

FOURIER INVERSION FOR UNIPOTENT INVARIANT INTEGRALS

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ABSTRACT. Consider G a semisimple Lie group and $\Gamma \subseteq G$ a discrete subgroup such that $\text{vol}(G/\Gamma) < \infty$. An important problem for number theory and representation theory is to find the decomposition of $L^2(G/\Gamma)$ into irreducible representations. Some progress in this direction has been made by J. Arthur and G. Warner by using the Selberg trace formula, which expresses the trace of a subrepresentation of $L^2(G/\Gamma)$ in terms of certain invariant distributions. In particular, measures supported on orbits of unipotent elements of G occur. In order to obtain information about representations it is necessary to expand these distributions into Fourier components using characters of irreducible unitary representations of G . This problem is solved in this paper for real rank $G = 1$. In particular, a relationship between the semisimple orbits and the nilpotent ones is made explicit generalizing an earlier result of R. Rao.

1. Introduction. Let G be a connected semisimple Lie group with finite center and Lie algebra \mathfrak{g} . Let q be any element of G and $O(q)$ be the orbit of q under the adjoint action of G . Then $O(q)$ carries a G -invariant measure which we denote by T_q . From the results in [11b] it follows that T_q has the important property that it defines a tempered distribution on G and is also a measure on G . Then it makes sense to try to obtain Fourier inversion formulas in the sense of Harish-Chandra for the distributions T_q . The problem considered in this paper is to find such explicit formulas. This means to express the distribution T_q in terms of a series of tempered invariant eigendistributions of G . These series include the characters of the discrete series representations (if G has a compact Cartan subgroup), characters of the unitary principal series and certain eigendistributions that can be interpreted as alternating sums of characters [7].

Such formulas have been obtained in [12] for q semisimple and G of real rank 1. For q regular and G of rank higher than 1, formulas are obtained in [6].

In this paper we derive an inversion formula for q unipotent in case G has real rank one. These distributions occur in the Selberg trace formula derived in [10]. The inversion formulas derived here are useful in obtaining

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information about the multiplicities of irreducible representations occurring in $L_0^2(\Gamma \backslash G)$, where Γ is a discrete subgroup of G with $\text{vol}(\Gamma \backslash G) < \infty$.

The method used for obtaining the formulas is an extension of some unpublished results of R. Rao [11a]. Let u be a unipotent element, $u = \exp X$ and $\{X, H, Y\}$ be a Lie triple. Then let $\mathfrak{u} = X + Z_Y$ where $Z_Y = \text{Cent}_q Y$. In [11a] it is shown that \mathfrak{u} is a transverse to the adjoint action of G on the Lie algebra \mathfrak{g} . Since both X and $X - Y$ are elements of \mathfrak{u} and $\bigcup_{t>0} O(t(X - Y))$ contains the orbit of X in its closure, the formula in Theorem 6.7 can be proved for a general class of nilpotents (such that $\text{ad } H$ has even eigenvalues only) by lifting the integrals to the transverse \mathfrak{u} . The main difficulties in using this method for an arbitrary nilpotent are the fact that \mathfrak{u}^G is not equal to the entire Lie algebra \mathfrak{g} , the fact that there may be several conjugacy classes of nilpotents conjugate under G_c but not under the real group G and finally convergence problems on \mathfrak{u} .

In order to avoid these difficulties we take a slightly different approach. In §§4 and 5 we construct transverses to the orbits of X and $Z = X - Y$ by using a decomposition theorem of Mostow [9] and lift the invariant measures T_X and T_Z to the corresponding transverses. Then it becomes apparent that we only need to consider the case $\text{su}(2, 1)$. In §6 the relation between T_Z and T_X is calculated. The central result is Theorem 6.7. In §7 the Fourier inversion formula for T_u is derived by differentiation from the formulas in [12], while in §8 the constants relevant to the results in [10] are computed explicitly. These results as well as the inversion for an arbitrary q are proved in a different way in [2].

I am indebted to Professor R. Rao for many helpful suggestions.

2. Notation and preliminary results. Let \mathfrak{g} be a real semisimple Lie algebra with complexification \mathfrak{g}_c . Let σ be the Lie conjugation of \mathfrak{g}_c defined by \mathfrak{g} . Fix once and for all a Cartan decomposition $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ of \mathfrak{g} . Then $\mathfrak{k} + \sqrt{-1} \mathfrak{p}$ is a compact real form of \mathfrak{g}_c . Denote by τ the conjugation defined by this real form. Then τ and σ commute and $\tau\sigma$ is a Cartan involution of \mathfrak{g} giving rise to the Cartan decomposition $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$. Choose a subspace \mathfrak{a} and \mathfrak{n} such that \mathfrak{a} is maximal abelian in \mathfrak{p} and $\mathfrak{g} = \mathfrak{k} + \mathfrak{a} + \mathfrak{n}$ is the Iwasawa decomposition of \mathfrak{g} .

Let G be a connected semisimple Lie group with finite center and Lie algebra \mathfrak{g} . If G_c is the connected simply connected group corresponding to \mathfrak{g}_c , we assume that G is the real subgroup of G_c corresponding to \mathfrak{g} . We also denote by K the subgroup corresponding to \mathfrak{k} , by A the subgroup corresponding to \mathfrak{a} and by N the subgroup corresponding to \mathfrak{n} .

Let $\mathfrak{j} \subseteq \mathfrak{g}$ be an Iwasawa Cartan subalgebra which is τ -invariant and has the property that $\mathfrak{j} \cap \mathfrak{p} = \mathfrak{a}$. Then there is a direct sum decomposition $\mathfrak{j} = \mathfrak{a}_K + \mathfrak{a}$ where $\mathfrak{a}_K = \mathfrak{j} \cap \mathfrak{k}$. If J is the Cartan subgroup with Lie algebra \mathfrak{j} ,

then $J = A \cdot A_K$ where $A = \exp \alpha$ and $A_K = J \cap K$.

Denote by $\Delta(\mathfrak{g}_c, \mathfrak{i}_c)$ the root system of \mathfrak{g}_c corresponding to \mathfrak{i}_c . We will drop the parentheses and write Δ whenever there is no danger of confusion. It is well known that if $\beta \in \Delta$ then $\sigma\beta \in \Delta$ where $\sigma\beta$ is defined by $(\sigma\beta)(H) = \overline{\beta(\sigma H)}$ (the bar denotes complex conjugation) for any $H \in \mathfrak{i}_c$.

We also denote by $\Delta(\mathfrak{g}, \alpha)$ the restricted root system and introduce compatible orderings on $\Delta(\mathfrak{g}_c, \mathfrak{i}_c)$ and $\Delta(\mathfrak{g}, \alpha)$ such that if $\beta > 0$ then $\sigma\beta > 0$ as well, provided $\beta + \sigma\beta \neq 0$. A root such that $\beta + \sigma\beta = 0$ is called imaginary and is characterized by the fact that $\beta|_\alpha = 0$. We denote the set of imaginary roots by Δ_0 . A root α is called real if $\alpha = \sigma\alpha$. It is characterized by the property that $\alpha|_{\alpha_K} = 0$.

Let B be the Cartan-Killing form of \mathfrak{g}_c and $B_\tau(X, Y) = B(X, \tau Y)$ be the associated negative definite form. The restriction of B to \mathfrak{i}_c will be denoted by $(,)$. Since B is nondegenerate we can define a form on the dual of \mathfrak{i}_c which will be denoted by $(,)$ as well. For $\beta \in \Delta$ define H_β and H'_β by the relations $(H, H_\beta) = \beta(H)$ for any H , and $H'_\beta = (2/(\beta, \beta))H_\beta$.

Let E_β be the root vector associated to the root β . Then $\sigma E_\beta = \rho_\beta E_{\sigma\beta}$ where $|\rho_\beta| = 1$. We use the Weyl normalization for these vectors so that $\tau E_\beta = -E_{-\beta}$. Then the vectors $E_\beta, E_{-\beta}$ and H'_β satisfy the relations

$$[H'_\beta, E_\beta] = 2E_\beta, \quad [H'_\beta, E_{-\beta}] = -2E_{-\beta}, \quad [E_\beta, E_{-\beta}] = -H'_\beta.$$

The following well-known facts are going to be used later.

LEMMA 2.1. *Let $\beta, \gamma \in \Delta$. Then there are integers $p, q \in \mathbb{N}$ such that $\gamma + s\beta$ is a root if and only if $-q \leq s \leq p$ and s is an integer. In addition $2(\beta, \gamma)/(\gamma, \gamma) = q - p$ and $\beta - 2(\beta, \gamma)\gamma/(\gamma, \gamma)$ is a root.*

PROOF. See [13, p. 41].

COROLLARY 2.2. *Let $\alpha, \beta \in \mathfrak{i}_c^*$, the dual of \mathfrak{i}_c . Then*

$$|(\alpha, \beta)|^2 \leq (\alpha, \alpha)(\beta, \beta).$$

If $\alpha, \beta \in \Delta$ then equality holds if and only if $\alpha = \pm \beta$.

PROOF. The first statement is the Cauchy-Schwartz inequality. The second is a consequence of the fact that the only multiples of β that are roots are $\pm\beta$.

LEMMA 2.3. *If β is a root then $\beta - \sigma\beta$ is not a root.*

PROOF. See [1, p. 6].

LEMMA 2.4. *Let β and γ be positive roots such that $\beta \neq \gamma$ and $\beta \neq \sigma\gamma$. If $\beta|_\alpha$ and $\gamma|_\alpha$ are multiples of each other then either $\beta - \gamma$ or $\beta - \sigma\gamma$ is a root.*

PROOF. We have the relation

$$(\beta, \gamma) + (\beta, \sigma\gamma) = (\beta, \gamma + \sigma\gamma) = (\beta, 2\gamma|_{\mathfrak{o}}) = 2(\beta|_{\mathfrak{o}}, \gamma|_{\mathfrak{o}}) > 0,$$

so either $(\beta, \gamma) > 0$ or $(\beta, \sigma\gamma) > 0$, say $(\beta, \gamma) > 0$. Then Lemma 2.1 implies that $\beta - \gamma$ must be a root and the proof is complete.

Next, we consider the conjugacy classes of Cartan subalgebras of \mathfrak{g} for the case when $\text{rank}(G/K) = 1$. It is well known that any Cartan subalgebra can be conjugated to a τ -invariant one. Let \mathfrak{h} be such a subalgebra and $\mathfrak{h} = \mathfrak{h}_k + \mathfrak{h}_p$ be its Cartan decomposition. Since $\text{rank}(G/K) = 1$ it follows that $\dim \mathfrak{h}_p \leq 1$. The Cartan subalgebras such that $\dim \mathfrak{h}_p = 1$ are all conjugate to the Iwasawa Cartan subalgebra \mathfrak{j} . The Cartan subalgebras such that $\dim \mathfrak{h}_p = 0$ are all conjugate. They are called fundamental or also compact Cartan subalgebras. The following result is well known. We include a proof for completeness.

PROPOSITION 2.5. *Let \mathfrak{g} be a real semisimple algebra such that $\text{rank}(G/K) = 1$. Then we have the following two cases.*

(1) *If $\text{rank } \mathfrak{g} > \text{rank } \mathfrak{k}$ then the fundamental and the Iwasawa Cartan subalgebra are the same. There is only one conjugacy class of Cartan subalgebras.*

(2) *If $\text{rank } \mathfrak{g} = \text{rank } \mathfrak{k}$ then there is a real root $\alpha \in \Delta(\mathfrak{g}_c, \mathfrak{j}_c)$. A representative for the fundamental Cartan subalgebra is $\mathfrak{b} = \mathfrak{a}_K + \mathbf{R}(E_\alpha - E_{-\alpha})$. The subalgebras \mathfrak{j} and \mathfrak{b} are not conjugate by G but the element $\nu = \exp[\pi\sqrt{-1}(E_\alpha + E_{-\alpha})/4]$ conjugates \mathfrak{j}_c into \mathfrak{b}_c .*

PROOF. The first statement is straightforward. Assume $\text{rank } \mathfrak{g} = \text{rank } \mathfrak{k}$. Let $\mathfrak{b} \subseteq \mathfrak{k}$ be a Cartan subalgebra. Since not all roots of $\Delta(\mathfrak{g}_c, \mathfrak{b}_c)$ are compact, there is a singular imaginary root $\tilde{\alpha}$. Select $Z \in \mathfrak{b}$ such that the only roots vanishing at Z are $\pm \tilde{\alpha}$. Let c_Z be the center of $\text{Cent}_{\mathfrak{g}} Z$ and l_Z be the derived algebra. Then l_Z is isomorphic to $\mathfrak{sl}(2, \mathbf{R})$ and we can select a basis X^*, H^*, Y^* such that $\tau X^* = -Y^*$ and $[X^*, Y^*] = H^*$, $[H^*, X^*] = 2X^*$, $[H^*, Y^*] = -2Y^*$. Then $\mathfrak{b} = c_Z + \mathbf{R}(X^* - Y^*)$. Define $\mathfrak{h} = \mathbf{R}H^* + c_Z$. Then \mathfrak{h} is an Iwasawa Cartan subalgebra and if $\mu = \exp[\sqrt{-1}\pi(X^* + Y^*)/4]$ then $\mathfrak{h}_c^\mu = \mathfrak{b}_c$. Furthermore $\tilde{\alpha}^\mu = \tilde{\alpha} \circ \mu$ is a real root of $\Delta(\mathfrak{g}_c, \mathfrak{h}_c)$ since μ leaves c_Z invariant and $\tilde{\alpha}$ vanishes on c_Z . The Cartan subalgebra \mathfrak{h} is conjugate to \mathfrak{j} by some element in G .

Conversely, let $\alpha \in \Delta(\mathfrak{g}_c, \mathfrak{j}_c)$ be a real root and $E_{\pm\alpha}$ be the corresponding root vectors. Since $|\rho_\alpha| = 1$ we can choose $t_\alpha \in \mathbf{R}$ such that $\exp[2\sqrt{-1}t_\alpha] = \rho_\alpha$. Let $h_\alpha = \exp[\sqrt{-1}t_\alpha H'_\alpha]$. Then $\tau(h_\alpha) = h_\alpha$. If we replace the root vectors E_β by $\text{Ad } h_\alpha(E_\beta)$ the Weyl normalization is preserved. In addition

$$\begin{aligned} \sigma(\text{Ad } h_\alpha E_\alpha) &= \sigma(\exp(\sqrt{-1}t_\alpha)E_\alpha) = \exp(-\sqrt{-1}t_\alpha)\rho_\alpha E_\alpha \\ &= \exp(\sqrt{-1}t_\alpha)E_\alpha = \text{Ad } h_\alpha E_\alpha. \end{aligned}$$

Similarly $\sigma(\text{Ad } h_\alpha E_{-\alpha}) = \text{Ad } h_\alpha E_{-\alpha}$ so $E_{\pm\alpha} \in \mathfrak{g}$. Then

$$\mathfrak{b} = \mathbf{R}(E_\alpha - E_{-\alpha}) + \alpha_K$$

is a compact Cartan subalgebra. Finally the fact that $\text{Ad } \nu$ maps \mathfrak{b}_c into \mathfrak{j}_c follows from an elementary calculation in $\mathfrak{sl}(2, \mathbf{C})$. This completes the proof.

