_2 to derive Theorem 1.1 without the technical result Proposition 3.4. But I use the latter because then I can prove the more general result, Theorem 2.3, giving necessary and sufficient conditions for the existence of a tractable thickening that involve known operations in the homotopy groups of spheres.

The case $n = m+1$. $K = S^m \cup_\alpha e^{m+1}$ and the attaching map $\alpha \in \pi_m S^m$ satisfies $\alpha = d\iota_m$ for some integer d. Consider equation (8) of Theorem 2.3. The element $\psi \in \pi_k(S^m \vee S^k)$ must be zero for degree reasons since it is in the cross-term. A consequence of (8) is equation (12) of Theorem 2.4, and, since $\psi_0 = 0$, (12) reduces to

$$S^{k-1}\alpha = (-1)^{mk} S^{m-1}\beta. \tag{40}$$

On the other hand, if we set ψ and all the γ_j's equal to zero, equation (40) implies

$$[i_1 \circ \alpha, i_2] = [i_2 \circ \beta, i_1]$$

and this is equation (8). Therefore there exist elements ψ, $\{\gamma_j\}$ satisfying (8) if and only if (40) is true. Now (40) just says $\beta = (-1)^{mk}\, d\iota_k$. Therefore there exist ψ and $\{\gamma_j\}$ satisfying (8) if and only if $\beta = (-1)^{mk} d\iota_k$.

Next I claim that K thickens in codimension 2. To see this, take $L = S^1$

$\cup_d e^2$, the pseudo projective plane. Then L embeds in R^4. The complex K, which is $S^{m-1}L$, embeds in $R^{m-1+4} = R^{m+3}$ and so thickens in codimension 2. Therefore K tractably thickens in every codimension greater than 2. It is easy to see by computing homology groups that if K tractably thickens in codimension $k + 1$ then the mapping cone of r: $W \to \partial W$ has the form

$$S^k \cup_\beta e^{k+1} \cup e^{k+m+1} \tag{41}$$

where $\beta = (-1)^{mk} d\iota_k$. Therefore K tractably thickens in codimension $k + 1$ with the mapping cone of r: $W \to \partial W$ having the form (41) if and only if $\beta = (-1)^{mk} d\iota_k$. This together with the previous paragraph proves Theorem 2.3 in the case $n = m + 1$.

PROOF OF LEMMA 3.2. I start from diagram (25) which gives a cell structure for diagram (21) for which K' has the form

$$K' = (K \vee S^k) \cup e^{k+n-m} \cup e^{n+k} \tag{24}$$

and $i'|K$ is the identity.

Step 1. I claim we can assume that $i'|S^k$ is null-homotopic. To prove this, suppose that $i'|S^k$ represents $\theta \in \pi_k(K)$. We will alter the cell structure on K' as follows: construct a homotopy equivalence

$$K \vee S^k \xrightarrow{h} K \vee S^k$$

satisfying $h|K = i_K$, $h|S^k = i_2 - i_K \circ \theta$, where i_K: $K \subseteq K \vee S^k$ and i_2: $S^k \subseteq K \vee S^k$ are the inclusions. Corresponding to the cells e^{k+n-m} and e^{n+k} of K', I attach cells to obtain a complex K'' and a homotopy equivalence

$$K'' = (K \vee S^k) \cup e^{k+n-m} \cup e^{n+k} \xrightarrow{h'} K',$$

which on $K \vee S^k$ is the map h. The map i': $K' \to K$ corresponds to a map $i'' = i' \circ h'$: $K'' \to K$, and now $i''|S^k = i' \circ h'|S^k = i' \circ (\iota_k - \theta) = \theta - \theta = 0$, as desired.

Assuming that $i'|S^k$ is null-homotopic, let γ in $\pi_{k+n-m-1}(K \vee S^k)$ denote the attaching map for the $(k + n - m)$-cell of K' (24). Let j: $S^m \vee S^k \subseteq K \vee S^k$ denote the inclusion. Remember that i_1: $S^m \subseteq S^m \vee S^k$ and i_2: $S^k \subseteq S^m \vee S^k$ are the inclusions.

Step 2. I claim there exists ψ in the cross-term of $\pi_{k+n-m-1}(S^m \vee S^k)$ such that $\gamma = j_*(i_2 \circ \beta + \psi)$. Here β occurs in the hypotheses of the present lemma as an attaching map for the mapping cone of r: $W \to \partial W$, as in (26) above.

PROOF OF STEP 2. Consider the diagram below:

$$\begin{array}{ccccc}
\pi_{k+n-m}(K \vee S^k, S^m \vee S^k) & \xrightarrow{\partial} & \pi_{k+n-m-1}(S^m \vee S^k) & \xrightarrow{j_*} & \\
i'_*\downarrow & & i'_*\downarrow & & \\
\pi_{k+n-m}(K, S^m) & \xrightarrow{\partial} & \pi_{k+n-m-1}(S^m) & \rightarrow & \\
 & & \pi_{k+n-m-1}(K \vee S^k) & \rightarrow & \pi_{k+n-m-1}(K \vee S^k, S^m \vee S^k) \\
 & & i'_*\downarrow & & i'_*\downarrow \\
 & & \pi_{k+n-m-1}(K) & \rightarrow & \pi_{k+n-m-1}(K, S^m)
\end{array}$$

The map i': $K \vee S^k \to K$ is defined as the restriction of i': $K' \to K$. Since i': $K \vee S^k \to K$ extends over the cell e^{k+n-m} whose attaching map is $\gamma \in \pi_{k+n-m-1}(K \vee S^k)$, $i'_*\gamma = 0$, and so γ must vanish in $\pi_{k+n-m-1}(K, S^m)$ in the above diagram. The map $(K \vee S^k, S^m \vee S^k) \to (K, S^m)$ induces isomorphisms of homology groups and so by the Blakers-Massey theorem [2] it induces isomorphisms of relative homotopy groups in degrees less than $n + k - 1$. Since $m \geqslant 2$, the vertical maps at the ends of the above diagram are isomorphisms. In particular, γ maps to zero in $\pi_{k+n-m-1}(K \vee S^k, S^m \vee S^k)$, and so lies in $\operatorname{Im} j_*$.

