TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 249, Number 1, April 1979

CELL-LIKE, 0-DIMENSIONAL DECOMPOSITIONS OF E^3

BY

MICHAEL STARBIRD[1]

ABSTRACT. Let G be a cell-like, 0-dimensional upper semicontinuous decomposition of E^3. It is shown that if Γ is a tame 1-complex which is a relatively closed subset of a saturated open set U whose boundary misses the nondegenerate elements of G, then there is a homeomorphism $h: E^3 \to E^3$ so that $h|E^3 - U = \text{id}$ and $h(\Gamma)$ misses the nondegenerate elements of G. This theorem implies a disjoint disk type criterion for shrinkability of G. This criterion in turn provides a direct proof of the recent result of Starbird and Woodruff that if G is an u.s.c. decomposition of E^3 into points and countably many cellular, tamely embedded polyhedra, then E^3/G is homeomorphic to E^3.

1. Introduction. Let G be an upper semicontinuous decomposition of E^3 into points and cell-like continua (see definitions below) so that the image A of the nondegenerate elements of G is a 0-dimensional subset of E^3/G. It is not assumed that the closure of A is 0-dimensional. In Theorem 3.1 a necessary and sufficient condition is presented for establishing that E^3/G is homeomorphic to E^3. In Theorem 4.1 this criterion is used to give an alternative proof of a recent result of Starbird and Woodruff [**6**, Theorem 1], namely, if G is an upper semicontinuous decomposition of E^3 into points and countably many cellular, tamely embedded polyhedra, then E^3/G is homeomorphic to E^3. This result extends Bing's theorem [**2**, Theorem 3] that E^3/G is homeomorphic to E^3 if G is an upper semicontinuous decomposition of E^3 into points and countably many tame arcs.

The shrinkability criterion for upper semicontinuous, 0-dimensional, cell-like decompositions G of E^3 found in Theorem 3.1 is proved by making use of a graph pushing property of G proved in §2. Specifically, Theorem 2.1 states that if G is any upper semicontinuous 0-dimensional, cell-like decomposition of E^3, U is a saturated open set with no nondegenerate element of G intersecting Bd U, and A is a tame, possibly infinite, graph in U which is a relatively closed subset of U, then there is a homeomorphism $H: E^3 \to E^3$ so that $H|E^3 - U = \text{id}$ and $H(A)$ misses all the nondegenerate elements of G. This theorem allows one to push the 1-skeleton of a triangulation of such a

Received by the editors September 12, 1977.

AMS (MOS) subject classifications (1970). Primary 57A10.

Key words and phrases. Cell-like, decompositions.

[1]Work on this paper was supported by NSF Grant MCS 76-07242-A01.

saturated open set U off the nondegenerate elements of G whether or not E^3/G is homeomorphic to E^3. A corollary (Corollary 2.5) of this theorem is that for every cell-like, 0-dimensional upper semicontinuous decomposition G of E^3, there is a cell-like, 0-dimensional upper semicontinuous decomposition G' of E^3 so that E^3/G is homeomorphic to E^3/G' and the union of the nondegenerate elements of G' is contained in a 1-dimensional subset of E^3.

The additional condition on G necessary to conclude that E^3/G is homeomorphic to E^3 is a condition which allows one, after pushing the nondegenerate elements of G off the 1-skeleton of a triangulation of U, to push U so that no nondegenerate element of G intersects two nonadjacent 2-simplexes of the triangulation of U. This procedure allows one to conclude that for any such saturated open set U, there is a homeomorphism of E^3 fixed outside U which shrinks each element of G in U small. This fact is sufficient to show that E^3/G is homeomorphic to E^3.

DEFINITIONS. 1. A decomposition G of E^3 is *upper semicontinuous* if and only if for each closed subset X in E^3, $\bigcup\{g \mid g \in G \text{ and } g \cap X \neq \varnothing\}$ is a closed set.

2. A continuum X in E^3 is *cell-like* if and only if for each open set U containing X, X is homotopic to a point in U.

2. Pushing graphs off the nondegenerate elements. Let G be a closed 0-dimensional, upper semicontinuous, cell-like decomposition of E^3. Then G is describable by cubes with handles [**3**, Theorem 1], [**4**, Theorem 1]. (*Note*. A decomposition G of E^3 is *closed* 0-*dimensional* if and only if the closure in E^3/G of the image of the nondegenerate elements of G is a 0-dimensional set.) Therefore, one can push a graph off the nondegenerate elements of such a closed 0-dimensional decomposition G by moving the graph off the cores of the cubes with handles in a stage of the defining sequence for G and then pushing away from the cores. This push moves the graph off all the cubes with handles at that stage and hence off all the nondegenerate elements.

In this section it is shown that graphs can be pushed off the nondegenerate elements of any 0-dimensional, cell-like, upper semicontinuous decomposition G of E^3 even when G is not closed 0-dimensional.

THEOREM 2.1. *Let G be a* 0-*dimensional upper semicontinuous decomposition of E^3 into points and cell-like continua. Let U be a saturated open set in E^3 so that no nondegenerate element of G intersects* Bd U. *Let A be a tame* (*possibly infinite*) *graph which is a relatively closed subset of U.*

Then there is a homeomorphism $H\colon E^3 \to E^3$ such that $H|E^3 - U = \mathrm{id}$ and $H(A)$ misses all the nondegenerate elements of G.

