A 3-LOCAL CHARACTERIZATION OF $L_4(2)$

BY

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Abstract. Recent work of Gorenstein and Lyons on finite simple groups has led to standard form problems for odd primes. The present paper classifies certain simple groups which have a standard 3-component of type $L_3(2)$.

Introduction. D. Gorenstein and R. Lyons [8] have recently shown that any "minimal unknown" simple group $G$ of characteristic 2 type with $e(G) > 4$ must satisfy one of three specific conditions. In [3], we consider a special case of one of those conditions. Here, we obtain characterizations of groups with $e(G) = 3$ which satisfy hypotheses analogous to those in that case.

We are concerned with the following hypothesis.

($\mathcal{K}_n$): $G$ is a group, $b$ is an element of $G$ of order 3, and $J = O^3(E(C(b)))$.

Furthermore, the following conditions hold.

(a) $J/Z(J) \cong L_n(2)$;
(b) $C(J)$ has cyclic Sylow 3-subgroups;
(c) $\langle b \rangle$ is not strongly closed in $C(b)$; and
(d) $m_{23}(G) = m_3(C(b))$.

Briefly, $\mathcal{K}_n$ says that $G$ has a standard 3-component of type $L_n(2)$ satisfying the Gorenstein-Lyons conditions. In the general case, the statement of the Gorenstein-Lyons conditions is somewhat more technical.

We also remark that by results of Schur [12] and Steinberg [13], either $J \cong L_n(2)$ or $n = 3$ or 4 and $J$ is the unique central extension $L_n(2)$ of $L_n(2)$ by $Z_2$.

The two main results in this paper are:

THEOREM A. Let $G$ satisfy $\mathcal{K}_4$ or $\mathcal{K}_5$ and assume that $F^*(G)$ is simple and that $b \notin GJ$. Then $b \notin F^*(G)$ and $F^*(G)$ has Sylow 3-subgroups of type $Z_3 \times Z_3$. In particular, $G$ is not simple.

THEOREM B. Let $G$ be a finite simple group of characteristic 2 type which satisfies $\mathcal{K}_5$. Then $G \cong L_4(2)$.

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1. Preliminary lemmas. In this section we collect some properties of \( L_4(2) \) and \( L_5(2) \) which will be useful in the proofs of Theorems A and B. We also derive some elementary consequences of \( \mathcal{X}_4 \) and \( \mathcal{X}_5 \).

**Lemma 1.1.** Let \( G = L_n(2) \), \( n > 4 \), and let \( t \) be an involution in the center of a Sylow 2-subgroup of \( G \). Then the following statements hold.

(a) \( C(t) = TL \) where \( T = O_2(C(t)) \) is extra-special of type \( 2^{2n-3} \) and \( L \cong L_{n-2}(2) \);
(b) \( T \) contains 2 elementary abelian \( L \)-invariant subgroups \( U \) and \( V \) of rank \( n - 1 \);
(c) \( L \) acts decomposably on \( U \) and on \( V \);
(d) \( C(t) \) acts on \( \text{Inv}(T) \) with the following orbits: \( \{t\} \), \( U \setminus \langle t \rangle \), \( V \setminus \langle t \rangle \), \( \text{Inv}(T) \setminus \text{Inv}(U) \cup \text{Inv}(V) \);
(e) \( t^G \cap T = U^T \cup V^T \cup \{t\} \);
(f) if \( H = \text{Aut} \, G \), then \( T = O_2(C_H(t)) \).

**Proof.** See Suzuki [14].

**Lemma 1.2.** Let \( H = \text{Aut}(L_n(2)) \), \( n = 4 \) or 5, and set \( G = H' \). Then \( G = L_n(2) \) and the following conditions hold:

(a) \( G \) contains a Sylow 3-subgroup \( P \) of type \( F_9 \);
(b) \( C_H(P) = P \times \langle \tau \rangle \) where \( \tau \in H \setminus G \) is an involution, \( C_G(\tau) = \Sigma_6 \), and \( H = G\langle \tau \rangle \);
(c) \( N_G(P) = D \) where \( D \) is dihedral of order 8;
(d) \( G \) has 2 classes of elements of order 3.

Letting \( \alpha \) and \( \beta \) be representatives of the two conjugacy classes of elements of order 3,

(e) \( N_H(\langle \alpha \rangle) = \langle \alpha, s \rangle \times K\langle \tau \rangle \) where \( \langle \alpha, s \rangle \cong L_2(2) \), \( K \cong L_2(4) \), and \( \langle \alpha, s \rangle = G \cdot C_K(\tau) \);
(f) \( N_H(\langle \beta \rangle) = \langle \beta, t \rangle \times L\langle \tau \rangle \) where \( \langle \beta, t \rangle \cong L_2(2) \), \( L \cong L_{n-2}(2) \) and \( \langle \beta, t \rangle = G \cdot C_L(\tau) \); and
(g) \( C_G(t) \) contains a Sylow 2-subgroup of \( G \) and \( C_G(s) \) does not contain a Sylow 2-subgroup of \( G \).

(h) \( \langle \alpha \rangle \) and \( \langle \beta \rangle \) are each contained in a subgroup of \( G \) of type \( L_2(4) \).

**Proof.** These results all follow from routine computations.

**Lemma 1.3.** Assume that \( G, b, \) and \( J \) satisfy \( \mathcal{X}_n \) where \( n = 4 \) or 5. Let \( B \in \text{Syl}_3(\text{C}(b)) \) and set \( B_0 = B \cap J \). Then \( B = \langle b \rangle \times B_0 \) and one of the following is true.

(i) \( b \notin GJ, N(B)/C^*(B) = \Sigma_4 \times Z_2 \) and \( \langle b \rangle^{N(B)} = \mathcal{S}_1(B) \setminus \mathcal{S}_1(B_0) \).
(ii) \( b \in GJ, N(B)/C^*(B) = \Sigma_4 \) and elements \( b_1, b_2 \in B_0 \) can be chosen so that \( B_0 = \langle b_1, b_2 \rangle \) and \( N(B)/C(B) \) acts as the full monomial group on \( B \) with respect to the basis \( \{b, b_1, b_2\} \).
Furthermore, in case (ii), $N(B)$ controls fusion in $B$.

**Proof.** $B = B_0 \times C_B(J)$ because $J \triangleleft C(b)$ and $\text{Out} J$ is a 3'-group. Setting $B_1 = C_B(J)$, we need to show that $B_1 = \langle b \rangle$. By assumption $B_1$ is cyclic, and $B_0$ is elementary abelian. Therefore $B$ is abelian and $\mathfrak{U}(B) = \mathfrak{U}(B_0)$. If $|B_1| > 3$, then $\langle b \rangle = \Omega_1(\mathfrak{U}(B))$ is strongly closed in $B$, contrary to hypothesis. Therefore $B_1 = \langle b \rangle$, and $B \simeq E_{27}$.

