CENTER-BY-METABELIAN GROUPS OF PRIME EXponent

BY

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Abstract. We show that a center-by-metabelian group of prime exponent p
is nilpotent of class at most p, and this result is best possible. The proof is
based on techniques dealing with varieties of groups.

1. Introduction. A classic result due to Meier-Wunderli [5] states that a
metabelian group of prime exponent p is nilpotent of class at most p. Thus a
center-by-metabelian group of exponent p is nilpotent of class at most p + 1.
We show below that this bound can be reduced to p. Because such groups do
exist for every prime p > 5 [4, Satz 6], the bound is best possible.

2. Notation and terminology. Our notation is generally the same as that in
[6], to which we also refer the reader for elementary results concerning
varieties of groups. We, however, use capital (small) italic letters for groups
(elements). The variety generated by a particular group G we denote var(G),
and {1} denotes the trivial group.

Commutators are left-normed, with (x1, x2, . . . , xn) an element in Gn, the
nth term of the descending central series of G. The nth Engel law is the
varietal law (x, ny), defined inductively:

\[(x, 0y) = x, \quad (x, ny) = ((x, (n - 1)y), y).\]

We abbreviate \((x, y), (u, v)\) by \((x, y; u, v)\) and \(((x, y; u, v), w)\) by \((x, y;
u, v; w)\). Thus G is metabelian if it satisfies the law \((x, y; u, v)\); and
center-by-metabelian, if it satisfies the law \((x, y; u, v; w)\). We call \((x, y; u, v)\)
a double commutator.

H is a factor group of G if H \cong G/N for some N < G. A is a factor of G
if A \cong H/K, where \{1\} < K < H < G. A finite group is critical if it is not
contained in the variety generated by its proper factors. A group is basic if it
is critical and generates a join-irreducible variety. The p-group G is regular if
for every a, b \in G, \((a, b)p = apbpcp\), where c is a commutator word in a and
b. A variety is regular if every finite group in it is regular.
3. The main result. Let $H$ be a center-by-metabelian group of prime exponent $p$. Then $\text{var}(H)$ has exponent $p$, it is regular and it is center-by-metabelian. To show that $H$ is nilpotent of class at most $p$, we first show that every basic group in $\text{var}(H)$ has that property. From here on, therefore, let $G$ denote a basic center-by-metabelian group of exponent $p$, $p > 5$, and assume that $G$ is nilpotent of the maximum possible class, $p + 1$. Also, let $F$ be a finitely generated relatively free group that generates $\text{var}(G)$.

**Lemma 3.1.** In $\text{var}(G)$, 2-generator groups have class at most $p$. Equivalently, 2-variable words of weight $p + 1$ are trivial.

**Proof.** We first note that $F'' < Z(F)$ because $\text{var}(G)$ is center-by-metabelian. $F/F''$ is a finite metabelian $p$-group such that every finite group in $\text{var}(F/F'')$ is regular. By [8, Theorem 1.4], every 2-generator subgroup of $F/F''$ has class less than $p$. In $F$, therefore, every commutator of weight $p$ in 2 variables is contained in $F''$. Since $F'' < Z(F)$, every commutator of weight $p + 1$ in 2 variables is trivial. Q.E.D.

**Proposition 3.2.** $\text{var}(G)$ satisfies the $(p - 1)$th Engel law.

**Proof.** Let $x$ and $y$ be elements of a free set of generators of $F$. Then $(x, (p - 1)y) \in F_{p+1}$ because $F$ has exponent $p$ [3, 18.4.13]. Thus in $F$, $(x, (p - 1)y)$ can be written as a product of simple commutators in $x$ and $y$, each of weight $p + 1$. By the lemma, each commutator in this product is trivial. Therefore $F$ and, hence, $\text{var}(G)$ satisfy the law $(x, (p - 1)y) = 1$. Q.E.D.

**Proposition 3.3.** Let $a$, $b$ and $y$ be elements of a free set $S$ of generators of $F$. Then in $F$, every simple commutator of the form $(a, ry, b, sy)$ is trivial, where $r + s = p - 1$.