From here on α will always denote a positive real root. Fix a positive restricted root $\lambda \in \Delta(\mathfrak{g}, \mathfrak{a})$ such that the only multiples of λ that are also roots are $\pm\lambda$ and possibly $\pm\frac{1}{2}\lambda$. Let

$$\begin{aligned} \Delta_0 &= \{ \beta \in \Delta(\mathfrak{g}_c, \mathfrak{j}_c) : \beta|_{\mathfrak{a}} = 0 \}, \\ P_1 &= \{ \beta \in \Delta(\mathfrak{g}_c, \mathfrak{j}_c) : \beta|_{\mathfrak{a}} = \frac{1}{2}\lambda \}, \\ P_2 &= \{ \beta \in \Delta(\mathfrak{g}_c, \mathfrak{j}_c) : \beta|_{\mathfrak{a}} = \lambda \}, \\ \Psi &= \{ \gamma \in \Delta_0 : \alpha \pm \gamma \text{ are not roots} \}. \end{aligned}$$

LEMMA 2.6. *If $P_1 \neq \emptyset$ then there is a real root $\alpha \in \Delta(\mathfrak{g}_c, \mathfrak{j}_c)$ such that $\alpha|_{\mathfrak{a}} = \lambda$. Also $H'_\lambda = H'_\alpha$ and $\text{ad } H'_\lambda$ has eigenvalues $0, \pm 1$ and ± 2 .*

PROOF. Let $\beta \in P_2$ and denote by α the functional on \mathfrak{j} which equals λ on \mathfrak{a} and 0 on α_K . We have to show that $\alpha \in \Delta$. Assume not. Then for any $\alpha_1 \in P_1$, the sum $\alpha_1 + \sigma\alpha_1 \notin \Delta$. Since $\alpha_1 - \sigma\alpha_1 \notin \Delta$ by Lemma 2.3, it follows that $(\alpha_1, \sigma\alpha_1) = 0$. Since $\alpha_1 + \sigma\alpha_1 = \alpha$ we also get $2(\alpha_1, \alpha_1) = (\alpha, \alpha)$. On the other hand

$$\frac{2(\beta, \alpha_1)}{(\alpha_1, \alpha_1)} + \frac{2(\beta, \sigma\alpha_1)}{(\sigma\alpha_1, \sigma\alpha_1)} = \frac{2(\beta, \alpha)}{(\alpha_1, \alpha_1)} = \frac{2(\alpha, \alpha)}{\frac{1}{2}(\alpha, \alpha)} = 4. \quad (1)$$

At least one of the summands must be larger than 2, say $2(\beta, \alpha_1)/(\alpha_1, \alpha_1) \geq 2$. Since β and α_1 cannot be multiples of each other, Lemma 2.1 implies that $\beta - \alpha_1$ is a root. Corollary 2.2 implies that

$$\frac{2(\beta, \alpha_1)}{(\alpha_1, \alpha_1)} \cdot \frac{2(\beta, \alpha_1)}{(\beta, \beta)} < 4$$

and since both numbers are integers, $2(\beta, \alpha_1)/(\alpha_1, \alpha_1) < 4$. Together with (1) this implies $(\beta, \sigma\alpha_1) > 0$ so $\beta - \sigma\alpha_1$ is also a root. Both $\beta - \alpha_1$ and $\beta - \sigma\alpha_1$ restrict to $\frac{1}{2}\lambda$ and they cannot be multiples of each other. Thus we can apply Lemma 2.4 to conclude that either $\beta - \alpha_1 - (\beta - \sigma\alpha_1) = \sigma\alpha_1 - \alpha_1$ or $\beta - \alpha_1 - (\sigma\beta - \alpha_1) = \beta - \sigma\beta$ is a root. This contradicts the statement of Lemma 2.3. Thus $\alpha \in \Delta$. Since $\alpha|_{\alpha_K} = 0$ it follows that $H'_\alpha = H'_\lambda$ and $\beta(H'_\lambda) = 2$, $\alpha_1(H'_\lambda) = 1$. This completes the proof.

Let B be the Cartan subgroup corresponding to \mathfrak{b} . We normalize the measures on $G, \mathfrak{g}, J, \mathfrak{j}, B$ and \mathfrak{b} as in [15a, §8.1.2]. Furthermore we normalize the Haar measures on any compact subgroup so that it has volume one.

Let \mathfrak{b}^* denote the dual to $\sqrt{-1} \mathfrak{b}$. Then we define a lattice L_B in \mathfrak{b}_c^* by

$$L_B = \left\{ \mu \in \mathfrak{b}_c^* : \frac{2(\mu, \beta)}{(\beta, \beta)} \in \mathbf{Z} \text{ for any } \beta \in \Delta(\mathfrak{g}_c, \mathfrak{b}_c) \right\}.$$

Then the element $\rho = \frac{1}{2} \sum_{\beta > 0} \beta$ is in this lattice. For any $\mu \in L_B$ we can define a character of B by the formula

$$\xi_\mu(\exp H) = \exp \mu(H) \quad \text{all } H \in \mathfrak{b}.$$

Let

$$B' = \{ h \in B : \xi_\beta(h) \neq 1 \text{ for any } \beta \in \Delta(\mathfrak{g}_c, \mathfrak{b}_c) \}$$

and

$$\Delta_B(h) = \xi_{-\rho}(h) \prod_{\beta > 0} (1 - \xi_\beta(h)) \quad \text{for } h \in B'.$$

For any $f \in C_c^\infty(G)$ we can define a function Φ_f^B by

$$\Phi_f^B(h) = \Delta_B(h) \int_{G/B} f(h^x) dx, \quad h \in B'.$$

This function has the following properties.

(a) $\Phi_f^B \in \mathcal{S}(B')$, the Schwartz space of B' ,

(b) if $W(G, B)$ denotes the Weyl group of G with respect to B then $\Phi_f^B(wh) = (\det w) \Phi_f^B(h)$ for any $w \in W(G, B)$.

The function Φ_f^B plays an important role in harmonic analysis on semi-simple groups. In particular the following result will be useful later.

LEMMA 2.7. *Let γ be an element of B . Let $P_\gamma = \{ \beta \in \Delta(\mathfrak{g}_c, \mathfrak{b}_c) \text{ such that } \xi_\beta(\gamma) = 1 \text{ and } \beta > 0 \}$ and $P_{g/\gamma} = \{ \beta > 0 : \beta \notin P_\gamma \}$. Define a differential operator $\bar{\omega}_\gamma$ by $\bar{\omega}_\gamma = \prod_{\beta \in P_\gamma} H_\beta$. Then*

$$\lim_{\substack{\tilde{\gamma} \rightarrow \gamma \\ \tilde{\gamma} \in B'}} \Phi_f^B(\tilde{\gamma}; \bar{\omega}_\gamma) = k_\gamma \int_{G/G_\gamma} f(\gamma^{\tilde{x}}) d\tilde{x}$$

where $k_\gamma = d_\gamma \xi_{-\rho}(\gamma) \prod_{\beta \in P_{g/\gamma}} (\xi_\beta(\gamma) - 1)$ and d_γ depends only on the measure $d\tilde{x}$.

PROOF. See [4d, p. 33].

To each element $\mu \in L_B$ there is, according to [4c, p. 281], associated an invariant central eigendistribution Θ_μ which is a locally summable function on G and analytic on the set of regular points. If Θ_μ is regular i.e. $w\mu \neq \mu$ for any $w \in W(G_c, B_c)$ then Θ_μ is the character of a discrete series representation of G . For the explicit form of Θ_μ for real rank 1 see [12].

Let W_I be the Weyl group corresponding to the imaginary roots of \mathfrak{j} . Then W_I can be identified with the Weyl group $W(M, A_K)$ where $M = \text{Cent}_K A$. A

unitary character $\chi \in \hat{A}_K$ is called regular if $w\chi \neq \chi$ for any $w \in W_I$. To each pair (χ, ν) where $\chi \in \hat{A}_K$ is regular and $\nu \in \hat{A} \approx \mathbf{R}$ there is associated a principal series character which we denote by $T^{(\chi, \nu)}$.

Let $q \in G$ be an arbitrary element. According to [3, p. 235], the group $G_q = \text{Cent}_{G,q}$ is unimodular so that the homogeneous space G/G_q carries an invariant measure. In [11b] it was proved that the integral

$$T_q(f) = \int_{G/G_q} f(q^x) dx$$

is convergent for any $f \in C_c(G)$. In particular, this implies that T_q is a tempered invariant distribution on G . We normalize the measures dx^* on G/G_q and dy on G_q so that

$$\int_G f(x) dx = \int_{G/G_q} \int_{G_q} f(x^*y) dx^* dy$$

for any $f \in C_c(G)$.

3. Conjugacy classes of nilpotent elements. An element $X \in \mathfrak{g}$ is called nilpotent if $\text{ad } X$ is a nilpotent transformation of \mathfrak{g} . A triple $\{X, H, Y\}$ is called a Lie triple if the following relations hold:

$$[H, X] = 2X, \quad [H, Y] = 2Y \quad \text{and} \quad [X, Y] = -H.$$

Then the real algebra generated by X, H and Y is isomorphic to $\mathfrak{sl}(2, \mathbf{R})$. The Jacobson-Morozov theorem [8a] implies that any nilpotent element X in a complex semisimple Lie algebra \mathfrak{g}_c can be embedded in a Lie triple $\{X, H, Y\}$. If in addition $X \in \mathfrak{g}$ then H and Y can be chosen to be in \mathfrak{g} as well.

Two Lie triples $\{X, H, Y\}$ and $\{X_1, H_1, Y_1\}$ are said to be conjugate by G or real conjugate if there is $x \in G$ such that $X^x = X_1$, $Y^x = Y_1$ and $H^x = H_1$.

The conjugacy classes of Lie triples and nilpotent elements are characterized in the next proposition.

PROPOSITION 3.1. *Two Lie triples, $\{X, H, Y\}$ and $\{X_1, H_1, Y_1\}$, are conjugate if and only if their nilpotent components X and X_1 are conjugate. The Lie triples are conjugate by G_c if and only if H and H_1 are conjugate by G_c . The Lie triples are conjugate by G if and only if $Z = X - Y$ and $Z_1 = X_1 - Y_1$ are conjugate.*

PROOF. The proof of the first two statements is contained in [8a]. The last statement was formulated by R. Rao and proved by B. Kostant based on [8b]. Since no proof seems to have appeared in print, we give one for completeness.

Assume that X_1 and X are conjugate by G . Then by the previous statement,

there is $x \in G$ such that $X_1^x = X$, $Y_1^x = Y$ and $H_1^x = H$. Thus $(X_1 - Y_1)^x = X - Y$. Conversely, assume that there is $x \in G$ such that $(X_1 - Y_1)^x = X - Y$. By Theorem 6 in [9] we may assume that the two Lie triples are normalized so that $\theta X = -Y$ and $\theta X_1 = -Y_1$. Then $X - Y$ and $X_1 - Y_1$ are in \mathfrak{k} and in addition we may assume that $x \in K$. We then replace the Lie triples X_1, H_1, Y_1 by X_1^x, H_1^x, Y_1^x and call it X_1, H_1, Y_1 once again. Thus we assume $X_1 - Y_1 = X - Y$. Since $X - Y$ is conjugate by $SL(2, \mathbb{C})$ to $\sqrt{-1} H$ we replace the Lie triples by the Lie triples

$$\left\{ \frac{1}{2}(X + Y + iH), i(X - Y), \frac{1}{2}(X + Y - iH) \right\}$$

and

$$\left\{ \frac{1}{2}(X_1 + Y_1 + iH_1), i(X_1 - Y_1), \frac{1}{2}(X_1 + Y_1 - iH_1) \right\}.$$

Let \mathfrak{k}_c and \mathfrak{p}_c be the complexifications of \mathfrak{k} and \mathfrak{p} . Furthermore let τ_c be the complex linear map defined by $\tau_c|_{\mathfrak{k}_c} = \text{id}$ and $\tau_c|_{\mathfrak{p}_c} = -\text{id}$. We then note that, due to the assumptions on G_c , the map τ_c extends to a map on G_c . Let K_τ be the set of fixed points of τ_c and \tilde{K}_τ the image of K_τ by the adjoint map. If \tilde{K} is the connected component of \tilde{K}_τ then by Proposition 1 in [8b] we can write $\tilde{K}_\tau = F \cdot \tilde{K}$ where F is the set of elements of order two in $\exp \text{ad } \mathfrak{a}_c$. Also, \tilde{K}_τ is the set of fixed points of τ_c in the adjoint group of \mathfrak{g}_c . In the terminology of [8b] the two Lie triples are called normal. Due to the remark following Proposition 4 in [8b] the two Lie triples are conjugate by some element $\tilde{x} \in \tilde{K}_\tau$. We note that $\tilde{K} = \text{Ad } K \exp \sqrt{-1} \text{ ad } \mathfrak{k}$. Since $K \subseteq G$ is connected we may assume $\tilde{x} = f \exp \sqrt{-1} \text{ ad } Z$ where $Z \in \mathfrak{k}$. Since $(X - Y)^{\tilde{x}} = X - Y$ we also have $\tau\{(X - Y)^{\tilde{x}}\} = \tau(X - Y)$ so $(X - Y)^{\tau(\tilde{x})} = X - Y$. But $\tau(\tilde{x}) = f \exp(-\sqrt{-1} \text{ ad } Z)$ so $\tilde{x}^{-1} \tau(\tilde{x}) = \exp 2\sqrt{-1} \text{ ad } Z$ also centralizes $X - Y$. But then

$$\begin{aligned} X - Y &= \exp 2\sqrt{-1} \text{ ad } Z(X - Y) = \cosh(2\sqrt{-1} \text{ ad } Z)(X - Y) \\ &\quad + \sinh(2\sqrt{-1} \text{ ad } Z)(X - Y). \end{aligned}$$

Since the first term on the right-hand side is in \mathfrak{k} and the second in $\sqrt{-1} \mathfrak{k}$ it follows that

$$\sinh(2\sqrt{-1} \text{ ad } Z)(X - Y) = 0.$$

But $2\sqrt{-1} \text{ ad } Z$ has real eigenvalues only, so $\text{ad } Z(X - Y) = 0$. This also implies $f \cdot (X - Y) = X - Y$ so f leaves the entire Lie triple fixed. Thus we also get $\exp(\sqrt{-1} \text{ ad } Z)H = H_1$. Again

$H_1 = \exp(\sqrt{-1} \text{ ad } Z)H = \cosh(\sqrt{-1} \text{ ad } Z)H + \sinh(\sqrt{-1} \text{ ad } Z)H$ so $\sinh(\sqrt{-1} \text{ ad } Z)H = 0$ since it is contained in $\sqrt{-1} \mathfrak{p}$ (while the other term is in \mathfrak{p}). But this implies $(\text{ad } Z)H = 0$ so $H = H_1$. Therefore the two Lie triples must coincide and the proof is complete.