Let $N = N(B)$ and $N = N/C^*(B)$. Then there is a natural injection $\bar{N} \hookrightarrow PGL(3, 3) \simeq SL(3, 3)$ so we can identify $\bar{N}$ with its image in $SL(3, 3)$. Let $\tau \in N_J(B)$ be an involution which inverts $B_0$, so that $\langle b \rangle = C_B(\tau)$. We have $C_{\bar{N}}(\tau) < N_{\bar{N}}(\langle b \rangle) = \overline{C(b) \cap N(B)}$, so that $C_{\bar{N}}(\tau) \simeq D_8$. Inspection of 2-local and 3-local subgroups of $SL(3, 3)$ yields that either $\bar{N} \simeq F_9 \cdot Z_8 \simeq S_4$ or $\bar{N} \simeq \Sigma_4$. In the former case $\bar{N}$ is the stabilizer of a hyperplane of $B$ which must be $B_0$, whence (a) holds.

Assume for the rest of the proof that $\bar{N} \simeq \Sigma_4$. Then $|\bar{N}| = 3$, so $\langle b \rangle$ has 3 $N$-conjugates. $C_{\bar{N}}(b)$ has orbits of lengths 1, 2, 2, 4, and 4 on $\mathfrak{U}_1(B)$ so $\langle b \rangle$ must fuse to exactly one of the orbits of length 2. Letting $\langle b_1 \rangle$ and $\langle b_2 \rangle$ be the groups in that orbit, we have $B_0 = \langle b_1, b_2 \rangle$. Since $C_{\bar{N}}(b)/C(B)$ acts as the full monomial group on $B_0$ with respect to $\{b_1, b_2\}$, we conclude that $N/C(B)$ acts as the full monomial group on $B$ with respect to $\{b, b_1, b_2\}$.

It remains to show that $N(B)$ controls fusion in $B$. If $P \in \text{Syl}_3(N(B))$, then $P \simeq Z_3 \times Z_3$ by the above paragraph, so $B = J(P)$. Therefore $B$ is weakly closed in $P$ and $N(P) < N(B)$. It follows that $P \in \text{Syl}_3(G)$ and that $N(B)$ controls fusion in $B$ with respect to $G$.

**Lemma 1.4.** Assume that $G, b,$ and $L$ satisfy $\mathfrak{H}_n$, where $n = 4$ or 5. Let $B \in \text{Syl}_3(C(b))$, and set $X = O_{3^+}(C(b))$. Assume that $X$ has odd order and that either $|C(B)|$ is odd or $n = 5$. Then $X$ is a normal Hall $\{2, 3\}$-subgroup of $C(B)$ and $X = O_{3^+}(C(A))$ for every group $A < B$ with $b \in_G A$. Finally, one of the following holds:

(i) $C(b) = \langle b \rangle \times J \times X$, or

(ii) $C(B)$ has even order.

**Proof.** It follows from Lemma 1.2(b) that $[C(B) : BX] < 2$. Therefore $X$ is a normal $\{2, 3\}$-complement for $C(B)$ and $X = O_{3^+}(C(a))$ for every $a \in b^G \cap C(b)$. To verify the second assertion, it suffices to assume that $A$ is an $E_2$-subgroup of $B$ containing $\langle b \rangle$. Then $C(A)$ normalizes $J$ and $O_{3^+}(C(A)/C(J) \cap C(J)) = 1$ by Lemma 1.2(b), (e), (f). Since $X = O_{3^+}(C(A) \cap C(J))$, we have $X = O_{3^+}(C(A))$. For the last assertion, set $C = C(b)$ and assume that $C(B)$ has odd order. Then $C_J(J) = \langle b \rangle \times X$ by transfer, so $C = N_C(J) = J \times \langle b \rangle \times X$ by Lemma 1.2(b).
Lemma 1.5. Let $J \cong L_5(2)$ act faithfully on $U \cong E_{25}$ with $C_U(J) \neq 1$. Then $U = [U, J] \times C_U(J)$.

Proof. Assume the contrary. Then $C_U(J) = \langle t \rangle$ has order 2 and $J$ acts semiregularly on the set $\Omega$ of complements to $\langle t \rangle$ in $U$. As $|\Omega| = 32$ and $J$ has a subgroup of order 31, it follows that $J$ is doubly transitive on $\Omega$. But $L_5(2)$ has no doubly transitive representations of degree 32 by [2], a contradiction.

Lemma 1.6. Assume that $J \cong L_5(2)$, that $\beta \in J$ has order 3 and that $J$ acts on the 2-group $T$ so that $C_T(\beta) < T_0$ where $T_0$ is $J$-invariant. Then $T = T_0$.

Proof. It suffices to assume that $T_0 = 1$. By Lemma 1.2(h), we can choose $\gamma \in J$ of order 5 so that $\langle \beta, \gamma \rangle \cong L_2(4)$. Then $T$ is the direct product of natural $L_2(4)$-modules by [10], so $C_T(\gamma) = 1$. Since $\langle \gamma \rangle$ acts fixed point-free on a subgroup $D$ of $J$ of order 31, we have $[D, T] = 1$. Thus $J$ centralizes $T$ and $T = 1$.

2. Groups of small 3-rank. In this section, we derive two propositions about configurations which arise in the proofs of Theorems A and B.

Proposition 2.1. Assume that $G = LB$ is a finite group such that $L = F^*(G)$ is simple and $B \not\subset L$ has order 3. Assume further that

(i) $C(B) = B \times K \times O_3\langle C(B) \rangle$ where $K \cong L_2(4)$.
(ii) If $A \in \text{Syl}_3(F)$, then $C(A)$ has odd order.
(iii) If $P = \langle B, A \rangle \in \text{Syl}_3(C(B))$ and $B_1 \in \mathfrak{S}_1(P) - \{A\}$, then $C(B_1) \cong C(B)$.
(iv) $m_{2,3}(L) = 1$.

Then $L \cong L_2(125), L_2(64)$ or $L_3(4)$.

Proof. Let $\mathfrak{S}_1(P) = \{A, B, B_1, B_2\}$, let $U \in \mathfrak{S}_R^*(P; 2)$ and let $U < T \in \mathfrak{S}_R^*(P, 2)$. Then $C_T(A) = 1$ so $T = C_T(B)C_T(B_1)C_T(B_2)$. Hypotheses (i) and (ii) imply that $U \in \text{Syl}_2(C(B))$, so $U = C_T(B)$ and $|C_T(B_i)| < 4$ for $i = 1, 2$ by hypothesis (iii). Either $U < Z(T)$ or $1 \neq C_T(B_i) < Z(T)$ for $i = 1, 2$. In the latter case, we may relabel $B$ and $B_i$ without affecting the hypotheses of the theorem to obtain $U < Z(T)$.

By the Frattini argument, $N(U) = C(U) \cdot (C(B) \cap N(U)) = C_L(U)(C(B) \cap N(U))$. Setting $C = C_L(U)$, we have $3 \mid |C|$ because $C_C(B)$ has 3'-order. Therefore $T \in \text{Syl}_2(C)$ and in fact $T \in \text{Syl}_2(N(U))$. By the preceding paragraph, $|T| = 4^n$ for $n = 1, 2, 3$. We consider each possibility in turn.