**Proof.** Since $F$ satisfies the $(p - 1)$th Engel law, both $(a, b, (p - 1)y)$ and $(a, (p - 1)y, b)$ are trivial.

As a metabelian group, $F/F''$ satisfies the Witt identity. Thus

$$(a, b, y)(b, y, a)(y, a, b) \in F''.$$

Since $F'' < Z(F)$, we get

$$((a, b, y)(b, y, a)(y, a, b), (p - 2)y) = 1.$$

Now

$$((a, b, y)(b, y, a)(y, a, b), y) = (a, b, y)(b, y, a)(y, a, b)g_7,$$

where $g_7 \in F_7$. 

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Proceeding by induction, we obtain

\[
1 = ((a, b, y)(b, y, a)(y, a, b), (p - 2)y)
\]

\[
= (a, b, y, (p - 2)y)(b, y, a, (p - 2)y)(y, a, b, (p - 2)y)g_{p+4},
\]

where \(g_{p+4} \in F_{p+4}\). Since \(F_{p+4} = \{1\}\) and \((a, b, (p - 1)y) = 1\), we obtain the following equation:

\[
(b, y, a, (p - 2)y) = (y, a, b, (p - 2)y)^{-1} = (a, y, b, (p - 2)y). \quad (A)
\]

Since 2-variable commutators of weight \(p + 1\) are trivial in \(F\),

\[
(x, y, x, (p - 2)y) = 1,
\]

where \(x \in S\). Replacing \(x\) by \(ab\) and expanding, we get

\[
(a, y, b, (p - 2)y)(b, y, a, (p - a)y) = 1.
\]

Equation (A) then implies that \((a, y, b, (p - 2)y)^2 = 1\). Since \(p > 2\), we have

\[
(a, y, b, (p - 2)y) = 1.
\]

Thus if \(r = 0, 1\) or \(p - 1\), \((a, ry, b, sy) = 1\) in \(F\), where \(r + s = p - 1\).

Continuing by induction, assume \(1 < r < p - 2\) and \((a, ry, b, sy) = 1\). Now

\[
((a, ry), y, b, (a, ry)) \in F''
\]

by the Witt identity, and since \(F'' \leq Z(F)\),

\[
(((a, ry)y, b)(y, b, (a, ry))(b, (a, ry), y), (s - 1)y) = 1.
\]

Reasoning as above, and expanding, this yields

\[
(a, ry, y, b, (s - 1)y)(y, b, (a, ry), (s - 1)y)(b, (a, ry), y, (s - 1)y) = 1.
\]

(B)

Since \((y, b, (a, ry)) \in F'' \leq Z(F)\), \((y, b, (a, ry), (s - 1)y) = 1\). Also

\[
(b, (a, ry), y, (s - 1)y) = ((a, ry), b, y, (s - 1)y)^{-1} = (a, ry, b, sy)^{-1},
\]

which is trivial by the induction hypothesis. Thus (B) becomes

\[
(a, (r + 1)y, b, (s - 1)y) = 1,
\]

and this induction step completes the proof. Q.E.D.

**Corollary 3.4.** In \(F\), \((a, ry; b, sy) = 1\), where \(r\) and \(s\) are positive integers such that \(r + s = p - 1\). Hence \((a, ry; b, sy)\) is a law in \(\text{var}(G)\).

**Proof.** Since \((a, ry; b, sy)\) is a double commutator of weight \(p + 1\), it can be written as a product of simple left-normed commutators of weight \(p + 1\), each of which contains exactly one \(a\), one \(b\), and \(p - 1\) \(y\)'s. By the proposition, each of these is trivial. Q.E.D.