We now determine the conjugacy classes of nilpotent elements in a semi-simple Lie algebra of real rank one. Let λ be the restriction of the real positive root α to \mathfrak{a} or in the absence of a real root let λ be the restriction of a simple positive root of $\Delta(\mathfrak{g}_c, \mathfrak{j}_c)$. Define $H'_\lambda = 2H_\lambda/(\lambda, \lambda)$ as in §2. Let $\mathfrak{g}_i = \{Z \in \mathfrak{g} : [H'_\lambda, Z] = iZ\}$.

THEOREM 3.2. *The real conjugacy classes of nilpotent elements are given by*

- (1) $\{X, H'_\lambda, Y\}$ where $X \in \mathfrak{g}_2$ and $\{-X, H'_\lambda, -Y\}$ when $\dim \mathfrak{g}_2 = 1$,
- (2) $\{X_1, 2H'_\lambda, Y_1\}$ where $X_1 \in \mathfrak{g}_1$.

PROOF. Let $\{X, H, Y\}$ be an arbitrary Lie triple. Since X, H and Y generate a semisimple algebra isomorphic to $\mathfrak{sl}(2, \mathbf{R})$, $\text{ad } H$ must have integral eigenvalues only. Furthermore Theorem 6 in [9] implies that we can normalize the Lie triple so that $\tau X = -Y$ and $\tau H = -H$. Since $H \in \mathfrak{p}$ we can conjugate the Lie triple by an element in K so that $H = \varepsilon H'$ where $\varepsilon > 0$. We note that $\text{ad } H'_\lambda$ has eigenvalues $0, \pm 2$ and possibly ± 1 . On the other hand, $\text{ad } H$ has an eigenvalue equal to $+2$ and thus we can only have $\varepsilon = 1$ or $\varepsilon = 2$. Furthermore, if $\varepsilon = 2$ then $\text{ad } H'_\lambda$ has to have eigenvalue $+1$ so in the notation of §2 we have $P_1 \neq \emptyset$. We thus have two types of Lie triples.

- (1) $\{X_1, 2H'_\lambda, Y_1\}$ where $X_1 \in \mathfrak{g}_1$,
- (2) $\{X, H'_\lambda, Y\}$ where $X \in \mathfrak{g}_2$.

Due to Proposition 3.1 the two types are in different conjugacy classes. By using Theorem 8.11.3 in [14] we conclude that the Lie triples of type (1) are one conjugacy class and the Lie triples of type (2) are one conjugacy class when $\dim \mathfrak{g}_2 > 1$. It remains to show that if $\dim \mathfrak{g}_2 = 1$ then $\{X, H, Y\}$ is not conjugate to $\{-X, H, -Y\}$. Suppose that there is $x \in G$ such that $X^x = -X, H^x = H$ and $Y^x = -Y$. Then $x \in M = \text{Cent}_K H$. Let \mathfrak{m} be the Lie algebra of M . Then $[\mathfrak{m}, X] = 0$, for otherwise $\dim \mathfrak{g}_2 > 1$. This implies that M_0 , the connected component of M , centralizes X . Let $G_c(H)$ be the centralizer of H in G_c and $\mathfrak{g}_c(H)$ its Lie algebra. Since $G_c(H)$ is the centralizer of a torus, it is connected. Its Lie algebra has Cartan decomposition

$$\mathfrak{g}_c(H) = (\mathfrak{m} + \sqrt{-1} \mathbf{R}H) + (\sqrt{-1} \mathfrak{m} + \mathbf{R}H).$$

Thus

$$G_c(H) = M \exp\{\sqrt{-1} \mathbf{R}H\} \exp\{\sqrt{-1} \mathfrak{m} + \mathbf{R}H\}.$$

Thus $M = M_0 \exp\{\sqrt{-1} \mathbf{R}H\} \cap K$. Suppose $h = \exp\{\sqrt{-1} tH\} \in K$ for some $t \in \mathbf{R}$. Then $\sigma(\exp\{\sqrt{-1} tH\}) = \exp\{\sqrt{-1} tH\}$ so $\exp\{\sqrt{-1} tH\} = \exp\{-\sqrt{-1} tH\}$ or $\exp\{2\sqrt{-1} tH\} = e$. Since H as an element of \mathfrak{g}_c is contained in a subalgebra isomorphic to $\mathfrak{sl}(2, \mathbf{C})$ it follows that $t = k\pi$ for some $k \in \mathbf{Z}$. But then

$$X^h = \exp\{2\sqrt{-1} t\} X = \exp\{2\sqrt{-1} k\pi\} X = X.$$

Thus there cannot be such $x \in M$ and the proof is complete.

EXAMPLE. Let $\mathfrak{g} = \mathfrak{su}(2, 1)$. Then representatives of all conjugacy classes are given by

$$X = \begin{bmatrix} 0, & 0, & 0 \\ 0, & i/2, & i/2 \\ 0, & -i/2, & -i/2 \end{bmatrix}, \quad H'_\lambda = \begin{bmatrix} 0, & 0, & 0 \\ 0, & 0, & -1 \\ 0, & -1, & 0 \end{bmatrix},$$

$$Y = \begin{bmatrix} 0, & 0, & 0 \\ 0, & i/2, & -i/2 \\ 0, & i/2, & -i/2 \end{bmatrix}. \quad (1)$$

In this case X and $-X$ are not conjugate.

$$X = \begin{bmatrix} 0, & 1, & 1 \\ -1, & 0, & 0 \\ 1, & 0, & 0 \end{bmatrix}, \quad 2H'_\lambda = \begin{bmatrix} 0, & 0, & 0 \\ 0, & 0, & -2 \\ 0, & -2, & 0 \end{bmatrix},$$

$$Y = \begin{bmatrix} 0, & 1, & -1 \\ -1, & 0, & 0 \\ -1, & 0, & 0 \end{bmatrix}. \quad (2)$$

REMARK 3.3. In the case when $P_1 \neq \emptyset$ let $\alpha_1 \in P_1$. Then the roots $\alpha_1, \sigma\alpha_1$ and α generate a subalgebra isomorphic to $\mathfrak{su}(2, 1)$. Thus we can select representatives for the conjugacy classes of nilpotents to be as in the example above. If $P_1 = \emptyset$ then the representatives can be chosen in $\mathfrak{sl}(2, \mathbf{R}) \subseteq \mathfrak{su}(2, 1)$ to have the same expression as X, H and Y in part (1) of the example.

4. The orbit of the semisimple element Z . With the notation as before let $\{X, H, Y\}$ be a Lie triple normalized as before and let $Z_0 = X - Y$. Let G_Z be the centralizer of Z_0 in G and \mathfrak{g}_Z its Lie algebra. Since Z_0 is contained in \mathfrak{k} , G_Z is connected and has a Cartan decomposition $G_Z = K_Z S_Z$. Let $\mathfrak{g}_Z = \mathfrak{k}_Z + \mathfrak{p}_Z$ be the corresponding decomposition for the Lie algebra. Let $L = \text{Cent}_G\{X, H, Y\}$ and \mathfrak{l} be its Lie algebra. If $g_i = \{Z: \text{ad } HZ = iZ\}$ then let $g_{\text{ev}} = \sum_{i \text{ even}} g_i$ and $g_{\text{od}} = \sum_{i \text{ odd}} g_i$. Then g_{ev} is a τ -invariant subalgebra and therefore reductive. Let $g_{\text{ev}} = \mathfrak{k}_{\text{ev}} + \mathfrak{p}_{\text{ev}}$ be the Cartan decomposition and \mathfrak{p}_o be the orthogonal complement of \mathfrak{p}_{ev} in \mathfrak{p} with respect to the Cartan-Killing form.

LEMMA 4.1. *Let (π, V) be an irreducible finite dimensional representation of $\mathfrak{sl}(2, \mathbf{C})$ of highest weight n . Let $V^\circ = \text{Cent } Z_0$ and $V_i = \{v: \pi(H)v = iv\}$. Then $V^\circ = \{0\}$ if n is odd and V° is isomorphic to V_n when n is even.*

PROOF. Let $v = v_n + v_{n-2} + \cdots + v_{-n}$ be an element of V° . From representation theory of $\mathfrak{sl}(2, \mathbf{C})$ e.g. [13, Lemma A and Theorem B, p. 28] we know that $\dim V_i = 1$ for $-n \leq i \leq n, i \equiv n \pmod{2}$, and $V_i = \{0\}$ other-

wise. Furthermore, $\pi(X)V_n = \pi(Y)V_{-n} = \{0\}$ and $\pi(X)$, $\pi(Y)$ are isomorphisms from V_i to V_{i+2} and V_i to V_{i-2} respectively. The equation $\pi(Z_0)v = 0$ decomposes according to eigenspaces of $\pi(H)$ in the following way. Let i be an eigenvalue of $\pi(H)$. Then the component of $\pi(Z_0)v$ in V_i gives the equation

$$\pi(X)v_{i-2} - \pi(Y)v_{i+2} = 0. \quad (*)$$

Let j be the largest integer so that $v_j \neq 0$. For V_{j+2} we get the equation $\pi(X)v_j = 0$. Thus $j = n$ if $v \neq 0$. Furthermore, for V_n we also get $\pi(X)v_{n-2} = 0$ so $v_{n-2} = 0$. Equation (*) implies $v_k = 0$ if $n - k \not\equiv 0 \pmod{4}$. Let k be the integer such that $v_{n-4k} \neq 0$ but $v_{n-4k-4} = 0$. Then $\pi(X)v_{n-4k-4} - \pi(Y)v_{n-4k} = 0$ so $\pi(Y)v_{n-4k} = 0$. Thus $n - 4k = -n$ or $n = 2k$. This proves the lemma.

COROLLARY 4.2. *The linear subspace $\mathbf{R}H + [H, \mathfrak{k}_Z]$ is orthogonal to \mathfrak{p}_Z . Let \mathfrak{p}_1 be the orthogonal complement of $\mathbf{R}H + [H, \mathfrak{k}_Z] + \mathfrak{p}_Z$ in \mathfrak{p} and $W \in \mathfrak{p}_1$, $W \neq 0$. Then $(\mathbf{R}W)^L = \mathfrak{p}_1$.*

PROOF. Let $Z_1 \in \mathfrak{k}_Z$ and $Z_2 \in \mathfrak{p}_Z$. Then $B_r([H, Z_1], Z_2) = B_r(H, [Z_1, Z_2])$. Then $[Z_1, Z_2] \in \mathfrak{g}_Z$. Let $Z \in \mathfrak{g}_Z$ be arbitrary. Then

$$B_r(H, Z) = B([X, Y], Z) = B(Y, [X, Z]) = B(Y, [Y, Z]) = 0$$

since $[Z_0, Z] = 0$ which is the same as $[X, Z] = [Y, Z]$. Thus $[H, \mathfrak{k}_Z]$ is orthogonal to \mathfrak{p}_Z and also H is orthogonal to \mathfrak{g}_Z . Thus the first statement in the corollary is proved. For the second we note that since $L \subseteq K$ and $\mathbf{R}H + [H, \mathfrak{k}_Z] + \mathfrak{p}_Z$ is L -invariant, \mathfrak{p}_1 is also L -invariant. Next we show that

$$(1) \quad [H, \mathfrak{k}_Z] + \mathfrak{p}_Z = \{Z - \tau Z : Z \in (\ker \operatorname{ad} X)^{\text{ev}}\}.$$

We recall that \mathfrak{g} is an algebra of real rank 1 and therefore \mathfrak{g}_Z is generated by elements of the form $v_4 + v_0 + v_{-4}$ and $v_2 + v_{-2}$ where v_2 and v_4 centralize $\operatorname{ad} X$. If $v_2 + v_{-2} \in \mathfrak{k}_Z$ or $v_4 + v_0 + v_{-4} \in \mathfrak{k}_Z$ then $[H, v_2 + v_{-2}] = 2v_2 - 2v_{-2}$ and $[H, v_4 + v_0 + v_{-4}] = 4v_4 - 4v_{-4}$ are in \mathfrak{p} and have the desired form. The statement follows from these two relations. This also implies that

$$(2) \quad \mathfrak{p}_1 = \{Z - \tau Z : Z \in \mathfrak{g}^+ \text{ and perpendicular to } (\ker \operatorname{ad} X)^{\text{ev}}\} \text{ where } \mathfrak{g}^+ = \sum_{i < 0} \mathfrak{g}_i.$$

Let $X_1 \in \mathfrak{g}^+$ be such that $W = X_1 - \tau X_1 \in \mathfrak{p}_1$. Then Theorem 8.11.3 in [14] has as a simple consequence the fact that the group L acts transitively on the unit sphere of the orthogonal of $(\ker \operatorname{ad} X)^{\text{ev}}$. Then in view of the description of \mathfrak{p}_1 given in (2) we get $(\mathbf{R}W)^L = \mathfrak{p}_1$.

PROPOSITION 4.3. *Let \mathfrak{p}_Z^\perp be the orthogonal complement of \mathfrak{p}_Z in \mathfrak{p} . Then the map $\psi: K_Z \times \mathbf{R}^2 \rightarrow \mathfrak{p}_Z^\perp$ given by $\psi(m, s_1, s_2) = (s_1 W + s_2 H)^m$ is onto \mathfrak{p}_Z^\perp .*

PROOF. We calculate the differential of ϕ . It equals

$$D\psi_{(m,s_1,s_2)}(Z, t_1, t_2) = \text{Ad } m(s_1[Z, W] + s_2[Z, H] + t_1W + t_2H).$$

In view of Corollary 4.2 this map is a submersion at any point (m, s_1, s_2) where $s_1, s_2 \neq 0$. Let \mathcal{D} be the set of points at which ψ is a submersion. Then \mathcal{D} is dense in $K_Z \times \mathbf{R}^2$ and $\psi(\mathcal{D})$ is open in \mathfrak{p}_Z^\perp . Since K_Z is a compact group, $\psi(K_Z \times \mathbf{R}^2)$ is also closed. We now examine the boundary of $\psi(\mathcal{D})$. It is contained in $\mathbf{R}W^{K_Z} \cup \mathbf{R}H^{K_Z}$ which is a union of two submanifolds namely $\mathbf{R}^xW^{K_Z}$ and $\mathbf{R}^xH^{K_Z}$, and $\{0\}$. If we can show that the two submanifolds have at least codimension 2 in \mathfrak{p}_Z^\perp then it follows that ψ is onto. Indeed, let $q \in \mathfrak{p}_Z^\perp \setminus \psi(K_Z \times \mathbf{R}^2)$. Then we can join q to any element of the interior of $\psi(K_Z \times \mathbf{R}^2)$ by a curve that does not intersect $\mathbf{R}W^{K_Z} \cup \mathbf{R}H^{K_Z}$ (see Proposition 9.4 in [5]). This is a contradiction so $\mathfrak{p}_Z^\perp = \psi(K_Z \times \mathbf{R}^2)$.