Case 1. $n = 1$. Then $U = T \in \text{Syl}_2(L)$, so $L \cong L_2(q)$ for some $q \equiv 3$ or 5 (mod 8) by Walter [15]. By elementary properties of $\text{Aut}(L_2(q))$, $C_L(B) \cong L_2(q^{1/3})$. Therefore $q = 125$, and the proposition holds.
Case 2. \( n = 2 \). Then \( T \) is elementary abelian of order 16 because \( C_T(A) = 1 \). Set \( N_1 = N(T) \cap N(P) \). Then \( A < N_1 \) and \( |C_{N_1}(A)| \) is odd by hypothesis, so \( |N_1 : P| < 2 \). Inspecting the subgroups of \( L_4(2) \), we then have \( |N(T) : C(T)|_2 < 2 \). Let \( S \in \text{Syl}_2(N(T)) \). Then either \( S = T \) or \( S \cong E_4 \sim Z_2 \). In the latter case, \( T = J(S) \), so \( S \in \text{Syl}_2(L) \) in either case. But no simple group has Sylow 2-subgroup of type \( E_4 \sim Z_2 \) by Corollary 6 of [6], so \( S = T \). But then [15] forces \( L \cong L_2(16) \), a contradiction as \( B \) must act as a group of outer automorphisms of \( L \). Thus Case 2 does not occur.

Case 3. \( n = 3 \). We argue that \( T \in \text{Syl}_2(L) \). It suffices to show that \( N(T) \) is 3-nilpotent since \( T \in \text{Nil}(P; 2) \). Set \( N = N(T) \). By hypothesis (i), \( N_n(B) < C(B) \cap N(U) \) has a normal 3-complement. Similarly, \( N_n(B_i) \) is 3-nilpotent for \( i = 1, 2 \) because \( C_T(B_i) \neq 1 \). This implies that \( \text{Aut}_n(P) \) is a 3-group, so \( N_n(P) \) is 3-nilpotent. If \( P \subseteq Q \subseteq \text{Syl}_3(N) \), then \( Q \cap L \) is cyclic by hypothesis (iv). It follows that \( P = \Omega_3(Q) \). Since \( A = \Omega_3(Q \cap L) \), \( N_n(A) \cap L \) is 3-constrained and \( N_n(A) = O_3(N_n(A))(N_n(A) \cap N_n(P)) \) by the Frattini argument. Therefore \( N_n(A) \) is 3-nilpotent and \( N \) is 3-nilpotent by the Frobenius transfer theorem.

If \( T \) is abelian, then \( G \cong L_2(64) \) by Walter [15]. Otherwise \( T \) is of type \( L_3(4) \) [7, p. 16] in which case \( L \cong L_3(4) \) by Collins [1]. The proof is complete.

**Lemma 2.2.** Let \( R \) be a solvable group with a normal subgroup \( S \) of index 2 such that \( O_3(R) \leq S \). Assume that \( T \cong Z_3 \) is a Sylow 3-subgroup of \( S \) and that \( x \in \text{Inv}(R) \setminus \text{Inv}(S) \). Then \( x \in R \cdot N(T) \).

**Proof.** Let \( R \) be a counterexample of minimal order and let \( N \) be a minimal normal subgroup of \( R \). Then \( N \) is an elementary abelian \( p \)-group for some \( p \neq 3 \) as \( R \) is solvable and \( O_3(R) = 1 \) by assumption. Setting \( R = R/N \) and applying induction, \( x \in R \cdot N(T) \). That is \( x \in R \cdot N \cdot N(T) \), so \( R = N \cdot N(T) \) by choice of \( R \). Furthermore \( p = 2 \), as otherwise \( N(T) \) contains a Sylow 2-subgroup of \( R \). Let \( x = nh \) for \( n \in N \) and \( h \in N(T) \setminus N_3(T) \). Our choice of \( R \) implies that \( R = \langle T, x \rangle = NT\langle h \rangle \). Evidently, \( h^2 \in N \), so \( R/N \cong \Sigma_3 \). Therefore \( N \cong E_4 \) and \( R \cong \Sigma_4 \) is not a counterexample.

**Proposition 2.3.** Let \( G \) be a finite group with an elementary abelian Sylow 3-subgroup \( P = \langle A, B \rangle \) of order 9. Assume that the following conditions are satisfied:

(i) \( E(C(B)) = K \cong L_2(4) \) with \( A < K \).

(ii) \( O_3(C(B)) \) has odd order.

(iii) One of the following holds:

(a) \( N(A) < N(B) \) and \( C(P) \) has odd order.

(b) \( N(A) < N(B) \) and \( O_3(C(B)) = 1 \).

(c) \( E(C(A)) \cong L_3(2) \) and \( O_3(C(A)) = O_3(C(B)) \).

Then \( G = O_3(G)N(K) \). 

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Proof. We first observe that $A$ and $B$ are strongly closed in $P$ with respect to $G$. In fact, it is evident from hypothesis (iii) that $A \neq G B$. Also, $C(B)$ has 3 orbits on $\mathfrak{S}_1(P)$. As $N(P)/P$ is a $3'$-group, it then follows that $A$ and $B$ are each normal in $N(P)$. But $N(P)$ controls fusion in $P$, so the assertion is proved.

Let $G$ be a counterexample of minimal order. Then $O_3(G) = 1$. It follows easily from assumption (i) that $O_3(G) = 1$. Thus $F^*(G) = E(G)$ is the direct product of simple groups. Set $E = E(G)$.

We argue that $P \leq E$. Set $Q = P \cap E$ and assume that $Q < P$. Then $Q \neq 1$ because $3 | |E|$. Since $Q \leq N(P)$, we have that $Q = A$ or $Q = B$. We shall use a 3-local characterization to obtain a contradiction. The claim is that either $|C_E(Q)| = 6$ or $|C_E(Q)|$ is odd. In fact, by transfer, $C_E(Q) < Q \cdot O_3(C(Q))$, hence $|C_E(Q)|$ is odd if $Q = B$ or if $Q = A$ and either (iii)(a) or (iii)(γ) holds. On the other hand, if $Q = A$ and (iii)(β) holds, then $|C_E(Q)| = 6$. So the claim is true in all cases. Therefore $E = L_2(q)$, $L_3(q)$ or $U_3(q)$ for appropriate $q$ by results of [4], [5] and [11]. In any case, $Out(E)$ is solvable, so $K < E$ which gives $Q = A$. If (iii)(γ) holds, then a similar argument shows that $Q = B$ which is absurd. If (iii)(β) holds, then $E = L_2(p)$ for some $p \in \{5, 7, 11, 13\}$ and $Out(E)$ is a $3'$ group. But $C(E) = 1$ and $P \leq N(E)$ then yield a contradiction. Therefore (iii)(α) must hold. Let $\tau$ be an involution in $N_K(A)$. Then $\tau \in E$, and $A = [C_E(A), \tau]$. It follows that $E = L_2(4)$, $L_3(2)$ or $U_3(4)$. But none of these groups admit an outer automorphism of order 3 in contradiction to $B < N(E)$ and $C(E) = 1$. This completes the argument that $P \leq E$.

