STARLIKE, CONVEX, CLOSE-TO-CONVEX, SPIRALLIKE, AND $\Phi$-LIKE MAPS IN A COMMUTATIVE BANACH ALGEBRA WITH IDENTITY

BY
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Abstract. Let $C(X)$ be the space of continuous functions on a compact $T_2$-space $X$ where each point of $X$ is a $G_6$. If $F: B \rightarrow C(X)$ is a biholomorphic (in the sense that $F$ and $F^{-1}$ are Fréchet differentiable) map of $B = \{ f \ | \ ||f|| < 1 \}$ onto a convex domain with $DF(0) = I$, then $F$ is Lorch analytic (i.e., $DF(f)(g) = ag$ for some $a \in C(X)$).

Let $R$ be a commutative Banach algebra with identity such that the Gelfand homomorphism of $R$ into $C(\mathfrak{M})$ is an isometry. Starlike, convex, close-to-convex, spirallike and $\Phi$-like functions are defined in $B = \{ x \in R \ | \ ||x|| < 1 \}$ for $L$-analytic functions in $B$ and they are related to associated complex-valued holomorphic functions in $\Delta = \{ z \in \mathbb{C} \ | \ |z| < 1 \}$.

Introduction. In §§2–7, let $R$ be a commutative Banach algebra over the complex numbers with identity (denoted by 1) and let $\mathfrak{M}$ be the space of maximal ideals in $R$. Then $\mathfrak{M}$ is a compact, $T_2$-space where the topology is the weakest topology on $\mathfrak{M}$ such that the Gelfand transformation $x(M)$ of $x$ is a continuous function on $\mathfrak{M}$. Assume further that the Gelfand homomorphism of $R$ into $C(\mathfrak{M})$ is an isometry; i.e., $\|x\| = \sup\{|x(M)| \ | M \in \mathfrak{M} \}$ for all $x \in R$. Let $B = \{ x \in R \ | \ ||x|| < 1 \}$ and $\Delta = \{ z \in \mathbb{C} \ | \ |z| < 1 \}$.

If $D$ is an open set in $R$, we say $F: D \rightarrow R$ is $L$-analytic in $D$ if for each $x \in D$, there is $F'(x) \in R$ such that

$$\lim_{h \rightarrow 0} \frac{\|F(x + h) - F(x) - hF'(x)\|}{\|h\|} = 0$$

[11]. Thus it is clear that $L$-analytic functions are Fréchet differentiable. If $F: B \rightarrow R$ is $L$-analytic in $B$, then for each $x \in B$, $F(x) = \sum_{n=0}^{\infty} a_n x^n$ where $a_n \in R$ and the series converges uniformly on $||x|| < \rho < 1$ [7, Theorems 3.19.1 and 26.4.1]. If $F: D \rightarrow R$ is $L$-analytic in $D$ and for each $y \in F(B)$, there is an open neighborhood $V$ of $y$ such that $F^{-1}$ exists and is $L$-analytic.

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in $V$, then we say that $F$ is locally bianalytic in $B$. If $F$ is univalent (one-to-one) and locally bianalytic in $B$, we say that $F$ is bianalytic in $B$. If $F$ is $L$-analytic in $B$, then for each $M \in \mathcal{M}$, there is an associated holomorphic function $F_M: \Delta \to \mathbb{C}$ defined by $F_M(z) \equiv F(z)(M)$ for all $z \in \Delta$. If $F(x) = \sum_{n=1}^{\infty} a_n x^n$ is $L$-analytic in $B$, then we write $F(x)/x$ for the $L$-analytic function $\sum_{n=1}^{\infty} a_n x^{n-1}$.

2. Preliminary lemmas.

**Lemma 2.1.** Let $V: B \times I \to B$ be $L$-analytic in $B$ for each $t \in I = [0, 1]$, $V(0, t) = 0$ for all $t \in I$, $V(x, 0) = x$ for all $x \in B$. If $\lim_{t \to 0+} (x - V(x, t))/(xt) = U(x)$ exists and is $L$-analytic in $B$, then $\Re U(x)(M) > 0$ for all $M \in \mathcal{M}$ and all $x \in B$.

**Proof.** For each $t \in I$, $V(x, t)$ satisfies Schwarz’ Lemma [17, Theorem A] so $\|V(x, t)\| < \|x\|$ for all $x \in B$. For all $M \in \mathcal{M}$ and all $x \in B$,

$$|V_M(x(M), t)| = |V(x, t)(M)| < \|V(x, t)\| < \|x\|.$$ 

For $z \in \Delta$, the choice $x = z1$ shows that $V_M(\cdot, t)$ satisfies Schwarz’ lemma. Now letting $z = x(M)$, we have $|V(x, t)/x(M)| = |V_M(x(M), t)/x(M)| < 1$ (where the limit value is to be taken when $x(M) = 0$) and taking the maximum over all $M \in \mathcal{M}$, we have $\|V(x, t)/x\| < 1$. Hence,

$$\Re \frac{x - V(x, t)}{xt}(M) > \frac{1 - \|V(x, t)/x\|}{t} > 0$$

for all $t \in I$. The lemma follows.