**Lemma 3.5.** Let \(\{a, b, x, y, y_1, y_2, \ldots, y_{p-3}\}\) be a free set of generators of \(F\). Then the following laws hold in \(F\):
(a, b; x, y, y_1, y_2, \ldots, y_{p-3}) = (a, b; x, y, y_{\sigma(1)}, y_{\sigma(2)}, \ldots, y_{\sigma(p-3)}), \quad (1)

where \( \sigma \) is any permutation of \( \{1, 2, \ldots, p-3\} \).

\[ (a, b; x, y, y_1, \ldots, y_{p-3})^{-1} = (b, a; x, y, y_1, \ldots, y_{p-3}) = (a^{-1}, b; x, y, y_1, \ldots, y_{p-3}) = \cdots = (a, b; x, y, y_1, \ldots, y_{p-3}^{-1}) = (x, y_1, \ldots, y_{p-3}; a, b). \quad (2) \]

\( (a, b; x, y, y_1, y_2, \ldots, y_{p-3})(a, b; y, x, y_1, x, y_2, \ldots, y_{p-3}) \times (a, b; y_1, x, y, y_2, \ldots, y_{p-3}) = 1. \quad (3) \]

Let \( d_1, d_2 \in F' \). Then \( (d_1, a, d_2) = (d_1; d_2, a^{-1}). \quad (4) \)

**Proof.** The following are well-known properties of metabelian groups:

(i) \( (x, y, y_1, y_2, \ldots, y_{p-3}) = (x, y, y_{\sigma(1)}, y_{\sigma(2)}, \ldots, y_{\sigma(p-3)}) \), where \( \sigma \) is any permutation of \( \{1, 2, \ldots, p-3\} \).

(ii) \( (a, b)^{-1} = (b, a) \).

(iii) \( (x, y, y_1)(y, x, y_1, x)(y_1, x, y) = 1 \) (the Witt identity).

With these, parts (1), (2) and (3) of the lemma follow easily since \( F \) is center-by-metabelian, and all double commutators are of maximal weight. Part (4) is Lemma 6.2 of [2]. Q.E.D.

**Proposition 3.6.** In \( F \), commutators of weight 2 commute with commutators of weight \( p - 1 \); i.e., \( (F_2, F_{p-1}) = 1 \).

**Proof.** Let \( \{a, b, x, y, y_1, y_2, \ldots, y_{p-2}\} \) be a free set of generators of \( F \). It suffices to show that \( (a, b; x, y_1, y_2, \ldots, y_{p-2}) \) is a law in \( F \).

When working with laws in \( F \), we shall use the notation "\( a \mapsto b \)" to mean that each occurrence of \( a \) is replaced by \( b \). Then "\( a \leftrightarrow b \)" will mean \( a \mapsto b \) and \( b \mapsto a \) simultaneously.

Also, we shall use the term "separation" to denote the following procedure: Assume \( 1 = f_1 f_2 \cdots f_r \) is a law in \( F \), where \( r < p \) and each \( f_i \) is a product of nontrivial commutators of maximal weight, each one containing exactly \( i \) occurrences of \( x \). Then each \( f_i \) is a law in \( F \), \( i = 1, 2, \ldots, r \). (See, e.g., [1, Corollary 1.1] for a more detailed description of this.)

We now proceed to the proof of the proposition. In the law

\[ 1 = (a, y; b, (p - 2)y), \]

which is valid by (3.4), let \( y \mapsto yy_1 \). After some expanding, we get

\[ 1 = (a, y; b, y, (p - 3)yy_1)(a, y_1; b, y, (p - 3)yy_1) \times (a, y; b, y_1, (p - 3)yy_1)(a, y_1; b, y_1, (p - 3)yy_1). \]

Using separation, we obtain from this a law written as the product of double commutators, in which exactly one \( y_1 \) occurs in each double commutator. We
thus have

$1 = (a, y; b, y, y_1, (p - 4)y)^{p - 3}(a, y; b, y, (p - 3)y)(a, y; b, y_1, (p - 3)y)$. 

Since $F$ has exponent $p$, this becomes

$1 = (a, y; b, y, y_1, (p - 4)y)^{3}(a, y_1; b, (p - 2)y)(a, y; b, y_1, (p - 4)y)$. 

