UNIQUELY ARCWISE CONNECTED PLANE CONTINUA
HAVE THE FIXED-POINT PROPERTY

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ABSTRACT. This paper contains a solution to a fixed-point problem of G. S. Young [17, p. 884] and R. H. Bing [4, Question 4, p. 124]. Let $M$ be an arcwise connected plane continuum that does not contain a simple closed curve. We prove that every continuous function of $M$ into $M$ has a fixed point.

1. Introduction. A space $S$ has the fixed-point property if for each continuous function $f$ of $S$ into $S$, there exists a point $x$ of $S$ such that $f(x) = x$. It is known that every arcwise connected plane continuum that does not separate the plane has the fixed-point property [6], [7]. In this paper we consider another class of arcwise connected plane continua.

A continuum $M$ is uniquely arcwise connected if for each pair $p$, $q$ of points of $M$, there exists exactly one arc in $M$ with endpoints $p$ and $q$. Note that a continuum is uniquely arcwise connected if and only if it is arcwise connected and does not contain a simple closed curve. The sin $1/x$ circle (Warsaw circle) is the simplest example of a uniquely arcwise connected plane continuum that separates the plane.

In [4, p. 123], Bing gave a dog-chases-rabbit argument that shows the sin $1/x$ circle has the fixed-point property. Recently L. Mohler [12] used the Markov-Kakutani theorem (measure theory) to prove that every homeomorphism of a uniquely arcwise connected continuum into itself has a fixed point.

We prove that every uniquely arcwise connected plane continuum has the fixed-point property. An example of Young [17, p. 884] shows that this theorem cannot be extended to all uniquely arcwise connected continua in Euclidean 3-space.
Our proof involves a continuous image of a ray defined by K. Borsuk [5] and a nested sequence of polygonal disks constructed by K. Sieklucki [15]. In [5], Borsuk established the fixed-point property for every hereditarily unicoherent arcwise connected continuum. Sieklucki [15] and H. Bell [2] proved that every nonseparating plane continuum with a hereditarily decomposable boundary has the fixed-point property.

2. Preliminaries. A continuum is a nondegenerate compact connected metric space.

Throughout this paper $R^2$ is the Cartesian plane with metric $\rho$. We denote the boundary, closure, and interior of a given set $K$ by $\text{Bd } K$, $\text{Cl } K$, and $\text{Int } K$ respectively. The union of a collection $\mathcal{K}$ of sets is denoted by $\text{St } \mathcal{K}$.

For each real number $\xi$, let $I(\xi)$ be the interval $\{(x, y) \in R^2: 0 < x < 1$ and $y = \xi\}$.

**Definition.** Suppose $A$ is an arc, $H$ is a continuum, and $A \cup H \subset R^2$. Then $H$ straddles $A$ if for each homeomorphism $h$ of $\text{St}\{I(\xi): -1 < \xi < 1\}$ into $R^2$ with $h[I(0)] = A$, there exists a positive real number $\eta$ such that $H \cap h[I(\xi)] \neq \emptyset$ when $|\xi| < \eta$.

Henceforth $M$ is a uniquely arcwise connected continuum in $R^2$.

**Notation.** If $u$ and $v$ are distinct points of $M$, then the arc, the half-open arc, and the arc segment (open arc) in $M$ with endpoints $u$, $v$ are denoted by $M[u, v]$, $M[u, v)$, and $M(u, v)$ respectively.

Let $P$ be the image in $M$ of the ray $[1, +\infty)$ under a one-to-one continuous function $\psi$. For each positive integer $n$, let $a_n = \psi(n)$. The function $\psi$ determines a linear ordering $\prec$ of $P$ with $a_1$ as the first point. In this section, if $u$ and $v$ are points of $P$, the notation $M[u, v]$, $M[u, v)$, and $M(u, v)$ will be used only when $u \prec v$.

**Notation.** For each point $u$ of $P$, let $P(u)$ denote $\{v \in P: u = v$ or $u \prec v\}$.

Let $L = \bigcap_{n=1}^{\infty} \text{Cl } P(a_n)$. In this section we assume $L$ is not degenerate. Hence $L$ is a continuum.

**Lemma 1.** Suppose $u$ and $v$ are distinct points of $M$ that belong to a complementary domain $D$ of $L$. Then $\text{Cl } D$ contains $M[u, v]$.

**Proof.** Assume there is a point $w$ of $M(u, v)$ in $R^2 \setminus \text{Cl } D$. Let $J$ be an arc in $D$ that is irreducible between $M[u, w]$ and $M[w, v]$. Every arc in $M$ that intersects $D$ and $R^2 \setminus \text{Cl } D$ is straddled by $\text{Bd } D$. Hence $\text{Bd } D$ intersects each complementary domain of $J \cup M[u, v]$. Since $\text{Bd } D \subset L$ and $M$ does not contain a simple closed curve, for each positive integer $n$, there is a point $z_n$ of $P(a_n)$ in $J$. Let $z$ be a limit point of $\{z_n\}_{n=1}^{\infty}$. It follows that $z \in J \cap L$, and this contradicts the fact that $J \subset R^2 \setminus L$. Hence $M[u, v] \subset \text{Cl } D$.

**Lemma 2.** There exist a complementary domain $D$ of $L$ and a positive integer $n$ such that $\text{Cl } D$ contains $P(a_n)$.

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Proof. Assume the contrary. Then by Lemma 1,
(1) for each complementary domain $D$ of $L$, there exists a positive integer $n$ such that $D \cap P(a_n) = \emptyset$.

Let $\{Y_n\}_{n=1}^{\infty}$ be the elements of a countable base for $M$ that intersect $L$. For each $n$, let $Z_n = \{z \in L \setminus \{a_1\}: Y_n \cap M \cup Z \setminus M \cap \{a_1, z\} = \emptyset\}$.

Note that $L \setminus \{a_1\} = \bigcup_{n=1}^{\infty} Z_n$. To see this let $z$ be a point of $L \setminus \{a_1\}$. If $L \subset M \cup Z \setminus M \cap \{a_1, z\}$, then $L$ is an arc and the closure of the complementary domain of $L$ (being $R^2$) contains $P$, contrary to our assumption. Thus $L \not\subset M \cup Z \setminus M \cap \{a_1, z\}$.

Hence there exists an integer $n$ such that $M \cup Z \setminus M \cap \{a_1, z\}$ contains an open subset $U$ of $L$. Let $Y$ and $Z$ be disjoint disks in $R^2$ such that
(2) $L \cap \text{Int } Y \neq \emptyset$ and $L \cap \text{Int } Z \neq \emptyset$ and
(3) $M \cap Y \subset Y$ and $L \cap Z \subset U$.

Notation. For each component $C$ of $P \setminus Y$ that misses $a_1$, let $\text{Dom } C$ denote the complementary domain of $C \cup Y$ that misses $a_1$.

Let $\mathcal{M}$ be the collection of all components $C$ of $P \setminus Y$ such that $a_1 \not\in C$ and $Z \cap \text{Dom } C \neq \emptyset$. Note that each element of $\mathcal{M}$ is an arc segment with both endpoints in $\text{Bd } Y$.

For each element $C$ of $\mathcal{M}$,
(4) $U \cap \text{Dom } C = \emptyset$ and
(5) $C \cap \text{Int } Z \neq \emptyset$.

Statement (4) is true; for otherwise, since $U \subset \text{Cl } Z_a$, $M \setminus Y_a$ contains an arc $A$ that runs from $a_1$ to $U \cap \text{Dom } C$, and $A \cup P$ contains a simple closed curve, violating the unique arcwise connectivity of $M$. Since $Z \cap \text{Dom } C \neq \emptyset$ and $L \cap \text{Int } Z$ is a nonempty subset of $U$, (5) follows immediately from (4).

Since $Y$ and $Z$ are disjoint, it follows from (2) that $\mathcal{M}$ is infinite. By (5), for each positive integer $n$,
(6) $P(a_n)$ contains all but finitely many elements of $\mathcal{M}$.

For each element $C$ of $\mathcal{M}$,
(7) $\text{Dom } C$ contains at most finitely many elements of $\mathcal{M}$ and
(8) only finitely many elements of $\mathcal{M}$ separate $a_1$ from $C$ in $R^2 \setminus Y$.

To verify (7) assume there exists an infinite subcollection $\mathcal{N}$ of $\mathcal{M}$ such that $\text{St } \mathcal{N} \subset \text{Dom } C$. By (5), each element of $\mathcal{N}$ intersects $Z$. Hence by (6), there is a point $t$ of $L$ in $Z \cap \text{Cl } \text{St } \mathcal{N}$. By (4), $t \not\in \text{Dom } C$. Therefore $t \in C$.

Let $J$ be an arc segment in $\text{Dom } C$ with endpoints in $C \setminus \{t\}$ such that the bounded complementary domain $K$ of $C \cup J$ has the following properties. The point $t$ belongs to $\text{Cl } K$ and $U$ contains $K \cap L$.

Since infinitely many elements of $\mathcal{N}$ intersect $K$, it follows from (1) and (6) that no complementary domain of $L$ contains $K \cap \text{St } \mathcal{N}$. Therefore $K \cap L \neq \emptyset$. But since $K \subset \text{Dom } C$, this contradicts (4). Hence (7) is true.
To establish (8) assume the contrary. By (7), there exists a sequence \{C_n\}_{n=1}^\infty \subseteq 91L such that for each \( n \), \( \text{Dom} \, C_n \subseteq \text{Dom} \, C_{n+1} \). By (1), (4), (5), (6), and Lemma 1, for some \( n \), \( L \) separates \( C_1 \) from \( C_n \) in \( Z \). Let \( A \) be an arc segment in \( Z \) such that \( \text{Cl} \, A \) is an arc irreducible between \( C_1 \) and \( C_n \). Since \( C_1 \subseteq \text{Dom} \, C_n \), \( A \subseteq \text{Dom} \, C_n \). But since \( A \cap L \neq \emptyset \), this contradicts (4). Hence (8) is true.