PROOF. The homeomorphism H is defined as the limit of a sequence of homeomorphisms. No point will be moved infinitely many times. The first

homeomorphism g is different from the remaining ones while the rest of the homeomorphisms are defined inductively and are similar to each other. The first homeomorphism, defined in Lemma 2.3 below, moves A to a position where each point of the moved A has a nice sequence of neighborhoods which are then used in defining the subsequent homeomorphisms.

It is sufficient to prove Theorem 2.1 in the case where $\overline{U}$ is compact, therefore we assume henceforth that $\overline{U}$ is compact.

Notation. For any set X in E^3, $S(X)$ is the saturation of X, i.e., the union of all elements of G which intersect X.

LEMMA 2.2. *Let G, A and U be defined as in Theorem* 2.1. *Let $\{B^\alpha\}_{\alpha\in\lambda}$ be an open cover of $S(A)$ so that $\bigcup_{\alpha\in\lambda}B^\alpha \subset U$.*

Then there is a homeomorphism f: $E^3 \to E^3$ so that $f|E^3 - \bigcup_{\alpha\in\lambda}B^\alpha = \mathrm{id}$ and for each point $x \in A$, $S(f(x))$ lies entirely in one B^α.

PROOF. Let $\{U_i\}_{i\in\omega}$ be a null sequence of disjoint saturated open sets so that the nondegenerate elements of G which intersect A are contained in $\bigcup_{i\in\omega}U_i$ and for each i, Bd U_i does not intersect any nondegenerate element of G and $\overline{U}_i \subset \bigcup_{\alpha\in\lambda}B^\alpha$. For each i in ω, let ε_i be a Lebesgue number associated with the cover of $\overline{U}_i$ by the B^α's. By the theorem stated in the first sentences of this section, there is a finite collection of disjoint cubes with handles in U_i whose union contains all the nondegenerate elements of G which are in U_i and have diameters bigger than ε_i. One can push $A \cap U_i$ off these cubes with handles by a homeomorphism fixed outside U_i. A homeomorphism f required in Lemma 2.2 is obtained by performing the above process on each U_i.

Definition of $\{B_1^i\}_{i\in\omega}$. Let T be a triangulation of U so that the mesh of T goes to 0 as one moves toward Bd U. Let $\{\tilde{B}_1^i\}_{i\in\omega}$ be the collection of all 3-cells obtained by taking the simplicial star about the barycenter of each simplex of T with respect to the second barycentric subdivision of T. Let $\{B_1^i\}_{i\in\omega}$ be a null sequence of 3-cells such that each B_1^i is a regular neighborhood of $\tilde{B}_1^i$ and if $B_1^i \cap B_1^j \neq \varnothing$, then $\tilde{B}_1^i \cap \tilde{B}_1^j \neq \varnothing$. Note that $\{\mathrm{Int}\, B_1^i\}_{i\in\omega}$ is an open cover of U.

LEMMA 2.3. *Let G, A, and U be defined as in Theorem* 2.1 *and $\{B_1^i\}_{i\in\omega}$ be as defined above. Then there is a homeomorphism g: $E^3 \to E^3$ with $g|E^3 - U = \mathrm{id}$ so that for each point x in A there are a saturated open set V, tame 3-cells B_2 and B_3, and an i in ω so that $g(x) \in V \subset B_3 \subset S(B_3) \subset \mathrm{Int}\, B_2 \subset S(B_2) \subset \mathrm{Int}\, B_1^i$.*

PROOF. Let f_1: $E^3 \to E^3$ be a homeomorphism obtained from Lemma 2.2 with respect to the cover $\{\mathrm{Int}\, B_1^i\}_{i\in\omega}$ of A. For each x in $S(f_1(A))$, there is a tame 3-cell $B_2(x)$ so that $x \in \mathrm{Int}\, B_2(x) \subset S(B_2(x)) \subset \mathrm{Int}\, B_1^i$ for some i. Now apply Lemma 2.2 to the open cover $\{\mathrm{Int}\, B_2(x) | x \in S(f_1(A))\}$ of the graph

$f_1(A)$ to obtain a homeomorphism f_2: $E^3 \to E^3$ so that $f_2|E^3 - \bigcup \{\text{Int } B_2(x) | x \in S(f_1(A))\} = \text{id}$ and for each $y \in S(f_2 \circ f_1(A))$, $S(y) \subset \text{Int } B_2(x)$ for some $x \in S(f_1(A))$.

One more step will do. That is, for each $y \in S(f_2 \circ f_1(A))$ find a tame 3-cell $B_3(y)$ so that $S(B_3(y)) \subset \text{Int } B_2(x)$ for some $x \in S(f_1(A))$. One last application of Lemma 2.2 with respect to the cover $\{\text{Int } B_3(y) | y \in S(f_2 \circ f_1(A))\}$ of $S(f_2 \circ f_1(A))$ yields a homeomorphism f_3. The homeomorphism $g = f_3 \circ f_2 \circ f_1$ is a homeomorphism satisfying the conclusion of Lemma 2.3 since for each $z \in f_3 \circ f_2 \circ f_1(A)$, there is a saturated open set $V(z)$ so that $V(z) \subset B_3(y) \subset B_2(x) \subset B_1^i$ for some $y \in S(f_2 \circ f_1(A))$, $x \in f_1(A)$, and $i \in \omega$.