It follows by diagram chase that there exists δ in $\pi_{k+n-m-1}(S^m \vee S^k)$ mapping to γ and satisfying $i'_*(\delta) = 0$, where i'_*: $\pi_{k+n-m-1}S^m \vee S^k \to \pi_{k+n-m-1}S^m$ is the homomorphism induced by i'. Since i' pinches S^k to a point and maps S^m by the identity, the first coordinate of δ in Hilton's representation (9) must be zero. It follows that

$$\delta = i_2 \circ \beta' + \psi \quad \text{for some elements } \beta' \in \pi_{k+n-m-1}S^k$$

and ψ in the cross-term of $\pi_{k+n-m-1}(S^m \vee S^k)$.

I complete the proof of Step 2 by showing that $\beta' = \beta$. Now β is defined to be the attaching map of the cell e^{k+n-m} in K'/K. The projection map $K' \to K'/K$ restricts to a map

$$(S^m \vee S^k) \cup_\delta e^{k+n-m} \to S^k \cup_\beta e^{k+n-m}$$

which simply pinches S^m to a point and is of degree one on the $(k + n - m)$-cells. It follows that δ maps to β under the pinch map $S^m \vee S^k \xrightarrow{p_2} S^k$. The cross-term vanishes under p_2, and $p_2 \circ i_2 = 1_{S^k}$, so

$$\beta = (p_2)_*\delta = (p_2)_*(i_2 \circ \beta' + \psi) = \beta'.$$

This completes Step 2.

Step 3. At this point we have the diagram

$$K' \overset{i'}{\supseteq} K, \tag{42}$$

K' is of the form

$$(S^m \vee S^k) \cup_{i_1 \circ \alpha} e^n \cup_\delta e^{k+n-m} \cup e^{n+k}, \qquad \delta = i_2 \circ \beta + \psi.$$

i': $K' \to K$ is a retraction onto the subcomplex K, and i' pinches S^k to a point. In this step I alter the cell structure on K' if necessary to ensure that the map i' carries the subcomplex $(S^m \vee S^k) \cup_\delta e^{k+n-m}$ into the sphere $S^m \subseteq K$. Let

$$Y = (S^m \vee S^k) \cup_\delta e^{k+n-m}, \qquad V = Y \cup_{i_1 \circ \alpha} e^n,$$

so I have inclusions $Y \subseteq V \subseteq K'$.

The map i' defines a map of pairs

$$(i'|V)\colon (V, S^m \vee S^k) \to (K, S^m).$$

Let $\delta' \in \pi_{k+n-m}(V, S^m \vee S^k)$ denote the homotopy class of the characteristic map for the $(n + k - m)$-cell of V. To ensure that i' is homotopic rel $K \vee S^k$ to a map carrying Y into S^m it is sufficient (in fact it is necessary too) to show that

$$(i'|V)_* \delta' = 0 \quad \text{in } \pi_{k+n-m}(K, S^m). \tag{43}$$

Since i' pinches S^k to a point it maps $i_2 \circ \beta$ and ψ to zero in S^m and so $(i'|S^m \vee S^k)_* \delta = 0$. Therefore

$$\partial\big((i'|V)_* \delta'\big) = (i'|S^m \vee S^k)_* \partial\delta' = (i'|S^m \vee S^k)_* \delta = 0.$$

Thus if j': $K \to (K, S^m)$ denotes the inclusion, $(i'|V)_* \delta'$ is in $\operatorname{Im} j'_*$. Choose $\theta \in \pi_{k+n-m}(K)$ such that

$$j'_*(\theta) = (i'|V)_* \delta'. \tag{44}$$

I show how to alter the given cell structure using this element θ so as to annihilate $(i'|V)_* \delta'$. I define a homotopy equivalence h: $V \to V$ as follows. The cell e^{k+n-m} of V is a top cell in the sense that no later cell is attached by a map meeting the interior of e^{k+n-m}. Pinching the boundary of a small ball B^{k+n-m} in the interior of e^{k+n-m} to a point defines a map

$$q\colon V \to V \vee S^{k+n-m}.$$

Define, for any $\gamma \in \pi_{k+n-m}(K)$, $h(\gamma)$ to be the composition

$$h(\gamma)\colon V \xrightarrow{q} V \vee S^{k+n-m} \xrightarrow{1 \vee i_K(\gamma)} V;$$

the second map is the identity on V and has the homotopy class $\gamma \in \pi_{k+n-m}(K)$ followed by the inclusion i_K: $K \subseteq V$ on the sphere S^{k+n-m}. If γ' is another element of $\pi_{k+n-m}(K)$,

$$h(\gamma + \gamma') = h(\gamma) \circ h(\gamma'),$$

so by taking $\gamma' = -\gamma$ and observing that $h(0) \simeq 1_V$, we see that $h(\gamma)$ is a

homotopy equivalence. ($h(\gamma)$ is defined by the action of the group $\pi_{k+n-m}(K)$ on the homotopy classes of maps $[V, V]$ which is obtained by taking the inclusion $(i_K)_*$: $\pi_{k+n-m}(K) \to \pi_{k+n-m}(V)$ and applying the usual Puppe sequence action of $\pi_{k+n-m}(V)$ on $[V, V]$.)