Let \( \emptyset = \{ C \in 91L : \text{no element of 91L separates} \, a_i \, \text{from} \, C \, \text{in} \, R^2 \setminus Y \} \). Since 91L is infinite, it follows from (7) and (8) that \( \emptyset \) is infinite.

Next we define a pair \( p, q \) of points of \( P \cap \text{Int} \, Y \). We consider two cases.

Case 1.1. Suppose \( L \cap P \cap \text{Int} \, Y \neq \emptyset \). Define \( p \) to be a point of \( L \cap P \cap \text{Int} \, Y \), and let \( q \) be a point of \( (P(p) \cap \text{Int} \, Y) \setminus \{ p \} \).

Case 1.2. Suppose \( L \cap P \cap \text{Int} \, Y = \emptyset \). Applying (1), we define \( p \) and \( q \) to be points of \( P \cap \text{Int} \, Y \) such that \( P(q) \) misses the \( p \)-component of \( R^2 \setminus L \).

An element \( M(u, v) \) of \( \emptyset \) in \( P(q) \), a point \( z \) of \( M(u, v) \cap \text{Int} \, Z \), and an arc segment \( I \) in \( \text{(Int} \, Z) \setminus \text{Cl} \, \text{Dom} \, M(u, v) \) exist such that

\[
\begin{align*}
(9) \, (\text{Cl} \, I) \cap (M[a_1, v] \cup \text{Bd} \, Z) &= \{ z \}, \\
(10) \, L \cap \text{Cl} \, I \neq \emptyset, \text{ and} \\
(11) \, \text{either} \, z \in L \, \text{or} \, P \cap \text{Cl} \, I = \{ z \}.
\end{align*}
\]

To verify this consider two cases.

Case 2.1. Suppose \( L \cap P(q) \cap \text{Int} \, Z \neq \emptyset \). Let \( z \) be a point of \( L \cap P(q) \cap \text{Int} \, Z \). Let \( M(u, v) \) be the element of 91L that contains \( z \). It follows from (4) that \( M(u, v) \in \emptyset \). Let \( I \) be an arc segment in \( (\text{Int} \, Z) \setminus \text{Cl} \, \text{Dom} \, M(u, v) \) that satisfies (9). Since \( z \in L \), (10) and (11) hold.

Case 2.2. Suppose \( L \cap P(q) \cap \text{Int} \, Z = \emptyset \). Note that \( (L \setminus P) \cap \text{Int} \, Z \neq \emptyset \). To see this assume otherwise. Since \( L \cap \text{Int} \, Z \neq \emptyset \) and \( L \cap P(q) \cap \text{Int} \, Z = \emptyset \), there exists a point \( a \) of \( M[a_1, q] \) in \( L \cap \text{Int} \, Z \). Let \( V \) be an open disk in \( Z \) such that \( a \in V \) and \( V \setminus M[a_1, q] \) has exactly two components. Since \( a \in L \), infinitely many elements of 91L intersect \( V \setminus M[a_1, q] \). But since both components of \( V \setminus M[a_1, q] \) are in \( R^2 \setminus L \), this contradicts (1) and (6).

Let \( t \) be a point of \( (L \setminus P) \cap \text{Int} \, Z \). Let \( W \) be an open disk in \( Z \) that contains \( t \) and misses \( M[a_1, q] \). By (1), an element \( M(w, x) \) of 91L and a point \( r \) of \( W \cap M(w, x) \) exist such that \( W \cap P(x) \) misses the \( r \)-component \( G \) of \( R^2 \setminus L \).

Let \( T \) be an arc in \( W \) that runs from \( r \) to \( t \). Define \( s \) to be the first point of \( T \) that belongs to \( L \). Let \( I \) be the arc segment in \( T \) that precedes \( s \) with the property that \( \text{Cl} \, I \) is irreducible between \( s \) and \( M[q, x] \). Define \( z \) to be the endpoint of \( I \) opposite \( s \). Let \( M(u, v) \) be the element of 91L that contains \( z \). It follows from (4) that \( s \notin \text{Cl} \, \text{Dom} \, M(u, v) \). Since \( z \in G \), \( P \cap \text{Cl} \, I = \{ z \} \). Hence by (4), \( M(u, v) \in \emptyset \). Clearly (9), (10), and (11) hold.

Let \( J \) be a polygonal arc segment in

\[
Z \setminus (M[a_1, v] \cup \text{Dom} \, M(u, v) \cup \text{Cl} \, I)
\]
with endpoints \( b \) and \( b_0 \) in \( M(u, v) \) such that \( z \in M(b, b_0) \subset Z \).

Define \( K_0 \) to be the component of \( R^2 \setminus (J \cup M[b, b_0]) \) that contains \( I \). Note that \( K_0 \subset Z \).

The following statements and definitions (12\( _n \))—(21\( _n \)) will be used inductively.

By (7), (8), and (10), for \( n = 1 \), there exists an element \( M(w_n, x_n) \) of \( \emptyset \) in \( P(v) \) such that

\[
(12\_n) \quad K_{n-1} \cap M(w_n, x_n) \neq \emptyset \text{ and }
\]

\[
(13\_n) \quad \text{no element of } \emptyset \text{ in } M(b_{n-1}, w_n) \text{ intersects } K_{n-1}.
\]

For \( n = 1 \),

\[
(14\_n) \quad H_n = M(b_{n-1}, x_n) \cap \text{Cl } K_{n-1},
\]

\[
(15\_n) \quad \text{let } J_n \text{ be the component of } I \setminus M(p, x_n) \text{ whose closure contains } z,
\]

\[
(16\_n) \quad \text{let } J' \text{ be the arc in Cl } J \text{ that is irreducible between } b \text{ and } M(w_n, x_n),
\]

and

\[
(17\_n) \quad \text{let } b_n \text{ be the endpoint of } J_n \text{ that belongs to } M(w_n, x_n).
\]

Let \( J' \) be the arc in Cl \( J \) that is irreducible between \( b_0 \) and \( M(w_1, x_1) \). Since \( Y \cup \text{Dom } M(u, v) \) misses \( M(w_1, x_1) \cup \text{Cl } K_0 \), there exists an arc segment \( /' \) in \( Y \cup \text{Dom } M(u, v) \) from \( p \) to \( z \) such that \( \text{Cl } I_1 \) and \( \text{Cl } I' \) abut on \( M[b, b_0] \) from opposite sides with respect to the simple closed curve \( \Gamma \) in \( J_1 \cup J' \cup M(b, b_0) \cup M(w_1, x_1) \) [13, Theorem 32, p. 181]. Since \( \Gamma \cap (I_1 \cup I' \cup \{ p \}) = \emptyset, J_1 \cup J' \cup M(u, x_1) \) separates \( p \) from \( I_1 \) in \( R^2 \). Since \( J_1 \cap J' = \emptyset \), either \( J_1 \cup M(u, x_1) \) or \( J' \cup M(u, x_1) \) separates \( p \) from \( I_1 \) in \( R^2 \) [13, Theorem 20, p. 173]. For convenience we assume that \( J_1 \cup M(u, x_1) \) separates \( p \) from \( I_1 \) in \( R^2 \).

For \( n = 1 \),

\[
(18\_n) \quad \text{let } K_n \text{ be the complementary domain of } J_n \cup M(b, b_n) \text{ that contains } I_n.
\]

Clearly, for \( n = 1 \),

\[
(19\_n) \quad p \notin K_n.
\]

By (10) and (11), for \( n = 1 \),

\[
(20\_n) \quad L \cap (K_n \cup \{ z \}) \neq \emptyset.
\]

Next we show that for \( n = 1 \),

\[
(21\_n) \quad K_n \cap Y \cap P(b_n) = \emptyset.
\]

Let \( A \) be an arc in \( J_1 \) such that \( A \cap M(p, x_1) = \{ b \} \). Since \( M(u, v) \) and \( M(w_1, x_1) \) belong to \( \emptyset \), \( M(u, v) \) and \( M(w_1, x_1) \) are not separated in \( R^2 \setminus Y \) by an element of \( \emptyset \). Hence there exists a polygonal arc segment \( B \) in \( R^2 \setminus (Y \cup M(p, x_1)) \) such that \( A \cup \{ b_1 \} \subset \text{Cl } B \). Let \( I'_1 \) be an arc in \( (\text{Cl } I_1) \setminus B \) that contains \( z \). Let \( K \) be the complementary domain of \( B \cup M[b, b_1] \) that intersects \( I'_1 \).

Note that

\[
(22) \quad K_1 \cap Y \subset K.
\]

To see this let \( y \) be a point of \( K_1 \cap Y \). We must show that \( y \in K \). Let \( F \) be a
polygonal arc in $K_1$ from $y$ to $I'_1$ such that $B \cap F$ is finite and $F$ crosses $B$ at each point of $B \cap F$. Clearly $y \in K$ if $B \cap F = \emptyset$, so we assume $B \cap F \neq \emptyset$.

Since $I'_1 \cup Y \cup M(z, v)$ misses $B \cup J_1$, one complementary domain of $B \cup J_1$ contains $I'_1 \cup \{y\}$. Since $F \cap J_1 = \emptyset$, it follows that $F$ crosses $B$ an even number of times. Therefore $F$ crosses $B \cup M(b, b_1)$ an even number of times. Thus $y \in K$. Hence (22) is true.

It follows from (22) that (211) can be established by proving

(23) $K \cap Y \cap P(b_1) = \emptyset$.

To verify (23) first note that since $p \notin K_1$ and $M[p, b] \cap (J_1 \cup M(b, b_1)) = \emptyset$, $I'_1$ and $M[p, b]$ abut on $A \cup M(b, v)$ from opposite sides with respect to a simple closed curve in $J_1 \cup M(b, b_1)$. Hence $I'_1$ and $M[p, b]$ abut on $A \cup M(b, v)$ from opposite sides with respect to $B \cup M(b, b_1)$ [13, Theorem 32, p. 181]. Since $M[p, b] \cap (B \cup M[b, b_1]) = \emptyset$, it follows that $p \notin K$. Let $Q$ be an arc from $p$ to $q$ in $\text{Int } Y$. Either $p \in L$ (Case 1.1) or $L \cap P \cap \text{Int } Y = \emptyset$ and $P(q)$ misses the $p$-component of $R^2 \setminus L$ (Case 1.2). Hence the $p$-component of $Q \setminus \text{Cl } K$ contains a point $r$ of $L$. Let $G$ be an open set in $Y \setminus \text{Cl } K$ that contains $r$.