The open sets $\{V(z) | z \in g(A)\}$ form an open cover of $g(A)$. Use the paracompactness of E^3/G to obtain an open cover $\{V_i\}_{i \in \omega}$ of $g(A)$ which is locally finite in U such that for each $i \in \omega$, $V_i \subset V(z)$ for some $z \in g(A)$. For each i, let $j(i)$ be an integer for which $V_i \subset B_3 \subset B_2 \subset B_1^{j(i)}$ as guaranteed in Lemma 2.3.

Recall that each B_1^i is associated with a simplex σ of the triangulation T in the sense that it is the regular neighborhood of a simplicial neighborhood of the barycenter of σ. Order the B_1^i's so that if B_1^i is associated with the simplex σ, then for each simplex of Bd σ, the B_1^j associated with it has $j \leqslant i$. Assume $\{B_1^i\}_{i \in \omega}$ is ordered in this fashion.

For each $k \in \omega$, only finitely many V_i's have the property that $j(i) = k$. Order the V_i's so that $i < s$ implies that $j(i) \leqslant j(s)$. Assume $\{V_i\}_{i \in \omega}$ is so ordered.

We are now ready to define homeomorphisms $\{h_i\}_{i \in \omega}$ inductively where h_i is associated with V_i. Each h_i will be the identity outside $B_1^{j(i)}$, hence $h = \lim_{i \to \infty} h_i$ will be a homeomorphism because the collection $\{B_1^i\}_{i \in \omega}$ is a null sequence and, owing to the ordering, each point in E^3 is moved only finitely often.

Suppose homeomorphisms $h_1, h_2, \ldots, h_{i-1}$ have been defined according to the following inductive lemma. Let $A' = h_{i-1} \circ \cdots \circ h_1 \circ g(A)$ and define h_i as follows.

LEMMA 2.4. *Given the situation described above there is a homeomorphism h_i: $E^3 \to E^3$ such that*

(i) $h_i | E^3 - B_1^{j(i)} = \text{id}$, *and*

(ii) *for each $x \in A'$, $h_i(x)$ is either a degenerate element of G or $h_i(x) \in \bigcup_{j > i} V_j$.*

Note that after Lemma 2.4 is proved, the homeomorphism $H = h \circ g$ will satisfy the conclusion of Theorem 2.1, that is, $H(A)$ misses the nondegenerate elements of G.

PROOF OF LEMMA 2.4. The strategy of the proof is to find a tame 2-sphere Σ in $B_1^{j(i)}$ which contains V_i in its interior and intersects the tame graph A' in a special way, namely, $A' \cap \Sigma$ is a finite number of points which lie on an arc C on Σ so that $C \subset (\bigcup_{j>i} V_j \cup$ (degenerate elements of G)). Then the homeomorphism h_i is defined with the aid of an R. L. Moore disk decomposition theorem so that h_i is the identity outside Σ and for each point x of A' in the interior of Σ, $h_i(x)$ lies near C and in $\bigcup_{j>i} V_j \cup$ (degenerate elements of G).

Let $V_i \subset B_3 \subset B_2 \subset B_1^{j(i)}$ be the sequence guaranteed by Lemma 2.3. Make a slight adjustment of Bd B_2 so that the B_t's still satisfy the conclusion of Lemma 2.3 and so that Bd $B_2 \cap A'$ is a finite number of points $\{p_v\}$.

For each pair of points p_v, p_w in Bd $B_2 \cap A'$ for which it is possible to do so, find an arc $C_{v,w}$ on Bd B_2 joining p_v and p_w so that $C_{v,w} \subset (\bigcup_{j>i} V_j \cup$ (degenerate elements of G)). Let $\{C_t\}_{t=1}^n$ be the components of $\bigcup_{v,w<m} C_{v,w}$. If n is larger than one, Moore's theorems imply that there are finitely many nondegenerate elements $\{g_r\}_{r=1}^k$ of G in $E^3 - \bigcup_{j>i} V_j$ so that for each r, g_r separates some pair of C_t's on Bd B_2, and each pair of C_t's is separated by some g_r. Note that each g_r is contained in Int $B_1^{j(i)} - B_3$ because $g_r \cap$ Bd $B_2 \neq \varnothing$ and $S(\text{Bd } B_2) \subset \text{Int } B_1^{j(i)} - B_3$. Also note that $g_r \cap A' = \varnothing$ since the inductive definition of the h_i's up to this point implies that $A' \subset (\bigcup_{j>i} V_j \cup$ (degenerate elements of G)) and $g_r \subset (E^3 - \bigcup_{j>i} V_j)$.

Let K be a PL arc from Bd B_3 to Bd $B_1^{j(i)}$ in general position with respect to Bd B_2 so that $K \cap (\bigcup_{r=1}^k g_r) = \varnothing$.

Let $\{N_1(g_r)\}_{r=1}^k$ be a collection of disjoint open sets such that, for each r, $g_r \subset N_1(g_r)$ and $N_1(g_r) \cap (B_3 \cup \text{Bd } B_1^{j(i)} \cup K \cup (\bigcup_{t=1}^n C_t)) = \varnothing$.

For each r, let $N_2(g_r)$ be a tame, compact, manifold neighborhood of g_r so that $N_2(g_r)$ is homotopic to a point inside $N_1(g_r)$ and $(\bigcup_{r=1}^k N_2(g_r)) \cap$ Bd B_2 is a finite union of simple closed curves $\{J_s\}_{s=1}^m$.