We now show that $E$ is simple. If not, then $E = E_1 \times E_2 \times \cdots \times E_n$, with $2 \leq n$ and $E_i$ simple, $1 \leq i \leq n$. As $C(P)$ is 3-solvable by hypotheses (i) and (ii), $E = E_1 \times E_2$ with $P \cap E_i \neq 1$, $i = 1, 2$. As $P \cap E_i$ is inverted in $E_i$, $I_1 = 1, 2$, it follows from $N_{E_i}(B) \leq C_{E_i}(B)$, $I_1 = 1, 2$, together with $O_3(C(B)) = C_E(B) = B \times K$ that, without loss, we may set $P \cap E_i = A$ and $P \cap E_2 = B$. But then $E_1 = K$ and therefore $E_1 \leq G$ by hypothesis (iii) which contradicts our choice of $G$. We conclude that $E$ is simple. Therefore $E = G$ by choice of $G$.

We shall now apply a result of G. Higman [7] to contradict the simplicity of $G$. Let $D \in \mathfrak{S}_1(P) \setminus \{A, B\}$ and for every subgroup $X$ of $G$, let $\mathfrak{S}(X)$ denote the set of involutions of $G$ which invert $X$. Higman's result asserts that if $t \in Inv(G)$, then two of the following three sets are nonempty: $\mathfrak{S}(A)^G \cap \{t\}$, $\mathfrak{S}(B)^G \cap \{t\}$, $\mathfrak{S}(D)^G \cap \{t\}$. In order to apply this result, we require some information about $\mathfrak{S}(A)$, $\mathfrak{S}(B)$ and $\mathfrak{S}(D)$.

We first claim that $\mathfrak{S}(D) \subseteq \mathfrak{S}(P)^G$. To see this, set $H = N(D)$, $H_0 = C(D)$ and $H = H/D$. By the first paragraph, $N_{H_0}(P) < N(B)$, so our hypothesis forces $N_{H_0}(P) = C_{H_0}(P)$. In particular, $H_0$ has a normal $3$-
complement $F$. If $F$ is not solvable, then $F$ has a chief section which is the direct product of one or more Suzuki groups. Therefore $C_F(A) = C_F(P)$ involves $S_2(2)$. But $C(P)$ contains no elements of order 4 by hypothesis, so $F$ is solvable and hence $H$ is solvable. As $F = O_3(\bar{H})$, it follows from Lemma 2.2 that $\bar{x} \in \bar{N}(\bar{P})$ for every $x \in \mathcal{G}(D)$. This in turn yields $\mathcal{G}(D) \subseteq \mathcal{G}(P)^G$ as claimed.

Let $r \in \mathcal{G}(A) \cap K$ and $s \in \mathcal{G}(B) \cap C(A)$ with $[r, s] = 1$. If $C(P)$ has even order, let $t \in \text{Inv}(C(P))$, otherwise, set $t = 1$. Then $\langle t \rangle \in \text{Syl}_2(C(P))$ by hypothesis (ii) and we may choose $t$ so that $\langle t, r, s \rangle$ is abelian. Observe that $K\langle s, t \rangle \cong \Sigma_5 \times \mathbb{Z}_2$ or $\Sigma_5 \times \mathbb{Z}_2$ and $K\langle s, t \rangle$ covers $N(B)/O_3(C(B))B$. By inspection, $B \in \text{Syl}_3(C(B) \cap C(r))$, hence $B \in \text{Syl}_3(C(r))$. Similarly $A \in \text{Syl}_3(C(s))$ and $P \in \text{Syl}_3(C(t))$. Since $A \neq B$, we see that $r, s$ and $t$ belong to different $G$-conjugacy classes. In particular $\mathcal{G}(D)^G = \mathcal{G}(P)^G = (rs)^G \cup \langle r, s \rangle^G \neq \text{Inv}(G)$.

As $K\langle s, t \rangle$ contains a Sylow 2-subgroup of $N(B)$, every involution of $K\langle s, t \rangle$ is $K$-conjugate to an element of $N(P)$. Therefore $g \in N(B) = N(P)$ for every $g \in \mathcal{G}(A)$. Similarly $x \in N(B) = N(P)$ for every $x \in \mathcal{G}(A)$. It follows that $\mathcal{G}(A)^G = r^G \cup (rt)^G \cup (rs)^G \cup (rst)^G$ and $\mathcal{G}(B)^G = s^G \cup (st)^G \cup (rs)^G \cup (rst)^G$.

If $|C(P)|$ is even, then $K\langle t \rangle \cong \Sigma_5$ by hypothesis (i), so $t = r st$. It follows that $\mathcal{G}(A)^G \cap \mathcal{G}(B)^G = \mathcal{G}(D)^G$ in any case. Let $x \in \text{Inv}(G) \setminus \mathcal{G}(D)^G$. Then $x$ belongs to at most one of $\mathcal{G}(A)^G$, $\mathcal{G}(B)^G$, $\mathcal{G}(D)^G$ contradicting Higman's result. Therefore our counterexample $G$ does not exist.

3. Proof of Theorem A. In this section, $G$, $b$ and $J$ satisfy the hypotheses of Theorem A. That is, $G$ is a finite group with $F^*(G)$ simple and $b \in G$ is an element of order 3 such that the following hold:

(a) $J$ is a normal subgroup of $C(b)$ of type $L_4(2)$, $L_4(2)$, or $L_5(2)$;
(b) $C(J)$ has cyclic Sylow 3-subgroups;
(c) $\langle b \rangle$ is not strongly closed in $C(b)$;
(d) $m_{23}(G) = 3$; and
(e) $b \notin G J$.

Choose $B \in \text{Syl}_3(C(b))$ and set $B_0 = B \cap J$. Then $B_0 \triangleleft N(B)$ by Lemma 1.3. We set $N = N(B)$ and $\bar{N} = N/O_3(N) \cdot B_0$.

**Lemma 3.1.** $\bar{N} = \langle \bar{b} \rangle \times \bar{A} \bar{D}$ or $\langle \bar{b}, \bar{t} \rangle \times \bar{A} \bar{D}$ where $\langle b, A \rangle = P \in \text{Syl}_3(N)$, $A$ is homocyclic abelian of order 34, $D \in \text{Syl}_2(N \cap J)$ is dihedral of order 8, $A$ is isomorphic to $B_0$ as a $GF(3)\bar{D}$-module and $t$, if it exists, is an involution which inverts $B$.

**Proof.** By Lemma 1.3, $\langle b \rangle \triangleleft \bar{N}$ and $C_{\bar{N}}(\bar{b})/\langle \bar{b} \rangle \cong N/C^*(B) \cong \Sigma_3 \ltimes \mathbb{Z}_2$. Let $P \in \text{Syl}_3(N)$ so that $\bar{P} = O_3(\bar{N})$. The action of $D$ on $P$ implies that $P$ is
either elementary abelian or extra-special of exponent 3. In the latter case, 
\( \text{Aut}_{C(b)}(\bar{P}) \) is isomorphic to a subgroup of \( SL_2(3) \) whereas \( \bar{D} \) acts faithfully on 
\( \bar{P} \). Therefore \( \bar{P} \cong E_3 \) and we may set \( \bar{P} = \langle \bar{b} \rangle \times \bar{A} \) where \( \bar{A} = [\bar{P}, Z(\bar{D})] \).