In the case at hand, take $h = h(-\theta)$, where θ is defined by (43). Note that h is then the identity map on the subcomplex $K \vee S^k$ of V. I have $K' = V \cup e^{n+k}$, so utilize the homotopy equivalence h: $V \to V$ to attach a corresponding cell e_1^{n+k} to V to obtain a new complex $K'' = V \cup e_1^{n+k}$ and a homotopy equivalence h': $K'' \to K'$ which restricts to h on the subcomplex V. Define a map i'': $K'' \to K$ by $i'' = i' \circ h'$ and consider the diagram

$$K'' \supseteq K \quad (i'')$$

Since h, and therefore h', maps the subcomplex $K \vee S^k$ of K'' by the identity map into K', the following properties of the new diagram are implied by the corresponding properties of the complex K' and the map i' stated at the beginning of this step:

(a) $i''|K$ is the identity.

(b) i'' pinches S^k to a point.

(c) $K'' = (S^m \vee S^k) \cup_{i_1 \circ \alpha} e^n \cup_\delta e^{k+n-m} \cup e_1^{n+k} = V \cup e_1^{n+k}$.

The map i'': $K'' \to K$ defines by restriction a map of pairs

$$i''|V\colon (V, S^m \vee S^k) \to (K, S^m).$$

As before, let $\delta' \in \pi_{k+n-m}(V, S^m \vee S^k)$ denote the characteristic map for the cell e^{k+n-m} of V. Recalling the discussion preceding (43), I can complete Step 3 by showing that in the new cell structure

$$(i''|V)_*(\delta') = 0.$$

Consider the diagram

$$\begin{tikzcd}
(V, S^m \vee S^k) \arrow[r, "q"] \arrow[drr, "i''|V"'] & (V \vee S^{k+n-m}, S^m \vee S^k) \arrow[r, "1 \vee i'_K(-\theta)"] & (V, S^m \vee S^k) \arrow[d, "i'|V"] \\
 & & (K, S^m)
\end{tikzcd}$$

The top row is the homotopy equivalence h relativized; i'_K: $K \subseteq (V, S^m \vee S^k)$ is the inclusion. The diagram is commutative because $i'' = i' \circ h'$ and $h'|V = h$. To show that $(i''|V)_*\delta' = 0$, compute what happens to δ' in the diagram. First

$$q_*(\delta') = \delta' + i_3$$

where I write δ' for its image in $(V \vee S^{k+n-m}, S^m \vee S^k)$ under inclusion and i_3: $S^{k+n-m} \to (V \vee S^{k+n-m}, S^m \vee S^k)$ for the inclusion. Now apply

$(1 \vee i'_K(-\theta))_*$; the result is

$$\delta' + (i'_K(-\theta))_*(i_2) = \delta' - (i'_K)_*\theta.$$

Finally apply $(i'|V)_*$, obtaining

$$\begin{aligned}(i''|V)_*\delta' &= (i'|V)_*\delta' - (i'|V)_*(i'_K)_*\theta \\ &= j'_*(\theta) - (i'|V)_*(i'_K)_*\theta \quad \text{by (44)} \\ &= 0 \quad \text{since } j' = (i'|V) \circ i'_K.\end{aligned}$$

This completes the proof of Step 3 and the lemma.

4. The reduced product space of a pair. Proof of Proposition 3.4. A fundamental problem in homotopy theory concerns the classification of finite CW complexes up to homotopy type. Since any CW complex is built up from vertices by attaching cells, and since all that matters, in so far as the homotopy type of the result is concerned, is the homotopy class of each attaching map, one method of attack is to compute all homotopy groups of all CW complexes. For the finite CW complexes, this reduces to the question: Suppose the homotopy groups of a space X are known. What happens when an element $\alpha \in \pi_{n-1}(X)$ is chosen and a single cell is attached to X by a map in the homotopy class of α? The resulting space is denoted by $X \cup_\alpha e^n$, and the relationship between the homotopy groups of X and those of $X \cup_\alpha e^n$ is tied by the exact sequence of homotopy groups of the pair $(X \cup_\alpha e^n, X)$ to the relative groups

$$\pi_i(X \cup_\alpha e^n, X). \tag{1}$$

For $i < n$, the groups (1) are zero. The first nonzero group, $\pi_n(X \cup_\alpha e^n, X)$ was studied by J. H. C. Whitehead [18]. He showed that the group in question is generated (over $\pi_1(X)$) by the characteristic map

$$\alpha' \colon (D^n, S^{n-1}) \to (X \cup_\alpha e^n, X)$$

for the n-cell. It follows that

$$\pi_i(X \cup_\alpha e^n) = \begin{cases} \pi_i(X), & i < n-1, \\ \pi_{n-1}(X)/\langle\alpha\rangle, & i = n-1, \end{cases}$$

where $\langle\alpha\rangle$ denotes the sub $\pi_1(X)$-module generated by α.