It remains to show that the boundary has codimension at least 2. By Corollary 4.2 we have the direct sum decomposition

$$(1) \mathbf{R}H + [H, \mathfrak{k}_Z] + \mathfrak{p}_Z + \mathfrak{p}_1 = \mathfrak{p}.$$

Consider the case where $\dim \mathfrak{p}_1 > 1$ first. Then $\mathbf{R}H + [H, \mathfrak{k}_Z]$ has codimension equal to $\dim \mathfrak{p}_1$ so $(\mathbf{R}^xH)^{K_Z}$ has codimension at least 2.

We have the following relations.

$$(2) B_\tau([W, \mathfrak{k}_Z], H) = 0,$$

$$(3) B_\tau([W, \mathfrak{k}_Z], \mathfrak{p}_Z) = 0,$$

$$(4) B_\tau([W, \mathfrak{k}_Z], W) = 0.$$

From (2) we obtain that H is orthogonal to $\mathbf{R}W + [W, \mathfrak{k}_Z]$. From relation (3) and $W \in \mathfrak{p}_1$ we get

$$(5) \mathbf{R}W + [W, \mathfrak{k}_Z] \subseteq [H, \mathfrak{k}_Z] + \mathfrak{p}_1.$$

In view of $\mathbf{R}W + [W, \mathfrak{k}_Z] = \mathfrak{p}_1$ let $[W, \mathfrak{k}_Z]^\perp$ be the perpendicular complement of \mathfrak{p}_1 in $\mathbf{R}W + [W, \mathfrak{k}_Z]$. Then $[W, \mathfrak{k}_Z]^\perp \subseteq [H, \mathfrak{k}_Z]$. If $[W, \mathfrak{k}_Z]^\perp \neq [H, \mathfrak{k}_Z]$ then we can find an element orthogonal to $\mathbf{R}W + [W, \mathfrak{k}_Z]$ and H so the codimension of $(\mathbf{R}^xW)^{K_Z}$ is at least 2.

If on the other hand $[W, \mathfrak{k}_Z]^\perp = [H, \mathfrak{k}_Z]$ we have equality in (5). Then $\mathbf{R}W + s_1[W, \mathfrak{k}_Z] = [H, \mathfrak{k}_Z] + \mathfrak{p}_1$ for $s_1 \neq 0$ so $(m, s_1, s_2) \in \mathcal{D}$ for any $s_1 \neq 0$. Evidently $\{0\}$ has codimension at least 2.

Next we consider the case $\dim \mathfrak{p}_1 = 1$. Then $W^\perp = W$ and $\mathbf{R}H + [H, \mathfrak{k}_Z]$ is the orthogonal of $\mathbf{R}W + \mathfrak{p}_Z$ in \mathfrak{p} so if $s_2 \neq 0$ then $(m, s_1, s_2) \in \mathcal{D}$. Again $\{0\}$ has codimension at least 2. The proof is now complete.

We now recall the following result.

PROPOSITION 4.4. *Let G be a connected semisimple Lie group and let $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ be its Cartan decomposition. Let \mathfrak{p}' be a linear subspace of \mathfrak{p} such that $[X, [X, Y]] \in \mathfrak{p}'$ for any $X, Y \in \mathfrak{p}'$. Let \mathfrak{p}^\perp be the orthogonal of \mathfrak{p}' in \mathfrak{p} . Then G decomposes topologically into $G = K \cdot F \cdot E$ where $F = \exp \mathfrak{p}^\perp$ and $E = \exp \mathfrak{p}'$.*

PROOF. See [9, Theorem 3, p. 40].

By applying this result to the case considered here we get the following corollaries.

COROLLARY 4.5. Let $F = \exp \mathfrak{p}_Z^\perp$ and $S_Z = \exp \mathfrak{p}_Z$. Then

$$G = KFS_Z = K \exp(\mathbf{R}W + \mathbf{R}H)G_Z.$$

PROOF. We note that \mathfrak{p}_Z is the \mathfrak{p} -component of the subalgebra \mathfrak{g}_Z so it satisfies the properties of \mathfrak{p}' in Proposition 4.4. Thus we can write $G = KFS_Z$. By Proposition 4.3 $\mathfrak{p}_Z^\perp = (\mathbf{R}H + \mathbf{R}W)^{K_Z}$ so

$$\exp \mathfrak{p}_Z^\perp \subseteq K_Z \exp(\mathbf{R}H + \mathbf{R}W)K_Z.$$

Therefore, $G = KK_Z \exp(\mathbf{R}H + \mathbf{R}W)K_Z S_Z = K \exp(\mathbf{R}H + \mathbf{R}W)G_Z$.

COROLLARY 4.6. Let $\mathfrak{g} = \mathfrak{su}(2, 1)$,

$$H = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \quad \text{and} \quad W = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}.$$

Let $A = \exp \mathbf{R}H$ and $F = \exp \mathbf{R}W$. Then the maps $\phi_1, \phi_2: A \times F \rightarrow \exp(\mathbf{R}H + \mathbf{R}W)$ given by $\phi_1(e, f) = fef$ and $\phi_2(e, f) = efe$ are isomorphisms.

PROOF. This corollary can be regarded as a particular case of Proposition 4.4. We give a sketch of the proof. Theorem 1 on p. 37 in [9] shows that ϕ maps $A \times F$ into $\exp(\mathbf{R}H + \mathbf{R}W)$. Let $p(t)$ be a path in $\exp(\mathbf{R}H + \mathbf{R}W)$ joining the origin to some arbitrary element x . We denote by $|x|$ the Riemannian distance between x and the origin under the metric given by $(ds/dt)^2 = \text{Tr}(p^{-1}\dot{p})$. Then the proof of Theorem 3 in [9] shows that $|efe| > \sup(|e|, |f|)$. If $e_n f_n e_n$ converges to x , $|e_n f_n e_n|$ is bounded so $|e_n|$ and $|f_n|$ are also bounded so e_n has a convergent subsequence and so does f_n . Let $e_n \rightarrow e \in A$ and $f_n \rightarrow f \in F$. Then $efe = x$. This shows that the image of ϕ is closed. A calculation of the Jacobian of ϕ shows that the image is also open. The rest of the proof is as in Theorem 1 in [9].

We now put all these results together in the following theorem.

THEOREM 4.7. Let G be a real semisimple group of rank one. With notation as before, $G = KA^+ \exp \mathbf{R}W G_Z$. Let L_0 be the normalizer of $\mathbf{R}W$ in L . Then the map $\psi: K/L_0 \times \mathbf{R}^+ \times \mathbf{R}^\times \times G_Z \rightarrow G$ defined by $\psi(k, s_1, s_2, m) = k \exp s_2 H \exp s_1 W m$ has finite fiber of constant cardinality.

PROOF. Let $x \in G$ be such that $x = ks$ where $k \in K$, $s \in S$ and $S = \exp \mathfrak{p}$. Let $x = k_1 z m$ be the decomposition in Corollary 4.5. By the remark at the end of §3 we may assume that the Lie triple $\{X, H, Y\}$ and W are contained in a τ -invariant subalgebra isomorphic to $\mathfrak{su}(2, 1)$. Let K' be the maximal compact subgroup of $SU(2, 1)$ and G_{Z_0}' be the centralizer of Z_0 in

$SU(2, 1)$. In this case, G'_Z is compact. We can then write

$$\begin{aligned} G &= (K/K')K' \exp(\mathbf{RH} + \mathbf{RW})G'_{Z_0}(G'_{Z_0} \setminus G_{Z_0}) \\ &= (K/K')SU(2, 1)(G'_{Z_0} \setminus G_{Z_0}). \end{aligned}$$

In view of this decomposition it is enough to show the result for $SU(2, 1)$. Assume that $x \in SU(2, 1)$. Then $\tau(x) = ks^{-1}$ and $\tau(x)^{-1}x = s^2$. Since in this case $G_{Z_0} = K_{Z_0}$, $\mathfrak{p}_Z^\perp = \mathfrak{p}$. Therefore, by Proposition 4.3, s^2 can be conjugated into $\exp(\mathbf{RH} + \mathbf{RW})$. Let $m \in G_Z$ and $Z \in \exp(\mathbf{RH} + \mathbf{RW})$ be such that $s = m^{-1}zm$. By Corollary 4.6 we can write $z^2 = fe^2f$ with $e \in A$ and $f \in \exp \mathbf{RW}$. Then $xm^{-1}f^{-1}e^{-1} \in K$ since

$$\begin{aligned} \tau(xm^{-1}f^{-1}e^{-1}) &= \tau(x)m^{-1}fe = ks^{-1}m^{-1}fe = km^{-1}z^{-1}fe \\ &= km^{-1}z^{-1}z^2f^{-1}e^{-1} = ksm^{-1}f^{-1}e^{-1} = xm^{-1}f^{-1}e^{-1}. \end{aligned}$$

Let $k' = xm^{-1}f^{-1}e^{-1}$. Then $x = k'efm$ so $G = KA \exp \mathbf{RW}G_Z$.

Suppose that $x = k_1e_1f_1m_1 = k_2e_2f_2m_2$. Then $\tau(x)^{-1}x = m_1^{-1}f_1e_1^2f_1m_1 = m_2^{-1}f_2e_2^2f_2m_2$. By Corollary 4.6, $f_1e_1^2f_1$ and $f_2e_2^2f_2$ are elements of $\exp(\mathbf{RH} + \mathbf{RW})$ so we have to investigate the relationship $(t_1W + t_2H)^m = s_1W + s_2H$ where t_1 and t_2 are nonzero. Due to the proof of Proposition 4.3, $(t_1W + t_2H)^{K_Z}$ is transverse to $\mathbf{RH} + \mathbf{RW}$ (t_1 and t_2 are fixed) since $[W, \mathfrak{k}_Z]$ and $[H, \mathfrak{k}_Z]$ are both orthogonal to $\mathbf{RH} + \mathbf{RW}$.

This also implies that the intersection has dimension 0. Since K_Z is compact, $(t_1W + t_2H)^{K_Z} \cap (\mathbf{RH} + \mathbf{RW})$ is a finite set. We have to show that it has constant cardinality and also compute the dimension of the centralizer of $t_1W + t_2H$ in K_Z . Let \mathfrak{L}_0 be the Lie algebra of L_0 . There are two cases.

Case I. ad H has odd eigenvalues. In this case K_Z leaves \mathfrak{p}_1 invariant since $\mathfrak{k}_Z \subseteq \mathfrak{g}^{\text{ev}}$ while \mathfrak{p}_1 is generated by elements of the form $X_1 - \tau X_1$ where $X_1 \in \mathfrak{g}^{\text{ev}}$. Let $m \in K_Z$. If $(t_1W + t_2H)^m = s_1W + s_2H$ then $t_1W^m = s_1W$ and $t_2H^m = s_2H$. This implies that the connected component of the centralizer of $t_1W + t_2H$ is contained in L_0 . Since L acts transitively on the unit sphere in \mathfrak{g} we can find $l \in L$ such that $W^l = -W$. Then $(t_1W + t_2H)^l = (-t_1W + t_2H)$. Also

$$\begin{aligned} &\text{card}\{(t_1W + t_2H)^{K_Z} \cap (\mathbf{RW} + \mathbf{RH})\} \\ &= \text{card}\{(-t_1W - t_2H)^{K_Z} \cap (\mathbf{RW} + \mathbf{RH})\}. \end{aligned}$$

Since the map $\phi: K_Z \times (\mathbf{R}^x)^2 \rightarrow \mathfrak{p}_Z^\perp$ is a submersion, ϕ has constant fibers on each connected component of $(\mathbf{R}^x)^2$.

Case II. ad H has even eigenvalues only. We calculate the dimension of $K/L_0 \times A \times \exp \mathbf{RW} \times G_Z$. We already know that the map ϕ is a submersion. We then have the following relations

- (1) $\dim \mathfrak{k} = \dim \mathfrak{g}_0 - 1 + \dim \mathfrak{g}_2 + \dim \mathfrak{g}_4$,
- (2) $\dim \mathfrak{g}_Z = \dim \mathfrak{g}_0$.

$$(3) \dim \mathfrak{L} - \dim \mathfrak{L}_0 = \dim \mathfrak{p}_1 - 1,$$

$$(4) \dim \mathfrak{p}_1 + \dim \mathfrak{p}_Z + \dim[H, \mathfrak{k}_Z] + 1 = \dim \mathfrak{p} = \dim \mathfrak{g}_2 + \dim \mathfrak{g}_4 + 1,$$

$$(5) \dim[H, \mathfrak{k}_Z] = \dim \mathfrak{k}_Z - \dim \mathfrak{L}.$$

Then relations (3), (4) and (5) give

$$\dim \mathfrak{L}_0 = \dim \mathfrak{L} - \dim \mathfrak{p}_1 + 1 = \dim \mathfrak{g}_0 - \dim \mathfrak{g}_2 - \dim \mathfrak{g}_4 + 1$$

so

$$\dim \mathfrak{k} - \dim \mathfrak{L}_0 + 2 + \dim \mathfrak{g}_Z = \dim \mathfrak{g}_0 + 2 \dim \mathfrak{g}_2 + 2 \dim \mathfrak{g}_4 = n.$$

To show that the fiber has constant cardinality we make use of the fact that the map ϕ mentioned in Case I is a submersion so its fiber has constant cardinality on connected components of $(\mathbf{R}^x)^2$, and the relation

$$\text{card}((-t_1W - t_2H)^{L_0}) = \text{card}((t_1W + t_2H)^{L_0}).$$

Let $X_2 = [X_1, X] \in \mathfrak{g}_4$ where $X_1 - \tau X_1 = W$. Then $X_2 \neq 0$ and there is l centralizing X_2 and H (by Theorem 8.11.3 in [14]) such that $X^l = -X$. But then $W^l = -W$. Let $(t_1W + t_2H)^m = s_1W + s_2H$ where $m \in L_0$. Then $l^{-1}ml \in L_0$ and

$$(-t_1W + t_2H)^{l^{-1}ml} = (t_1W + t_2H)^{ml} = (s_1W + s_2H)^l = -s_1W + s_2H.$$

Thus the fiber has constant cardinality. The proof of the proposition is complete once we note that in $\text{su}(2, 1)$ the element

$$\begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & +1 \end{bmatrix} \in L_0$$

maps H into $-H$.

We now derive an integration formula corresponding to the decomposition in Theorem 4.7.