Clearly \( \bar{D} \) acts faithfully on \( \bar{A} \). Since \( [b, P] = B_0 \), \( A \) and \( B_0 \) are isomorphic as 
\( GF(3)\bar{D} \)-modules. Also \( Z(\bar{D}) \) acts regularly on \( \bar{A} \) and \( B_0 \) then yields that \( A \) is 
homocyclic of order \( 3^4 \). Finally if \( b \) is inverted in \( G \), hence necessarily in \( N \), 
then \( t \) may be chosen as described.

**Lemma 3.2.** Either \( G \) satisfies the conclusion of Theorem A or the following 
conditions hold:

(a) \( A \cong E_3 \); 
(b) \( C(B) \) has odd order, in particular \( |O_3(C(b))| \) is odd; 
(c) \( J = O^2(C(b)) \); and 
(d) \( P \) has exponent 3.

**Proof.** By Lemma 3.1, \( A \) is homocyclic abelian of order \( 3^4 \). Assume first 
that \( A \cong Z_3 \times Z_3 \). Since \( A \) and \( F_0 \) are isomorphic as \( GF(3)\bar{D} \)-modules, an 
 easy argument yields \( B = \Omega_3(P) \). Therefore \( B \) is weakly closed in \( P \) with 
respect to \( G \) and consequently \( P \in Syl_3(G) \). Suppose \( b = \bar{g}b^{-1} \) so that 
\( \bar{N} = \langle \bar{b}, i \rangle \times AD \) as in Lemma 3.1. Then, assuming that \( t \in N(A) \), as we 
may, \( t \) centralizes \( A/B_0 \) and inverts \( B_0 \), an obvious contradiction. Hence 
\( b \notin N' \). But \( N(P) < N \), and \( P \) has no \( Z_3 \triangleleft Z_3 \) homomorphic image. A recent 
transfer theorem of Yoshida [16] implies that \( A = P \cap N(P)' \). Thus \( A \in Syl_3(P^*(G)) \) because \( A = [A, D] \), and \( G \) satisfies the conclusion of Theorem A.

Now assume that \( A \) is elementary abelian of rank 4. By Burnside's transfer 
theorem, \( C(B) \) has a normal 3-complement \( X \). Clearly \( A \triangleleft N(X) \). Since 
\( m_{2,3}(G) = 3 \), we have \( |X|_2 = |C(B)|_2 = 1 \), so (b) holds. By Lemma 1.4, this 
implies that \( J = O^2(C(b)) \). Finally, \( P \) has exponent 3 since \( P = \Omega_3(P) \) and \( P \) 
has class 2. This completes the proof of Lemma 3.2.

Now assume until a contradiction is reached that \( G \) does not satisfy the 
conclusion of Theorem A. Therefore conditions (a), (b), (c) and (d) of Lemma 
3.2 hold.

**Lemma 3.3.** Choose \( A^* \in Syl_3(C(A)) \) so that \( b \in N(A^*) \). Then \( A^* \) is abelian 
and \( A^* \langle b \rangle \in Syl_3(G) \).

**Proof.** Set \( N_1 = N(A) \). By Lemma 3.1, we may assume that \( C_{N_1}(b) = 
O_3(C_{N_1}(b)) \langle b \rangle \times B_0 \). Set \( Y = C_{N_1}(Z(D)) \). Then the regular action of 
\( Z(D) \) on \( B_0 \) implies that \( \langle b \rangle \) is a Sylow 3-subgroup of \( Y \). As \( m_{2,3}(G) = 3 \), 
\( C(A) \) has odd order, and \( Z(D) \) must act regularly on some Sylow 3-subgroup 
of \( C(A) \). Therefore \( A^* \) is abelian. As \( Z(D) \) inverts \( A \), \( C(A)Z(D) \triangleleft N_1 \);
hence, by the Frattini argument, we have that $N_1 = C(A)Y$. Thus $A^* \langle b \rangle \in \text{Syl}_3(N_1)$. Now $Z(D)$ normalizes $\langle C(A), b \rangle$, so $Z(D)$ normalizes a Sylow 3-subgroup of $\langle C(A), b \rangle$ containing $\langle b \rangle$. Without loss, we may then assume that $Z(D)$ normalizes $\langle A^*, b \rangle$. If $Y^* = C_{N(A^*)}(Z(D))$, then we may argue as before to conclude that $N(A^*) = C(A^*)Y^*$ where $\langle b \rangle \in \text{Syl}_3(Y^*)$. But $C(A^*) < C(A)$ yields $A^* \langle b \rangle \in \text{Syl}_3(N(A^*))$ and as $A^*$ is characteristic in $A^* \langle b \rangle$, we conclude that $A^* \langle b \rangle \in \text{Syl}_3(G)$.

**Lemma 3.4.** Choose $a \in B_0$ so that $K = E(C_2(a)) \cong L_2(4)$. Then $C_A(K) \cap N(\langle a, b \rangle) = U \cong E_9$, $b \in N(U)$ and $O_{3,E}(C(U))/O_{3}(C(U)) \cong L_3(4)$.

**Proof.** Set $V = N_A(\langle a, b \rangle)$. Then $V \cong E_3$ as $[A, b] = B_0$ and $A \cong E_3$. $V$ normalizes $K = E(C(\langle a, b \rangle))$, so $U = C_F(K) \cong E_9$. Clearly $[b, U] < [b, V] = \langle a \rangle$, so $b \in N(U) \setminus C(U)$.

Set $M = N(U)$, $C = C(U)$ and $\bar{M} = M/O_{3}(C)$. For the proof of this lemma, we are interested only in the structure of $\bar{M}$; therefore we shall abuse notation and identify elements and subgroups of $M$ with their images in $\bar{M}$. If $R = O_{3}(C)$, then $C_R(b) = \langle a \rangle$. Since $[a, K] = 1$, the $P \times Q$ lemma applied to $\langle b \rangle \times K$ acting on $R$ shows that $[R, K] = 1$. This implies that $E = E(C) \neq 1$. By Lemma 3.3, $A^* \in \text{Syl}_3(C)$; so $O_{3}(E) = 1$, and the components of $E$ are all simple. As $m(R) > 2$ and $m_{3}(G) = 3$, $E$ has at most 2 components, each of which must be normalized by $b$. Since $m_{3}(C_{E}(b)) = 1$, $E$ is simple, and $C_{E}(b) = O_{3}(C_{E}(b)) \times K$.

We argue that $E \langle b \rangle$ satisfies the hypotheses of Proposition 2.1. Since $U \triangleleft C(E)$, $m_{3}(E) = 1$. Furthermore, if $\langle a^* \rangle \in \text{Syl}_3(K)$, then $\langle a^* \rangle \in \text{Syl}_3(C_{E}(b))$. As $N_{A}(\langle b, a^* \rangle)$ acts transitively on $\delta_{1}(\langle b, a^* \rangle)\setminus \langle a^* \rangle$, $C_{E}(b)$ is a 3-group, so $C^*$ is 3-constrained. In particular, if $R^*$ is a $\langle b \rangle$-invariant Sylow 3-subgroup of $C^*$, then $R^*$ is abelian, and $O_{3,3}(C^*) = O_{3}(C)R^*$. Since $C_{R}(b) = \langle a^* \rangle$, $m(R^*) < 3$. Since $C^*/O_{3}(C)R^*$ acts faithfully on $\Omega_{1}(R^*)$ and fixes $\langle a^* \rangle$, it follows that $C^*$ is 3-nilpotent. This implies that $|C^*|$ is odd and hence $E \langle b \rangle$ satisfies the hypotheses of Proposition 2.1.