A substantial result on the groups (1) was obtained by Blakers and Massey [2] as a consequence of their theorem on homotopy groups of a triad. The characteristic map α' referred to above defines a square

$$\alpha' \colon \begin{array}{ccc} S^{n-1} & \to & X \\ \cap & & \cap \\ D^n & \to & X \cup_\alpha e^n \end{array}$$

and induces isomorphisms of relative homology groups. Assume that X is $(k-1)$-connected, where $k \geqslant 2$, then the theorem of Blakers and Massey states that

THEOREM 4.1. *The map α' induces isomorphisms of relative homotopy groups in degrees less than $n+k-1$, so that every element of $\pi_i(X \cup_\alpha e^n, X)$ can be uniquely represented as a composition $\alpha' \circ \theta$ for some $\theta \in \pi_i(D^n, S^{n-1})$, assuming that $i < n+k-1$.*

Notice that this theorem is not an improvement of Whitehead's result unless $k > 2$. As far as I know, the groups $\pi_i(X \cup_\alpha e^n, X)$ are not well understood in the case where X is allowed to be non-simply-connected, even for values of i such as $n+1$ and $n+2$.

Blakers and Massey also showed that in the next degree, the relative group is generated by compositions and relative Whitehead products $[\alpha', \theta]$, $\theta \in \pi_k(X)$.

I also wish to mention that Blakers and Massey's techniques allow one to compute relative groups when one attaches two or more cells simultaneously. The discussion in §3 on the relative homotopy groups of the pair $(V, S^m \vee S^k)$ can easily be generalized to show:

THEOREM 4.2. *Let X be 1-connected $\alpha \in \pi_{i-1}(X)$, $\beta \in \pi_{j-1}(X)$; then there is a splitting*

$$\pi_k\big(X \cup_\alpha e^i \cup_\beta e^j, X\big) \approx \pi_k\big(X \cup_\alpha e^i, X\big) \oplus \pi_k\big(X \cup_\beta e^j, X\big)$$

induced by inclusion, valid for $k < i+j-2$.

I now describe a technique that permits one to compute the groups in degrees less than $2n-2$ for many spaces X. This technique will be applied to prove Proposition 3.4.

Let X be a CW complex. In [11], James defined a space X_∞ called the reduced product space of X. The reduced product space is a cellular approximation to ΩSX, and it is filtered by subcomplexes

$$X = X_1 \subseteq X_2 \subseteq \cdots \subseteq X_n \subseteq \cdots \subseteq X_\infty = \bigcup_i X_i$$

such that if X is $(k-1)$-connected the inclusion of X_{n-1} in X_n is $(nk-1)$-connected. James obtained important results concerning the homotopy groups of spheres by means of this construction, for he was able to define maps $(S^n)_\infty \to (S^{2n})_\infty$ that have interesting homotopy theoretic properties and therefore relate $\pi_*(S^{n+1})$ and $\pi_*(S^{2n+1})$.

Recently, Brayton Gray [7], and, independently, the present author (unpublished) defined a relativized version of the James reduced product

space. Given a pair (X, A) of CW complexes with base point, we define a relative reduced product space $(X, A)_\infty$ having the following properties:

(a) $(X, A)_\infty$ is a CW complex. It is a covariant functor of pairs. If $A = X$ $(X, A)_\infty = X_\infty$.

(b) $(X, A)_\infty$ is filtered by subcomplexes $(X, A)_n$ and $(X, A)_\infty = \bigcup_n (X, A)_n$. The nth stage $(X, A)_n$ is an appropriate quotient of

$$\underbrace{A \times \cdots \times A}_{n-1 \text{ factors}} \times X.$$

In particular $(X, A)_1 = X$ and

$$(X, A)_2 = A \times X / (a, *) \sim (*, a).$$

(c) Suppose that A is $(k-1)$-connected and X is $(l-1)$-connected. The inclusion

$$(X, A)_{n-1} \subseteq (X, A)_n$$

is $(l + (n-1)k - 1)$-connected.

(d) Let $X \cup CA$ denote the mapping cone of the inclusion of A in X. There is a natural projection

$$q: X \cup CA \to SA,$$

defined by pinching X to a point. The diagram

$$\begin{array}{ccc} X & \to & * \\ \cap\!\!\!| & & \cap\!\!\!| \\ X \cup CA & \xrightarrow{q} & SA \end{array} \tag{2}$$

is thus commutative. The most important property of the relative reduced product space $(X, A)_\infty$ is that it is homotopy equivalent to the fibre of q. More precisely, convert q into a fibration, and let $F(q)$ denote the fibre. Then there is a natural homotopy equivalence

$$h: (X, A)_\infty \to F(q).$$

There results a map of $(X, A)_\infty \xrightarrow{g} X \cup CA$, and the following diagram is commutative:

$$\begin{array}{ccccc} (X, A)_1 & = & X & \to & * \\ \cap\!\!\!| & & \cap\!\!\!| & & \cap\!\!\!| \\ (X, A)_\infty & \xrightarrow{g} & X \cup CA & \xrightarrow{q} & SA \end{array} \tag{3}$$

Diagram (3) can be regarded as asserting that the map of pairs given by diagram (2) has as relative fibre the pair $((X, A)_\infty, (X, A)_1)$.