**Definition 2.2.** Let $F$ be a domain in $\mathcal{M}$. If $U: D \to F$ is $L$-analytic in $D$ and $\Re U(x)(M) > 0$ for each $M \in \mathcal{M}$ and each $x \in D$, then we say $U$ has positive real part in $D$.

**Example 1.** Let $X = \{1, 2, \ldots, n\}$ with the discrete topology. Then $C(X) = \mathbb{C}^n$ with the multiplication $(a_1, a_2, \ldots, a_n) \cdot (b_1, b_2, \ldots, b_n) = (a_1 b_1, a_2 b_2, \ldots, a_n b_n)$ and the unit ball $B$ is the polydisk $\{(z_1, z_2, \ldots, z_n): |z_j| < 1, 1 < j < n\}$. Therefore, $L$-analytic functions on $B$ are functions $F(z_1, z_2, \ldots, z_n) = (F_1(z_1), F_2(z_2), \ldots, F_n(z_n))$ where each $F_j$ is analytic in the unit disk $\Delta$. There are $n$ maximal ideals $M_1, M_2, \ldots, M_n$ in $C(X)$ given by $M_j = \{(z_1, z_2, \ldots, z_n) \in \mathbb{C}^n: z_j = 0\}$. It follows that $U = (U_1, U_2, \ldots, U_n): B \to \mathbb{C}^n$ has positive real part if and only if $\Re U_j(z_j) > 0$ whenever $|z_j| < 1$ (where it is assumed that $U$ is $L$-analytic in $B$). Note that if $U$ has positive real part, then $zU$ is in the class $\mathcal{M}$ defined in [15].

**Example 2.** Let $X = [0, 1]$ with the usual topology and $R = C(X)$. Then $L$-analytic functions on $B$ are the power series $F(f) = \sum_{n=0}^{\infty} a_n x^n$ where $a_n \in C(X)$ with $\lim \sup \|a_n\|^{1/n} < 1$. The maximal ideals are the sets $M_x = \{f \in C(X): f(x) = 0\}$ for some $x$, $0 < x < 1$. Therefore, if $U$ is $L$-analytic
in $B$, $U$ has positive real part if and only if $\text{Re } U(f)(x) > 0$ for all $f \in B$ and $x \in [0, 1]$.

**Example 3.** Let $R = H^\infty(\Delta)$. In this case, one needs to modify Definition 2.2 to replace $\mathfrak{N}$ by $\mathfrak{N}' = \text{cl}(M_\zeta \in \mathfrak{N} : f \in M_\zeta \Rightarrow \Re f(z) = 0, |z| < 1)$. The theory in the remainder of this paper can then be applied to this space. Thus $U : B \to H^\infty(\Delta)$ has positive real part if $\text{Re } U(f)(z) > 0$ for all $f \in H^\infty$ and $z \in \Delta$. For example, $U(f) = (1 + f)(1 - f)^{-1} = 1 + 2f + 2f^2 + \ldots$ has positive real part.

**Lemma 2.3.** Let $F : B \to R$ be bianalytic in $B$. Let $G : B \times I \to R$ be $L$-analytic for each $t \in I$, $G(x, 0) = F(x)$, for each $x \in B$, $G(0, t) = F(0)$ for each $t \in I$, and $G(B, t) \subset F(B)$ for each $t \in I$. If $\lim_{t \to 0^+} (G(x, 0) - G(x, t))/t = xH(x)$ exists and is $L$-analytic, then $H(x) = F'(x)U(x)$ where $U$ has positive real part in $B$.

**Proof.** We will show that $V(x, t) = F^{-1}(G(x, t))$ satisfies Lemma 2.1. Fix $x \in B$, $x \neq 0$, and expand $G(x, t)$ about $x$,

$$G(x, t) = F(V(x, t)) = F(x) + F'(x)(V(x, t) - x) + K(V(x, t), x)$$

where $\|K(y, x)\|/\|y - x\| \to 0$ as $\|y - x\| \to 0$. Therefore,

$$\frac{G(x, 0) - G(x, t)}{t} = F'(x) \frac{x - V(x, t)}{t} - \frac{K(V(x, t), x)}{t}.$$

If we show $K(V(x, t), x)/t \to 0$ as $t \to 0^+$, then

$$\lim_{t \to 0^+} \frac{x - V(x, t)}{xt} = [F'(x)]^{-1}H(x)$$

and the lemma follows by Lemma 2.2.