PROPOSITION 4.8. *Let $\psi: K/L_0 \times A^+ \times \exp \mathbf{R}W \times G_Z \rightarrow G$ be the map given by $\psi(k, a, w, m) = kawm$. Let $g(t_1, t_2) = |\det D\psi_{k,a,w,m}|$ where $k \in K/L_0$, $a = \exp t_2H$, $w = \exp t_1W$ and $m \in G_Z$. Let c_{Z_0} be the cardinality of the fiber of ψ at any point where $t_1, t_2 \neq 0$. Then*

$$\int_{G/G_{Z_0}} f(Z_0^x) dx = \frac{1}{c_{Z_0}} \int_{K/L_0} \int_{A^+} \int_{\mathbf{R}} g(t_1, t_2) f(Z_0^{kaw}) dt_1 da dk.$$

Furthermore $g(t_1, t_2) = |\det \phi|$ where ϕ is the linear map

$$\phi(Z_1, s_2, s_1, Z_2) = \text{Ad } a^{-1}Z_1 + s_2H + s_1W + \text{Ad } w^{-1}Z_2$$

for $Z_1 \in \mathfrak{k}/\mathfrak{L}_0$, $Z_2 \in \mathfrak{g}_Z$.

PROOF. In order to establish the integral formula it is enough to calculate the pullback under ψ of the Haar measure dx of G .

By Theorem 4.7, $K/L_0 \times A^+ \times \exp \mathbf{R}^\times W \times G_Z$ is a covering space of an open dense subspace of G and the fiber has constant cardinality. Then the following formula holds for any $f \in C_c^\infty(G)$.

$$\int_G f(x) dx = \frac{1}{c_{Z_0}} \int_{K/L_0} \int_{A^+} \int_{\mathbf{R}} \int_{G_Z} f(kawm) J_{k,a,w,m} dk da dw dm$$

where J is the Jacobian. Due to the fact that the measures involved are all left and right invariant we can express $J_{k,a,w,m}$ as a function of a and w alone. We get

$$D\psi_{k,a,w,m}(Z_1, s_2, s_1, Z_2) = L_{kawm}^*(\text{Ad}(m^{-1}w^{-1}a^{-1})Z_1 \\ + s_2 \text{Ad}(m^{-1}w^{-1})H + s_1 \text{Ad}(m^{-1})W + Z_2)$$

where L_g^* is the differential of left multiplication by g . By using the fact that dx is left invariant and that $|\det \text{Ad } x| = 1$ for any $x \in G$ and $|\det_{g_Z} \text{Ad}(m)| = 1$ for any $m \in G_Z$ we get $|J_{k,a,w,m}| = |\det(\text{Ad } a^{-1}Z_1 + s_2H + s_1W + \text{Ad } wZ_2)|$. This is exactly the expression for $|\det \phi|$ in the statement of the proposition. The integral formula now follows from the normalization of the measures so that $dx = d_{G/G_Z}x^* \cdot d_{G_Z}\dot{x}$.

5. The orbit of the nilpotent element X . We now find a decomposition and an integral formula similar to Theorem 4.7 and Proposition 4.8 for a nilpotent element X . Let $\{X, H, Y\}$ be a Lie triple as before.

LEMMA 5.1. *Let $N_0 = \text{Cent}_N X$. If G_X is the centralizer of X in G then $G_X = LN_0$. Furthermore, if $y \in G_X$ is semisimple then y can be conjugated into L by some element in N_0 .*

PROOF. From representation theory of $\mathfrak{sl}(2, \mathbf{C})$ it follows that $\text{Cent}_g X = \mathfrak{L} + \mathfrak{n}_0$ where \mathfrak{n}_0 is the Lie algebra of N_0 . Let $y \in G$ be such that $X^y = X$. Then $H^y - H \in \text{Cent}_g X$. Therefore, $H^y = H + m_0 + n_0$ where $m_0 \in \mathfrak{L}$ is semisimple and $n_0 \in \mathfrak{n}_0$. For any invariant polynomial p we have the relation $p(H) = p(H^y) = p(H + m_0 + n_0) = p(H + m_0)$. But $\text{ad } m_0$ has imaginary eigenvalues and $[H, m_0] = 0$. It follows that $m_0 = 0$. By a well-known argument, $H + n_0 = H^{n_1}$ for some $n_1 \in N_0$. Thus $H^y = H^{n_1}$ so $yn_1^{-1} \in \text{Cent}_G H \cap \text{Cent}_G X = L$. This establishes the decomposition $G_X = LN_0$.

Let now $y \in G_X$ be semisimple. $\text{Ad } y$ normalizes N_0 so

$$\mathfrak{n}_0 = \text{Im}(-\text{Ad } y + 1) \oplus \ker(-\text{Ad } y + 1).$$

Then $H^y = H + Z_0 + Z_1$ where $Z_0 = (-\text{Ad } y + 1)Z_2$ and $(-\text{Ad } y + 1)Z_1 = 0$. Then

$$(H + Z_2)^y = H + Z_0 + Z_1 + Z_2^y = H + Z_2 + Z_1$$

so that

$$(\text{Ad } y - 1)^2(H + Z_2) = (\text{Ad } y - 1)Z_1 = 0.$$

Since $\text{Ad } y - 1$ is semisimple $(\text{Ad } y - 1)(H + Z_2) = 0$. As $H + Z_2 = H^n$ for some $n \in N_0$ it follows that $H^n = H + Z_2 = (H + Z_2)^y = H^{ny}$ or $ny n^{-1} \in \text{Cent}_G H \cap \text{Cent}_G X = L$.

This completes the proof.

LEMMA 5.2. *Let \mathfrak{n}_0 be the Lie algebra of N_0 and \mathfrak{n}_1 its orthogonal complement in \mathfrak{n} . If $N_1 = \exp \mathfrak{n}_1$ then $N = N_1 N_0$ and $G = K/LAN_1 G_X$.*

PROOF. Let $\mathfrak{n}^{(k)} = \sum_{i \geq k} \mathfrak{g}_i$. Then $\mathfrak{n}^{(k)}$ is an ideal in \mathfrak{n} , $\mathfrak{n}^{(k)} \supseteq \mathfrak{n}^{(k+1)}$ and $\mathfrak{n}^{(1)} = \mathfrak{n}$. Furthermore, since $\text{ad } H \mathfrak{n}_0 \subseteq \mathfrak{n}_0$ and $\text{ad } H \mathfrak{n}_1 \subseteq \mathfrak{n}_1$ we can decompose $\mathfrak{n}^{(k)}$ into $\mathfrak{n}^{(k)} = \mathfrak{n}^{(k)} \cap \mathfrak{n}_0 + \mathfrak{n}^{(k)} \cap \mathfrak{n}_1$. Thus Lemma 1 on p. 736 in [4a] applies so $N = N_1 N_0$. To show that this decomposition is unique it is enough to show that, if $\exp X_2 = \exp(-X_1)\exp Y_1$ where $X_2 \in \mathfrak{n}_0$ and $X_1, Y_1 \in \mathfrak{n}_1$, then $X_2 = 0$. The Campbell-Baker-Hausdorff formula states that

$$\begin{aligned} \log[\exp(-X_1)\exp Y_1] &= -X_1 + Y_1 + \frac{1}{2}[-X_1, Y_1] \\ &\quad + \frac{1}{12}[-X_1, [-X_1, Y_1]] + \cdots \end{aligned}$$

Suppose that $X_1 = Z + X'_1$ where $X'_1 \in \mathfrak{n}^{(s+1)}$ and $Y_1 = Z + Y'_1$ where $Y'_1 \in \mathfrak{n}^{(s+1)}$. Then $[X_1, Y_1] = [Z, Y'_1] + [X'_1, Z] + [X'_1, Y'_1] \in \mathfrak{n}^{(s+2)}$. Therefore

$$\log[\exp(-X_1)\exp Y_1] \equiv Y'_1 - X'_1 \pmod{\mathfrak{n}^{(2+s)}}.$$

On the other hand $\log[\exp(-X_1)\exp Y_1] \equiv X_2 \pmod{\mathfrak{n}^{(s+2)}}$ which is possible only if $X'_1 \equiv Y'_1 \pmod{\mathfrak{n}^{(s+2)}}$ and $X_2 \equiv 0 \pmod{\mathfrak{n}^{(s+2)}}$. A simple induction argument then shows that $X_2 = 0$. The Iwasawa decomposition transforms into

$$G = KAN = KAN_1 N_0 = K/L LAN_1 N_0 = K/L AN_1 LN_0 = K/L AN_1 G_X$$

and the proof is complete.

REMARK. The previous two lemmas are true for G of arbitrary rank with minor modifications.

In accordance with §4 let $W = X_1 - \tau X_1$ where $X_1 \in \mathfrak{n}$.

PROPOSITION 5.3. *Let G be a semisimple Lie group of real rank one. Then $G = KA \exp \mathbf{R} X_1 G_X$ and the decomposition is unique up to $L_0 = \text{Norm}_L \{\mathbf{R} X_1\} = \text{Norm}_L \{\mathbf{R} W\}$.*

PROOF. By the proof of Corollary 4.2 we know that $\mathfrak{n}_1 = (\mathbf{R} X_1)^L$ so $N_1 = (\exp \mathbf{R} W_1)^L$. We substitute this into the decomposition of Lemma 5.2.

$$G = K/LA(\exp \mathbf{R} X_1)^L G_X = KA \exp \mathbf{R} X_1 G_X.$$

Suppose $k_1 a_1 \exp t_1 X_1 l_1 n_1 = k_2 a_2 \exp t_2 X_1 l_2 n_2$. Then

$$k_1 l_1 a_1 \exp t_1 X_1^{l_1^{-1}} n_1 = k_2 l_2 a_2 \exp t_2 X_1^{l_2^{-1}} n_2.$$

Since this decomposition is unique, $a_1 = a_2$, $k_1 l_1 = k_2 l_2$, $\exp t_1 X_1^{l_1^{-1}} = \exp t_2 X_1^{l_2^{-1}}$ or $(t_1 X_1)^{l_1^{-1} l_2} = t_2 X_1$ and $n_1 = n_2$. Thus $l_1 l_2^{-1} \in \text{Norm}_L(\mathbf{R}W_1)$ so $k_2^{-1} k_1 \in \text{Norm}_L(\mathbf{R}X_1)$. Thus the decomposition $G = K/L_0 A \exp t_1 X_1 G_X$ is unique.

PROPOSITION 5.4. *Let $\rho(H) = \frac{1}{2} \text{tr ad } H|_{\mathfrak{n}}$. Then*

$$\int_{G/G_X} f(X^x) dx = \int_{K/L_0} \int_A e^{2\rho(\log a)} \int_{\mathbf{R}} t^{\dim n_1 - 1} f(X^{ka \exp t_1 X_1}) dt_1 da dk$$

for any $f \in C_c^\infty(\mathfrak{g})$.

PROOF. Let $\psi: K/L_0 \times A \times \exp t_1 W \times G_X \rightarrow G$ be the map given by $\psi(k, a, w, m) = kawm$ where $w = \exp t_1 X_1$ and $m \in G_X$.

As in Proposition 4.8 we have to calculate the pullback under ψ of the Haar measure dx of G . The map ψ is an isomorphism so for any $f \in C_c^\infty(G)$,

$$\int_G f(x) dx = \int_{K/L_0} \int_A \int_{\mathbf{R}} \int_{G_X} f(kawm) J_{k,a,w,m} dk da dt_1 dm$$

where J is the Jacobian. We obtain the formula

$$D\psi_{k,a,w,m}(Z_1, s_2, s_1, Z_2) = L_{kawm}^*(\text{Ad}(m^{-1}w^{-1}a^{-1})Z_1 + s_2 \text{Ad}(m^{-1}w^{-1})H + s_1 \text{Ad}(m^{-1})X_1 + Z_2).$$

By using the fact that G_X is unimodular the same argument as in Proposition 4.8 implies

$$J_{k,a,w,m} = |\det(\text{Ad } a^{-1}Z_1 + s_2 H + s_1 X_1 + \text{Ad } wZ_2)|.$$

We observe that

$$\text{Ad } w = e^{t_1 \text{ad } X_1} = I + t_1 \text{ad } X_1 + t_1^2 (\text{ad } X_1)^2 / 2 + \dots$$

so in view of the fact that X_1 is nilpotent, we have

- (1) $\text{Ad } w|_{\mathfrak{e}_0} = \text{id}$,
- (2) $\text{Ad } wZ_2 = Z_2 + t_1[Z_2, X_1] + \text{higher order terms}$ ($Z_2 \in \mathfrak{e}/\mathfrak{e}_0$),
- (3) $\text{Ad } wZ_2 = Z_2 + \text{higher order terms}$ if $Z_2 \in \mathfrak{n}_0$.

Then $\text{Ad } w$ maps $\mathfrak{e} + \mathfrak{n}_0$ into $\mathfrak{e} + \mathfrak{n}$ and is given by upper triangular matrices.

On the other hand \mathfrak{k} has a basis formed by vectors $Z + \tau Z$ where $Z \in \mathfrak{n}$ and vectors $Z' \in \text{Cent}_{\mathfrak{k}} H$. Also the map $Z_2 \rightarrow [Z_2, X_1]$ is onto the orthogonal complement of X_1 in \mathfrak{n}_1 . By putting all these facts together we can calculate the determinant of $D\psi$. It equals $t_1^{\dim n_1 - 1} e^{2\rho(\log a)}$ where the power of t_1 comes from relation (2) and $e^{2\rho(\log a)}$ comes from the action $\text{Ad } a^{-1}$ on \mathfrak{k} .

Finally the formula in the proposition is a consequence of the normalization of the measures so that $dx = d_{G/G_x} \dot{x} d_x x^*$.

6. The relationship between the unipotent and semisimple integrals. We use the same notation as before. Let $G_{Z_0} = \text{Cent}_G Z_0$ and $G_X = \text{Cent}_G X$. The goal of this section is to prove the formula in Theorem 6.5.

By Remark 3.3 we may assume that $X, H, Y, W \in \mathfrak{su}(2, 1) \subseteq \mathfrak{g}$. According to the nature of the eigenvalues of $\text{ad } H$ there are two cases.

Case I. $\text{ad } H$ has even eigenvalues only. This is a special case of some results on even nilpotent elements of R. Rao. We give an independent argument. We may set

$$X = \begin{bmatrix} 0 & -i & -i \\ -i & 0 & 0 \\ +i & 0 & 0 \end{bmatrix}, \quad Y = \begin{bmatrix} 0 & -i & i \\ -i & 0 & 0 \\ -i & 0 & 0 \end{bmatrix},$$

$$H = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -2 \\ 0 & -2 & 0 \end{bmatrix} \quad \text{and} \quad W = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}.$$

We obtain the following relation

$$Z_0^{\exp t_1 W} = i \begin{bmatrix} 2 \sinh 2t_1 & 0 & -2 \cosh 2t_1 \\ 0 & 0 & 0 \\ 2 \cosh 2t_1 & 0 & -2 \sinh 2t_1 \end{bmatrix}.$$

Then

$$Z_0^{\exp t_1 W} = \cosh 2t_1 Z_0 + \sinh 2t_1 X_0 + \sinh 2t_1 (X_2 - \tau X_2), \quad (1)$$

where

$$X_0 = \begin{bmatrix} 2i & 0 & 0 \\ 0 & -i & 0 \\ 0 & 0 & -i \end{bmatrix}$$

is in \mathfrak{g}_0 and

$$X_2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & -i/2 & i/2 \\ 0 & -i/2 & i/2 \end{bmatrix}$$

is in \mathfrak{g}_4 . Then

$$t Z_0^{\exp t_1 W} = \left[\cosh 2t_1 (X - t^2 Y) + t \sinh 2t_1 X_0 \right. \\ \left. + \sinh 2t_1 \left(\frac{1}{t} X_2 - t^3 \tau X_2 \right) \right]^{\exp \theta H} \quad (2)$$

where $\theta = \frac{1}{2} \log t$.