(e) The left square of diagram (3) defines a map

$$G: ((X, A)_\infty, (X, A)_1) \to (X \cup CA, X).$$

Restrict G to the pair $((X, A)_2 : (X, A)_1)$:

$$G_2 : ((X, A)_2, (X, A)_1) \to (X \cup CA, X).$$

Now suppose that A and X are suspensions, so that $A = SA'$, $X = SX'$. The product $A \times X$ can be obtained from the wedge $A \vee X$ by attaching a cone on the join $A' * X'$ via the Whitehead product of the inclusions $i_A : A \subseteq A \vee X$ and $i_X : X \subseteq A \vee X$:

$$A \times X \simeq (A \vee X) \cup_{[i_A, i_X]} C(A' * X'). \tag{4}$$

See [21] for a definition of the generalized Whitehead product required for this fact.

Now according to (b), $(X, A)_2$ is the quotient of $A \times X$ obtained by identifying the factor $A \times *$ with the subset $* \times X$ defined by inclusion i: $A \subseteq X$. It follows from this that the "cell" structure in (4) defines a "cell" structure for $(X, A)_2$:

$$(X, A)_2 \simeq X \cup_{[i, 1_X]} C(A' * X'); \tag{5}$$

the cone on $(A' * X')$ is attached by the Whitehead product of the identity and the inclusion i: $A \subseteq X$. Recalling that X is identified with $(X, A)_1$ in (b), it follows from (5) that the map G_2 defined above induces a map

$$(X \cup_{[i, 1_X]} C(A' * X'), X) \xrightarrow{G_2'} (X \cup CA, X) \tag{6}$$

which is the identity on X. The characteristic map of the cone $C(A' * X')$ for the pair on the left in (6) is a map denoted j'

$$j' : (C(A' * X'), A' * X') \to (X \cup_{[i, 1_X]} C(A' * X'), X).$$

The characteristic map of the cone CA for the pair on the right in (6) is a map denoted i':

$$i' : (CA, A) \to (X \cup CA, X).$$

The final result required is that the composition $G_2' \circ j'$ is the relative Whitehead product $[i', 1_X]$:

$$G_2' \circ j \sim [i', 1_X]. \tag{6a}$$

The generalized relative Whitehead product is defined in [20].

The above properties of the relative reduced product construction will be sufficient to prove Proposition 3.4. I would like to remark in passing that the approach used by Gray clarifies property (d). First, $(X, A)_\infty$ is defined for any NDR pair [19]; secondly, Gray constructs a natural map

$$(X, A)_\infty \to X/A \tag{7}$$

which is a quasifibration with fibre A_∞. The map (7) is equivalent to the map

$g\colon (X, A)_\infty \to (X \cup CA)$ used in (d), recalling that

$$X \cup CA \simeq X/A$$

for an NDR pair. Gray's quasifibration

$$A_\infty \to (X, A)_\infty \to X/A$$

fits nicely into the fibration Puppe sequence for the projection $X \cup CA \to^q SA$, according to the following diagram (here $E(q)$ denotes the total space obtained by converting q into a fibration):

$$\begin{array}{ccccccc}
\Omega SA & \to & F(q) & \to & E(q) & \to & SA \\
 & & & & \downarrow\simeq & & \| \\
\uparrow h_A & & \uparrow h & & X \cup CA & \xrightarrow{q} & SA \\
 & & & & \downarrow\simeq & & \\
A_\infty & \to & (X, A)_\infty & \to & X/A & &
\end{array}$$

The vertical map h is defined in (d); the map h_A is the map h for the pair (A, A) and was defined by James in showing that A_∞ is a CW approximation to ΩSA.

I now prepare to apply the above results on the relative reduced product space to prove Proposition 3.4.

Let r be a given integer and let $\gamma\colon S^{r-1} \to S^m \vee S^k$ denote an arbitrary continuous map. Form the complex

$$Y_\gamma = (S^m \vee S^k) \cup_\gamma e^r,$$

and let $\gamma'\colon (D^r, S^{r-1}) \to (Y_\gamma, S^m \vee S^k)$ denote the characteristic map for the r-cell.

If $i_1\colon S^m \to S^m \vee S^k$ and $i_2\colon S^k \to S^m \vee S^k$ are the inclusions, there are spaces

$$Y_{[\gamma,i_1]} = (S^m \vee S^k) \cup_{[\gamma,i_1]} e^{m+r-1},$$
$$Y_{[\gamma,i_2]} = (S^m \vee S^k) \cup_{[\gamma,i_2]} e^{k+r-1},$$

and characteristic maps

$$[\gamma, i_1]' \in \pi_{m+r-1}\big(Y_{[\gamma,i_1]}, S^m \vee S^k\big),$$
$$[\gamma, i_2]' \in \pi_{k+r-1}\big(Y_{[\gamma,i_2]}, S^m \vee S^k\big).$$

THEOREM 4.3. *Assume $m, k \geqslant 2$. There are maps $h_1\colon Y_{[\gamma,i_1]} \to Y_\gamma$ and $h_2\colon Y_{[\gamma,i_2]} \to Y_\gamma$ which both map the subcomplex $S^m \vee S^k$ by the identity, such that*

(a) $h_1 \circ [\gamma, i_1]' = [\gamma', i_1]$, *the relative Whitehead product,* $h_2 \circ [\gamma, i_2]' = [\gamma', i_2]$.