To show that $K(V(x, t), x)/t \to 0$ as $t \to 0^+$, observe that $\|(x - V(x, t))/t\|$ is bounded as $t \to 0^+$; otherwise, for some sequence $(t_n)$, $t_n \to 0$ and $\|(x - V(x, t_n))/t_n\| \to \infty$. In this case,

$$xH(x) = \lim_{n \to \infty} \left[ F'(x) \frac{x - V(x, t_n)}{\|x - V(x, t_n)\|} - \frac{K(V(x, t_n), x)}{\|x - V(x, t_n)\|} \right] \frac{\|x - V(x, t_n)\|}{t_n}$$

so that

$$F'(x) \frac{x - V(x, t_n)}{\|x - V(x, t_n)\|} \to 0 \quad \text{as } n \to \infty.$$

But this implies that $F'(x)$ is a generalized divisor of zero which contradicts the $L$-analyticity of $F^{-1}$. Thus we have shown

$$\lim_{t \to 0^+} \frac{K(V(x, t), x)}{t} = \lim_{t \to 0^+} \frac{K(V(x, t), x)}{\|V(x, t) - x\|} \frac{\|V(x, t) - x\|}{t} = 0.$$
Lemma 2.4. Let $U$ have positive real part.

1. If $M \in \mathcal{M}$, then

$$\frac{1 - \|x\|}{1 + \|x\|} \text{Re} \, U(0)(M) < \text{Re} \, U(x)(M) < \frac{1 + \|x\|}{1 - \|x\|} \text{Re} \, U(0)(M)$$

for all $x \in B$;

and so $\text{Re} \, U(0)(M) > 0$ if and only if $\text{Re} \, U(x)(M) > 0$ for all $x \in B$.

2. $\text{Re} \, U(0)(M) > 0$ for all $M \in \mathcal{M}$ implies $U(x)$ is nonsingular for all $x \in B$.

Proof. For $M \in \mathcal{M}$ and $0 \neq x \in B$, let $p(\lambda) = U(\lambda x/\|x\|)(M)$ for $\lambda \in \Delta$. Since $p$ is holomorphic in $\Delta$ and $\text{Re} \, p(\lambda) > 0$, by the classical inequality,

$$\frac{1 - |\lambda|}{1 + |\lambda|} \text{Re} \, p(0) < \text{Re} \, p(\lambda) < \frac{1 + |\lambda|}{1 - |\lambda|} \text{Re} \, p(0)$$

so, $\lambda = \|x\|$ yields (1). (2) follows from (1) and the fact that $\text{Re} \, U(x)(M) > 0$ for all $M \in \mathcal{M}$ implies $U(x) \notin M$ for any $M \in \mathcal{M}$ and, hence, $U(x)$ is nonsingular.

Definition 2.5. If $U$ has positive real part in a domain $D \subset R$ and $\text{Re} \, U(x)(M) > 0$ for all $M \in \mathcal{M}$ and all $x \in D$, then we write $U \in \mathcal{P}(D)$. If $D = B$, then we write $\mathcal{P}$ for $\mathcal{P}(B)$.

Lemma 2.6. Let $P \in \mathcal{P}$. Then, for each $x \in B$, the initial value problem

$$\frac{dw}{dt} = -wP(w), \quad w(0) = x,$$

has a unique solution $V(t) = V(x, t)$ defined on $t > 0$. For fixed $t > 0$, $V(x) = V(x, t)$ is $L$-analytic and univalent in $B$ and

$$\|V(x, t)\| < \|x\|\exp\left( -\frac{1 - \|x\|}{1 + \|x\|} \delta t \right)$$

for all $t > 0$ and all $x \in B$ where $\delta = \min\{\text{Re} \, P(0)(M)|M \in \mathcal{M}\}$.

Proof. The proof of the existence and uniqueness of the solution is covered in [12]. If (1) holds, then the solution can be continued to obtain a solution for all $t > 0$. The univalence of solution follows from the uniqueness of the solution, and the $L$-analyticity of $V(x, t)$ in $B$ for each $t > 0$ follows from the equilocal boundedness of the successive approximations $V_m(x, t)$ of $V(x, t)$ and Theorem 8.4.3 [6, p. 272].

We now show (1). For each $M \in \mathcal{M}$, $V(t)(M)$ is the solution of the initial value problem

$$\frac{du}{dt} = -uP_M(u), \quad u(0) = V(0)(M) = x(M).$$

By [1, Lemma 1], $|V(t)(M)| < |V(0)(M)|$ for all $t > 0$. Differentiating $|V(t)(M)|^2 = V(t)(M)\overline{V}(t)(M)$, we get
\[
\frac{1}{|V(t)(M)|} \frac{d|V(t)(M)|}{dt} = - \operatorname{Re} P_M(V(t)(M))
\]
\[
\leq - \frac{1 - |V(t)(M)|}{1 + |V(t)(M)|} \operatorname{Re} P_M(0)
\]
\[
\leq - \frac{1 - |V(0)(M)|}{1 + |V(0)(M)|} \operatorname{Re} P_M(0) \leq - \frac{1 - \|x\|}{1 + \|x\|} \delta
\]

and (1) follows.

**3. Starlike functions.** In $\mathbb{C}$, if $f(z) = \sum_{n=1}^{\infty} a_n z^n$, $a_1 \neq 0$, is holomorphic in $\Delta$, then $f$ is starlike in $\Delta$ if $(1 - t)f(\Delta) \subset f(\Delta)$ for all $t \in I = [0, 1]$ which is equivalent to $\operatorname{Re}(zf'(z)/f(z)) > 0$ for all $z \in \Delta$. We will define starlike functions in $R$ and relate them to starlike function in $\mathbb{C}$.