(A)

Using 3.5(3) and 3.5(2), we get that

$(a, y; b, y_1, (p - 4)y) =$

Using the Witt identity, we have

$1 = (a, y; b, y_1, (p - 4)y) = (y_1, a, y; b, (p - 3)y)(a, y; y_1, b, (p - 4)y) =$

Taking inverses and rearranging yields

$1 = (a, y; y_1, b, (p - 4)y)(y_1, a, y; b, (p - 3)y)(a, y; y_1, b, (p - 4)y) =$

(C)

Now from (3.4), $1 = (a, y; b, (p - 3)y)$. Letting $y \mapsto xy$, we get, after some expanding,

$1 = (a, x; b, x, (p - 3)xy)(a, y; b, y, (p - 3)xy)$

$\times (a, x; b, y, (p - 3)xy)(a, y; b, x, (p - 3)xy)$. 

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Using separation, we obtain from this a law in which each double commutator contains exactly one \( x \). We get

\[
1 = (a, y; b, y, x, (p - 4)y)^{p-3}(a, x; b, y, (p - 3)y)(a, y; b, x, (p - 3)y)
= (a, y; b, y, (p - 4)y, x)^{p-3}(a, x; b, (p - 3)y, y)(a, y; b, x, (p - 3)y)
= (a, y, x^{-1}; b, (p - 3)y)^{p-3}(a, x, y^{-1}; b, (p - 3)y)(a, y; b, x, (p - 3)y),
\]
or

\[
1 = (a, y, x; b, (p - 3)y)^{p}(a, y; b, (p - 3)y)(a, y; b, x, (p - 3)y). \quad (D)
\]

Using the Witt identity again, we have

\[
(x, a, y; b, (p - 3)y) = (y, a, x; b, (p - 3)y)(x, y, a; b, (p - 3)y).
\]

Substituting this into (D) yields

\[
1 = (a, y; x; b, (p - 3)y)^{p-1}(x, y, a; b, (p - 3)y)(a, y; x, (p - 3)y).
\]

Letting \( x \leftrightarrow b \), we get

\[
1 = (a, y, b; x, (p - 3)y)^{p-1}(b, y, a; x, (p - 3)y)(a, y; x, (p - 3)y).
\]

Multiplying these last two laws yields

\[
1 = (a, y, x; b, (p - 3)y)^{p-1}(a, y, b; x, (p - 3)y)^{p-1}
\times (x, y, a; b, (p - 3)y)(b, y, a; x, (p - 3)y). \quad \text{(E)}
\]

We now note the following:

\[
(b, y, a; x, (p - 3)y) = (b, y; x, (p - 3)y, a^{-1})
= (b, y; x, y, a^{-1}, (p - 4)y)
= (b, y, y^{-1}; x, y, a^{-1}, (p - 5)y) = \cdots
= (b, y, (p - 4)y^{-1}; x, y, a^{-1})
= (b, y, (p - 4)y; x, y, a^{-1})^{(p-4)}
= (b, (p - 3)y; x, y, a).
\]

Using this in equation (E) yields

\[
1 = [(a, y, x; b, (p - 3)y)(a, y, b; x, (p - 3)y)]^{2}.
\]