Now suppose that (23) is false. Since $r \in L$, there exists an arc $M[s, t]$ in $P(b_1)$ such that $s \in K \cap Y$ and $t \in G$.

Let $M(s', t')$ be an arc segment in $M[s, t] \setminus Y$ such that $s' \in K \cap \text{Bd } Y$ and $t' \in (\text{Bd } Y) \setminus K$. A component of $(B \cup M[b, b_1]) \setminus \text{Int } Y$ separates $s'$ from $t'$ in $R^2 \setminus \text{Int } Y$ [13, Theorem 27, p. 177]. Since $B \cap M(b, b_1) = \emptyset$ and $M(s', t') \cap (Y \cup M(b, b_1)) = \emptyset$, it follows that $B \cup M[b, v] \cup M[w_1, b_1]$ separates $s'$ from $t'$ in $R^2 \setminus \text{Int } Y$. Hence $(v, w_1)$ separates $s'$ from $t'$ in $\text{Bd } Y$. Thus $M(s', t')$ is an element of $K$ that separates $M(u, v)$ from $M(w_1, x_1)$ in $R^2 \setminus Y$ [13, Theorem 30, p. 158], and this contradicts the fact that $M(u, v)$ and $M(w_1, x_1)$ belong to $\emptyset$. Hence (23) is true. Consequently (211) is true.

Proceeding inductively, for each integer $n > 1$, we define $M(w_n, x_n)$, $H_n$, $I_n$, $J_n$, $b_n$, and $K_n$ satisfying (12n) – (21n). For $n > 1$, (12n) and (13n) follow from (7), (8), and (20n–1). To verify (19n) for $n > 1$, note that by (19n–1), there exists an arc $A$ in $J_n$ such that $M[p, b]$ and $\text{Cl } I_n$ abut on $A \cup M[b, v]$ from opposite sides with respect to a simple closed curve in $J_n \cup M(b, b_n)$ [13, Theorem 32, p. 181]. The arguments given for (20n) and (21n) when $n = 1$ hold when $n > 1$.

Since $Y \cap Z = \emptyset$, for each positive integer $m$,

(24) there exists an integer $n$ such that $a_m \notin P(x_m)$.

Let $H$ be the limit superior of $\{H_n\}_{n=1}^{\infty}$. By (24), $H \subset L$.

Since $K_0 \cap Y = \emptyset$, it follows from (21n) that

(25) $Y \cup \bigcup_{n=1}^{\infty} H_n = \emptyset$.

Since $(J_n)_{n=1}^{\infty}$ is a nested sequence of arcs, $(b_n)_{n=1}^{\infty}$ converges to a point $c$ of $H \cap J_1$. For each positive integer $n$, let $B_n$ be the polygonal arc in $J_n$ with endpoints $c$ and $b_n$.
Since \( J_1 \cap M[a_1, u] = \emptyset \), it follows from (13\(_n\)) and (16\(_n\)) that \((J_1 \setminus B_i) \cap \bigcup_{n=1}^{\infty} H_n = \emptyset\). For each positive integer \( n \), every component of \( H_{n+1} \) intersects \( B_n \). Hence \( H \) is connected.

For each component \( C \) of \( P(u) \setminus Y \),
\[
(26) \quad H \cap \text{Int}(Y \cup \text{Dom } C) = \emptyset.
\]
To see this let \( x \) be the last point of \( \text{Cl } C \) with respect to the ordering of \( P \). By (13\(_n\)) and (16\(_n\)), \( c \notin M(b, x) \). Let \( i \) be a positive integer such that \( B_i \cap M(b, x) = \emptyset \). Since \( c \in U \), it follows from (4) that \( B_i \cap \text{Dom } C = \emptyset \). Since \( P \) does not contain a simple closed curve, \( C \cap \bigcup_{n=1}^{\infty} H_n = \emptyset \). Since \( B_i \) intersects each component of \( \bigcup_{n=1}^{\infty} H_n \), it follows from (25) that \( \bigcup_{n=1}^{\infty} H_n \) misses \( Y \cup \text{Dom } C \) [13, Theorem 28, p. 156]. Hence (26) is established.

Next we prove that Knaster's chainable indecomposable continuum with one endpoint [11, Example 1, p. 204] is a continuous image of \( H \). We use a result [8, Theorem 1] that was derived from an argument of D. P. Bellamy [3].

According to Theorem 1 of [8], \( H \) can be mapped continuously onto Knaster's continuum if there exists a sequence \( \{G_n\}_{n=1}^{\infty} \) of nonempty open sets in \( H \) such that \((\text{Cl } G_1) \cap \text{Cl } G_2 = \emptyset \) and for each \( n \),
\[
(27) \quad G_{2n+1} \cup G_{2n+2} \subset G_{2n-1}
\]
and
\[
(28) \quad \text{there exists a separation } E_n \cup F_n \text{ of } M \setminus G_{2n} \text{ such that } G_{2n+1} \subset E_n \text{ and } G_{2n+2} \subset F_n.
\]
To establish the existence of \( \{G_n\}_{n=1}^{\infty} \), order \( B_1 \) so that \( b_1 \) is its first point. Let \( c_1 \) be the first point of \( B_1 \cap L \) with respect to the ordering of \( B_1 \). By (1) and (24), \( c_1 \neq c \).

If \( b_1 \neq c_1 \), define \( C_1 \) to be the arc in \( B_1 \) from \( b_1 \) to \( c_1 \). If \( b_1 = c_1 \), let \( C_1 = \{c_1\} \).

Note that
\[
(29) \quad c_1 \in H.
\]
To see this consider two cases.

Case 3.1. Suppose \( c_1 \in P(z) \). Let \( m \) be an integer such that \( J_m \cap M[z, c_1] = \emptyset \). For each integer \( n > m \), \( M[z, c_1] \subset \text{Cl } K_n \). It follows from (7) that \( c_1 \in H \).

Case 3.2. Suppose \( c_1 \notin P(z) \). For some positive integer \( n \), \( C_1 \cap P(a_n) = \emptyset \); for otherwise, by (1), \( L \cap (C_1 \setminus \{c_1\}) = \emptyset \), and this contradicts the definition of \( c_1 \). Let \( d \) be the last point of \( C_1 \cap P \) that precedes \( c_1 \) with respect to the ordering of \( B_1 \). Since \( C_1 \cap M[a_1, z] = \emptyset \), \( d \in P(z) \).

Let \( \Delta \) be the arc in \( C_1 \) from \( d \) to \( c_1 \). Let \( M(w, x) \) be the \( d \)-component of \( P \setminus Y \). Since \( c_1 \in U \) and \( \Delta \cap P = \{d\} \), it follows from (4) that \( M(w, x) \in \emptyset \).

Let \( A \) be an arc in \( J_1 \setminus \Delta \) such that \( A \cap M(p, x) = \{b\} \). Observe that
\[
(30) \quad A \text{ and } \Delta \text{ abut on } M[u, x] \text{ from the same side.}
\]
To verify (30) first note that, by (4), \( \Delta \cap \text{Dom } M(w, x) = \emptyset \). Hence there exists a polygonal arc segment \( B \) in \( R^2 \setminus (Y \cup M(p, x)) \) such that \( A \cup \Delta \subset \text{Cl } B \). Let \( F \) be an arc in \( R^2 \setminus M(p, x) \) from \( p \) to \( x \) such that \( B \cap F \) is finite.
and $F$ crosses $B$ at each point of $B \cap F$.

Suppose (30) is false. Then $F$ crosses $B$ an odd number of times. It follows that $p$ and $x$ are separated in $R^2$ by $B \cup M[b, d]$. By the argument for (23), there is a component of $P(z) \setminus Y$ that separates $M(u, v)$ from $M(w, x)$ in $R^2 \setminus Y$, and this contradicts the fact that $M(u, v)$ and $M(w, x)$ belong to $\emptyset$. Hence (30) is true.

Let $i$ be a positive integer such that $J_i \cap M(b, x) = \emptyset$. Let $E$ be an arc in $C_l I_i$ such that $E \cap M[u, x] = \{z\}$. Since $E$ and $A$ abut on $M[u, x]$ from the same side, it follows from (30) that $E$ and $\Delta$ abut on $M[b, x]$ from the same side. Since $\Delta \cap P = \{d\}$, for each integer $j > i$, $c_i \in K_j$.

Let $V$ be a disk in $K_j$ such that $c_i \in \text{Int} V$. Since $c_i \in L$, $V \cap P(b_i) \neq \emptyset$. For each integer $j > i$, since $c_i \in K_j$, if $V \cap M(b_j, b_j) = \emptyset$, then $V \subset K_j$. Hence for some $j > i$, $H_j$ contains the first point of $V \cap P(b_i)$ with respect to the ordering of $P$ (recall (14)). It follows that $c_i \in H$. Thus (29) is established.

Let $D_1$ and $D_2$ be open disks in $R^2$ such that $B_1 \subset D_1$, $C_l I_1 \subset D_2$, and $(C_l D_1) \cap C_l D_2 = \emptyset$. Let $i_1 = 1$.

Let $j_1$ be an integer greater than 1 such that
\[(31) C_1 \cap P(w_j) = \emptyset \quad \text{and} \quad J_i \cap M[z, b_j] = \emptyset.\]
By (7), (8), (10), and (11), there exists an integer $i_2 > j_1$ such that $D_2 \cap K_1 \cap M(w_i, x_i) \neq \emptyset$.