For each s, J_s bounds a disk D_s on Bd B_2 so that D_s has algebraic intersection number 0 with K. Also J_s bounds a singular disk E_s in some $N_1(g_r)$. So $E_s \cap K = \varnothing$. Since the singular 2-sphere $D_s \cup E_s$ has algebraic intersection number 0 with K, it does not link the 0-sphere Bd K homologically. Since $H_2(B_1^{j(i)} - B_3)$ is isomorphic to $\pi_2(B_1^{j(i)} - B_3)$, we see that E_s is homotopic to D_s in $B_1^{j(i)} - B_3$ by a homotopy which keeps Bd $E_s = J_s =$ Bd D_s fixed.

Let $D_{s(1)} \subset D_{s(2)} \subset \cdots \subset D_{s(u)}$ be a maximal sequence of nested D_s's. Let F be the closure of the component of Bd $B_2 - (D_{s(u)} \cup (\bigcup_{s=1}^m J_s))$ so that Bd $D_{s(u)}$ is contained in Bd F.

Note that F is a disk with holes so that Cl(Bd $B_2 - F$) is a union of disjoint D_s's. Note also that F contains at most one component C_t. Call it C_t'.

Let Σ' be the singular 2-sphere obtained by replacing each of the D_s's

which make up Cl(Bd $B_2 - F$) by E_s's. Note also that Σ' is homotopic to Bd B_2 in $B_1^{j(i)} - B_3$ and that $(\Sigma' \cap A') \subset (F \cap A') \subset C_t'$, since for each s, $E_s \cap A' = \varnothing$.

Use Dehn's Lemma or the Sphere Theorem to obtain a 2-sphere Σ such that $C_t' \subset \Sigma$, $(\Sigma \cap A') \subset C_t'$, and Σ separates B_3 from Bd $B_1^{j(i)}$.

Next find an arc C on Σ so that C contains $\Sigma \cap A'$ and $C \subset (\bigcup_{j>i} V_j \cup$ (degenerate elements of G)). Let $\tilde{\Sigma}$ be the tame 3-cell bounded by Σ.

Let M be a tame disk properly embedded in $\tilde{\Sigma}$ such that $C \subset$ Bd M. Let π: $\tilde{\Sigma} \to M$ be a projection map such that $\pi|\tilde{\Sigma} \cap A'$ is a regular projection. Notice that up to this point, A' has not been moved; however, it is about to be moved.

Let h_i': $E^3 \to E^3$ be a homeomorphism such that $h_i'|E^3 - \tilde{\Sigma} = \text{id}$ and $h_i'(\tilde{\Sigma} \cap A') \subset (M \cup (\bigcup_{t=1}^k F_t))$ where $\{F_t\}_{t=1}^k$ is a collection of disjoint tame disks in Int $\tilde{\Sigma}$, $F_t \cap M =$ Bd $F_t \cap M =$ an arc $\hat{F}_t$, (Bd $F_t - \hat{F}_t) \cap h_i'(A') = \varnothing$, and $(M \cap h_i'(A')) \cup \bigcup_{t=1}^k \hat{F}_t$ is a finite graph Γ on M.

By Moore's theorems [5], there is a homeomorphism of M to itself fixed on Bd M which takes Γ into the set $\bigcup_{j>i} V_j \cup$ (degenerate elements of G). Let h_i'': $E^3 \to E^3$ be a homeomorphism which extends the above homeomorphism of M and has the property that $h_i''|E^3 - \tilde{\Sigma} = \text{id}$.

For each t, there is a homeomorphism of $h_i''(F_t)$ to itself fixed on the boundary such that $h_i''(F_t \cap h_i'(A'))$ is moved into the set $\bigcup_{j>i} V_j \cup$ (degenerate elements of G). The homeomorphisms thus defined on $\bigcup_{t=1}^k F_t$ can be extended to a homeomorphism h_i''': $E^3 \to E^3$ so that $h_i'''|(E^3 - \tilde{\Sigma}) \cup M = \text{id}$.

The homeomorphism $h_i = h_i''' \circ h_i'' \circ h_i'$ is a homeomorphism required by Lemma 2.4.

As was mentioned right after the statement of Lemma 2.4, the homeomorphism $h = \lim_{i\to\infty} h_i$ has the property that $H = h \circ g$ is a homeomorphism of E^3 fixed outside U such that $H(A)$ misses the nondegenerate elements of G. Thus Theorem 2.1 is proved.

COROLLARY 2.5. *Let G be a cell-like, 0-dimensional upper semicontinuous decomposition of E^3. Then there is another cell-like, 0-dimensional, upper semicontinuous decomposition G' of E^3 so that E^3/G is homeomorphic to E^3/G' and the union of the nondegenerate elements of G' is contained in a 1-dimensional subset of E^3.*

PROOF. Let $T_0 = \varnothing$ and for each $i = 1, 2, \ldots,$ let T_i be the 1-skeleton of a triangulation of E^3 of mesh less than $1/i$ so that $T_i \subset T_{i+1}$. Our plan is to push the nondegenerate elements of G off $\bigcup T_i$. To accomplish this we construct inductively a sequence $\{W_i\}_{i\in\omega}$ of open covers of the nondegenerate elements of G and a sequence $\{h_i\}_{i\in\omega}$ of homeomorphisms of E^3. Let $W_0 = \{E^3\}$ and $h_0 = \text{id}$. Suppose W_i and h_i have been defined, then define W_{i+1} and h_{i+1} to satisfy the following conditions.