(b) *the homotopy homomorphisms induced by*

$$\gamma' \colon (D^r, S^{r-1}) \to (Y_\gamma, S^m \vee S^k),$$
$$h_1 \colon (Y_{[\gamma,i_1]}, S^m \vee S^k) \to (Y_\gamma, S^m \vee S^k),$$
$$h_2 \colon (Y_{[\gamma,i_2]}, S^m \vee S^k) \to (Y_\gamma, S^m \vee S^k)$$

are isomorphisms onto subgroups of $\pi_i(Y_\gamma, S^m \vee S^k)$ *which split the latter into a direct sum, as long as* $1 \leqslant 2r - 3$.

COROLLARY 4.4. *The relative homotopy groups* $\pi_i(Y_\gamma, S^m \vee S^k)$, *in degrees* $i \leqslant 2r - 3$, *are generated under composition by the characteristic map* $\gamma' \in \pi_r(Y_\gamma, S^m \vee S^k)$ *and iterated relative Whitehead products of* γ' *with the elements* $i_1 \in \pi_m(S^m \vee S^k)$ *and* $i_2 \in \pi_k(S^m \vee S^k)$. *Assuming* $i \leqslant 2r - 3$, *any element* $\zeta \in \pi_i(Y_\gamma, S^m \vee S^k)$ *has a unique representation in the form*

$$\zeta = \gamma' \circ \zeta_0' + [\gamma', i_1] \circ \zeta_1' + [\gamma', i_2] \circ \zeta_2' + \cdots, \tag{8}$$

where the sum continues over all possible iterated relative Whitehead products of γ' *with* i_1 *and* i_2.

PROOF OF PROPOSITION 3.4 FROM THE COROLLARY. For Proposition 3.4, we assume $k, m \geqslant 2$, and $n \geqslant 2m$. The cell attached has dimension $r = k + n - m$. The desired relative homotopy group has degree $n + k - 1$. The corollary applies to $\pi_i(Y_\delta, S^m \vee S^k)$ as long as i is no greater than

$$2r - 3 = 2k + 2n - 2m - 3 \geqslant k + n + (n - 2m) + k - 3 \geqslant k + n - 1.$$

Therefore the corollary applies to $\pi_{n+k-1}(Y_\delta, S^m \vee S^k)$, taking $\gamma = \delta$. The conclusion of the corollary is that any element of π_{n+k-1} is uniquely represented in the form (8); the only iterated product involving i_1 is $[\delta', i_1]$, for degree reasons:

$$\text{degree}[\delta', i_1] = r + m - 1 = n + k - 1,$$

and so other iterated products with i_1 have degree greater than $n + k - 1$. The result is that in the case at hand, (8) reduces to

$$\zeta = s[\delta', i_1] + \delta' \circ \zeta_0' + [\delta', i_2] \circ \zeta_1' + [[\delta_1', i_2], i_2] \circ \zeta_2' + \cdots$$

and this is the desired conclusion of Proposition 3.4.

PROOF OF COROLLARY 4.4 FROM THEOREM 4.3. The conclusion of 4.3 asserts that, for i in the desired range, every element ζ of $\pi_i(Y_\gamma, S^m \vee S^k)$ has a unique expression in the form

$$\zeta = \gamma' \circ \zeta_0' + h_1 \circ \eta_1 + h_2 \circ \eta_2 \tag{9}$$

where

$$\zeta_0' \in \pi_i(D^r, S^{r-1}), \quad \eta_1 \in \pi_i(Y_{[\gamma,i_1]}, S^m \vee S^k), \quad \eta_2 \in \pi_i(Y_{[\gamma,i_2]}, S^m \vee S^k).$$

Apply 4.3 again to the space $Y_{[\gamma,i_1]}$. Since the dimension of the cell attached to

$S^m \vee S^k$ to form $Y_{[\gamma,i_1]}$ is $r + m - 1$, the theorem applies to an even greater range of dimensions. The conclusion is that there are maps

$$h_3\colon Y_{[[\gamma,i_1],i_1]} \to Y_{[\gamma,i_1]}, \qquad h_4\colon Y_{[[\gamma,i_1],i_2]} \to Y_{[\gamma,i_1]},$$

satisfying conditions (a) and (b) with γ replaced by $[\gamma, i_1]$. Thus η_1 can be expressed uniquely as a sum

$$\eta_1 = [\gamma, i_1]' \circ \zeta_1' + h_3 \circ \eta_3 + h_4 \circ \eta_4 \tag{10}$$

for some elements

$$\zeta_1' \in \pi_i(D^{r+m-1}, S^{r+m-2}),$$
$$\eta_3 \in \pi_i\big(Y_{[[\gamma,i_1],i_1]}, S^m \vee S^k\big),$$
$$\eta_4 \in \pi_i\big(Y_{[[\gamma,i_1],i_2]}, S^m \vee S^k\big).$$

Similarly, by applying Theorem 4.3 to the space $Y_{[\gamma,i_2]}$ and the element η_2, there will be maps

$$h_5\colon Y_{[[\gamma,i_2],i_1]} \to Y_{[\gamma,i_2]}, \qquad h_6\colon Y_{[[\gamma,i_2],i_2]} \to Y_{[\gamma,i_2]}$$

and a decomposition

$$\eta_2 = [\gamma, i_2]' \circ \zeta_2' + h_5 \circ \eta_5 + h_6 \circ \eta_6 \tag{11}$$

for appropriate elements ζ_2', η_5, and η_6.