Let $\Lambda$ be an arc segment in
\[((Bd Y) \cap \text{Cl Dom } M(w_i, x_i)) \setminus M[p, x_i]\]
that has $x_i$ as an endpoint. Let $\Lambda_1$ be a polygonal arc segment in $\text{Dom } M(w_i, x_i) \setminus M[b, b_i]$ from a point $e$ of $\Lambda$ to $D_2 \cap K_1$ such that $B_1 \cap \Lambda_1$ is finite and $\Lambda_1$ crosses $B_1$ at each point of $B_1 \cap \Lambda_1$.

By (21), $e \in K_1$.

Let $\Pi_1$ be the arc in $B_1$ from $c_1$ to $c$. Since $\Lambda_1$ misses $M[b, b_i] \cup (J_i \setminus \Pi_1)$, it follows that $\Lambda_1$ crosses $\Pi_1$ an odd number of times.

By (26), there exists a simple closed curve $\Sigma_1$ in
\[Y \cup \Lambda_1 \cup D_2 \cup \text{Dom } M(u, v)\]
such that $\Lambda_1 \subset \Sigma_1$, $H \cap \Sigma_1 \subset D_2$, and $\Pi_1 \cap \Sigma_1 \subset \Lambda_1$. Since $\Pi_1$ crosses $\Sigma_1$ an odd number of times, $\Sigma_1$ separates $c$ from $c_1$ in $R^2$.

Define $\Omega_1$ to be the $c$-component of $R^2 \setminus \Sigma_1$. Note that $B_1 \subset \Omega_1 \subset R^2 \setminus C_1$.

Let $E_1 = \Omega_1 \cap (H \setminus D_2)$ and $F_1 = H \setminus (D_2 \cup E_1)$.

For $i = 1$ and 2, let $G_i = D_i \cap H$. Note that $E_1 \cup F_1$ is a separation of $H \setminus G_2$.

Let $c_2$ be the first point of $L \cap B_i$ with respect to the ordering of $B_1$. By (1) and (24), $c_2 \neq c$.

If $b_i \neq c_2$, define $C_2$ to be the arc in $B_1$ from $b_i$ to $c_2$. If $b_i = c_2$, let $C_2 = \{c_2\}$. By the argument for (29), $c_2 \in H$.
Let $D_3$ and $D_4$ be open disks such that $B_i \subset D_3 \subset D_1 \cap \Omega_1$ and $C_1 \subset D_4 \subset D_1 \setminus \Omega_1$. Note that $D_3 \cap H \subset E_1$ and $D_4 \cap H \subset E_1$.

It follows from (31) and the arguments for Cases 3.1 and 3.2 that $c_1 \in \text{Cl } K_i$
for each integer $i > i_2$.

Proceeding inductively, we let $n$ be an integer greater than 1. We assume that for each integer $m$ $(1 < m < n)$, an integer $i_m$, a point $c_m$ of $H \setminus \{c\}$, a subset $C_m$ of $B_1$, and disjoint open disks $D_{2m-1}$, $D_{2m}$ have been defined such that

(32m) $c_m$ is the first point of $L \cap B_m$ with respect to the ordering of $B_1$,
(33m) $C_m$ is a minimal connected set containing $\{b_m, c_m\}$,
(34m) $B_m \subset D_{2m-1}$,
(35m) $C_{m-1} \subset D_{2m}$, and
(36m) $c_{m-1} \in \text{Cl } K_i$ for each integer $i \geq i_m$.

For each integer $i$ $(2 < i < 2n)$, let $G_i = D_i \cap H$. We assume that for each positive integer $i$ less than $n$, $(27_i)$ and $(28_i)$ are satisfied.

Let $j_n$ be an integer greater than $i_n$ such that $C_n \cap P(w_j) = \emptyset$ and $J_{j_n} \cap M[z, b_{j_n}] = \emptyset$. Define $i_{n+1}$ to be an integer greater than $j_n$ such that

$$D_{2n} \cap K_i \cap M(w_{i_n+1}, x_{i_n+1}) \neq \emptyset.$$

Let $\Pi_n$ be the arc in $B_1$ from $c_n$ to $c$. Define $\Lambda_n$ to be a polygonal arc segment in $\text{Dom } M(w_{i_n+1}, x_{i_n+1})$ from $(\text{Bd } Y) \setminus H$ to $D_{2n} \cap K_i$ that crosses $\Pi_n$ an odd number of times.

Let $\Sigma_n$ be a simple closed curve in

$$Y \cup \Lambda_n \cup D_{2n} \cup \text{Dom } M(w_{i_n+1}, x_{i_n+1})$$
such that $\Lambda_n \subset \Sigma_n$, $H \cap \Sigma_n \subset D_{2n}$, and $\Pi_n \cap \Sigma_n \subset \Lambda_n$. Since $\Pi_n$ crosses $\Sigma_n$ an odd number of times, $\Sigma_n$ separates $c$ from $c_n$ in $\mathbb{R}^2$.

Define $\Omega_n$ to be the $c$-component of $\mathbb{R}^2 \setminus \Sigma_n$. Let $E_n = \Omega_n \cap (H \setminus D_{2n})$ and $F_n = H \setminus (D_{2n} \cup E_n)$.

To complete the inductive step define $c_{n+1}$, $C_{n+1}$, $D_{2n+1}$, $D_{2n+2}$ satisfying $(32_{n+1}) - (36_{n+1})$, $(27_n)$, and $(28_n)$ when $G_{2n+1} = D_{2n+1} \cap H$ and $G_{2n+2} = D_{2n+2} \cap H$. It follows from the existence of $\{G_n\}_{n=1}^\infty$ that Knaster's continuum is a continuous image of $H$ [8, Theorem 1].

Since Knaster's continuum is indecomposable, $H$ contains an indecomposable continuum $\Phi$ [11, Theorem 4, p. 208]. According to a theorem of J. Krasinkiewicz [10, Theorem 3.1], $\Phi$ has a composant $\Psi$ with the property that

(37) no arc segment in $\mathbb{R}^2 \setminus \Psi$ has an endpoint in $\Psi$.

Note that

(38) $\Psi \cap P(u) = \emptyset$.

To see this assume the contrary. Let $t$ be a point of $\Psi \cap P(u)$. Since $\Psi \subset H \subset M \setminus \text{Int } Y$, it follows from (37) that $t \notin Y$. Let $C$ be the $t$-component of $P(u) \setminus Y$. By (37), $\Psi \cap \text{Dom } C \neq \emptyset$, and this contradicts (26). Hence (38) is true.
Since $M$ is arcwise connected, there exists an arc $A$ in $M$ that intersects both $\Psi$ and $M \setminus \Psi$. By (37), there exist points $r$ and $s$ of $A \cap \Psi$ such that $M(r, s) \not\subset \Psi$. Let $B$ be a continuum in $\Psi$ that contains $\{r, s\}$. Note that $A \cup B$ separates $\mathbb{R}^2$ [13, Theorem 22, p. 175]. By (37), each component of $\mathbb{R}^2 \setminus (A \cup B)$ intersects $\Psi$. Since $\Psi \subset L$, it follows from (38) that $A \cup \varphi$ contains a simple closed curve, and this violates the unique arcwise connectivity of $M$. This contradiction completes the proof of Lemma 2. □

**Lemma 3.** For each positive integer $n$, $P(a_n)$ intersects $M \setminus L$.

**Proof.** Suppose $P(a_n) \subset L$ for some positive integer $n$. Assume without loss of generality that $P \subset L$. It follows from the proof of Lemma 2 (with Cases 1.2, 2.2, and 3.2 deleted) that this assumption involves a contradiction. □

We assume without loss of generality that $D$ (in Lemma 2) is the unbounded complementary domain of $L$ and $P \subset \text{Cl} \ D$. We also assume without loss of generality that $a_1 \in D$ (Lemma 3).

Let $L'$ be a subcontinuum of $L$. Define $D'$ to be the unbounded complementary domain of $L'$. Note that $D \subset D'$.

Let $X$ be the nonseparating plane continuum $\mathbb{R}^2 \setminus D'$. Since $M$ is arcwise connected, there is an arc segment $S$ in $M \cap D'$ with an endpoint $s$ in $X$. Since $M$ does not contain a simple closed curve, we can assume without loss of generality that $P \cap \text{Cl} \ S = \emptyset$.

Sieklucki [15, Lemma 5.5, p. 267] proved that $X$ has the following properties. There exists a sequence $\{Q_n\}_{n=1}^\infty$ of disks in $\mathbb{R}^2$ such that $X = \cap_{n=1}^\infty Q_n$ and for each $n$,

(i) $Q_{n+1} \subset \text{Int} \ Q_n$,

(ii) the boundary $B_n$ of $Q_n$ is a polygonal simple closed curve with consecutive vertices $b_n^1, b_n^2, \ldots, b_n^{\mu_n}, b_n^{\mu_n+1} = b_n^1$, and

(iii) for $j = 1, 2, \ldots, \mu_n$, the interval in $B_n$ from $b_n^j$ to $b_n^{j+1}$ has diameter less than $2^{-n}$.

For every $b_n^j$ ($n = 1, 2, \ldots$ and $j = 1, 2, \ldots, \mu_n$) there exists a vertex $b_{n+1}^{s(j)}$ such that

(iv) the interval $N_n^j$ in $\mathbb{R}^2$ from $b_n^j$ to $b_{n+1}^{s(j)}$ has diameter less than $2^{-n}$,

(v) $N_n^j \setminus \{b_n^j, b_{n+1}^{s(j)}\} \subset (\text{Int} \ Q_n) \setminus Q_{n+1}$, and

(vi) $N_n^j \cap N_n^k = \emptyset$ for each integer $k \neq j$ ($1 \leq k \leq \mu_n$).

Let $N = \cup_{n=1}^\infty \cup_{j=1}^{\mu_n} N_n^j$. Note that each component of $N$ is a half-open arc in $\mathbb{R}^2 \setminus X$ with an endpoint in $X$.