(i) W_{i+1} is a disjoint open cover of the nondegenerate elements of G;

(ii) for each V in W_{i+1}, there is a V' in W_i so that $\bar{V} \subset V'$ and $\bar{V}$ is homotopic to a point in V';

(iii) for each V in W_{i+1}, there is an element g of G so that $V \subset N(g, 1/(i+1)) =$ the $1/(i+1)$ neighborhood of g;

(iv) $T_i \cap h_i \circ \cdots \circ h_1(\bigcup W_{i+1}) = \varnothing$;

(v) h_{i+1} is the identity outside $h_i \circ \cdots \circ h_1(\bigcup W_{i+1})$;

(vi) for each g in G, $h_{i+1}(g) \cap T_{i+1} = \varnothing$.

It is an easy matter to define W_i's and h_i's with these properties with Theorem 2.1 being used to obtain condition (vi).

For each element g of G for which $g \subset \bigcap_{i\in\omega}(\bigcup W_i)$, define g' to be $\bigcap_{i\in\omega}\{h_i \circ \cdots \circ h_1(\bar{V}_g) | g \subset V_g \in W_i\}$. For each other element g of G, g is a single point which lies in $E^3 - \bigcup W_i$ for some i. So define g' to be $h_i \circ \cdots \circ h_1(g)$. Let $G' = \{g' | g \in G\}$. It is a straightforward matter to check that G' is a cell-like, 0-dimensional upper semicontinuous decomposition of E^3 whose nondegenerate elements lie in the 1-dimensional set $E^3 - \bigcup_{i\in\omega} T_i$ and that the natural correspondence between E^3/G and E^3/G' is a homeomorphism.

3. Criteria for deciding when $E^3/G \approx E^3$. It is well known that a cell-like, 0-dimensional, upper semicontinuous decomposition G of E^3 is shrinkable, i.e., that E^3/G is homeomorphic to E^3, if and only if for any saturated open set U in E^3 with Bd U missing the nondegenerate elements of G and $\varepsilon > 0$, there is a homeomorphism h: $E^3 \to E^3$ so that $h|E^3 - U = \mathrm{id}$ and for each element X of G in U, the diameter of $h(X)$ is less than ε [**1**, Theorem 1], [**2**, Proof of Theorem 1].

We would like to describe properties of G which imply that such a homeomorphism h could be produced. The idea behind their definitions is the following method of producing an h. Take a triangulation T of U with small mesh and produce a homeomorphism h of U so that for each X of G in U, $h(X)$ does not intersect two disjoint 2-simplexes of T.

There are two disjoint disk properties of G listed below, either of which implies that such a homeomorphism h can be produced. The following examples are intended to illustrate the appropriateness of these definitions.

Consider the nondegenerate element X of G pictured above. Think of it as a feeler shaped disk which intersects the disjoint disks D_1 and D_2 as pictured. Think of D_1 and D_2 as disjoint 2-simplexes of a triangulation. One can think of at least two methods of replacing D_1 and D_2 by D_1' and D_2' so that X does not intersect both. One method, labeled I in Figure 3.1, is to replace subdisks of D_1 and D_2 by new subdisks which miss X. The second method, labeled II, is to push parts of D_1 and D_2 over the boundary of X, moving only points which are originally near X.

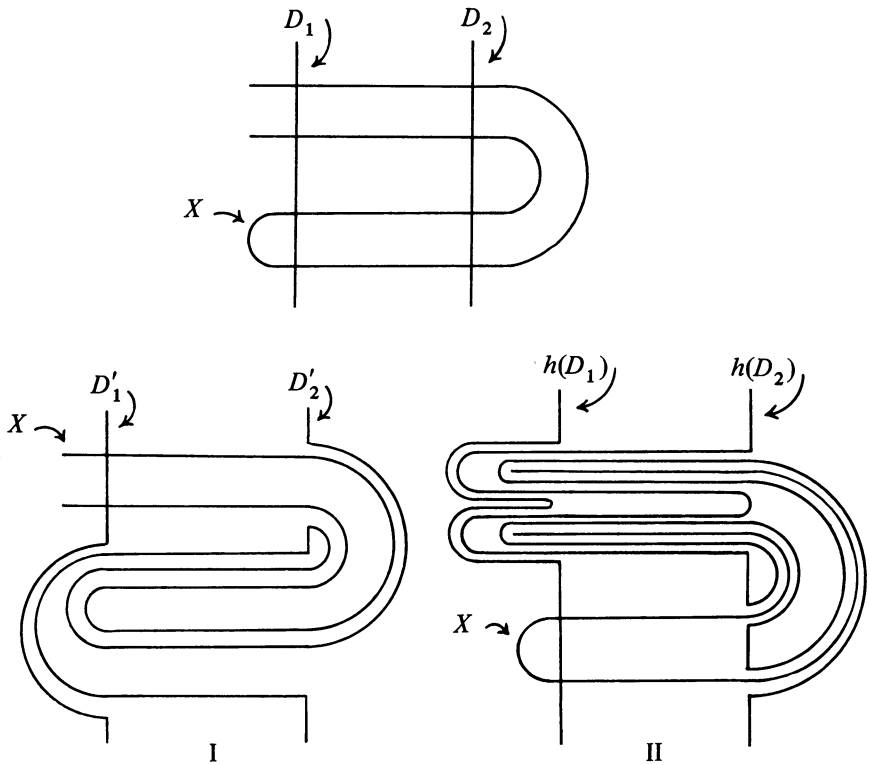

FIGURE 3.1

Each of these two methods gives rise to a disjoint disk property of G defined below. It is shown in Theorem 3.1 that E^3/G is homeomorphic to E^3 if and only if G satisfies either one of the disjoint disk properties defined below.