**Definition 3.1.** A bianalytic map $F: F \to F$ is said to be starlike in $F$ if $F(0) = 0$ and $(1 - t)F(B) \subset F(B)$ for all $t \in I$.

**Theorem 3.2.** Let $F(x) = \sum_{n=1}^{\infty} a_n x^n$ be locally bianalytic in $B$. Then $F$ is starlike in $B$ if and only if $F_M(z) = \sum_{n=1}^{\infty} a_n(M) z^n$ is starlike in $\Delta$ for all $M \in \mathcal{M}$.

**Proof.** Assume $F$ is starlike in $B$ and set $G(x, t) = (1 - t)F(x)$. Lemma 2.3 applies with $xH(x) = F(x)$ so that $F(x) = xF'(x)U(x)$ where $U$ has positive real part. However, $U(0) = 1$ by equating coefficients, so by Lemma 2.4, $U \in \mathcal{P}$. Setting $x = z e$, we conclude

\[
\operatorname{Re} \frac{\sum_{n=1}^{\infty} a_n(M) z^{n-1}}{\sum_{n=1}^{\infty} na_n(M) z^{n-1}} > 0 \quad \text{for } z \in \Delta
\]

and, hence, $F_M$ is starlike for each $M \in \mathcal{M}$.

Conversely, if $F_M$ is starlike for every $M \in \mathcal{M}$, then for fixed $x \in B$, the function $V(x, t) = F^{-1}(e^{-t}F(x))$, defined near $t = 0$, satisfies the initial value problem

\[
\frac{\partial V(x, t)}{\partial t} = - \left[ \frac{F(V(x, t))}{V(x, t)F'(V(x, t))} \right] V(x, t), \quad V(x, 0) = x.
\]

Set $P(w) = F(w)/wF'(w)$ for all $w \in B$. By hypothesis, $P \in \mathcal{P}$ and so by Lemma 2.6, $V(x, t)$ is the unique solution of the initial value problem

\[
dw/dt = -wP(w), \quad w(0) = x.
\]

Then $\|V(x, t)\| < \|x\| < 1$ and $F(V(x, t)) = e^{-t}F(x)$, $t > 0$. This implies that $(1 - t)F(B) \subset F(B)$ for $0 < t < 1$. To see the univalence of $F$ in $B$, let $x_1, x_2 \in B$ such that $F(x_1) = F(x_2)$. Suppose $V_x(t) = V(x_1, t)$ is the unique solution of
\[ \frac{dw}{dt} = -wP(w), \quad w(0) = x_i, \]

and let \( W_i(t) = F(V_i(t)), \ i = 1, 2. \) For small \( t > 0, \) \( W_i(t) \) satisfies the initial value problem

\[ \frac{dw}{dt} = -w, \quad w(0) = F(x_i), \]

which has a unique solution \( W_i(t) = F(x_i)e^{-t} \) for \( t > 0. \) Since \( F(x_i) = F(x_2), \) \( W_i(t) = W_2(t) \) for \( t > 0. \) Since \( W_i(t) \to 0 \) as \( t \to +\infty, \) and since \( F \)
has a local inverse in an open neighborhood of 0, \( V_{x_i}(t) = V_{x_2}(t) \) for all \( t > M > 0. \) Then \( V_{x_1}(t) = V_{x_2}(t) \) for all \( t > 0; \) in particular, \( x_1 = V_{x_1}(0) = V_{x_2}(0) = x_2 \) and \( F \)
is univalent in \( B. \)

**Example 1.** Let \( F: B \to R \) be given by \( F(x) = x(1 - ax)^{-2} \) where \( ||a|| < 1. \) Let \( M \in \mathbb{N} \) and set \( a(M) = a. \) Then \( |a| < 1 \) and \( F_M(z) = z/(1 - az)^2, \)
which is known to be starlike. Therefore \( F \) is starlike. If \( X = \{1, 2, \ldots, n\} \) so that \( C(X) = C = \mathbb{R}, \) \( F \) has the form

\[ F(z_1, z_2, \ldots, z_n) = \left( z_1/ (1 - a_1z)^2, \ldots, z_n/ (1 - a_nz)^2 \right) \]

where \( |a_j| < 1, 1 < j < n. \)

If \( R = C[0, 1], \)

\[ F(f)(x) = f(x)/(1 - a(x)f(x))^2, \quad 0 < x < 1. \]

If \( R = H^\infty(\Delta), \)

\[ F(f)(z) = f(z)/(1 - a(z)f(z))^2, \quad |z| < 1. \]

**Example 2.** Other choices for \( F: B \to R \) that will make \( F \) starlike are

\[ F(x) = x + ax^2, \quad a \in R, \ ||a|| < \frac{1}{2}, \]

where each \( a_j > 0 \) and \( \sum_{j=1}^{\infty} a_j < 2 \) with \( ||a_j|| < 1 \) for each \( j. \)