In the notation of Proposition 4.8 we consider

$$\int_0^\infty \int_0^\infty g(t_1, t_2) f(tZ_0^{aw}) dt_1 dt_2 \quad (3)$$

and by using relations (1) and (2) we make the change of variables $t_2^1 = t_2 + \theta$ and $t_1^1 = t_1/t$. The integral transforms into

$$\int_0^\infty \int_{(1/2)\log t}^\infty tg(tt_1, t_2 - \theta) f(t, t_1, t_2) dt_1 dt_2 \quad (4)$$

where

$$f(t, t_1, t_2) = f\left(\frac{1}{t} \sinh 2tt_1 X_2 + Z_{t,t_1}\right)^{\exp t_2 H}. \quad (5)$$

The term Z_{t,t_1} is equal to

$$Z_{t,t_1} = \cosh 2tt_1 X + t \sinh 2tt_1 X_0 - t^2 \cosh 2tt_1 Y - t^3 \sinh 2tt_1 \tau X_2.$$

LEMMA 6.1. *The function $f(t, t_1, t_2)$ has compact support in t_1 and $t_2 \geq 0$ independent of t .*

PROOF. From formula (2) we see that Z_{t,t_1} is orthogonal to X_2 with respect to $-B_\tau$. Then

$$\left\| \left(\frac{1}{t} \sinh 2tt_1 X_2 + Z_{t,t_1} \right)^{\exp t_2 H} \right\|^2 \geq e^{8t_2} \frac{1}{t^2} (\sinh 2tt_1)^2 \|X_2\|^2. \quad (6)$$

Since $((tt_1)^{-1} \sinh 2tt_1)^2 \geq C > 0$ for any t and t_1 we can write

$$\left\| \left(\frac{1}{t} \sinh 2tt_1 X_2 + Z_{t,t_1} \right)^{\exp t_2 H} \right\|^2 \geq Ct_1^2 e^{8t_2} \quad (7)$$

for any t . Since f has compact support, the proof is complete.

We rewrite the integral in (3) as

$$\begin{aligned} & t \int_0^\infty \int_{(1/2)\log t}^0 g(tt_1, t_2 - \theta) f(t, t_1, t_2) dt_1 dt_2 \\ & + t \int_0^\infty \int_0^\infty g(tt_1, t_2 - \theta) f(t, t_1, t_2) dt_1 dt_2. \end{aligned} \quad (8)$$

LEMMA 6.2. *Let $r = \dim \text{Cent } X$ and $n - \dim g$. Then*

$$2\rho(H) = n - r + 2(\dim \mathfrak{L} - \dim \mathfrak{L}_0 + 1) = n - r + 2 \dim \mathfrak{p}_1.$$

PROOF. We have $2\rho(H) = 2 \dim g_2 + 4 \dim g_4$. Since the map $\text{ad } W: \mathfrak{L} \rightarrow \mathfrak{p}_1$ is onto the orthogonal complement of W we get

$$\dim \mathfrak{L} - \dim \mathfrak{L}_0 + 1 = \dim \mathfrak{p}_1.$$

Also

$$\begin{aligned} n &= \dim g_0 + 2(\dim g_2 + \dim g_4), \\ r &= \dim \mathcal{L} + \dim g_2 - \dim \mathfrak{p}_1 + \dim g_4 = \dim g_0 \quad \text{and} \\ &\quad \dim g_4 = \dim \mathfrak{p}_1 \end{aligned}$$

since $\text{ad } X: g_2 \rightarrow g_4$ is onto and $\text{ad } H$ is even. Thus

$$\begin{aligned} n - r + 2 \dim \mathfrak{p}_1 &= \dim g_0 + 2(\dim g_2 + \dim g_4) - \dim g_0 + 2 \dim \mathfrak{p}_1 \\ &= 2 \dim g_2 + 4 \dim g_4 \end{aligned}$$

and the proof of the lemma is complete.

PROPOSITION 6.3. For any $f \in C_c(g)$,

$$\lim_{t \rightarrow 0^+} t^{(n-r)/2} \int_{G/G_0} f(tZ_0^x) dx = c_X \int_{G/G_X} f(X^x) dx$$

where c_X depends only on the conjugacy class of X .

PROOF. We note that in formula (3) $g \neq 0$ so we may replace g by $\det \phi$ where ϕ has the expression given in Proposition 4.8. The same holds for formula (8). We note that

$$\lim_{t \rightarrow 0} \left(\frac{1}{t} \sinh 2tt_1 X_2 + \cosh 2tt_1 X + Z'_{t,t_1} \right) = X + 2t_1 X_2 = X^{\exp t_1 X_1}$$

where X_1 is defined by $X_1 - \tau X_1 = W$, $X_1 \in g_2$ and

$$Z'_{t,t_1} = Z_{t,t_1} - \cosh 2tt_1 X.$$

We then have to calculate $\lim_{t \rightarrow 0^+} t^{(n-r)/2+1} g(tt_1, t_2 - \theta)$. We choose a basis of $\mathfrak{k}/\mathcal{L}_0$ consisting of $X_i + \tau X_i$ and Y_j , where $X_i \in g_i$ with $i > 0$ and $Y_j \in \mathcal{L}$ orthogonal to \mathcal{L}_0 . For g_Z we choose an orthogonal basis consisting of Y_j and Z_k where Y_j are as before. Then

$$\begin{aligned} &t^{(n-r)/2+1} g(tt_1, t_2 - \theta) \\ &= t^{(n-r)/2+1} \det \left(e^{-it_1 t^{1/2} X_i} + e^{it_1 t^{-1/2} \tau X_i}, Y_j, H, W, \text{Ad } w Y_j, \text{Ad } w Z_k \right) \\ &= t_1^{\dim \mathfrak{p}_1 - 1} \det \left(e^{-it_1 t^{1/2} X_i} + e^{it_1 \tau X_i}, Y_j, H, W, \frac{\text{Ad } w Y_j - Y_j}{tt_1}, \text{Ad } w Z_k \right) \end{aligned}$$

by using the formula for $n - r$ in Lemma 6.2. Since $w = \exp tt_1 W$,

$$\lim_{t \rightarrow 0^+} \frac{\text{Ad } w Y_j - Y_j}{tt_1} = [W, Y_j].$$

Thus

$$\begin{aligned} &\lim_{t \rightarrow 0} t^{(n-r)/2+1} g(tt_1, t_2 - \theta) \\ &= t_1^{\dim \mathfrak{p}_1 - 1} e^{2t_2 \rho(H)} \det(\tau X_i, Y_j, H, W, [W, Y_j], Z_k) \\ &= \text{const } t_1^{\dim \mathfrak{p}_1 - 1} e^{2t_2 \rho(H)}. \end{aligned}$$

Since $f(t, t_1, t_2)$ has compact support in t_1 and $t_2 > 0$ independent of t , the second integral in (8) converges to

$$\int_0^\infty \int_0^\infty e^{2\rho(\log a)} t_1^{\dim p_1 - 1} f(X^{\exp t_1 X_1}) dt_1 dt_2$$

by the dominated convergence theorem. For the term for which $t_2 < 0$, we expand the determinant into powers of e^{t_2} . For positive powers of e^{t_2} we get terms of the form $e^{jt_2(2\rho(H)-j)/2}$ where $0 < j \leq 2\rho(H)$. All these terms are absolutely integrable on $(-\infty, 0)$ so we may take the limit inside the integral sign. All terms that have $j < 2\rho(H)$ converge to 0 while for $j = 2\rho(H)$ we get

$$e^{2t_2\rho(H)} t_1^{\dim p_1 - 1} \det(\tau X_i, Y_j, H, W, [W, Y_j], Z_k).$$

For strictly negative powers of e^{t_2} we make a change of variables $u = t_2 - \log t$. The integral becomes a sum of terms

$$\int_0^\infty \int_{-(1/2)\log t}^\infty t_1^{\dim p_1 - 1} e^{ju t_1(2\rho(H)+j)/2} \cdot \det\left(\tilde{X}_i, Y_j, H, W, \frac{\text{Ad } w Y_j - Y_j}{t t_1}, \text{Ad } w Z_k\right) \tilde{f}(t, t_1, t_2) dt_1 dt_2$$

where \tilde{X}_i are either X_i or τX_i . Since $j < 0$, e^{ju} is absolutely integrable and $\tilde{f}(t, t_1, t_2)$ is bounded in the interval $(0, \infty)$, all the terms converge to 0 (we note that $-\frac{1}{2} \log t \rightarrow +\infty$).

Finally the term for $j = 0$ can be estimated by

$$t^{\rho(H)} \int_0^\infty \int_{-(1/2)\log t}^0 f(t, t_1, t_2) dt_1 \leq C t^{\rho(H)} |\log t|.$$

Since $\rho(H) \geq 1$ this term also converges to 0.

Putting all these facts together we get the formula

$$\begin{aligned} \lim_{t \rightarrow 0^+} t^{(n-r)/2} \int_0^\infty \int_0^\infty g(t_1, t_2) f(t Z_0^{aw}) dt_1 dt_2 \\ = \int_0^\infty \int_{-\infty}^\infty e^{2\rho(\log a)} t_1^{\dim p_1 - 1} f(X^{\exp t_1 X_1}) dt_1 dt_2. \end{aligned} \quad (9)$$

In Proposition 4.8 we can split the right-hand side of the formula into a sum of two integrals each of which converges to the corresponding expression to the right-hand side of (9). We thus get

$$\begin{aligned} \lim_{t \rightarrow 0^+} t^{(n-r)/2} \int_{G/G_{Z_0}} f(tZ_0^x) dx \\ = \frac{1}{c_{Z_0}} \int_{K/L_0} \int_A \int_{\mathbf{R}} e^{2\rho(\log a)} t_1^{\dim p_1 - 1} f(X^{kaw}) dt_1 dt_2 \\ = c_X \int_{G/G_X} f(X^x) dx \end{aligned}$$

and $c_X = \text{const}/c_{Z_0}$ and Z_0 depend only on the conjugacy class of X by Proposition 3.1.

This completes Case I.

Case II. $\text{ad } H$ has even as well as odd eigenvalues. We can identify $X, H, Y, W \in \mathfrak{su}(2, 1) \subseteq \mathfrak{g}$ with

$$\begin{aligned} X = \begin{bmatrix} 0 & 0 & 0 \\ 0 & i/2 & i/2 \\ 0 & -i/2 & -i/2 \end{bmatrix}, \quad Y = \begin{bmatrix} 0 & 0 & 0 \\ 0 & +i/2 & -i/2 \\ 0 & +i/2 & -i/2 \end{bmatrix}, \\ H = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{bmatrix} \end{aligned}$$

and

$$W = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}.$$

Then

$$Z_0 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & i & 0 \\ 0 & 0 & -i \end{bmatrix}.$$

Let $w = \exp t_1 W$ and $a = \exp t_2 H$. Then we have the relation

$$Z_0^w = i \begin{bmatrix} \sinh^2 t_1 & 0 & -\sinh t_1 \cosh t_1 \\ 0 & 1 & 0 \\ \sinh t_1 \cosh t_1 & 0 & -\cosh t_1 \end{bmatrix}. \quad (10)$$

Then

$$tZ_0^w = \left[\left(\frac{1 + \cosh^2 t_1}{2} \right) (X - t^2 Y) + Z_{t,t_1} \right]^{\exp \theta H} \quad (11)$$

where $\theta = \frac{1}{2} \log t$ and

$$Z_{t,t_1} = -\frac{1}{2} \sinh 2t_1 (t^{1/2} X_1 + t\tau X_1) + \frac{1}{2} t \sinh^2 t_1 X_0. \quad (12)$$

Here $X_1 \in \mathfrak{g}_1$ and $X_0 \in \mathfrak{g}_0$ are given by

$$X_1 = \begin{bmatrix} 0 & i/2 & i/2 \\ i/2 & 0 & 0 \\ -i/2 & 0 & 0 \end{bmatrix}, \quad X_0 = \begin{bmatrix} 2i & 0 & 0 \\ 0 & -i & 0 \\ 0 & 0 & -i \end{bmatrix}.$$

From formula (12) we see that $Z_{t,t_1} \rightarrow 0$ when $t \rightarrow 0$.

In the notation of Proposition 4.8 we consider

$$\int_0^\infty \int_0^\infty g(t_1, t_2) f(t Z_0^{aw}) dt_1 dt_2. \quad (13)$$

By using relations (10) and (11) we make the change of variables $t'_2 = t_2 + \theta$. The integral in (12) becomes

$$\int_0^\infty \int_{(1/2)\log t}^\infty g(t_1, t_2 - \theta) f(t, t_1, t_2) dt_1 dt_2 \quad (14)$$

where

$$f(t, t_1, t_2) = f \left[\left(\frac{1 + \cosh^2 t_1}{2} (X - t^2 Y) + Z_{t,t_1} \right)^a \right]. \quad (15)$$

LEMMA 6.4. *The function $f(t, t_1, t_2)$ has compact support in t_1 and $t_2 \geq 0$ independent of t .*

PROOF. Since X is orthogonal to Y and Z_{t,t_1} we have the relation

$$\left\| \left\{ \left(\frac{1 + \cosh^2 t_1}{2} \right) (X - t^2 Y) + Z_{t,t_1} \right\}^{\exp t_2 H} \right\|^2 \geq \left(\frac{1 + \cosh^2 t_1}{2} \right)^2 e^{4t_2} \|X\|^2. \quad (16)$$

The statement in the lemma now follows from inequality (16) and the fact that f has compact support.