We conclude that $E \cong L_{2}(125)$, $L_{2}(64)$ or $L_{3}(4)$. If $E \cong L_{2}(125)$ or $L_{2}(64)$, let $a^* \in S \in \text{Syl}_3(E)$ where $S$ is $\langle b \rangle$-invariant. Then $S$ is cyclic of order 9 and $[S, b] = \langle a^* \rangle$. But this implies that $S \triangleleft N(B)$ contradicting Lemma 3.2. Hence $E \cong L_{3}(4)$ as required.

**Lemma 3.5.** $A$ contains $E_9$ subgroups $U_1$ and $U_2$ satisfying

(i) $b \in N(U_1) \cap N(U_2)$;

(ii) If $L_i = O_{3,E}(C(U_i))$, then $L_i/O_{3}(L_i) \cong L_{3}(4)$ and $U_i \triangleleft L_{3-i}$ for $i = 1, 2$; and

(iii) $C_{L_i}(b) = O_{3}(C_{L_i}(b)) \times K_i$ where $O^{2}(C_{L_i}(b)) = K_i \cong A_5$, $i = 1, 2$. 

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Proof. Choose a and U as in Lemma 3.4 and set $U_1 = U$ and $L_1 = O_{3,1}(C(U_1))$. Let $U_2 = A \cap L_1$ so that $U_2 \in \text{Syl}_3(L_1)$ and $A = U_1 \times U_2$. By properties of $L_3(4)$, $U_2$ is inverted by $\sigma \in \text{Inv}(C(<U_1, b>))$. Recall from Lemmas 3.1 and 3.2 that a Sylow 2-subgroup $D$ of $C(b) \cap N(A)$ is dihedral of order 8 and $Z(D) = <\tau>$ inverts $A$. Assume that $\sigma \in D$, as we may. Then for some $d \in \text{Inv}(D)$, $\sigma^d = \tau z$. As $U_2 = [A, \sigma]$ and $z$ inverts $A$, $U_2 = [A, \sigma^d] = [A, \tau z] < U_1$; hence, $U_2 = U_1$. Since $N(U_1)$ satisfies (iii) and $d$ interchanges $U_1$ and $U_2$ under conjugation, the result follows.

We are now in position to obtain our final contradiction. Let $L_i^* = <\langle L_i, b \rangle$ and set $L_i^* = L_i^* / O_{3}(L_i)$. Then $N_{L_i^*}(U_i) = \overline{U_i} \overline{Q_i} < b, i \neq j$, where $\overline{Q_i} \simeq Q_i$. As $C_{L_i^*}(b) = K_i \simeq A_4$ and all involutions of $L_i$ are conjugate, we may assume that $b$ normalizes $\overline{Q_i}$. Now $O_{3}(L_i)$ is a $\{2, 3\}$-group, so we may assume that $N_{L_i^*}(U_i)$ contains a $Q_8$-subgroup $Q_i$ and that $b$ normalizes $Q_1, 1 < i \neq j < 2$.

Let $M_i = L_i^* Q_i < b>$ and set $M_j = M_i / O_{3}(L_i) = L_i \overline{Q_i} < b>$. Since $\overline{Q_i}$ normalizes $\overline{L_i}$ and $[Q_i, U_j] = 1$, $i \neq j$, it follows from $|C_{\Aut(\overline{L_i})}(U_i)| = 2 \cdot 3^2$ that $[\overline{Q_i} : C_{\overline{Q_i}}(L_i)] < 2$. But $b$ acts regularly on $\overline{Q_i} / Z(\overline{Q_i})$ then yields $\overline{Q_i} \overline{L_i} = \overline{Q_i} \times L_i$. Therefore $C_{\overline{Q_i}}(b) = Z(\overline{Q_i}) \times K_i \simeq Z_2 \times L_2(4)$ and this in turn implies that $O_{3}^{2}(C(b))$ contains a subgroup isomorphic to $Z_2 \times L_2(4)$. But by Lemma 3.2, $O_{3}^{2}(C(b)) = J \simeq L_2(2)$ or $L_3(2)$ and no involution of $J$ centralizes an $L_3(2)$-subgroup. With this contradiction, the proof of Theorem A is complete.

4. Proof of Theorem B. In this section, $G, b, J$ satisfy the hypotheses of Theorem B. Thus $G$ is a finite simple group of characteristic 2 type such that:

(a) $b \in G$ has order 3;
(b) $J$ is a normal subgroup of $C(b)$ of type $L_3(2)$;
(c) $\langle b \rangle$ is not strongly closed in $C(b)$; and
(d) $C(J)$ has cyclic Sylow 3-subgroups.

By Theorem A, we may assume that $b \in G$. As before, we choose $B \in \text{Syl}_3(C(b))$ and set $B_0 = B \cap J$. By Lemma 1.3, $B_0 = \langle b_1, b_2 \rangle$ where $\langle b \rangle N(B) = \{\langle b \rangle, \langle b_1 \rangle, \langle b_2 \rangle \}$. We shall show that $G \cong L_7(2)$ by constructing the centralizer of a central involution. In order to do this, we show in Lemma 4.2 that the 3-fusion pattern in $G$ is the same as that in $L_7(2)$. First, we show that $O_{3}(C(b))$ has odd order.

Lemma 4.1. $O_{3}(C(b))$ has odd order.

Proof. Set $X = O_{3}(C(b))$ and let $V \in \text{Syl}_2(X)$. Since $G$ has characteristic 2 type, it suffices to show that $\langle b \rangle$ centralizes $V^* = O_2(N(V))$.

By Lemma 1.2, $X = O_{3}(C(<b, b^*>) for every $b^* \in J$ of order 3. Since $b^{N(B)} = (b)^* \cup (b^{N(B)} \cap J)$, it follows that $X = O_{3}(C(b^*))$ for all $b^* \in b^{N(B)}$. Therefore $X = N(B)$, so that $N(B) = X \cdot N_{N}(b)(V)$ by the Frattini argument. This implies that $b = N_{N}(b)$, so $C_{V^*}(b) = V = C_{V^*}(b_1)$. Since
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We have that \( V^* = V \) by Lemma 1.6. Thus \( \langle b \rangle \) centralizes \( V^* \), as required.

Now set \( \beta = b_1 b_2 \) and \( \gamma = b_1^2 b_2 \), so that \( B_0 = \langle \beta, \gamma \rangle \). Also set \( H^* = C(\beta) \) and \( H = O^3(H^*) \). We use Proposition 2.3 to determine the structure of \( H \).

**LEMMA 4.2.** \( H^* = \langle \beta \rangle \times H \) and \( H \) has the following properties:

(i) \( \langle b, \gamma \rangle \in \text{Syl}_3(H) \).