**4. Convex functions.** In \( C, \) if \( f(z) = \sum_{n=1}^{\infty} a_n z^n, a_1 \neq 0, \) is holomorphic in \( \Delta, \)
then \( f \) is convex in \( \Delta \) if \( f(\Delta) \) is a convex domain. This is equivalent to \( \text{Re}(1 + zf''(z)/f'(z)) > 0 \) for all \( z \in \Delta. \) We will define convex functions in \( R \)
and relate them to convex functions in \( C. \)

**Definition 4.1.** A bianalytic map \( F: B \to R \) is said to be convex in \( B \) if \( F(B) \) is a convex domain.

**Theorem 4.2.** Let \( F(x) = \sum_{n=1}^{\infty} a_n x^n \) be locally bianalytic in \( B. \) \( F \) is convex in \( B \) if and only if \( F_M(z) = \sum_{n=1}^{\infty} a_n(M) z^n \) is convex in \( \Delta \) for each \( M \in \mathbb{N}. \) Thus the Alexander relation (\( F \) is convex in \( B \) if and only if \( G \) is starlike in \( B \)
where \( G(x) = xF'(x) \) for all \( x \in B \) holds.
Proof. Assume \( F \) is convex in \( B \). Set \( G(x, t) = \frac{1}{2}(F(e^{it}x) + F(e^{-it}x)) \) and apply Lemma 2.3. Expanding \( F(e^{\pm it}x) \) about \( x \), we have

\[
\frac{1}{2}(F(e^{it}x) + F(e^{-it}x)) = \frac{1}{2} \left[ F(x) + F'(x)(e^{it}x - 1) + \frac{1}{2} F''(x)(e^{it}x - 1)^2 \right] + o(t)
\]

Therefore,

\[
\lim_{t \to 0^+} \frac{G(x, 0) - G(x, t)}{t} = xF'(x) \lim_{t \to 0^+} \frac{1 - \cos t}{t} + x^2 F''(x) \lim_{t \to 0^+} \cos \sqrt{t} \left( \frac{1 - \cos \sqrt{t}}{t} \right)
\]

\[
= \frac{1}{2} \left[ xF'(x) + x^2 F''(x) \right].
\]

Therefore \( F'(x)U(x) = \frac{1}{2} [F'(x) + xF''(x)] \) where \( U \) has positive real part. Equating coefficients, we conclude that \( U(0) = \frac{1}{2} \) and so \( U \in \mathfrak{P} \). This means

\[
\text{Re} \left[ 1 + \frac{\sum_{n=1}^{\infty} n^2 a_n(M) z^{n-1}}{\sum_{n=1}^{\infty} n a_n(M) z^{n-1}} \right] > 0 \quad \text{for all } M \in \mathfrak{M}
\]

so that \( F_M \) is convex in \( \Delta \).

Suppose \( F_M \) is convex in \( \Delta \) for each \( M \in \mathfrak{M} \) and let \( x, y \in B_r = \{ x \in \mathbb{R} \mid \|x\| < r \}, r < 1 \). Since \( F_M \) is univalent, \( F \) is bianalytic in \( B \). Let \( V(t) = F^{-1}(tF(x) + (1 - t)F(y)) \). Then for all \( M \in \mathfrak{M} \),

\[
F_M(V(t)(M)) = tF_M(x(M)) + (1 - t)F_M(y(M)).
\]

Since \( F_M \) is convex in \( |z| < r \), \( |V(t)(M)| < r \). Choose \( M \in \mathfrak{M} \) such that \( \|V(t)\| = |V(t)(M)| < r \) and the convexity of \( F \) follows.

Example. (i) \( x(1 - ax)^{-1} \) when \( \|a\| < 1 \) is convex.

(ii) \( \log((1 + x)(1 - x))^{-1} \) is convex.

5. Close-to-convex functions. In \( \mathbb{C} \), a holomorphic function \( f : \Delta \to \mathbb{C} \) is said to be close-to-convex in \( \Delta \) if there is a convex function \( g : \Delta \to \mathbb{C} \) such that \( \text{Re}(f'(z)/g'(z)) > 0 \) for all \( z \in \Delta \). In [9], it is shown that every close-to-convex function is univalent. We define close-to-convex functions in \( R \) and show that every close-to-convex function in \( B \) is univalent. Compare [13] and [17].
Definition 5.1. Suppose $F: B \to R$ is $L$-analytic in $B$. We say that $F$ is close-to-convex if $F_M: \Delta \to C$ is close-to-convex in $\Delta$ for all $M \in \mathcal{M}$.

Clearly if $G: B \to R$ is convex in $B$, $U \in \mathcal{P}$, and $F'(x) = G'(x)U(x)$ for all $x \in B$, then $F$ is close-to-convex in $B$.

Theorem 5.2. If $D$ is a convex domain in $R$ and $G: D \to R$ is such that $G' \in \mathcal{P}(D)$, then $G$ is univalent in $D$.