Substitute (10) and (11) into (9), obtaining

$$\begin{aligned}\zeta &= \gamma' \circ \zeta_0' + \big(h_1 \circ [\gamma, i_1]' \circ \zeta_1'\big) + (h_1 \circ h_3 \circ \eta_3) + (h_1 \circ h_4 \circ \eta_4)\\ &\quad + \big(h_2 \circ [\gamma, i_2]' \circ \zeta_2'\big) + (h_2 \circ h_5 \circ \eta_5) + (h_2 \circ h_6 \circ \eta_6)\\ &= \gamma' \circ \zeta_0' + [\gamma', i_1] \circ \zeta_1' + [\gamma', i_2] \circ \zeta_2'\\ &\quad + h_1 \circ h_3 \circ \eta_3 + h_1 \circ h_4 \circ \eta_4 + h_2 \circ h_5 \circ \eta_5 + h_2 \circ h_6 \circ \eta_6.\end{aligned}$$

by condition (a) of the theorem. Continue in this manner, applying the theorem to the elements η_3, η_4, η_5, η_6 and obtaining a representation of ζ as a sum of compositions with γ' and iterated relative Whitehead products of γ' with i_1 and i_2 of length at most 3. Repeat indefinitely to deduce the corollary. □

I now prove Theorem 4.3 using the relative reduced product space and its properties (a) through (e).

We are given r, $\gamma \in \pi_{r-1}(S^m \vee S^k)$, and $Y = Y_\gamma = (S^m \vee S^k) \cup_\gamma e^r$. Let $M = M_\gamma$ denote the mapping cylinder for γ. Identify S^{r-1} with the top of the mapping cylinder M. The mapping cone of the inclusion $i\colon S^{r-1} \subseteq M$ is just $M \cup_i CS^{r-1} = Y$. There is a homotopy equivalence of pairs

$$\big(M \cup_i CS^{r-1}, M\big) \simeq \big(Y, S^m \vee S^k\big).$$

Apply the relative reduced product construction for the pair (M, S^{r-1}).

According to part (d) above, the map of pairs

$$(Y, M) = \left(M \cup_i CS^{r-1}, M\right) \xrightarrow{q} (S^r, *)$$

defined by pinching M to a point has as relative fibre the pair $((M, S^{r-1})_\infty, (M(S^{r-1})_1)$. See diagram (3) above. Therefore there is a long exact sequence of relative homotopy groups:

$$\xrightarrow{\partial} \pi_*\left((M, S^{r-1})_\infty, (M, S^{r-1})_1\right) \xrightarrow{G_*} \pi_*(Y, M) \xrightarrow{q_*} \pi_*(S^r, *) \xrightarrow{\partial} \quad (12)$$

The map G is referred to at the beginning of (e) above.

Step 1. Replace the infinite reduced product space in (12) by the second stage $(M, S^{r-1})_2$. I need to find out how connected the pair

$$(M, S^{r-1})_2 \subseteq (M, S^{r-1})_\infty \quad (13)$$

is. This is a consequence of property (c) of the reduced product construction. The sphere S^{r-1} is $(r-2)$-connected. The mapping cylinder M is equivalent to $S^m \vee S^k$ and so is $(\min\{m, k\} - 1)$-connected. Therefore the pair (13) is

$$\min\{m, k\} + 2(r-1) - 1 = \min\{m, k\} + 2r - 3 \quad (14)$$

connected, by property (c). Since $m, k \geqslant 2$, the bound in (14) is at least $2r - 1$. Therefore the inclusion

$$\left((M, S^{r-1})_2, (M, S^{r-1})_1\right) \subseteq \left((M, S^{r-1})_\infty, (M, S^{r-1})_1\right)$$

induces isomorphisms of relative homotopy groups in dimensions less than $2r - 1$. So for the purposes of proving the theorem I replace (12) by the following sequence, valid and exact in degrees less than $2r - 1$. The map G_2 is simply the restriction of G, as in (e) above.

$$\xrightarrow{\partial} \pi_*\left((M, S^{r-1})_2, (M, S^{r-1})_1\right) \xrightarrow{(G_2)_*} \pi_*(Y, M) \xrightarrow{q^*} \pi_*(S^r, *) \xrightarrow{\partial}. \quad (15)$$

Step 2. Obtain a reasonable cell structure for $(M, S^{r-1})_2$ and study the map G_2. According to property (e) of the reduced product space construction, $(M, S^{r-1})_1$ may be identified with M, and since $M \simeq S^m \vee S^k = S(S^{m-1} \vee S^{k-1})$ and $S^{r-1} = S(S^{r-2})$, the second stage $(M, S^{r-1})_2$ may be identified with

$$M \cup C\left(S^{r-2} * (S^{m-1} \vee S^{k-1})\right), \quad (16)$$

where the attaching map for the cone is the Whitehead product of the inclusion $S^{r-1} \subseteq M$ and the homotopy equivalence $S^m \vee S^k \to M$. The join $S^{r-2} * (S^{m-1} \vee S^{k-1})$ is homotopy equivalent to $S^{r+m-2} \vee S^{r+k-2}$; the homotopy equivalence $M \to S^m \vee S^k$ already referred to takes the homotopy class of i: $S^{r-1} \subseteq M$ to that of $\gamma \in \pi_{r-1}(S^m \vee S^k)$. It follows that (16) may

be replaced by

$$(S^m \vee S^k) \cup_{[\gamma, i_1]} e^{r+m-1} \cup_{[\gamma, i_2]} e^{r+k-1}, \tag{17}$$

which I will denote by Z. According to the discussion in part (e) above, the map G_2 occurring in (15) defines via the equivalence of $(M, S^{r-1})_2$ with (16) a map