Because \( p \) is odd, we get

\[
1 = (a, y, x; b, (p - 3)y)(a, y, b; x, (p - 3)y). \quad (I)
\]
From equation (C), with $a \leftrightarrow b$ and $y, \rightarrow x$, we obtain
\[
1 = (b, y; x, y, a, (p - 4)y)(y, b, x; a, (p - 3)y)(y, x, b; a(p - 3)y) = (b, y, a^{-1}; x, (p - 3)y)(y, b, x; a, (p - 3)y)(y, x, b; a, (p - 3)y).
\]
Taking inverses yields
\[
1 = (b, y, a; x, (p - 3)y)(b, y, x; a, (p - 3)y)(x, y, b; a, (p - 3)y). \tag{J}
\]
Now from (I), with $a \leftrightarrow b$, we get
\[
1 = (b, y, x; a, (p - 3)y)(b, y, a; x, (p - 3)y).
\]
Using this, we eliminate the first two factors in (J), getting
\[
1 = (x, y, b; a, (p - 3)y). \tag{K}
\]
Now using the Witt identity one last time, we have
\[
1 = (x, y, b; a, (p - 3)y)(y, b, x; a, (p - 3)y)(b, x, y; a, (p - 3)y).
\]
Therefore, by using (K) and then taking inverses, we get
\[
1 = (b, y, x; a, (p - 3)y)(x, b, y; a, (p - 3)y).
\]
From this, and from (K) with $b \leftrightarrow x$, we obtain
\[
1 = (x, b, y; a, (p - 3)y) = (x, b; a, (p - 3)y, y^{-1}).
\]
Letting $a \leftrightarrow x$ and taking inverses yields
\[
1 = (a, b; x, (p - 2)y).
\]
Thus, $F/Z(F')$ satisfies the law $(x, (p - 2)y)$. Since $F'' < Z(F')$ and $F_p < Z(F')$, $F/Z(F')$ is a metabelian group of class at most $p - 1$. By a well-known result [1, Corollary 2.1], any metabelian group of small class satisfying the law $(x, (p - 2)y)$ also satisfies the law $(x, y_1, y_2, \ldots, y_{p-2})$. Therefore $F/Z(F')$ satisfies this law, and $(a, b; x, y_1, y_2, \ldots, y_{p-2})$ is a law in $F$. Q.E.D.

**Theorem 3.7.** Let $G$ be a basic center-by-metabelian group of prime exponent $p$. Then $G$ is nilpotent of class at most $p$.

**Proof.** Let $F$ be a relatively free group that generates $\text{var}(G)$, having \{a, b, x, y_1, y_2, \ldots, y_{p-2}\} as a free set of generators. It is known that $F$ is nilpotent of class at most $p + 1$; so assume $F$ has class exactly $p + 1$. From the above proposition, we have that $1 = (a, b; x, y_1, y_2, \ldots, y_{p-2})$. By 3.5(4), we get $1 = (a, b, y_{p-2}; x, y_1, y_2, \ldots, y_{p-3})$, and taking inverses yields $1 = (a, b, y_{p-2}; x, y_1, y_2, \ldots, y_{p-3})$.

Proceeding by induction, we see that every double commutator of weight $p + 1$ is trivial. Because $F/F''$ is a metabelian group of exponent $p$, $F_{p+1} < F''$ [5]. Thus the simple commutator $(a, b, x, y_1, y_2, \ldots, y_{p-2})$ can be written as a product of double commutators, each of weight $p + 1$ and containing all
the generators of $F$. Since these are all trivial, so is $F_{p+1}$. Therefore, $F$ has class at most $p$; and, hence, so does $G$. Q.E.D.

**Theorem 3.8.** Let $H$ be a center-by-metabelian group of prime exponent $p$. Then $H$ is nilpotent of class at most $p$.

**Proof.** Since groups of exponent 2 are abelian, and groups of exponent 3 are nilpotent of class at most 3 [6, Theorem 34.32], $H$ is metabelian if $p < 5$. Thus we may assume that $p > 5$ and $H$ is not metabelian.

Since $H$ is center-by-metabelian but not metabelian, $Z(H) > \{1\}$ and $H'' < Z(H)$. Therefore $H$ and, hence, var($H$) are solvable. Because a solvable group of exponent $p$ is locally finite [7, Theorem 7.16], var($H$) is a locally finite variety. Hence var($H$) is generated by its basic groups, and it remains only to show that those are nilpotent of class at most $p$.

Let $G$ be a basic group in var($H$). If $G$ is abelian, it has class 1; and if $G$ is metabelian, it has class at most $p$ since every metabelian group of exponent $p$ does.

If $G$ is not metabelian, it must be a basic center-by-metabelian group of exponent $p$. Hence it has class at most $p$ by 3.7. Therefore all the basic groups in var($H$) have class at most $p$. Hence $H$ itself is nilpotent of class at most $p$. Q.E.D.

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**Bibliography**


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