(ii) If \( K = E(C_H(b)) \) and \( L = E(C_H(\gamma)) \), then \( K \cong L_2(4) \), \( L \cong L_3(2) \), \( b \in L \) and \( \gamma \in K \).

(iii) \( H = C(B)E(H) \) where \( E(H) = K \times L \) and \( C(B) \) is 2-nilpotent with \( O_3(C(B)) = O_3(H) = O_3(C(b)) \).

**PROOF.** We first argue that \( H \) satisfies the hypotheses of Proposition 2.3. By Lemma 1.3, \( B \in \text{Syl}_3(N_{H^*}(B)) \) and \( \langle b, \gamma \rangle = B \cap N_{H^*}(B)' \), so Grün’s theorem implies that \( \langle b, \gamma \rangle \in \text{Syl}_3(H) \). Now \( B_0 = \langle b, b_1 \rangle \). Thus \( L = E(C_H(\gamma)) \cong E(C(\langle b, b_1 \rangle)) \) and \( b \in L \). Furthermore \( K = E(C_H(b)) = E(C(\langle b, \beta \rangle)) \) and \( \gamma \in K \). Also, by Lemma 1.4, \( O_3(C_H(\gamma)) = O_3(C(b)) = O_3(C_H(b)) \) has odd order. Since either \( K \cong L_2(4) \), \( L \cong L_3(2) \) or \( K \cong L_3(2) \), \( L \cong L_2(4) \), it follows from Proposition 2.3, that \( H = N_{H^*}(K)O_3(H) \) or \( N_{H^*}(L)O_3(H) \) according to whether \( K \cong L_2(4) \) or \( L \cong L_3(2) \), respectively.

Assume first that \( K \cong L_2(4) \). Set \( H_0 = O_3(H) \). Then \( C_{H_0}(b) < O_3(C(b)) \) implies that \( C_{H_0}(b) = C_{H_0}(L) \). As \( L \cong L_3(2) \), we have \( [L, H_0] = 1 \) whereupon \( H_0 < O_3(C(b)) \). But then \( [K, H_0] = 1 \), and \( K \vartriangleleft H \). An easy argument shows that \( H_0 = O_3(C(b)) \) and \( E(H) = K \times L \). By the Frattini argument, \( H = C(B)E(H) \) proving (iii). On the other hand, if \( K \cong L_3(2) \), then \( C_{H_0}(\gamma) < O_3(C(B_0)) = O_3(C(b)) \); hence, \( C_{H_0}(K) = C_{H_0}(\gamma) \). Therefore \( [K, H_0] = 1 \) and \( H_0 < O_3(C(B_0)) \). This in turn yields \( [L, H_0] = 1 \), whereupon \( L \vartriangleleft H \). As before, \( H_0 = O_3(C(b)) \), \( E(H) = K \times L \) and \( H = C(B)E(H) \). Thus (iii) is true in either case.

In order to complete the proof of (ii), assume that \( K \cong L_3(2) \) and \( L \cong L_2(4) \) for purpose of a contradiction. Let \( U \) be a \( B \)-invariant fours subgroup of \( L \) and set \( V = O_2(N(U)) \). Then \( [B, U] = U \) gives \( U \vartriangleleft Z(V) \). Now \( C_V(\beta) = V \cap H < O_2(C_H(U)) \), so \( [K, C_V(\beta)] \vartriangleleft K \cap O_2(C_H(U)) = 1 \). But \( C_H(K) < C(B_0) \), and \( C(B_0) \) has dihedral Sylow 2-subgroups. It follows that \( U = C_V(\beta) \). As \( \beta = N_{B_0}(\gamma) \) and \( L = E(C(B_0)) \), an application of the Frattini argument yields \( \beta = N_{B_0}(\gamma) \). Hence \( U = C_V(\gamma) = C_V(K) \) whereupon \( [K, V] = 1 \). This contradicts \( V = F^*(N(U)) \).

**LEMMA 4.3.** \( C(\beta) = \langle \beta \rangle \times O_3(C(\beta)) \times K \times L \) and \( C(b) = \langle b \rangle \times O_3(C(b)) \times J \).

**PROOF.** By Lemma 4.2(iii) and Lemma 1.4, it suffices to show that \( C(B) \) has odd order. Assume not and let \( \tau \in \text{Inv}(C(B)) \). Then \( J\langle \tau \rangle \cong \text{Aut}(L_3(2)) \),
so \( O_2(C(⟨b, τ⟩)) = 〈τ⟩ \) by Lemma 1.2(b). Since \( b = _N(B) b_i \), \( i = 1, 2 \), and \( 〈τ⟩ \in \text{Syl}_2(C(B)) \), we have \( O_2(C(⟨b, τ⟩)) = 〈τ⟩ \) for \( i = 1, 2 \). By Lemma 1.2(e), (f), \( O_2(C_κ(τ)) = O_2(C_κ(τ)) = 1 \), so \( O_2(C_κ(τ)) = 1 \). As \( KL⟨τ⟩ \) contains a Sylow 2-subgroup of \( C(β) \) and \( β = _N(B) γ \), we have \( O_2(C(⟨β, τ⟩)) = O_2(C(⟨γ, τ⟩)) = 〈τ⟩ \). It follows from the action of \( B_9 \) on \( O_2(C(τ)) \) that \( O_2(C(τ)) = 〈τ⟩ \), a contradiction since \( C(τ) \) is 2-constrained.

For the remainder of the section, let \( t \) be an involution in \( C(B_0) \) which inverts \( b \). Set \( C = C(t) \) and \( T = O_2(C) \). We shall show that \( C \) is isomorphic to the centralizer of an involution in \( L_7(2) \).

**Lemma 4.4.** \( J < C, T = C_T(b_1) \cdot C_T(b_2) \) and \( T \) is extra-special of type \( 2_+^{11} \). Furthermore \( C_T(b_i) = O_2(C(t) \cap E(C(b_i))) \).

**Proof.** By Lemmas 4.2 and 4.3, \( C(⟨b_1, τ⟩) \) contains a subgroup isomorphic to \( L_3(2) \) which is centralized by \( b \). Therefore \( t \) centralizes \( J \) by Lemma 1.2(b), whence \( J \subset C \).

Observe that \( T = 〈C_T(x) : x \in B_0^2⟩ < 〈O_2(C(⟨x, τ⟩)) : x \in B_0^2⟩ \). Setting \( D = O_2(C(⟨β, τ⟩)) \), it follows from Lemma 4.3 that \( D < C(B_0) \) and that \( D \) is dihedral of order 8. This implies that \( C_T(β) = C_T(B_0) = D \cap T \). Since \( β = _T γ \), we have that \( T = C_T(b_1) \cdot C_T(b_2) \).