Let $m$ be a given positive integer. Define $N_m$ to be the union of all components of $N$ that intersect $B_m$. Let $O_m$ be a subset of $N_m$ that is maximal with respect to the property that each component of $O_m$ is a component of $N_m$ and each pair of components of $O_m$ with a common endpoint is separated in $Q_m \setminus X$ by another pair of components of $O_m$.  

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Let $c_m^1, c_m^2, \ldots, c_m^x, c_m^{x+1} = c_m^1$ denote the consecutive vertices of $B_m$ that belong to $O_m$. Since $X$ is not degenerate, we can assume without loss of generality that $\xi_m > 3$. Assume without loss of generality that $B_m \cap S \neq \emptyset$.

Let $n$ be an integer greater than $m$. Define $E$ to be the closure of a component of $Q_n \setminus (O_m \cup X)$. We call $E$ an $(m, n)$-link on $X$. The polygonal arc $B_n \cap E$ is called the bottom of $E$. The two components of $E \cap O_m$ are called the sides of $E$. Note that the sides of $E$ are half-open arcs in $Q_n \setminus X$ with distinct endpoints in $X$. The diameter of the union of the sides of $E$ is less than $2^{3-m}$.

For $j = 1, 2, \ldots, \xi_m$, let $E_j$ be the $(m, n)$-link whose sides are contained in the components of $O_m$ that intersect $\{c_m^j, c_m^{j+1}\}$.

Suppose there exist two $(m, n)$-links $E$ and $F$ that have a common side such that $Q_n \cap S \subset E \cup F$ and $\text{Cl} S$ misses the closure of each uncommon side of $E$ and $F$. Change the indexing of the $(m, n)$-links (if necessary) so that $E = E_1$, $F = E_{\xi_m}$, and each pair of consecutive links has a common side.

Define $F_1$ to be the closure of the component of $(E_1 \cup E_{\xi_m}) \setminus (S \cup X)$ that contains a side of $E_2$. Let $F_j = E_j$ for $1 < j < \xi_m$. Define $F_{\xi_m}$ to be the closure of the component of $(E_1 \cup E_{\xi_m}) \setminus (S \cup X)$ that contains a side of $E_{\xi_m-1}$. We call $\mathcal{F} = \{F_j : 1 < j < \xi_m\}$ an $m$-chain on $(X, S)$. We call $F_1$ and $F_{\xi_m}$ the end links of $\mathcal{F}$. Each $F_j$ ($1 < j < \xi_m$) is called an interior link of $\mathcal{F}$. Let $T$ be the arc in $\text{Cl} S$ that is irreducible between $s$ and $B_n$. The half-open arc $T \setminus \{s\}$ is called the common side of $F_1$ and $F_{\xi_m}$.

Since $S$ is an arc segment, for each positive integer $m$, there exists an $m$-chain on $(X, S)$.

**Lemma 4.** For each positive integer $i$, there exists an $m$-chain $\mathcal{F}$ on $(X, S)$ such that $m > i$ and no pair of consecutive links of $\mathcal{F}$ contains $\text{Bd} X$ in its union.

**Proof.** Let $m$ and $m'$ be integers such that $0 < m < m'$. Suppose $\mathcal{F}$ is an $m$-chain on $(X, S)$ and $U$ is the union of a pair of consecutive links of an $m'$-chain on $(X, S)$. Then the union of some pair of consecutive links of $\mathcal{F}$ contains $U \cap \text{Bd} X$. Hence it is sufficient to show that there exists an $m$-chain $\mathcal{F}$ on $(X, S)$ such that no pair of consecutive links of $\mathcal{F}$ contains $\text{Bd} X$ in its union.

Assume that for each positive integer $m$, every $m$-chain on $(X, S)$ has a pair of consecutive links whose union contains $\text{Bd} X$. Then for each $m$, there exist a positive number $\epsilon_m$, a pair of consecutive links $E_m, F_m$ of an $m$-chain on $(X, S)$, and an arc segment $A_m$ in $B_m \setminus O_m \cup S$ such that $\{\epsilon_m\}_{m=1}^\infty$ converges to zero, $\text{Bd} X \subset E_m \cup F_m$, $A_m$ has diameter less than $\epsilon_m$ and contains the uncommon sides of $E_m$ and $F_m$, and $A_m \cup X$ separates $(\text{Int} E_m) \setminus X$ from $R^2 \setminus Q_m$ in $R^2$.

For each positive integer $m$, let $x_m$ and $y_m$ be the endpoints of $A_m$. Note
that for each $m$, $\{x_m, y_m\} \subseteq \text{Bd } X$. For each $m$, let $W_m$ be the complementary domain of $A_m \cup X$ whose closure contains $E_m \cup F_m$. Let $y$ be a limit point of $\{y_m\}_{m=1}^{\infty}$.

The continuum $\text{Bd } X$ is nonaposyndetic at $y$ with respect to each point of $(\text{Bd } X) \setminus \{y\}$ [9]. For assume otherwise. Then a continuum $Y$, an open disk $G$, and a point $z$ of $(\text{Bd } X) \setminus \text{Cl } G$ exist such that

$$y \in G \cap \text{Bd } X \subset Y \subset (\text{Bd } X) \setminus \{z\}.$$  

Let $Z$ be an open disk such that $z \in Z \subset R^2 \setminus (G \cup Y)$.

Let $i$ be an integer such that $B_i \cap Z \neq \emptyset$. Define $m$ to be an integer greater than $i$ such that $A_m \subset G$. Let $p$ be a point of $Z \cap (Q_i \setminus Q_m)$. Let $q$ be a point of $W_m \cap Z$.

There exists a polygonal arc $I$ in $Q_i \setminus X$ from $p$ to $q$ such that $A_m \cap I$ is finite and $I$ crosses $A_m$ at each point of $A_m \cap I$. Since $A_m$ separates $p$ from $q$ in $Q_i \setminus X$, $I$ crosses $A_m$ an odd number of times. It follows that $I \cup Z$ contains a simple closed curve that separates $x_m$ from $y_m$ in $R^2$. Since $\{x_m, y_m\} \subset Y \subset R^2 \setminus (I \cup Z)$, this violates the connectivity of $Y$. Hence $\text{Bd } X$ is nonaposyndetic at $y$ with respect to each point of $(\text{Bd } X) \setminus \{y\}$.

According to a theorem of H. E. Schlais [14, Theorem 9], [8, Theorem 4], $\text{Bd } X$ contains an indecomposable continuum $\Phi$. Let $\Psi$ be a composant of $\Phi$ with the property that no arc segment in $R^2 \setminus \Psi$ has an endpoint in $\Phi$. Let $^n$ be an arc in $\Psi$ that is irreducible between $M[a, v]$. Since $M[a, v] \cap \text{Int } F = \emptyset$, and each complementary domain of $J \cup M[a, v]$ intersects $\Psi$, for each positive integer $n$, $J \cap P(a_n) \neq \emptyset$. Thus $J \cap L \neq \emptyset$, and this contradicts the definition of $J$. Hence $\Psi \cap \text{P } = \emptyset$.

From the last paragraph in the proof of Lemma 2, we see that the existence of $\Psi$ implies that $M$ contains a simple closed curve. This contradiction completes the proof of Lemma 4.

**Lemma 5.** Suppose $F$ is an element of an $m$-chain $\mathcal{F}$, $u$ is a point of $(\text{P } \cap \text{Int } F) \setminus X$, $v$ is a point of $P \setminus F$, and $\text{St } \mathcal{F}$ contains $M[u, v]$. Then $M[u, v]$ intersects the closure of a side of $F$.

**Proof.** Assume $M[u, v]$ misses the closure of each side of $F$. By Lemma 3, there is a point $y$ of $P(v)$ in $D$. Let $x$ be a point of $X \cap M[u, v] \cap \text{Bd } F$ such that every arc in $M[u, x]$ from $P \setminus X$ to $x$ intersects $\text{Int } F$, and every arc in $M[x, y]$ from $x$ to $P \setminus X$ intersects $P \setminus F$. Let $J$ be an arc in $D$ that is irreducible between $M[a, x]$ and $M[x, y]$.

The continuum $X \cap \text{Bd } F$ straddles every arc in $P$ that contains $x$ and has both endpoints in $D$. Consequently each complementary domain of $J \cup M[a, y]$ intersects $X \cap \text{Bd } F$. Since $M[a, y] \cap P(y) = \emptyset$, it follows that
$J \cap L \neq \emptyset$, and this contradicts the definition of $J$. Hence $M[u, v]$ intersects the closure of a side of $F$. \hfill \Box

In the remaining part of this section we assume $L = L'$. Hence $D = D'$ and $L$ is the boundary of the nonseparating plane continuum $X$.

Let $\mathcal{F} = \{F_j: 1 < j < \xi_m\}$ be an $m$-chain on $(X, S)$. Let $\mathcal{G}$ be the set of all elements $F_j$ of $\mathcal{F}$ such that for each point $u$ of $P$, $P(u)$ intersects $(\text{Int } F_j) \setminus X$. It follows from Lemma 3 that $\mathcal{G}$ is not empty. Let $F_i$ and $F_k$ be the first and last links, respectively, of $\mathcal{F}$ that belong to $\mathcal{G}$.

**Lemma 6.** Suppose $B$ is an arc segment in $F_j \setminus (X \setminus F_l) (l < j < k)$ that has an endpoint in $X$. Then there exists a point $u$ of $P$ such that each arc in $P(u)$ that intersects both $(\text{Int } F_j) \setminus X$ and $(\text{Int } F_k) \setminus X$ also intersects $B$.

**Proof.** Let $\mathcal{K}$ be an $m$-chain on $(X, S)$ such that $B$ intersects the bottom of a link of $\mathcal{K}$. Define $u$ to be a point of $P$ such that $P(u) \subset (\text{St } \mathcal{K}) \setminus ((\text{Cl } B) \setminus B)$.

Suppose there is an arc $M[v, w]$ in $P(u) \setminus B$ that intersects both $(\text{Int } F_j) \setminus X$ and $(\text{Int } F_k) \setminus X$. Assume without loss of generality that $v \in (\text{Int } F_i) \setminus X$ and $w \in (\text{Int } F_k) \setminus X$.