DEFINITIONS. A tame disk D' is obtained from a tame disk D by a *simple replacement of subdisks* $\{F_i\}_{i=1}^k$ if and only if $\{F_i\}_{i=1}^k$ is a collection of disjoint subdisks in Int D and D' is obtained by replacing each F_i by some tame disk F_i' called a *replacement subdisk*. A disk D' is obtained from D by *replacement of subdisk* if and only if there is a sequence of disks, each obtained from the previous one by a simple replacement of subdisks, starting at D and ending at D'.

More definitions. Let G be a decomposition of E^3.

HYPOTHESIS $*$. The sets D_1 and D_2 are disjoint tame disks in E^3 so that Bd $D_1 \cup$ Bd D_2 misses the nondegenerate elements of G, and V is an open set which contains all the elements of G which intersect D_1 and D_2.

DEFINITION I. The decomposition G has *disjoint disk property* I if and only if for every D_1, D_2, and V satisfying Hypothesis $*$ there are a homeomorphism g: $E^3 \to E^3$ such that $g|E^3 - V = \text{id}$ and disks D_1' and D_2' obtained

from $g(D_1)$ and $g(D_2)$, respectively, by replacement of subdisks so that each replacement subdisk used in getting from $g(D_i)$ to D_i' ($i = 1, 2$) lies in V and so that no element of G intersects both D_1' and D_2'.

DEFINITION II. The decomposition G has *disjoint disk property* II if and only if for every D_1, D_2 and V satisfying Hypothesis $*$ there is a homeomorphism h: $E^3 \to E^3$ such that $h|E^3 - V = \text{id}$ and no element of G intersects both $h(D_1)$ and $h(D_2)$.

Although property I above is an apparently weaker condition than II, it is shown below that the properties are equivalent for decompositions considered in this paper.

THEOREM 3.1. *Let G be a cell-like, 0-dimensional, upper semicontinuous decomposition of E^3. Then the following are equivalent*:

I. *G has disjoint disk property* I,

II. *G has disjoint disk property* II, *and*

III. *E^3/G is homeomorphic to E^3.*

PROOF. We will prove that I $\Rightarrow$ II $\Rightarrow$ III $\Rightarrow$ I.

Recall that $S(A)$ denotes the union of all elements of G which intersect the set A.

I $\Rightarrow$ II. Let D_1, D_2, and V satisfy Hypothesis $*$. We seek to produce a homeomorphism h: $E^3 \to E^3$ satisfying the conclusions of disjoint disk property II. Since an open set contained in V and satisfying Hypothesis $*$ would have the following properties, we assume that V has the properties that V is saturated, $V \cap (\text{Bd } D_1 \cup \text{Bd } D_2) = \varnothing$, no nondegenerate element of G intersects Bd V, and each tame 2-sphere in V bounds a 3-cell in V. This last condition can be obtained by making each component of V have connected boundary.

Let A_1 and A_2 be 1-complexes so that A_i ($i = 1, 2$) is a relatively closed subset of $D_i \cap V$, $(D_i \cap V) - A_i$ is simply connected, and each component of A_i has a limit point in $D_i - V$. Note that this last condition implies that every tame 2-sphere in $V - (A_1 \cup A_2)$ bounds a 3-cell in $V - (A_1 \cup A_2)$.

Let h_1: $E^3 \to E^3$ be a homeomorphism obtained from Theorem 2.1 so that $h_1(A_1 \cup A_2)$ misses the nondegenerate elements of G and $h_1|E^3 - V = \text{id}$. Note that $S(h_1(D_1)) \cap S(h_2(D_2)) \subset (V - h_1(A_1 \cup A_2))$.

Apply disjoint disk property I to $h_1(D_1)$, $h_1(D_2)$, and $V' = V - h_1(A_1 \cup A_2)$ to obtain a homeomorphism g: $E^3 \to E^3$ with $g|E^3 - V' = \text{id}$ and disks D_1' and D_2'. Since $h_1(D_i) \cap V'$ is simply connected ($i = 1, 2$), the replaced subdisks as well as the replacement subdisks lie in V'. This fact together with the fact that V' has the property that each tame 2-sphere in V' bounds a tame 3-cell in V' implies that there is a homeomorphism h_2: $E^3 \to E^3$ so that $h_2|E^3 - V' = \text{id}$ and $D_i' = h_2 \circ g \circ h_1(D_i)$ ($i = 1, 2$). Thus the homeomorphism $h = h_2 \circ g \circ h_1$ proves that G has disjoint disk property II.

II $\Rightarrow$ III. Next we assume that the decomposition G has disjoint disk property II and prove that E^3/G is homeomorphic to E^3. It is sufficient to prove the following lemma.

LEMMA 3.2. *Let G be a cell-like, 0-dimensional, upper semicontinuous decomposition with disjoint disk property* II. *Let U be a bounded saturated open set so that no nondegenerate element of G intersects* Bd U *and ε be larger than zero.*

Then there is a homeomorphism g: $E^3 \to E^3$ so that $g|E^3 - U = \mathrm{id}$ and for each element X of G in U, $g(X)$ has diameter less than ε.

PROOF OF LEMMA 3.2. Let T be a triangulation of U so that the simplicial star of each 2-simplex in T has diameter less than ε. Theorem 2.1 implies that there is a homeomorphism fixed outside of U which moves the elements of G so that no moved nondegenerate element intersects $T^{(1)}$, the 1-skeleton of T. We assume, therefore, without loss of generality, that $T^{(1)}$ misses the nondegenerate elements of G.