Proof. Let $x_1, x_2 \in D$, $x_1 \neq x_2$. Since $D$ is convex, $\{tx_2 + (1 - t)x_1 | t \in I\} \subset D$. We have

$$\frac{d}{dt} G(tx_2 + (1 - t)x_1) = G'(tx_2 + (1 - t)x_1)(x_2 - x_1)$$

so that

$$G(x_2) - G(x_1) = (x_2 - x_1) \int_0^1 G'(tx_2 + (1 - t)x_1) \, dt.$$  

Let $M \in \mathcal{M}$ be such that $\|x_2 - x_1\| = \|(x_2 - x_1)(M)\|$. Then

$$|(G(x_2) - G(x_1))(M)| = \|x_2 - x_1\| \left| \int_0^1 G'(tx_2 + (1 - t)x_1)(M) \, dt \right| \geq \|x_2 - x_1\| \int_0^1 \text{Re} \ G'(tx_2 + (1 - t)x_1)(M) \, dt > 0$$

and hence $G(x_2) \neq G(x_1)$.

Theorem 5.3. If $F$ is close-to-convex in $B$, then $F$ is univalent in $B$.

Proof. If there is a convex function $G: B \to R$ such that $F'(x) = G'(x)U(x)$ for some $U \in \mathcal{P}$, we may apply Theorem 5.2 to $F \circ G^{-1}$: $G(B) \to R$ to conclude that $F$ is univalent.

Otherwise, let $x_1, x_2 \in B$, $x_1 \neq x_2$ and choose $M \in \mathcal{M}$ such that $|(x_2 - x_1)(M)| = \|x_2 - x_1\|$. Since $F_M$ is close-to-convex in $\Delta$, there is a convex function $g: \Delta \to C$ such that $\text{Re}(F_M(z)/g(z)) > 0$ for all $z \in \Delta$. Define $G: B \to R$ by $G(x) = \sum_{k=1}^{\infty} (b_k)x^k$ where $g(z) = \sum_{k=1}^{\infty} b_kz^k$. Then $G$ is convex (in particular, bianalytic) in $B$. Consider $H \equiv F \circ G^{-1}$: $G(B) \to R$ and let $y_1 = G(x_1)$ and $y_2 = G(x_2)$. As in the proof of Theorem 5.2, we have

$$F(x_2) - F(x_1) = H(y_2) - H(y_1) = \int_0^1 H'(ty_2 + (1 - t)y_1)(y_2 - y_1) \, dt$$
so that

\[
|\{F(x_2) - F(x_1)\}(M)| = |\{H(y_2) - H(y_1)\}(M)|
\]

\[
= \left| \int_0^1 H'(y_2 + (1 - t)y_1)(M) \, dt \right| |(y_2 - y_1)(M)|
\]

\[
> \int_0^1 \text{Re} \, H'(y_2 + (1 - t)y_1)(M) \, dt |(y_2 - y_1)(M)|
\]

\[
= \int_0^1 \text{Re} \, \frac{F'(G^{-1}(y_2 + (1 - t)y_1)(M))}{g'(G^{-1}(y_2 + (1 - t)y_1)(M))} \, dt |(y_1 - y_2)(M)| > 0
\]

if \(|y_2 - y_1)(M)| \neq 0\). But

\[
\|x_2 - x_1\| = |(x_2 - x_1)(M)|
\]

\[
= \left| \int_0^1 (G^{-1})'(y_2 + (1 - t)y_1)(M) \, dt \right| |(y_2 - y_1)(M)|
\]

so the desired result follows.

**Example.** (i) \(F(x) = x(1 - ax)(1 - x)^{-2}\) is close-to-convex in \(B\) where \(\|a - \frac{1}{2}\| < \frac{1}{2}\) because \(F_M(z)\) is known to be close-to-convex for every \(M \in \mathfrak{M}\).

(ii) Every starlike function is close-to-convex.

(iii) Every convex function is close-to-convex.

**6. Spirallike functions.** In \(C\), if \(f(z) = \sum_{n=1}^{\infty} a_nz^n\), \(a_1 \neq 0\), is holomorphic in \(\Delta\), then \(f\) is spirallike in \(\Delta\) if \(\text{Re}(e^{-ia}\bar{z}f'(z)/f(z)) > 0\) for all \(z \in \Delta\) where \(a \in (-\pi/2, \pi/2)\). If \(f\) is spirallike in \(\Delta\), then \(f\) is univalent in \(\Delta [14]\). We will define spirallike functions in \(R\) and prove that they are also univalent in \(B\).