Let $V$ be the $t$-component of $(\text{St } \mathcal{K}) \setminus (B \cup S \cup X)$. Note that $V$ misses $F_k$ and contains $(\text{St } \mathcal{K}) \cap ((\text{Int } F_j) \setminus X)$. Since $M[v, w] \cap \text{Cl}(B \cup S) = \emptyset$, the continuum $X \cap \text{Bd } V$ straddles each subarc of $M[v, w]$ that has one endpoint in $V$ and the other endpoint in $M \setminus \text{Cl } V$.

Since $F_i \in \mathcal{G}$, there is a point $z$ of $P(w)$ in $(\text{Int } F_i) \setminus X$. Let $J$ be an arc in $V$ that is irreducible between $M[v, w]$ and $M[w, z]$. Each complementary domain of $J \cup M[v, z]$ intersects $X \cap \text{Bd } V$. Since $M[v, z] \cap P(z) = \emptyset$, it follows that $J \cap L \neq \emptyset$, and this contradicts the definition of $J$. Hence each arc in $P(u)$ that intersects $(\text{Int } F_i) \setminus X$ and $(\text{Int } F_k) \setminus X$ intersects $B$. \hfill \Box

It follows from Lemma 6 that $\mathcal{G}$ is the subchain $\{F_j: i < j < k\}$ of $\mathcal{F}$.

**Definition.** Suppose $K$ is an arcwise connected subset of $M$ that is contained in $(\text{St } \mathcal{G}) \setminus \text{Bd}(X \cup \text{St } \mathcal{F})$ and intersects $\text{Int } F_i$ and $\text{Int } F_k$. Then $K$ is a *trace* of $\mathcal{G}$ if for each arc $A$ in $K$, there exists a function $g$ of $A$ into $\mathcal{G}$ with the following properties:

1. For each point $a$ of $A$, $a \in g(a)$.
2. If $a$ and $b$ are points of $A$ and $g(a) \neq g(b)$, then $M[a, b]$ intersects a side of $g(a)$ and the interior of each link of $\mathcal{G}$ between $g(a)$ and $g(b)$ (with respect to the index ordering of $\mathcal{G}$).

**Definition.** An arcwise connected set $K$ agrees with $\mathcal{G}$ if $K$ is a trace of $\mathcal{G}$, $\mathcal{G} \setminus \{F_i\}$, $\mathcal{G} \setminus \{F_k\}$, or $\mathcal{G} \setminus \{F_i, F_k\}$.

**Lemma 7.** There exists a point $u$ of $P \cap \text{Int } F_i$ such that $P(u)$ is a trace of $\mathcal{G}$.

**Proof.** Let $W = \{x \in X: x$ is an endpoint of a side of a link of $\mathcal{G}\}$. Define $u$ to be a point of $P \cap ((\text{Int } F_i) \setminus X)$ such that $P(u)$ is contained in $\text{St } \mathcal{F}$ and
misses $W \cup \text{Bd}(X \cup \text{St } F)$ and $\bigcup \{(\text{Int } F_j) \setminus X: 1 \leq j < i \text{ or } k < j < \xi_m\}$.

Using Lemma 5, we define a function $g^*$ of $P(u)$ onto $\mathcal{G}$ such that
(i) $v \in g^*(v)$ for each point $v$ of $P(u)$ and
(ii) if $v$ and $w$ are points of $P(u)$ and $g^*(v) \neq g^*(w)$, then $M[v, w]$ intersects a side of $g^*(v)$ and the interior of each link of $\mathcal{G}$ between $g^*(v)$ and $g^*(w)$.

By considering the restriction of $g^*$ to each arc in $P(u)$, we see that $P(u)$ is a trace of $\mathcal{G}$.

3. Principal result.

Theorem. If $M$ is a uniquely arcwise connected plane continuum, then $M$ has the fixed-point property.

Proof. Assume there exists a continuous function $f$ of $M$ into $M$ that moves each point of $M$. Let $\epsilon$ be a positive number such that
(1) $\rho(z, f(z)) > \epsilon$ for each point $z$ of $M$.

According to Borsuk [5], there exists a sequence $\{a_n\}_{n=1}^{\infty}$ of points of $M$ such that for each $n$,
(2) $\rho(a_n, a_{n+1}) = \epsilon/3$ [5, p. 19, (4), n],
(3) if $z \in M(a_n, a_{n+1})$, then $\rho(a_n, z) < \epsilon/3$ [5, p. 19, (5), n],
(4) $M[a_1, a_n] \cap M[a_n, a_{n+1}] = \{a_n\}$ for $n > 1$ [5, p. 19, (11)], and
(5) $a_n, a_{n+1} \subset M[a_1, f(a_n)]$ [5, p. 19, (7), (13)].

For each positive integer $n$, let $\psi_n$ be a homeomorphism of the half-open real line interval $[n, n + 1)$ onto $M[a_n, a_{n+1})$. For each point $x$ of $[1, + \infty)$, let $\psi(x) = \psi_n(x)$ if $n < x < n + 1$.

Let $P = \bigcup_{n=2}^{\infty} M[a_1, a_n]$. It follows from (4) that $\psi$ is a one-to-one continuous function of $[1, + \infty)$ onto $P$.

The function $\psi$ determines a linear ordering $\prec$ of $P$ with $a_1$ as the first point. As in §2, for each point $u$ of $P$, we let $P(u)$ denote $\{v \in P: u = v \text{ or } u \prec v\}$.

For each point $u$ of $P$,
(6) $u \in M[a_1, f(u)]$.
To see this assume $u \notin M[a_1, f(u)]$. Suppose $u \in M[a_n, a_{n+1})$. Since $M$ does not contain a simple closed curve, $a_{n+1} \notin M[a_1, f(u)]$. By (1), (2), and (3), $a_{n+1} \notin f[M[u, a_{n+1})]$. Thus $M[a_1, f(u)] \cup f[M[u, a_{n+1}]]$ misses $a_{n+1}$ and contains an arc that runs from $a_1$ to $f(a_{n+1})$, and this contradicts (5). Hence (6) holds.

By (2), $P \not\subset M[a_1, f(a_1)]$. Since $M$ does not contain a simple closed curve, there exists a point $a$ of $P$ such that $P(a) \cap M[a_1, f(a_1)] = \{a\}$.

Next we prove that
(7) $a \in f[M[a_1, a]]$.

Statement (7) is obviously true if $f(a_1) = a$, so we assume $f(a_1) \neq a$. Suppose $a \in M[a_n, a_{n+1})$. By (5), $a_2 \in M[a_1, f(a_1)]$. Hence $M[a_1, a_2] \subset$
$M[a_1, f(a_1)]$ and $n > 1$. Note that $a_{n+1} \notin M[a_1, f(a_1)]$ and $f(a_1) \notin M[a_1, a_{n+1}]$. By (5), $a_{n+1} \in M[a_1, f(a_n)]$. Since $M$ does not contain a simple closed curve, it follows that $M[a, f(a_n)] \cap M[a, f(a_1)] = \{a\}$.

Suppose (7) is false. Then $a \notin f[M[a_1, a_n]]$. Consequently

$$M[a, f(a_1)] \cup M[a, f(a_n)] \cup f[M[a_1, a_n]]$$

contains a simple closed curve, and this violates the unique arcwise connectivity of $M$. Hence (7) is true.

For each point $c$ of $P(a)$,

(8) $c \in f[M[a_1, c]]$.

To establish (8) assume the contrary. Let $y$ be the point of $P(a)$ that is the greatest lower bound of $\{z \in P(a): z \notin f[M[a_1, z]]\}$ relative to $\prec$. By (7) and the continuity of $f$, there exists a point $x$ of $M[a_1, y)$ such that $f(x) = y$. Assume without loss of generality that $y \notin f[M(x, y)]$.

Suppose $x \in M[a_1, a_{n+1})$ and $y \in M[a_n, a_{n+1})$. By (1), (2), and (3), $n > i + 1$. Since $M$ does not contain a simple closed curve, $M[y, a_{n+1}] \cap f[M[x, a_{n+1}]] = \{y\}$. By (5), $a_{n+1} \in M[a_1, f(a_n))$. Therefore $M[y, f(a_n)] \cap M[y, f(a_{n+1})] = \{y\}$. Since $y \notin f[M(x, y)]$, it follows that

$$M[y, f(a_n)] \cup M[y, f(a_{n+1})] \cup f[M[a_{n+1}, a_n]]$$

contains a simple closed curve, and this violates the unique arcwise connectivity of $M$. Hence (8) is true.

For each integer $i > 1$,

(9) there exists a positive integer $n$ such that $P(a_n) \cap f[M[a_1, a_i]] = \emptyset$.

To see this assume there exists an integer $i > 1$ such that for each positive integer $n$, $P(a_n) \cap f[M[a_1, a_i]] \neq \emptyset$. Since $M$ does not contain a simple closed curve, there exists a positive integer $j$ such that $P(a_j) \subset f[M[a_1, a_i]]$. Since $f[M[a_1, a_i]]$ is a dendrite, $\overline{P(a_j)}$ is a dendrite [11, Theorem 4, p. 301], and this contradicts (2) (see Theorem 5 of [11, p. 302]). Therefore (9) is true.

Let $L = \cap_{n=1}^{\infty} \overline{P(a_n)}$. It follows from (2) that $L$ is not degenerate. Hence $L$ is a continuum.

According to Lemma 2, there exist a complementary domain $D$ of $L$ and a positive integer $\alpha$ such that $P(a_\alpha) \subset \overline{D}$. Assume without loss of generality that $P \subset \overline{D}$, $a_1 \in D$ (Lemma 3), and $D$ is the unbounded complementary domain of $L$.

Let $X$ be the continuum $R^2 \setminus D$. Since $M$ is arcwise connected, there is an arc segment $S$ in $M \cap D$ with an endpoint in $X$. Since $M$ does not contain a simple closed curve, we can assume without loss of generality that $P \cap \overline{S} = \emptyset$.