Let $\{W_i\}_{i\in\omega}$ be a null sequence of disjoint open sets in $U - T^{(1)}$ such that each nondegenerate element of G lies in $\bigcup_{i\in\omega} W_i$, for each i, $\overline{W}_i \subset U$, each nondegenerate element of G of diameter greater than or equal to ε lies in W_1, and for each $i = 2, 3, \ldots,$ W_i has diameter less than ε.

The strategy of the proof is to produce a homeomorphism h: $E^3 \to E^3$ so that $h|E^3 - \bigcup_{i\in\omega} W_i = \mathrm{id}$, so for each i, $h(W_i) = W_i$, and so that no element of G in W_1 intersects two disjoint 2-simplexes of T. The conditions on the sizes of the W_i's ($i > 1$) and the mesh of T guarantee that such a homeomorphism is one satisfying the conclusions of Lemma 3.2.

Let $\{(\sigma_{i(k)}, \sigma_{j(k)})\}_{k=1}^{m}$ be an ordering of all pairs of disjoint 2-simplexes of T such that for each k, $\sigma_{i(k)} \cap W_1 \neq \varnothing$ and $\sigma_{j(k)} \cap W_1 \neq \varnothing$. We will define homeomorphisms $h_1, h_2, \ldots, h_m$ inductively using the hypothesis that G has disjoint disk property II.

Suppose $h_1, h_2, \ldots, h_{k-1}$ have been defined; then h_k is defined as follows. Let $D_1 = h_{k-1} \circ \cdots \circ h_1(\sigma_{i(k)})$, $D_2 = h_{k-1} \circ \cdots \circ h_1(\sigma_{j(k)})$, and V be an open set containing $S(D_1) \cap S(D_2)$ so that $V \subset \bigcup_{i\in\omega} W_i$ and no component of V contains points of both $S(h_{k-1} \circ \cdots \circ h_1(\sigma_{i(r)}))$ and $S(h_{k-1} \circ \cdots \circ h_1(\sigma_{j(r)}))$ for any $r < k$.

Let h_k be a homeomorphism guaranteed by disjoint disk property II as applied to D_1, D_2 and V. Since h_k is set-wise invariant on each component of V and h_k is the identity outside of V, we have that for each $r \leqslant k$, $S(h_k \circ \cdots \circ h_1(\sigma_{i(r)})) \cap S(h_k \circ \cdots \circ h_1(\sigma_{j(r)})) = \varnothing$.

Let $h = h_1^{-1} \circ h_2^{-1} \circ \cdots \circ h_m^{-1}$. This homeomorphism satisfies the conclusions of Lemma 3.2 for the reasons given above. Therefore, Lemma 3.2 is proved; hence II $\Rightarrow$ III is proved.

III $\Rightarrow$ I. We are given that E^3/G is homeomorphic to E^3 and wish to prove that G has disjoint disk property I.

Let D_1, D_2, and V satisfy Hypothesis $*$ and assume that no nondegenerate element of G intersects Bd V.

Since E^3/G is homeomorphic to E^3, there is a homeomorphism g^{-1}: $E^3 \to E^3$ so that $g^{-1}|E^3 - V = \text{id}$ and for every element X of G in V, $g^{-1}(X)$ has diameter less than the distance from D_1 to D_2. Therefore $D_1' = g(D_1)$ and $D_2' = g(D_2)$ do not intersect the same element of G, so G has disjoint disk property I. Since no replacement of subdisks was necessary, we have actually shown that G has disjoint disk property II.

Therefore III $\Rightarrow$ I is proved and Theorem 3.1 is also proved.

4. Decompositions with only countably many nondegenerate elements. In [**6**, Theorem 1] it is shown that if G is an upper semicontinuous decomposition of E^3 into points and countably many cellular, tame polyhedra, then E^3/G is homeomorphic to E^3. Here a shrinkability criterion in Theorem 3.1 is used to give a new proof of this result, which, unlike the previous proof, does not make use of Woodruff's 2-sphere Theorem [**7**].

THEOREM 4.1. *Let G be an upper semicontinuous decompositon of E^3 into points and countably many cellular, tame polyhedra. Then E^3/G is homeomorphic to E^3.*

PROOF. By Theorem 3.1, it is only necessary to prove that G has disjoint disk property I. Let D_1, D_2, and V satisfy Hypothesis $*$ of disjoint disk property I and let $\{P_i\}_{i\in\omega}$ be the nondegenerate elements of G which intersect both D_1 and D_2. Assume $V \cap (\text{Bd } D_1 \cup \text{Bd } D_2) = \varnothing$. Find a homeomorphism g: $E^3 \to E^3$ so that $g|E^3 - V = \text{id}$ and each intersection of $g(D_1 \cup D_2)$ with P_i is a general position intersection for each i. We assume for simplicity of notation that $g = \text{id}$.