We rewrite the integral in (14) as

$$\int_0^\infty \int_{(1/2)\log t}^0 g(t_1, t_2 - \theta) f(t, t_1, t_2) dt_1 dt_2 + \int_0^\infty \int_0^\infty g(t_1, t_2 - \theta) f(t, t_1, t_2) dt_1 dt_2. \quad (17)$$

LEMMA 6.5. *Let $r = \dim \text{Cent}_{\mathfrak{g}} X$ and $n = \dim \mathfrak{g}$. Then*

$$2\rho(H) = n - r. \quad (18)$$

PROOF. The lemma is an immediate consequence of the relations

$$\begin{aligned} 2\rho(H) &= \dim \mathfrak{g}_1 + 2 \dim \mathfrak{g}_2, \\ n &= \dim \mathfrak{g}_0 + 2 \dim \mathfrak{g}_1 + 2 \dim \mathfrak{g}_2, \\ r &= \dim \mathfrak{L} + \dim \mathfrak{g}_1 + \dim \mathfrak{g}_2, \quad \text{and} \\ \dim \mathfrak{g}_0 - \dim \mathfrak{L} &= \dim \mathfrak{g}_2. \end{aligned}$$

PROPOSITION 6.6. *Let $f \in C_c^\infty(\mathfrak{g})$. Then*

$$\lim_{t \rightarrow 0^+} t^{(n-r)/2} \int_{G/G_{Z_0}} f(tZ_0^x) dx = c_X \int_{G/G_X} f(X^x) dx \quad (19)$$

and c_X depends only on the conjugacy class of X in \mathfrak{g} .

PROOF. We note that in formula (17) $g \neq 0$ over the range of the integral as we may take $\pm \det \phi$ in the notation of Proposition 4.8. From formula (11) we conclude that

$$\lim_{t \rightarrow 0} \left\{ \left(\frac{1 + \cosh^2 t_1}{2} \right) (X - t^2 Y) + Z_{t,t_1} \right\}^a = \frac{1 + \cosh^2 t_1}{2} X^a.$$

As in Case I we select a basis of $\mathfrak{k}/\mathfrak{L}_0$ consisting of vectors $X_i + \tau X_i$ and Y_j where $X_i \in \mathfrak{g}_i$ with $i > 0$ and $Y_j \in \mathfrak{L}$ orthogonal to \mathfrak{L}_0 . For \mathfrak{g}_Z we select a basis consisting of Y_j and Z_k . Then

$$\begin{aligned} & t^{(n-r)/2} g(t_1, t_2 - \theta) \\ &= t^{(n-r)/2} \det(e^{-it_2 t^i/2} X_i + e^{it_2 t^{-i/2}} \tau X_i, Y_j, H, W, \text{Ad } wY_j, \text{Ad } wZ_k) \\ &= \det(e^{-it_2 t^i} X_i + e^{it_2 \tau} X_i, Y_j, H, W, \text{Ad } wY_j, \text{Ad } wZ_k) \end{aligned} \quad (20)$$

by multiplying the t inside the determinant. Using formula (18) this expression converges to

$$e^{2\rho(\log a)} c(t_1) = e^{2\rho(\log a)} \det(\tau X_i, Y_j, H, W, \text{Ad } wY_j, \text{Ad } wZ_k). \quad (21)$$

Due to Lemma 6.4 we can use the dominated convergence for the second term in (17). Its limit is

$$\int_0^\infty \int_0^\infty e^{2\rho(\log a)} c(t_1) f\left(\frac{1 + \cosh^2 t_1}{2} X^a\right) dt_1 dt_2.$$

For the first term in (17) we expand in powers of e^{t_2} . For $0 < j \leq 2\rho(H)$ we get $e^{jt_2(2\rho(H)-j)/2} \det(\tilde{X}_i, Y_j, H, W, \text{ad } wY_j, \text{Ad } wZ_k)$ where \tilde{X}_i is either X_i or τX_i . Then $e^{jt_2} f(t, t_1, t_2)$ is bounded by an absolutely integrable function for any t_1 and t_2 so we can use the bounded convergence theorem for each term. Only the term for $j = 2\rho(H)$ contributes anything. We get

$$\int_0^\infty \int_{-\infty}^0 e^{2\rho(\log a)c(t_1)} f\left(\frac{1 + \cosh^2 t_1}{2} X^a\right) dt_1 dt_2.$$

For $-2\rho(H) \leq j < 0$ we make the change of variables $u = t_2 - \log t$. The integrals transform into

$$\int_0^\infty \int_{-(1/2)\log t}^\infty e^{ju_t(2\rho(H)+j)/2} \cdot \det(\tilde{X}_i, Y_j, H, W, \text{Ad } wY_j, \text{Ad } Z_k) \tilde{f}(t, t_1, u) dt_1 dt_2$$

where \tilde{X}_i is either X_i or τX_i and $\tilde{f}(t, t_1, u) = f(t, t_1, u + \log t)$ is a bounded function in (t, t_1, u) . We can apply the bounded convergence theorem. Since $-\frac{1}{2} \log t \rightarrow +\infty$ the limit of any of these integrals is 0. Remains to consider the case $j = 0$. The same argument as in Case I applies. We have obtained the formula

$$\begin{aligned} \lim_{t \rightarrow 0^+} t^{(n-r)/2} \int_0^\infty \int_0^\infty g(t_1, t_2) f(tZ_0^{aw}) dt_1 dt_2 \\ = \int_0^\infty \int_{-\infty}^\infty e^{2\rho(\log a)c(t_1)} f\left(\frac{1 + \cosh^2 t_1}{2} X\right) dt_1 dt_2. \end{aligned} \quad (22)$$

We make the change of variables $t' = t + \frac{1}{2} \log((1 + \cosh^2 t_1)/2)$ in the right-hand side of (22) to get

$$\int_0^\infty \frac{c(t_1)}{((1 + \cosh^2 t_1)/2)^{\rho(H)}} dt_1 \int_{-\infty}^\infty e^{2\rho(\log a)} f(X^a) dt_2. \quad (23)$$

The proof of the proposition is completed in the same manner as in Case I. We now put these results together in the following theorem.

THEOREM 6.7. *Let $f \in C_c^\infty(G)$. Then*

$$\lim_{t \rightarrow 0^+} t^{(n-r)/2} \int_{G/G_{z_0}} f(z_t^x) dx = c_X \int_{G/G_u} f(u^x) dx \quad (24)$$

where $u = \exp X$, $z_t = \exp tZ_0$, $r = \dim \text{Cent}_g X$ and $\{X, H, Y\}$ is an arbitrary Lie triple.

PROOF. Let $\mathcal{O} = \{Z \in \mathfrak{g} : \text{ad } Z \text{ has eigenvalues } \lambda \text{ with } |\text{Im } \lambda| < \pi\}$. It is well known that the map $\exp: \mathcal{O} \rightarrow G$ is a diffeomorphism onto an open set \mathcal{P} . If $u = \exp X$ and $z_t = \exp tZ_0$ then u and z_t are contained in \mathcal{P} for t small enough. In addition, \mathcal{O} and \mathcal{P} are invariant by the adjoint action so the orbits of u and z_t are contained in \mathcal{P} . Without loss of generality we may assume that $\text{supp } f \subseteq \mathcal{P}$. Let $F = f \circ \exp$. If F does not have compact support we can always find F_1 with support in a set \mathcal{O}_f such that

- (1) $\mathcal{O}_f \subseteq \mathcal{O}$ and $\text{supp } f \subseteq \exp \mathcal{O}_f$,
- (2) $F_1 \in C_c^\infty(\mathfrak{g})$,
- (3) $F_1|_{\mathcal{O}_f} = f \circ \exp$.

By applying Propositions 6.3 or 6.6 to F_1 and $\{X, H, Y\}$ we get

$$\lim_{t \rightarrow 0^+} t^{(n-r)/2} \int_{G/G_{Z_0}} F_1(tZ_0^x) dx = c_X \int_{G/G_X} F_1(X^x) dx.$$

We now note that $G_X = G_u$ and $G_{Z_0} = G_{z_t}$ for t small enough. The first statement follows from Lemma 5.1. If t is small enough the eigenvalues of tZ_0 have absolute value less than π . Then $\text{Cent}_{G_t} tZ_0 = \text{Cent}_{G_t} z_t$ is a connected group. Thus we can write

$$\lim_{t \rightarrow 0^+} t^{(n-r)/2} \int_{G/G_{Z_0}} f(z_t^x) dx = c_X \int_{G/G_u} f(u^x) dx$$

and the proof is complete.

We rewrite this formula in terms of the distribution Φ_f^B defined in §1.

COROLLARY 6.8. *Under the assumptions of Theorem 6.7 let $p = r - s$ where $s = \dim \text{Cent}_{\mathfrak{g}} H$. Then*

$$\left(\frac{d}{dt} \right)^p \Big|_{t=0} \Phi_f^B(z_t; \bar{\omega}_{Z_0}) = \Phi_f^B(1; \bar{\omega}_{Z_0} Z_0^p) = c_u \int_{G/G_u} f(u^x) dx.$$

PROOF. Fix a connected component $B^+ \subseteq B'$ such that $z_t \in \text{cl}(B^+)$. By the work of Harish-Chandra it is known that $\Phi_f^B(z_t; \bar{\omega}_{Z_0})$ is well defined independent of B^+ . By Lemma 2.7 applied to $\gamma = z_t$ we get

$$\Phi_f^B(z_t; \bar{\omega}_{Z_0}) = k_t \int_{G/G_{Z_0}} f(z_t^x) dx$$

where

$$k_t = d_{Z_0} \xi_{-\rho}(z_t) \prod_{\beta \in P_{\mathfrak{g}/Z_0}} (\xi_{\beta}(z_t) - 1).$$

Then

$$\lim_{t \rightarrow 0^+} t^{(n-r)/2} k_t^{-1} \Phi_f^B(z_t; \bar{\omega}_{Z_0}) = c_X \int_{G/G_u} f(u^x) dx$$

by Theorem 6.7. Since $[P_{\mathfrak{g}/Z_0}] = (n - s)/2$ we can write

$$\lim_{t \rightarrow 0^+} t^{(n-s)/2} k_t^{-1} = \frac{1}{d_{Z_0}} \prod_{\beta \in P_{\mathfrak{g}/Z_0}} \beta(Z_0)^{-1}.$$

Thus, since Φ_f^B is a Schwartz function, we can write

$$\lim_{t \rightarrow 0^+} t^{-p} \Phi_f^B(z_t; \bar{\omega}_{Z_0}) = \text{const} \int_{G/G_u} f(u^x) dx.$$

The proof will be complete if we show that

$$\begin{aligned} \lim_{t \rightarrow 0^+} (p!)t^{-p} \Phi_f^B(z_i; \bar{\omega}_{z_0}) &= \lim_{t \rightarrow 0^+} \Phi_f^B(z_i; \bar{\omega}_{z_0} Z_0^p) \\ &= \Phi_f^B(1; \bar{\omega}_{z_0} Z_0^p) = \left(\frac{d}{dt} \right)^p \Big|_{t=0} \Phi_f^B(z_i; \bar{\omega}_{z_0}). \end{aligned}$$

This is a consequence of the elementary Lemma 6.9. Putting all the coefficients together we get the following expression for c_u .

$$c_u = (p!)d_{z_0}c_X \prod_{\beta \in P_{g/z_0}} (\beta, Z_0). \tag{25}$$

LEMMA 6.9. *Let $F \in C^\infty(0, \epsilon]$ be such that for any integer k , $D_t^k F$ has a limit when $t \rightarrow 0$ ($D_t = d/dt$). Then if $\lim_{t \rightarrow 0^+} t^{-p} F(t)$ exists, it equals $(p!)^{-1} D_t^p F(0)$.*

PROOF. Extend F to a C^{p+2} function on $(-\epsilon, \epsilon)$. The Taylor series expansion of F is

$$F(t) = F(0) + \frac{F'(0)}{1!} t + \dots + \frac{F^{(p)}(0)}{p!} t^p + O(t^{p+1}).$$

If $\lim_{t \rightarrow 0^+} t^{-p} F(t)$ exists, then it is clear that

$$F(0) = \dots = F^{(p-1)}(0) = 0 \text{ and } \lim_{t \rightarrow 0^+} t^{-p} F(t) = (p!)^{-1} D_t^p F(0).$$

7. The Fourier transform of the unipotent integral. In this section we calculate the Fourier transform of the distribution $T_u(f)$ for $u \in G$. This will be done by using the results in §6 and [13].

We use the notation in §2. In order to use the results in [12] we explain the relevant notation used there. Let B be the compact CSG. Then $A_K = Z(A)A_K^0$ where A_K^0 is the connected component of A_K and $Z(A) = \{1, v\}$ where $v = \exp(\sqrt{-1} \pi H_\alpha) = \exp(\pi(E_\alpha - E_{-\alpha}))$.

We let $\mathfrak{b}_2 = \mathbf{R}(E_\alpha - E_{-\alpha})$ and \mathfrak{b}_1 its orthogonal complement in \mathfrak{b} . Let B_1 and B_2 be the corresponding subgroups. Let now $Z \in \mathfrak{b}$ and $z = \exp Z$. Set P_Z as in §2, Lemma 2.7. Let $W_z(G, B)$ be the subgroup of $W(G, B)$ generated by the compact roots in P_Z . Choose $w_1 = 1, \dots, w_n$ such that

$$W(G, B) = \bigcup_{i=1}^N W_z(G, B)w_i \text{ (disjoint union).}$$

Then if $w \in W(G, B)$, $w = w_2 w_i$. Moreover

$$w_i^{-1} z = z_1(w_i) z_2(w_i), \quad z_1 \in B_1, z_2 \in B_2.$$

Since this decomposition is unique only up to $\{1, v\}$ we normalize z_2 such that

$$z_2(w_i) = \exp \theta_{w_i}(E_\alpha - E_{-\alpha}), \quad -\pi/2 \leq \theta_{w_i} < \pi/2.$$

We will take $Z = tZ_0$ for t small enough so that P_Z and P_z are identical.

We define the functions

$$\begin{aligned}
 F_X^\pm(w; \nu; z) &= \overline{w\chi(z)} \left\{ e^{\mp \nu\pi/2} (w\xi_z^\pm(z))^{\sqrt{-1}\nu/2} - e^{\pm \nu\pi/2} (w\xi_z^\pm(z))^{-\sqrt{-1}\nu/2} \right\} \\
 &= \left\{ e^{\mp \nu\pi/2} \exp \left[\left(\log \bar{\chi} + \frac{\sqrt{-1}\nu}{2} \tilde{\alpha}, w^{-1}Z \right) \right] \right. \\
 &\quad \left. - e^{\pm \nu\pi/2} \exp \left[\left(\log \bar{\chi} - \frac{\sqrt{-1}\nu}{2} \tilde{\alpha}, w^{-1}Z \right) \right] \right\}, \\
 G_X^\pm(w; \nu; z) &= \overline{\pm w\chi(z)} \left\{ e^{\mp \nu\pi/2} (w\xi_z^\pm(z))^{\sqrt{-1}\nu/2} \right. \\
 &\quad \left. + e^{\pm \nu\pi/2} (w\xi_z^\pm(z))^{-\sqrt{-1}\nu/2} \right\} \\
 &= \pm \left\{ e^{\mp \nu\pi/2} \left(\log \bar{\chi} + \frac{\sqrt{-1}\nu}{2} \tilde{\alpha}, w^{-1}Z \right) \right. \\
 &\quad \left. + e^{\pm \nu\pi/2} \left(\log \bar{\chi} - \frac{\sqrt{-1}\nu}{2} \tilde{\alpha}, w^{-1}Z \right) \right\}.
 \end{aligned}$$

Let $r = [P]$, $r_z = [P_z]$ and $r_I = [P_I]$. We also define $\hat{A}_K^\pm = \{\chi: \chi(v) = \pm 1\}$ and $\varepsilon(\chi) = \text{sgn} \prod_{\alpha \in P_I} (\log \chi, \alpha)$.