Using Sieklucki's nested sequence of polygonal disks (described in §2 above), define a sequence $\{\mathscr{T}_m\}_{m=1}^{\infty}$ with the property that for each $m$, $\mathscr{T}_m$ is an $m$-chain on $(X, S)$ refined by $\mathscr{T}_{m+1}$.
For each positive integer \( m \), let \( \mathcal{G}_m \) be the set of all elements \( F \) of \( \mathcal{F}_m \) such that for each point \( u \) of \( P \), \( P(u) \) intersects \( (\text{Int} \, F) \setminus X \). By Lemma 6, for each \( m \), \( \mathcal{G}_m \) is a subchain of \( \mathcal{F}_m \). Note that if \( m \) and \( n \) are integers and \( 0 < m < n \), then \( \mathcal{G}_n \) refines \( \mathcal{G}_m \) and each end link of \( \mathcal{G}_m \) contains an end link of \( \mathcal{G}_n \).

For each positive integer \( m \), let \( G_1^m, G_2^m, \ldots, G_{\lambda_m}^m \) be the consecutive links of \( \mathcal{G}_m \).

By Lemma 7, for each positive integer \( m \), there exists a point \( u_m \) of \( P \cap \text{Int} \, G_m^m \) such that \( P(u_m) \) is a trace of \( \mathcal{G}_m \). Hence for each \( m \), \( L \subset \text{St} \, \mathcal{G}_m \).

Since \( M \) does not contain a simple closed curve, it follows from the proof of Lemma 6 that for each \( m \),

(10) no arc segment in \((M \setminus X) \cap \text{St}\{G_i^m: 1 < i < \lambda_m\}\) has an endpoint in \( L \).

For each positive integer \( m \), there exists an arc \( B_m \) in \( P(u_m) \) such that \( B_m \subset (\text{St} \, \mathcal{G}_m) \setminus X \) and \( B_m \) is a trace of \( \mathcal{G}_m \). To see this let \( v \) be a point of \( P(u_m) \cap \text{Int} \, G_m^m \). By (8), (9), and (10), there exist points \( w \) and \( x \) of \( P(v) \) such that \( f(w) \in P(w) \cap \text{Int} \, G_m^m \), \( f(x) \in P(x) \cap \text{Int} \, G_m^m \), and the arc \( M[f(w), f(x)] \) is between \( L \) and \( M[u_m, v] \) in \( \text{St} \, \mathcal{G}_m \). By (6), \( M[u_m, v] \cap f[M[w, x]] = \emptyset \). Since \( M \) does not contain a simple closed curve, it follows from (10) that there exists a subarc \( B_m \) of \( M[w, x] \) such that \( f[B_m] \subset (\text{St} \, \mathcal{G}_m) \setminus X \) and \( f[B_m] \) is a trace of \( \mathcal{G}_m \).

Note that \( X \) has the following property:

**Reduction Property.** The continuum \( X \) does not separate \( R^2 \) and there exist an arc segment \( S \) in \( M \setminus X \) with an endpoint in \( X \), a sequence \( \{A_m\}_{m=1}^\infty \) of arcs in \( P \) converging to \( \text{Bd} \, X \), and a sequence \( \{\mathcal{G}_m\}_{m=1}^\infty \) of chains such that for each \( m \),

(i) \( \mathcal{G}_m \) is a subchain of an \( m \)-chain on \((X, S)\),

(ii) \( \mathcal{G}_{m+1} \) refines \( \mathcal{G}_m \),

(iii) each end link of \( \mathcal{G}_m \) contains an end link of \( \mathcal{G}_{m+1} \),

(iv) \( A_m \) agrees with \( \mathcal{G}_m \), and

(v) either \( f[A_m] \subset (\text{St} \, \mathcal{G}_m) \setminus X \) or there exists a subarc \( B_m \) of \( A_m \) such that \( f[B_m] \subset (\text{St} \, \mathcal{G}_m) \setminus X \) and \( f[B_m] \) agrees with \( \mathcal{G}_m \).

Next we prove that \( X \) contains a continuum that is irreducible with respect to the Reduction Property. Assume \( \{X_n\}_{n=1}^\infty \) is a nested sequence of nonseparating plane continua in \( X \). For each \( n \), assume there exist an arc segment \( S_n \) in \( M \setminus X_n \) that has an endpoint in \( X_n \), a sequence \( \{A_n\}_{n=1}^\infty \) of arcs in \( P \) converging to \( \text{Bd} \, X_n \), and a sequence \( \{\mathcal{G}_n\}_{n=1}^\infty \) of chains with the following property. For each \( m \), \( \mathcal{G}_n^m \) is a subchain of an \( m \)-chain on \((X_n, S_n)\), \( \mathcal{G}_{n+1}^m \) refines \( \mathcal{G}_n^m \), each end link of \( \mathcal{G}_n^m \) contains an end link of \( \mathcal{G}_{n+1}^m \), \( A_n^m \) agrees with \( \mathcal{G}_n^m \), and either \( f[A_n^m] \subset (\text{St} \, \mathcal{G}_n^m) \setminus X_n \) or there exists a subarc \( B_n^m \) of \( A_n^m \) such that \( f[B_n^m] \subset (\text{St} \, \mathcal{G}_n^m) \setminus X_n \) and \( f[B_n^m] \) agrees with \( \mathcal{G}_n^m \). Let \( X_0 = \cap_{n=1}^\infty X_n \).

According to the Brouwer reduction theorem \[16, \text{p. 17}\], it is sufficient to prove that \( X_0 \) is a continuum with the Reduction Property.
Since \( f \) is continuous, for each positive integer \( n \), either \( f[\partial X_n] \subseteq \partial X_n \) or \( \partial X_n \subseteq f[\partial X_n] \). Since \( \{\partial X_n\}_{n=1}^{\infty} \) converges to \( \partial X_0 \), it follows that \( f[\partial X_0] \subseteq \partial X_0 \) or \( \partial X_0 \subseteq f[\partial X_0] \). Since \( f \) moves each point of \( M \), \( \partial X_0 \) is not degenerate. Hence \( X_0 \) is a continuum.

Since \( R^2 \setminus X_0 = \bigcup_{n=1}^{\infty} R^2 \setminus X_n \) and for each \( n \), \( R^2 \setminus X_n \) is connected, it follows that

\[(11) \ R^2 \setminus X_0 \text{ is connected.}\]

Since \( M \) is arcwise connected, there is an arc segment \( S_0 \) in \( M \setminus X_0 \) with an endpoint in \( X_0 \). Assume without loss of generality that \( P \cap \text{Cl} \ S_0 = \emptyset \).

Define a sequence \( \{\gamma_m^0\}_{m=1}^{\infty} \) with the property that for each \( m \), \( \gamma_m^0 \) is an \( m \)-chain on \( (X_0, S_0) \) refined by \( c_{\gamma_m^0} \).

There exists a sequence \( \{\gamma_m^0\}_{m=1}^{\infty} \) of chains such that for each \( m \),

\[(12) \ \gamma_m^0 \text{ is a subchain of } \gamma_m^0,\]

\[(13) \ \gamma_m^{0+1} \text{ refines } \gamma_m^0,\]

\[(14) \text{ each end link of } \gamma_m^0 \text{ contains an end link of } \gamma_m^{0+1},\]

\[(15) \text{ there exist integers } i_m \text{ and } j_m \text{ (} j_m > m \text{) such that}\]

\[\text{(i) } \text{St } \gamma_i^m \subseteq (\text{St } \gamma_m^0) \setminus \text{Bd}(X_0 \cup \text{St } \gamma_m^0),\]

\[\text{(ii) the interior of each interior link of } \gamma_m^0 \text{ contains the sides of two consecutive links of } \gamma_i^m,\]

\[\text{(iii) no endpoint of a side of a link of } \gamma_m^0 \text{ belongs to } A_{i_m}^0 \cup f[A_{i_m}^0] \text{ (recall (6)), and}\]

\[\text{(iv) the Hausdorff distance [11, p. 47] from } A_{i_m}^0 \text{ to } \text{Bd } X_{j_m} \text{ is less than } m^{-1}.\]

For each positive integer \( m \), let \( A_m^0 = A_{i_m}^0 \). Since \( \{\partial X_n\}_{n=1}^{\infty} \) converges to \( \partial X_0 \), it follows from (15(iv)) that

\[(16) \ \{A_m^0\}_{m=1}^{\infty} \text{ converges to } \partial X_0.\]

By (15(i)–(iii)) and Lemma 5, for each positive integer \( m \),

\[(17) \ A_m^0 \text{ agrees with } \gamma_m^0,\]

\[(18) \ f[A_m^0] \subseteq (\text{St } \gamma_m^0) \setminus X_0,\]

\[(19) \text{ there exists a subarc } B_m^0 \text{ of } A_m^0 \text{ (} B_m^0 = B_{i_m}^0 \text{) such that } f[B_m^0] \subseteq (\text{St } \gamma_m^0) \setminus X_0 \text{ and } f[B_m^0] \text{ agrees with } \gamma_m^0.\]

It follows from (11)–(14) and (16)–(19) that \( X_0 \) has the Reduction Property. Hence there exists a subcontinuum of \( X \) that is irreducible with respect to the Reduction Property.

For convenience we assume that

\[(20) \text{ no proper subcontinuum of } X_0 \text{ has the Reduction Property.}\]

According to Lemma 4, there exists a positive integer \( \beta \) such that \( \epsilon > 2^{1-\beta} \) and no pair of consecutive links of \( \gamma_m^0 \) contains \( \partial X_0 \) in its union.

By (6), \( S_0 \cap f[P(u)] = \emptyset \) for some point \( u \) of \( P \). Assume without loss of generality that for each integer \( m > \beta \),

\[(21) S_0 \cap f[A_m^0] = \emptyset.\]
Let $G_1, G_2, \ldots, G_\gamma$ be the consecutive links of $\mathcal{G}_\beta$. For $i = 1, 2, \ldots, \gamma - 1$, let $V_i = \bigcup_{j=i}^{i+1} G_j$, and let $W_i$ be the common side of $G_i$ and $G_{i+1}$.