We now show that \( T \) is extra-special. Set \( T_i = C_T(b_i) \) and \( R_i = O_2(C(⟨b_i, τ⟩)) \) so that \( T_i = T \cap R_i \), \( i = 1, 2 \). By Lemma 1.2(g), \( C \cap E(C(b_i)) = R_i L_i \) where \( L_i \cong L_3(2) \), \( b_i \in L_{i-1} \), \( R_i \) is extra-special of type \( 2_+ \) and \( D \leq R_1 \cap R_2 \). Setting \( \overline{C} = C/⟨t⟩ \) and observing that \( \overline{T} = F^*(\overline{C}) \), we have \( \overline{R} \) is abelian, hence \( D \leq C_{\overline{C}}(\overline{T}) = Z(\overline{C}) \). Since \( D = C_T(β) \), Lemma 1.6 implies that \( \overline{T} \) is elementary abelian. As \( C(β) \cap Z(T) = ⟨t⟩ \), we have \( Z(T) = ⟨t⟩ \) by the same lemma. Thus \( T \) is extra-special.

By Lemma 4.3, \( K \) centralizes \( D \), so \( γ \) acts regularly on the \( K \)-invariant section \( T/D \). Therefore \( |T/D| = 2^r \) for some \( r > 1 \). But \( |T| = |T_1T_2| = |T_1||T_2|/|D| < |R_1||R_2|/|D| = 2^{11} \) gives \( r < 2 \). On the other hand, \( J \) acts faithfully on \( T \) forcing \( r > 1 \). We conclude that \( r = 2 \) whereupon \( |T| = 2^{11} \) and \( T_i = R_i, i = 1, 2 \).

Since \( T_1 \) is extra-special, \( T = T_1 \cdot C_T(T_1) \). As \( C_T(T_1) \cap C(b_i) = C_T(T_1) \cap T_1 = 〈t⟩ \), we have \( C_T(T_1) < T_2 \). But \( C_T(T_1) = [C_T(T_1), b_i] < [T_2, b_i] \) which is extra-special of type \( 2_+^5 \) so that \( T = T_1 \cdot [T_2, b_i] \) is extra-special of type \( 2_+^{11} \). This completes the proof.

**Lemma 4.5.** \( TJ \) is isomorphic to the centralizer of a central involution of \( L_7(2) \).

**Proof.** Recall from Lemma 4.4, that \( C \cap E(C(b_i)) = T_i L_i \) where \( T_i = C_T(b_i) \) is extra-special of type \( 2_+^7 \), \( L_i \cong L_3(2) \) and \( b_j \in L_i, 1 < i \neq j < 2 \). If we set \( S_i = [T_{3-i}, b_i] \), then \( T = T_1 \cdot S_1, S_1 = [T, b_1] \) and \( S_i \) is extra-special of
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As $S_i$ does not admit the faithful action of $L_i$, we have $[S_i, L_i] = 1$, $i = 1, 2$. By Lemma 1.1(b), $T_i$ contains precisely 2 $L_i$-invariant $E_{16}$-subgroups, say $U_i$ and $V_i$, with $U_i \cap V_i = \langle t \rangle$. Recall from the proof of Lemma 4.4 that $D = C_T(B_0)$ is dihedral of order 8 and $D = T_1 \cap T_2$. It is easy to check that $D \cap U_i = C_{U_i}(b_{3-i})$ has order 4. Relabelling $U_2$ and $V_2$, if necessary, we may assume that $D \cap U_1 = D \cap U_2$. Set $U = U_1U_2$ and $W_i = \{U_i, b_{3-i} \}$. Then $U_i = W_i \times (D \cap U_i)$ implies that $U$ has order $2^6$ and that $U = U_1W_2 = U_2W_1$. Since $W_{3-i} < S_i$, $L_4 < C(W_{3-i}) \cap N(U_i) < N(U)$. Therefore $J = \langle L_1, L_2 \rangle < N(U)$. As $|U| = 2^6$, it follows immediately that $U$ is elementary abelian. Similarly $V = V_1V_2$ is a $J$-invariant $E_{26}$-subgroup of $T$. Furthermore $J$ acts trivially on $U \cap V$, then gives $T = UV$ and $\langle t \rangle \cap U \cap V = \langle t \rangle$.

By Lemma 1.5, $U = \langle t \rangle \times U_0$ where $U_0$ is $J$-invariant. Clearly $U_0 \cap C_T(V) = U_0 \cap V = 1$, so $U_0J$ acts faithfully on $V$. Since $L_3(2)$ contains a subgroup isomorphic to the split extension of $V$ by Aut($V$), we may embed $VU_0J = TJ$ in $L_3(2)$. But then consideration of orders verifies that $TJ$ is isomorphic to the centralizer of a central involution of $L_3(2)$.

**PROPOSITION 4.6.** $C(t) = TJ$.

**Proof.** As in the proof of the previous lemma, let $U$ and $V$ be the $J$-invariant $E_{26}$-subgroups of $T$. Then $N_C(U) = C_C(U)TJ$ because Aut$_C(U) = \text{Aut}_T(J(U))$. Furthermore, the $P \times Q$ lemma applied to $O^2(C_C(U)) \times U$ acting on $T$ gives $O^2(C_C(U)) < T$ since $C_T(U) = U$. Therefore $C_C(U)$ is a 2-group and $O_2(N_C(U)) = C_C(U)T$. We shall argue that $N_C(U) = TJ$ and that $N_C(U) < C$.

We claim that $t^G \cap T = U^* \cup V^*$. First, we show that $U^* \cup V^* \subseteq t^G \cap T$. In fact, $TJ$ has 2 orbits on $U^*$, namely $\{t\}$ and $U \setminus \langle t \rangle$. Since $D \cap U$ is a fours group and Inv($D$) $\subseteq t^G$ (recall that $D < E(C(B_0)) \approx L_3(2)$ by Lemma 4.3), we have $U^* \subseteq t^G$. Similarly $V^* \subseteq t^G$. As $C_T(b_i) = T_1 = O_2(C \cap E(C(b_i)))$ by Lemma 4.4, it follows from Lemma 1.1(e) that $T$ contains an involution $x$ with $x \in_C C(\langle b_i, bb_2 \rangle)$. Certainly $x \neq t$ because $\langle b_1, bb_2 \rangle \neq B_0 \in \text{Syl}_3(C)$. But an easy argument shows that $TJ$ acts transitively on Inv($T$) $\setminus (U^* \cup V^*)$. Hence $t^G \cap T = U^* \cup V^*$ as claimed.

If $W = U^* < T$, then $U$ and $V$ are the unique $E_{26}$-subgroups of $T$ generated by conjugates of $t$ implies that $W = U$ or $W = V$. Therefore $[C : N_C(U)] < 2$ and $N_C(U) < C$. Thus $T = O_2(N_C(U))$, $U = C_C(U)$ and $N_C(U) = TJ$.

We have $C = TJ \cdot N_C(B_0)$ by the Frattini argument. Also, by Lemmas 4.3 and 4.4, $C_C(B_0)$ is 2-closed and $O_2(C_C(B_0)) < T$. As $|\text{Aut}_C(B_0)| < |\text{Aut}_G(B_0)| = 8$ by Lemma 1.4 and $|\text{Aut}_T(B_0)| = 8$ by Lemma 1.2(c), $TJ$ contains a Sylow
2-subgroup of $N_C(B_0)$. Therefore $C = TJ$, as required.

By Suzuki's theorem [14], we conclude from Proposition 4.6 that $G \cong L_3(2)$. This completes the proof of Theorem B.

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