$$G_2'\colon \big(M \cup C(S^{r-2} * (S^{m-1} \vee S^{k-1})), M\big) \to (Y, M) \tag{18}$$

which carries the characteristic map of the cone in (16) to the relative Whitehead product $[i', 1_M]$ according to (6a). Here $i'\colon (D^r, S^{r-1}) \to (Y, M)$ is the characteristic map for the mapping cone of the inclusion $S^{r-1} \subseteq M$. It follows that when I replace (16) by (17) and M by its equivalent $S^m \vee S^k$, the resulting map

$$G_2''\colon (Z, S^m \vee S^k) \to (Y, S^m \vee S^k)$$

carries the characteristic maps $[\gamma, i_1]'$ and $[\gamma, i_2]'$ of the cells e^{r+m-1} and e^{r+k-1} of Z as follows:

$$(G_2'')_*[\gamma, i_1]' = [\gamma', i_1], \qquad (G_2'')_*[\gamma, i_2]' = [\gamma', i_2]. \tag{19}$$

Note that under the homotopy equivalence of the mapping cylinder M with $S^m \vee S^k$, the inclusion $i\colon S^{r-1} \subseteq M$ corresponds to the map $\gamma\colon S^{r-1} \to S^m \vee S^k$.

I now have, in place of (15), the following exact sequence:

$$\xrightarrow{\partial} \pi_*(Z, S^m \vee S^k) \xrightarrow{G_2''} \pi_*(Y, S^m \vee S^k) \xrightarrow{q_*} \pi_*(S^r, *) \xrightarrow{\partial} \tag{20}$$

where Z has the form (17) and G_2'' is the identity on $S^m \vee S^k$ and satisfies (19).

Step 3. The sequence (20) breaks up into split short exact sequences according to the following argument. The characteristic map

$$\gamma'\colon (D^r, S^{r-1}) \to (Y, S^m \vee S^k)$$

for the r-cell of Y, when composed with the pinching map

$$q\colon (Y, S^m \vee S^k) \to (S^r, *)$$

is the map

$$(D^r, S^{r-1}) \to (S^r, *)$$

obtained by pinching S^{r-1} to a point. The latter is a factor of the suspension homomorphism:

$$\pi_{i-1}(S^{r-1}) \underset{\approx}{\xleftarrow{\partial}} \pi_i(D^r, S^{r-1}) \xrightarrow{(q \circ \gamma')_*} \pi_i(S^r, *).$$

By the Freudenthal suspension theorem, $(q \circ \gamma')_*$ is an isomorphism for

$i \leqslant 2r - 3$ and onto for $i = 2r - 2$. Therefore

$$(\gamma')_*: \pi_i(D^r, S^{r-1}) \to \pi_i(Y, S^m \vee S^k)$$

splits q_* as long as $i \leqslant 2r - 3$. It follows that the homomorphisms $(G_2'')_*$ of sequence (20) and $(\gamma')_*$ are isomorphisms onto complementary direct summands in degrees $i \leqslant 2r - 3$.

Step 4. Apply Theorem 4.2 to split $\pi_*(Z, S^m \vee S^k)$ as a direct sum. The domain of G_2'', Z, which has the form (17), is obtained from $S^m \vee S^k$ by attaching two cells simultaneously. The subcomplexes

$$Y_{[\gamma,i_1]} = (S^m \vee S^k) \cup_{[\gamma,i_1]} e^{r+m-1} \quad \text{and}$$
$$Y_{[\gamma,i_2]} = (S^m \vee S^k) \cup_{[\gamma,i_2]} e^{r+k-1}$$

reside in a diagram of inclusions:

$$\begin{tikzcd}
(Y_{[\gamma,i_1]}, S^m \vee S^k) \arrow[dr, "j_1"] & \\
& (Z, S^m \vee S^k) \\
(Y_{[\gamma,i_2]}, S^m \vee S^k) \arrow[ur, "j_2"'] &
\end{tikzcd}$$

According to Theorem 4.2, $(j_1)_*$ and $(j_2)_*$ are isomorphisms onto complementary direct summands of $\pi_i(Z, S^m \vee S^k)$ as long as $i < (r + m - 1) + (r + k - 1) - 2$. The latter bound is at least $2r$, since $m, k \geqslant 2$, so the splitting of $\pi_i(Z, S^m \vee S^k)$ is valid in the range $i \leqslant 2r - 3$ required for Theorem 4.3.

I can now complete the proof of Theorem 4.3. Define

$$h_1 = G_2'' \circ j_1: Y_{[\gamma,i_1]} \to Y_\gamma, \qquad h_2 = G_2'' \circ j_2: Y_{[\gamma,i_2]} \to Y_\gamma.$$

Equations (19) imply the desired conclusion (a) of the theorem. It follows from Steps 3 and 4 that

$$(\gamma')_*: \pi_i(D^r, S^{r-1}) \to \pi_i(Y_\gamma, S^m \vee S^k),$$
$$(h_1)_*: \pi_i(Y_{[\gamma,i_1]}, S^m \vee S^k) \to \pi_i(Y_\gamma, S^m \vee S^k),$$
$$(h_2)_*: \pi_i(Y_{[\gamma,i_2]}, S^m \vee S^k) \to \pi_i(Y_\gamma, S^m \vee S^k),$$

are isomorphisms onto complementary direct summands as long as $i \leqslant 2r - 3$. $\square$

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