LEMMA 4.2. *Given the above situation it is possible to define inductively on j ($j = 0, 1, 2, \ldots$) disjoint tame disks D_1^j and D_2^j and open sets W_j so that*

(i) *$D_1^j \cup D_2^j$ is in general position with respect to each P_i;*

(ii) *W_j is a saturated open set whose boundary misses all the nondegenerate elements of G;*

(iii) *$P_j \subset W_j$;*

(iv) *$D_1^0 = D_1$, $D_2^0 = D_2$, and $W_0 = \varnothing$;*

(v) *D_i^j is obtained from D_i^{j-1} by a simple replacement of subdisks where each replacement subdisk is in V ($i = 1, 2; j > 0$);*

(vi) *$D_i^j \cap (\bigcup_{k=1}^{j} \overline{W}_k) = \varnothing$ ($i = 1, 2; j > 0$);*

(vii) *for any element P of G, if P intersects both D_1^j and D_2^j, then P also intersects both D_1^{j-1} and D_2^{j-1}.*

PROOF. Suppose D_1^{j-1}, D_2^{j-1}, and W_{j-1} have been defined. If $(D_1^{j-1} \cup D_2^{j-1}) \cap P_j = \varnothing$, let $D_i^j = D_i^{j-1}$ ($i = 1, 2$) and pick an appropriate W_j.

Suppose therefore that $(D_1^{j-1} \cup D_2^{j-1}) \cap P_j \neq \varnothing$.

Claim. There is a tame 3-cell neighborhood N of P_j equal to the disjoint union of Bd $N \times (0, 1]$ and P_j so that

(i) $N \cap (\bigcup_{k=1}^{j-1} \overline{W}_k) = \varnothing$, and $N \subset V$;

(ii) for each $n = 1, 2, \ldots,$ Bd $N \times \{1/n\}$ is in general position with respect to each P_i;

(iii) $S(\text{Bd } N \times \{1/n\}) \cap S(\text{Bd } N \times \{1/(n+1)\}) = \varnothing$; and

(iv) let π: Bd $N \times (0, 1] \to$ Bd N be the projection map. Then for each x in $(0, 1]$, $\pi(\text{Bd } N \times \{x\}) = \pi(\text{Bd } N \times \{1\})$.

Proof of Claim. Incorporate $D_1^{j-1} \cup D_2^{j-1} \cup P_j$ into a triangulation T of E^3. Let N be the simplicial neighborhood of P_j in the second barycentric subdivision of T. Assume that the mesh of T is small enough that condition (i) of the Claim holds. It is not difficult to parametrize $N - P_j$ as Bd $N \times (0, 1]$ so that it satisfies all the conditions of the Claim.

We now return to the proof of Lemma 4.2. Recall that disks D_1^{j-1} and D_2^{j-1} and open sets $\{W_i\}_{i=1}^{j-1}$ have been defined. Let N be a neighborhood of P_j obtained from the Claim.

Let E_1 and E_2 be the disk with holes components of $D_1^{j-1} - \text{Int } N$ and $D_2^{j-1} - \text{Int } N$, respectively, so that Bd $D_1^{j-1} \subset E_1$ and Bd $D_2^{j-1} \subset E_2$. Let $\{J_i\}_{i=1}^{k}$ be the components of $(\text{Bd } N \times \{1\}) \cap (E_1 \cup E_2)$ ordered so that J_i bounds a disk F_i on Bd $N \times \{1\}$ so that $F_i \cap (\bigcup_{r=1}^{i-1} J_r) = \varnothing$.

Let D_i^j $(i = 1, 2)$ be the disk equal to E_i union disks defined as follows. For each boundary component J_r of E_i $(i = 1, 2)$, let F_r^+ be the disk equal to $(J_r \times [1/r, 1]) \cup (F_j \times \{1/r\})$.

Note that if a nondegenerate element P of G intersects F_r^+ and F_s^+, then P intersects $J_r \times [1/(r+1), 1]$ and $J_s \times [1/(s+1), 1]$. Hence, if a nondegenerate element P intersects D_1^j and D_2^j, then P intersects D_1^{j-1} and D_2^{j-1}. Let W_j be a saturated open set as required containing P_j and with $\overline{W}_j \cap (D_1^j \cup D_2^j) = \varnothing$.

The sets D_1^j, D_2^j, and W_j constructed above complete the proof of Lemma 4.2.

Since $\bigcup_{i \in \omega} P_i$ is compact and $\{W_j\}_{j \in \omega}$ covers it, there is an integer k so that $(D_1^k \cap D_2^k) \cap (\bigcup_{i \in \omega} P_i) = \varnothing$ and, because of condition (vii) of Lemma 4.2, $S(D_1^k) \cap S(D_2^k) = \varnothing$. Therefore the disks $D_1' = D_1^k$ and $D_2' = D_2^k$ prove that G has disjoint disk property I and hence prove that E^3/G is homeomorphic to E^3.

References

1. S. Armentrout, *Shrinkability of certain decompositions of E^3 that yield E^3*, Illinois J. Math. **13** (1969), 700–706.

2. R. H. Bing, *Upper semicontinuous decompositions of E^3*, Ann. of Math. (2) **65** (1957), 363–374.

3. H. W. Lambert and R. B. Sher, *Point-like 0-dimensional decompositions of S^3*, Pacific J. Math. **24** (1968), 511–518.

4. D. R. McMillan, Jr., *A criterion for cellularity in a manifold*, Ann. of Math. (2) **79** (1964), 327–337.

5. R. L. Moore, *Concerning upper semicontinuous collections of continua*, Trans. Amer. Math. Soc. **27** (1925), 416–428.

6. M. Starbird and E. Woodruff, *Decompositions of E^3 with countably many nondegenerate elements*, Proc. Georgia Topology Conf. (to appear).

7. E. Woodruff, *Decomposition spaces having arbitrarily small neighborhoods with 2-sphere boundaries*, Trans. Amer. Math. Soc. **232** (1977), 195–204.

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF TEXAS, AUSTIN, TEXAS 78712