We are now ready to state the main result of this section.

THEOREM 7.1. *Let $u = \exp X$ be unipotent. Then the distribution $c_u T_u(f)$ equals*

$$\begin{aligned}
 &(-1)^{r+r_z} \sum_{\mu \in L_B} \left[\prod_{\beta \in P_{z_0}} (\mu, \beta) \right] [\overline{(\mu, Z_0)}]^p \Theta_\mu(f) \\
 &+ (\sqrt{-1}/4) (-1)^{r_I} [W_{Z_0}(G, B)] / [W(G, A)] \sum_{\chi \in \hat{A}_K^+} \varepsilon(\chi) \\
 &\times \left\{ \sum_{\substack{w_i \\ 0 < \theta_{w_i} < \pi/2}} \det(w_i) \int_{-\infty}^{+\infty} T^{(x, \nu)}(f) [F_X^+(w_i; \nu; 1; \bar{\omega}_{Z_0} Z_0^p) / \sinh(\nu\pi/2)] dv \right. \\
 &\quad \left. + \sum_{\substack{w_i \\ -\pi/2 < \theta_{w_i} < 0}} \det(w_i) \int_{-\infty}^{+\infty} T^{(x, \nu)}(f) [F_X^-(w_i; \nu; 1; \bar{\omega}_{Z_0} Z_0^p) / \sinh(\nu\pi/2)] dv \right\} \\
 &+ (\sqrt{-1}/4) (-1)^{r_I} [W_{Z_0}(G; B)] / [W(G, A)] \sum_{\chi \in \hat{A}_K^-} \varepsilon(\chi) \\
 &\times \left\{ \sum_{\substack{w_i \\ 0 < \theta_{w_i} < \pi/2}} \det(w_i) \int_{-\infty}^{+\infty} T^{(x, \nu)}(f) [G_X^+(w_i; \nu; 1; \bar{\omega}_{Z_0} Z_0^p) / \cosh(\nu\pi/2)] dv \right. \\
 &\quad \left. + \sum_{\substack{w_i \\ -\pi/2 < \theta_{w_i} < 0}} \det(w_i) \int_{-\infty}^{+\infty} T^{(x, \nu)}(f) [G_X^-(w_i; \nu; 1; \bar{\omega}_{Z_0} Z_0^p) / \cosh(\nu\pi/2)] dv \right\}.
 \end{aligned}$$

PROOF. This formula follows immediately by applying Theorem 6.7 to Theorems 3.19 and 5.21 in [13], substituting $y = z_t = \exp tZ_0$ and differentiating. Lemma 68 on p. 315 in [5c] for the discrete series part and the form of $T^{(\chi, \nu)}$ given in [13] can be used to justify term by term differentiation.

The formulas for $F_{\bar{\chi}}^{\pm}$ and $G_{\bar{\chi}}^{\pm}$ occurring in the statement are given below

$$F_{\bar{\chi}}^{\pm}(w: \nu: 1; \bar{\omega}_{Z_0} Z_0^p) = \left\{ c_1 e^{\mp \nu \pi / 2} \left[\left(\overline{\log \bar{\chi}} + \frac{\sqrt{-1} \nu}{2} \bar{\alpha}, w^{-1} Z_0 \right) \right]^p \right. \\ \left. - c_2 e^{\pm \nu \pi / 2} \left[\left(\log \bar{\chi} - \frac{\sqrt{-1} \nu}{2} \bar{\alpha}, w^{-1} Z_0 \right) \right]^p \right\}$$

$$G_{\bar{\chi}}^{\pm}(w: \nu: 1; \bar{\omega}_{Z_0} Z_0^p) = \left\{ c_1 e^{\mp \nu \pi / 2} \left[\left(\log \bar{\chi} + \frac{\sqrt{-1} \nu}{2} \bar{\alpha}, w^{-1} Z_0 \right) \right]^p \right. \\ \left. + c_2 e^{\pm \nu \pi / 2} \left[\left(\log \bar{\chi} - \frac{\sqrt{-1} \nu}{2} \bar{\alpha}, w^{-1} Z_0 \right) \right]^p \right\},$$

where

$$c_1 = \prod_{\beta \in P_{Z_0}} \left(w \left(\log \bar{\chi} + \frac{\sqrt{-1} \nu}{2} \bar{\alpha} \right), \beta \right),$$

$$c_2 = \prod_{\beta \in P_{Z_0}} \left(w \left(\log \bar{\chi} - \frac{\sqrt{-1} \nu}{2} \bar{\alpha} \right), \beta \right).$$

REMARK. These formulas become simpler when Z_0 is regular, i.e. $\mathfrak{g} = \mathfrak{sl}(2, \mathbf{R})$ or $\mathfrak{g} = \mathfrak{su}(2, 1)$ as Theorem 3.19 in [13] suggests. Similar formulas for $u \in G$ arbitrary were calculated in [2] using Jordan decomposition.

8. The constant c_u . In this section we evaluate the constant c_u relevant to the calculations in [10]. In view of this we note that if the discrete series part of the distribution $T_u(f)$ is $\sum a(\mu) \theta_{\mu}(f)$ then up to a constant factor,

$$a(\mu) = (\overline{\mu, Z_0})^p \prod_{\beta \in P_{Z_0}} (\mu, \beta) / \prod_{\beta \in P_{\mathfrak{g}/Z_0}} (\beta, Z_0).$$

LEMMA 8.1. *Suppose that there is an element $x \in G$ such that $X^x = -X$. Then the discrete series part in Theorem 7.1 is equal to zero.*

PROOF. If $X^x = -X$ then there is $w \in W(G, B)$ such that $Z_0^w = -Z_0$ by the results in §3. Furthermore we must have $\dim \mathfrak{g}_2 > 1$. By the formulas for θ_{μ} in [12] we know that $\theta_{w\mu} = \varepsilon(w) \theta_{\mu}$. Both θ_{μ} and $\theta_{w\mu}$ occur in the sum. We

compute $a(w\mu)$. We have

$$\begin{aligned} a(w\mu) &= (\overline{w\mu}, Z_0)^p \prod_{\beta \in P_{Z_0}} (w\mu, \beta) / \prod_{\beta \in P_{g/Z_0}} (\beta, Z_0) \\ &= (-1)^p \varepsilon_{Z_0}(w) (\overline{\mu}, Z_0)^p \prod_{\beta \in P_{Z_0}} (\mu, \beta) / \prod_{\beta \in P_{g/Z_0}} (\beta, Z_0) \\ &= (-1)^p \varepsilon_{Z_0}(w) a(\mu) \end{aligned}$$

where $\varepsilon_{Z_0}(w)$ is the number of roots in P_{Z_0} mapped into $-P_{Z_0}$ by w . We recall that $p = (r - s)/2$ also equals $\frac{1}{2} \dim g_1$. Assume $p > 0$. We will show that p is even or equivalently that $\dim g_1 \equiv 0 \pmod{4}$. We recall that the complexification of g_1 is generated by root vectors of roots α_1 such that $\alpha_1(H) = 1$. Then $\sigma\alpha_1 \neq \alpha_1$ and $\sigma\alpha_1(H) = 1$. Since $\dim g_2 > 1$ let $\beta \in P$ be a root such that $\beta \neq \alpha$ and $\beta(H) = 2$. By Lemma 2.4 either $\beta - \alpha_1$ or $\beta - \sigma\alpha_1$ is a root, say $\beta - \alpha_1 = \beta_1$ is a root. Clearly $\beta - \sigma\alpha_1$ cannot also be a root since otherwise either $(\beta - \alpha_1) - (\beta - \sigma\alpha_1) = \sigma\alpha_1 - \alpha_1$ or $(\beta - \alpha_1) - \sigma(\beta - \sigma\alpha_1) = \beta - \sigma\beta$ would be a root. Thus P_1 splits into disjoint sets of the form $\{\alpha_1, \sigma\alpha_1, \beta_1, \sigma\beta_1\}$. Thus $[P_1] \equiv 0 \pmod{4}$. Therefore $a(w\mu) = \varepsilon_{Z_0}(w)a(\mu)$. Finally we show that $\varepsilon_{Z_0}(w)\varepsilon(w) = -1$. We may assume that x is such that $X^x = -X$, $Y^x = -Y$ and $H^x = H$. Let $\mathfrak{m} = \text{Cent}_K H$ and $\mathfrak{m}_{Z_0} = \text{Cent}_{\mathfrak{m}} Z_0$. Then $\mathfrak{m}_{Z_0}^x = \mathfrak{m}_{Z_0}$ is a compact subalgebra containing \mathfrak{a}_K as a Cartan subalgebra. Thus we may replace x with $w \in W(G, B)$ with the same properties. In addition, we may consider w as an element of $W(G, J)$ as well. But w maps Z_0 into $-Z_0$ so it maps P_{g/Z_0} into $-P_{g/Z_0}$. Since $[P_{g/Z_0}] = \dim g_1 + \dim g_2$ is odd and the complement of P_{g/Z_0} in P is P_{Z_0} we have $\varepsilon(w) = (-1)\varepsilon_{Z_0}(w)$. This completes the proof of the lemma.

Thus the only cases when the unipotent orbit has a nontrivial discrete series contribution is when $\dim g_2 = 1$. Then $\dim g_4 = 0$ and the only cases are the groups $\text{su}(n, 1)$ for $n \geq 1$. The case $n = 1$ is well known. Thus we assume $\dim g_1 > 0$. This is Case II in §6. Let $w = \exp t_1 W$ and X, H, Y, W be as in that case. From the proof of Proposition 6.6 we have to calculate $c(t_1)$ defined by (20). Let $Z_1 \in g^- \oplus \mathfrak{L}/\mathfrak{L}_0$ where $g^- = \sum_{i < 0} g^i$ and $Z_2 \in g_Z$. Then we have to evaluate the determinant of $\phi: g^- \oplus \mathfrak{L}/\mathfrak{L}_0 \times \mathbf{R}^2 \times g_Z \rightarrow g$ given by $\phi(Z_1, s_2, s_1, Z_2) = Z_1^{a^{-1}} + s_2 H + s_1 W + Z_2^w$. By choosing a basis as in Case II we get $e^{2\rho(\log a)} \det(\tau X_i, Y_j, H, W, \text{Ad } w Y_j, \text{Ad } w Z_k)$. We note that $g_Z = \mathfrak{L} + \mathbf{R}Z_0$ in this case. Also W and H are conjugate in $\text{su}(2, 1)$ by some element $x = s_{\alpha_1}$ where s_{α_1} is the reflection in $W(G, B)$ corresponding to a simple root $\alpha_1 \in \Delta(g_c, j_c)$. Suppose $E_{\beta_1 - \alpha_1} \in \mathfrak{L}_c$ is a root vector. Then

$$wE_{\beta_1 - \alpha_1} = wE_{\tilde{\beta}_1 - \tilde{\alpha}_1} = s_{\tilde{\alpha}_1} \exp t_1 H s_{\tilde{\alpha}_1} E_{\tilde{\beta}_1 - \tilde{\alpha}_1} = s_{\alpha_1} \exp t_1 H E_{\tilde{\beta}_1}.$$

Thus it is enough to calculate $\exp t_1 H E_{\tilde{\alpha}_1}$ since $\tilde{\alpha}_1$ and $\tilde{\beta}_1$ generate a subalgebra isomorphic to $\text{su}(2, 1)$. We get $\cosh t_1 E_{\tilde{\beta}_1} - \sinh t_1 E_{\sigma\tilde{\beta}_1}$ and $s_{\tilde{\alpha}_1}$ maps

them into

$$\cosh t_1 E_{\beta_1 - \tilde{\alpha}_1} - \sinh t_1 E_{\tilde{\beta}_1} = \cosh t_1 E_{\beta_1 - \alpha_1} - \sinh t_1 (E_{\beta_1} \pm E_{-\sigma\beta_1}).$$

Furthermore, by relations (10) and (11),

$$Z_0^w = \frac{1 + \cosh^2 t_1}{2} Z_0 - \frac{1}{2} \sinh 2t_1 (X_1 + \tau X_1) + \frac{1}{2} \sinh^2 t_1 X_0.$$

We write

$$Z_0^w = \frac{1 + \cosh^2 t_1}{2} Z_0 + Z_{t_1}.$$

Substituting into the expression for φ we get

$$e^{2\rho(\log a)} \det(\tau X_i, Y_j, H, W, \text{Ad } w a_k, \cosh t_1 E_{\beta_1 - \alpha_1} - \sinh t_1 (E_{\beta_1} \pm E_{-\sigma\beta_1}), Z_0^w).$$

By elementary column operations this reduces to

$$2^{p-1} e^{2\rho(\log a)} (\sinh t_1)^{2p-2} \det_{\text{su}(2,1)}$$

where $\det_{\text{su}(2,1)}$ is the corresponding determinant for $g = \text{su}(2, 1)$. But this can be calculated explicitly and equals $\sinh 2t_1$. Thus $c(t_1) = (\sinh t_1)^{2p-2} \sinh 2t_1$ and the integral to be evaluated is

$$2^p \int_0^\infty \frac{(\sinh t_1)^{2p-2} \sinh 2t_1}{((1 + \cosh^2 t_1)/2)^{\rho(H)}} dt_1$$

which is finite since $2p + 2 = 2\rho(H)$. We get by setting $u = \sinh t_1$

$$\begin{aligned} \int_0^\infty \frac{(\sinh t_1)^{2p-1} \cosh t_1}{\left(1 + \frac{\sinh^2 t_1}{2}\right)^{p+1}} dt_1 &= \int_0^\infty \frac{u^{2p-1}}{\left(1 + \frac{u^2}{2}\right)^{p+1}} du \\ &= 2^{p-1} \int_1^\infty \frac{(v-1)^{p-1}}{v^{p+1}} dv = 2^{p-1} (-1)^{p-1} \int_0^1 (1-v)^{p-1} v^4 dv \\ &= 2^{p-1} (-1)^{p-1} B(p-1, 4) \end{aligned}$$

where $v = 1 + u^2/2$ in the third equality. Putting these calculations together we get the following proposition.

PROPOSITION 8.2. *In the case when the discrete series contribution of T_u is not zero, the constant c_u can be expressed as*

$$c_u = 2^{2p-1} (-1)^{p-1} B(p-1, 4) \cdot c_{Z_0}^{-1} \cdot d_{Z_0} \cdot (p!) \prod_{\beta \in P_{g/Z_0}} (\beta, Z_0)$$

where c_{z_0} is defined in Proposition 4.8 and d_{z_0} in Lemma 2.7.

PROOF. The statement follows easily from the previous calculation and Corollary 6.8.

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