Let $\mathcal{W} = \{W_i: 2 \leq i \leq \gamma - 2\}$. Note that each element of $\mathcal{W}$ has diameter less than $\varepsilon$.

For $m = \beta, \beta + 1, \ldots$ and $i = 2, 3, \ldots, \gamma - 2$, let $W_i^m = A_0^m \cap W_i$.

**Definition.** A point $x$ of $W_i^m$ is sent back by $f$ if $f(x) \in V_i$; otherwise, $x$ is sent forward by $f$.

**Definition.** The arc $A_0^m$ has the switch property if a component of $A_0^m \setminus \operatorname{St} \mathcal{W}$ has endpoints in $\operatorname{St} \mathcal{W}$ that are sent in opposite directions by $f$.

Statement (18) is true for only finitely many integers $m > \beta$. To see this assume the contrary. Suppose without loss of generality that (18) is true for each integer $m > \beta$.

For each integer $m > \beta$, if $f$ sends two points of $\bigcup_{i=2}^{\gamma-2} W_i^m$ in opposite directions, then, by (1), (18), and (21), $A_0^m$ has the switch property.

Suppose that for infinitely many integers $m > \beta$, two points of $\bigcup_{i=2}^{\gamma-2} W_i^m$ are sent in opposite directions by $f$. Then infinitely many elements of $\{A_0^m\}_{m=\beta}^{\infty}$ have the switch property. Assume without loss of generality that there exists a component $G$ of $(\operatorname{St} \mathcal{G}_\beta) \setminus (S_0 \cup X_0 \cup \operatorname{St} \mathcal{W})$ such that for each integer $m > \beta$, $A_0^m$ has the switch property on a component $T_m$ of $A_0^m \setminus \operatorname{St} \mathcal{W}$ that is contained in $\operatorname{Cl} G$.

For each integer $m > \beta$, we have three cases.

**Case 1.** Suppose $f[\operatorname{Cl} T_m] \subset G$. Then $\operatorname{Cl} G$ is a link of $\mathcal{G}_0^m$ and $T_m$ has an endpoint in each side of $\operatorname{Cl} G$.

**Case 2.** Suppose $f[\operatorname{Cl} T_m]$ intersects two components of $(\operatorname{St} \mathcal{G}_\beta) \setminus (G \cup S_0 \cup X_0)$. Then $\operatorname{Cl} G$ is a link of $\mathcal{G}_0^m$ and there exists an arc $A$ in $\operatorname{Cl} T_m$ such that $f[A] \subset \operatorname{Cl} G$ and $f[A]$ intersects each side of $\operatorname{Cl} G$.

**Case 3.** Suppose $f[\operatorname{Cl} T_m]$ intersects only one component of $(\operatorname{St} \mathcal{G}_\beta) \setminus (G \cup S_0 \cup X_0)$. Then there exist an element $W_i$ of $\mathcal{W}$ in $\operatorname{Cl} G$ and an arc $A$ in $\operatorname{Cl} T_m$ with an endpoint in $W_i$ such that $f[A] \subset \operatorname{Cl} G$ and $W_i \cap f[A] \neq \emptyset$.

Since one of these three cases holds for infinitely many elements of $\{T_m\}_{m=\beta}^{\infty}$, there is a continuum $Y$ in $X_0 \cap \operatorname{Cl} G$ with the following properties.

A sequence $\{H_m\}_{m=1}^{\infty}$ of arcs in $P$ converging to $Y$ and a sequence $\{\mathcal{K}_m\}_{m=1}^{\infty}$ of chains exist such that for each positive integer $m$,

(22) $\operatorname{St} \mathcal{K}_m \subset \operatorname{Cl} G$,
(23) $\mathcal{K}_m$ is a subchain of $\mathcal{G}_0^m$,
(24) $\mathcal{K}_{m+1}$ refines $\mathcal{K}_m$,
(25) each end link of $\mathcal{K}_m$ contains an end link of $\mathcal{K}_{m+1}$,
(26) $H_m$ agrees with $\mathcal{K}_m$, and
(27) either $f[H_m] \subset (\operatorname{St} \mathcal{K}_m) \setminus X_0$ or there exists a subarc $I_m$ of $H_m$ such that $f[I_m] \subset (\operatorname{St} \mathcal{K}_m) \setminus X_0$ and $f[I_m]$ agrees with $\mathcal{K}_m$.

Let $U$ be the complementary domain of $Y$ that contains $D$. Let $X'$ be the nonseparating plane continuum $R^2 \setminus U$. Note that $Y = \operatorname{Bd} X'$. 

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Since $M$ is arcwise connected, there exists an arc segment $S'$ in $M \setminus X'$ with an endpoint in $X'$. Since $M$ does not contain a simple closed curve, we can assume without loss of generality that $P \cap \text{Cl } S' = \emptyset$.

Define a sequence \( \{ \mathcal{K}_m \}_{m=1}^{\infty} \) with the property that for each $m$, $\mathcal{K}_m$ is an $m$-chain on $(X', S')$ refined by $\mathcal{K}_{m+1}$.

There exists a sequence \( \{ \mathcal{K}'_m \}_{m=1}^{\infty} \) of chains such that for each $m$,

(28) $\mathcal{K}'_m$ is a subchain of $\mathcal{K}_m$,

(29) $\mathcal{K}'_{m+1}$ refines $\mathcal{K}_m$,

(30) each end link of $\mathcal{K}_m$ contains an end link of $\mathcal{K}_{m+1}$, and

(31) there exists an integer $i_m$ such that

(i) $\text{St } \mathcal{K}_m \subset (\text{St } \mathcal{K}'_m) \setminus \text{Bd}(X' \cup \text{St } \mathcal{K}_m)$,

(ii) the interior of each interior link of $\mathcal{K}_m$ contains the sides of two consecutive links of $\mathcal{K}'_m$, and

(iii) no endpoint of a side of a link of $\mathcal{K}_m$ belongs to $H_{i_m} \cup f[H_{i_m}]$.

It follows from (26)–(31) that $X'$ has the Reduction Property. Since $\text{Cl } G$ is either a link or the union of two consecutive links of $\mathcal{G}_\beta$, $\text{Bd } X_0 \subset \text{Bd } X'$. But $X'$ and $X_0$ are nonseparating plane continua and $\text{Bd } X' \subset \text{Bd } X_0$. Consequently $X'$ is a proper subcontinuum of $X_0$, and this contradicts (20). Hence for all but finitely many integers $m > \beta$, $f$ sends each point of $\bigcup_{i=2}^{\infty} W_i^m$ in the same direction.

Assume without loss of generality that for each integer $m > \beta$, every point of $\bigcup_{i=2}^{\infty} W_i^m$ is sent back by $f$.

The set \( \{ m : f[A_m \cap (G_1 \cup G_2)] \subset G_1 \cup G_2 \} \) is finite; for otherwise, Case 3 (with $\text{Cl } G = G_1 \cup G_2$ and $W_i = W_2$) holds for infinitely many elements of $\{ A_m \}_{m=\beta}^{\infty}$, and we have shown that this is impossible. Hence we can assume without loss of generality that for each integer $m > \beta$,

(32) $f[A_m \cap (G_1 \cup G_2)] \subset G_1 \cup G_2$.

For each positive integer $m$, let $\mathcal{H}_m$ be the chain consisting of all links of $\mathcal{G}_m + \beta$ that intersect $\text{Int}(G_1 \cup G_2)$, and let $H_m$ be an arc in $A_m^{0+\beta} \cap (G_1 \cup G_2)$ that intersects $W_2$ and agrees with $\mathcal{H}_m$.

The sequence \( \{ H_m \}_{m=1}^{\infty} \) converges to a continuum in $X_0 \cap (G_1 \cup G_2)$. For each positive integer $m$, $\text{St } \mathcal{H}_m \subset G_1 \cup G_2$ and conditions (23)–(26) are satisfied. By (18) and (32), for each positive integer $m$, $f[H_m] \subset (\text{St } \mathcal{H}_m) \setminus X_0$. According to the argument following (27), a proper subcontinuum of $X_0$ has the Reduction Property, and this contradicts (20). Hence (18) is true for at most finitely many integers.

Assume without loss of generality that (19) holds for each integer $m > \beta$.

By the preceding argument, for infinitely many integers $m > \beta$, $A_m^{0}$ does not have the switch property on a component of $A_m^{0} \setminus \text{St } \mathcal{H}^{0}$ that is contained in $B_m^{0}$. Hence we can assume without loss of generality that for each integer $m > \beta$, every point of $B_m^{0} \cap \bigcup_{i=2}^{\infty} W_i^m$ is sent forward by $f$. It follows from a similar argument that for infinitely many integers $m > \beta$,
(33) for each point \(x\) of \(B_0^0 \setminus V_i\) \((2 \leq i \leq \gamma - 2)\), \(f(x) \notin V_i\).

We assume without loss of generality that (33) holds for each integer \(m \geq \beta\).

Since for each positive integer \(m\), \(B^0_{m+\beta}\) has the properties given in (19) and (33), there exist a sequence \(\{H_m\}_{m=1}^\infty\) of arcs in \(P\) converging to a continuum in \(X_0 \cap (G_1 \cup G_2)\) and a sequence \(\{\gamma_m\}_{m=1}^\infty\) of chains with the following properties. For each positive integer \(m\), \(\text{St} \gamma_m \subset G_1 \cup G_2\), conditions (23)–(26) are satisfied, \(H_m \cap W_2 \neq \emptyset\), and there exists a subarc \(I_m\) of \(H_m\) such that \(f[I_m] \subset (\text{St} \gamma_m) \setminus X_0\) and \(f[I_m]\) agrees with \(\gamma_m\).

By the argument following (27), a proper subcontinuum of \(X_0\) has the Reduction Property, and this contradicts (20). Hence every continuous function of \(M\) into \(M\) has a fixed point. \(\square\)

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