**Definition 6.1.** Suppose \(F(x) = \sum_{n=1}^{\infty} a_nx^n\) is locally biaalytic in \(B\). We say that \(F\) is spirallike in \(B\) if there exists \(a \in R\) such that \(\text{Re} \, a(M) > 0\) for all \(M \in \mathfrak{M}\) and \(U \in \mathfrak{P}\) such that

\[
a \frac{F(x)}{x} = F'(x)U(x) \quad \text{for all} \quad x \in B \left( \text{where} \quad \frac{F(x)}{x} = \sum_{n=1}^{\infty} a_nx^{n-1} \right). \quad (1)
\]

From (1), we see that \((F(x)/x)(M) \neq 0\) whenever \(M \in \mathfrak{M}\) and so \(F(x)/x\) and \(a\) are nonsingular. It is clear that (1) can be replaced by the condition \(\text{Re}(bxF'(x)/F(x))(M) > 0\) for all \(M \in \mathfrak{M}\) where \(b = a^{-1}\) and \(xF'(x)/F(x)\) means \((F(x)/x)^{-1}F(x)\).

**Theorem 6.2.** Every spirallike function in \(B\) is univalent in \(B\). Furthermore, if \(F\) is spirallike in \(B\), then \(F_M\) is spirallike in \(\Delta\) for each \(M \in \mathfrak{M}\).
Proof. Since \( F'(x) \) is nonsingular for each \( x \in B \), \( F \) is locally bianalytic in \( B \). For fixed \( x \in B \) and \( t \) near zero, set \( V(x, t) = F^{-1}(e^{-at}F(x)) \). Then \( V(x, t) \) is a solution of the initial value problem
\[
dw/dt = -wP(w), \quad w(0) = x,
\]
where \( P(w) = aF(w)/wF'(w) \). By (1), \( P \in \mathcal{P} \) and so by Lemma 2.6, \( V(x, t) \) is the unique solution for \( t > 0 \) and \( V(x, t) \to 0 \) as \( t \to \infty \). Let \( x_1, x_2 \in B \) such that \( F(x_1) = F(x_2) \) and let \( V_{x_i}(t) = V(x_i, t) \) be the unique solution of the initial value problem
\[
dw/dt = -wP(w), \quad w(0) = x_i, \quad i = 1, 2.
\]
Let \( W_{x_i}(t) = F(V_{x_i}(t)) \) for all \( t > 0, i = 1, 2 \). For small \( t > 0 \), \( W_{x_i}(t) \) satisfies the initial value problem
\[
dw/dt = -aw, \quad w(0) = F(x_i),
\]
which has a unique solution \( W_{x_i}(t) = F(x_i)e^{-at} \) for \( t > 0 \). Since \( F(x_1) = F(x_2) \), \( W_{x_i}(t) = W_{x_i}(t) \) for all \( t > 0 \). Since \( W_{x_i}(t) \to 0 \) as \( t \to +\infty \), we conclude that \( x_1 = x_2 \) as in Theorem 3.2.

That \( F \) is spirallike in \( \Delta \) follows from the equation
\[
a(M)[F(x)/x](M) = F'(x)(M)U(x)(M), \quad \text{and writing } x = z1 \text{ gives } a(M)[F_M(z)/z] = F_M'(z)U_M(z)
\]
where \( Re U_M(z) > 0 \). Write \( a(M) = \alpha e^{i\beta} \) where \( \alpha \in (-\pi/2, \pi/2) \).

Example. Let
\[
F(x) = x(1 - ax)^{-((1 + b))} = \sum_{n=1}^{\infty} \frac{(b + 1)(b + 2 \cdot 1) \cdots (b + n \cdot 1)}{n!} a^n x^n
\]
where \( ||a|| < 1 \), \( ||b|| < 1 \) and \( -1 < Re b(M) < 1 \) for all \( M \in \mathcal{M} \). For example, one might take \( b = \rho e^{ia} \cdot 1 \) where \( \alpha \) is real, \( 0 < |a| < \pi \) and \( 0 < \rho < 1 \). Then

\[
(F(x)/x) \cdot (F'(x))^{-1} \equiv (1 - ax)(1 + abx)^{-1},
\]
and setting \( U(x) = (1 + b)(F(x)/x)(F'(x))^{-1} \) yields

\[
Re U(x)(M) = Re \left[ (1 + b(M))(1 - a(M)x(M))(1 + a(M)b(M)x(M))^{-1} \right].
\]

With \( z = a(M)x(M) \) and \( \beta = b(M) \),
\[
U(x)(M) = (1 + \beta)(1 - z)/(1 + \beta z),
\]
and it is easy to show \( Re U(x)(M) > 0 \).

7. \( \Phi \)-like functions. See [1] for the definitions of a \( \Phi \)-like function and a \( \Phi \)-like domain in \( C \).

Definition 7.1. Let \( F: B \to R \) be a locally bianalytic function in \( B \), and \( F(0) = 0 \). If \( \Phi: F(B) \to R \) is \( L \)-analytic, then we say \( F \) is \( \Phi \)-like in \( B \) if there is \( U \in \mathcal{P} \) such that \( \Phi(F(x)) = xF'(x)U(x) \) for all \( x \in B \).
Since $F(0) = 0$, $\Phi(0) = 0$. Letting $\alpha \to 0$ in $\Phi(F(ax))/\alpha = xF'(ax)U(ax)$, we have $\Phi'(0)x = xU(0)$ for all $x \in B$. Setting $x = \alpha 1$, where $0 < |\alpha| < 1$, we see that $\Phi'(0) = U(0)$.

**Definition 7.2.** Let $D$ be a domain in $R$ which contains 0 and let $P \in \mathcal{P}(D)$. If, for each $\alpha \in D$, the initial value problem

$$\frac{dw}{dt} = -\Phi(w), \quad w(0) = \alpha,$$

where $\Phi(w) = wP(w)$, has a unique solution $w = W(t) \in D$ for all $t > 0$ and $W(t) \to 0$ as $t \to +\infty$, then $D$ is said to be $\Phi$-like.

Note that starlike, convex, close-to-convex and spirallike functions are $\Phi$-like for appropriate choices of $\Phi$.

**Theorem 7.3.** If $F$ is $\Phi$-like in $B$ for $\Phi(v) = vP(v)$ where $P \in \mathcal{P}(F(B))$, then $F$ is univalent in $B$ and $F(B)$ is $\Phi$-like.

**Proof.** The proof follows along the lines of the proof of Theorem 1 in [5].

**Theorem 7.4.** If $F: B \to R$ is bianalytic in $B$ with $F(0) = 0$ and $F(B)$ is $\Phi$-like, then $F$ is $\Phi$-like in $B$.

Since our proof uses Lemma 2.3, and therefore is shorter than Theorem 2 [5], we will give our proof.

**Proof.** Since $F(B)$ is $\Phi$-like, for each $x \in B$, let $W_x(t)$ be the unique solution of $\frac{dw}{dt} = -\Phi(w)$, $w(0) = F(x)$ where $\Phi(w) = wP(w)$, $P \in \mathcal{P}(F(B))$. Since $F$ is bianalytic in $B$, set $V_x(t) = F^{-1}(W_x(t))$ for all $t > 0$. Then $V_x(0) = x$ and

$$F'(V_x(t))V_x'(t) = W_x'(t) = -W_x(t)P(W_x(t)) = -F(V_x(t))P(F(V_x(t)))$$

for all $t > 0$. Letting $t = 0$, we have $-F'(x)V_x'(0) = \Phi(F(x))$. To show that $-V_x'(0) = xU(x)$ for some $U \in \mathcal{P}$, let $G(x, t) = W_x(t) = F(V_x(t))$ in Lemma 3.2. Then

$$\lim_{t \to 0^+} \frac{G(x, 0) - G(x, t)}{t} = \lim_{t \to 0^+} \frac{F(V_x(0)) - F(V_x(t))}{t} = -F'(x)V_x'(0) = xH(x)$$

is $L$-analytic in $B$ and so $H(x) = F'(x)U(x)$ where $U$ has positive real part. Hence $xU(x) = -V_x'(0)$. Since $xF'(x)U(x) = \Phi(F(x))$, we have $U(0) = \Phi(0) = P(0)$, and so $Re U(0)(M) = Re P(0)(M) > 0$ for all $M \in \mathcal{P}$ and, by Lemma 2.4, $U \in \mathcal{P}$.

**8. Convex $F$-holomorphic functions in $C(X)$.** If $D$ is an open set in the Banach space $\mathcal{B}$ and $F: D \to \mathcal{B}$, then $F$ is said to be $F$-holomorphic in $D$ if for each $x \in D$, there is a bounded linear map $DF(x): \mathcal{B} \to \mathcal{B}$ such that

$$\lim_{h \to 0} \frac{\|F(x + h) - F(x) - DF(x)(h)\|}{\|h\|} = 0.$$
such that $F^{-1}: V \to D$ is $F$-holomorphic in $V$, then we say $F$ is locally biholomorphic in $B$. If $F$ is univalent and locally biholomorphic in $B$, we say that $F$ is biholomorphic in $B$. If $F$ is $F$-holomorphic in $D$, then for each $x_0 \in D$ there is a disk in $D$ with center at $x_0$ such that $F(x) = \sum_{n=0}^{\infty} \frac{1}{n!} D^n F(x_0)((x - x_0)^n)$ where $D^n F(x_0) \in L_n(\mathbb{B}, \mathbb{B})$, space of all continuous symmetric $n$-linear maps of $\mathbb{B}$ into $\mathbb{B}$ and the series converges uniformly in this disk. If $F: B \to \mathbb{B}$ is $F$-holomorphic in $B$, then $F(x) = \sum_{n=0}^{\infty} \frac{1}{n!} D^n F(0)(x^n)$ for all $x \in B$ \cite[Theorem 3.16.2]{7}. If $\mathbb{B}$ is a commutative Banach algebra with identity, it is clear that every $L$-analytic function is $F$-holomorphic; however, not every $F$-holomorphic function is $L$-analytic \cite[p. 115]{7}. We will prove that $F$-holomorphic implies $L$-analytic in the special case in which $F$ is a biholomorphic map of the unit ball of $C(X)$ onto a convex domain in $C(X)$.

Assume $X$ is a compact $T_2$-space such that each point of $X$ is a $G_\delta$; i.e., each $x \in X$ is the intersection of a countable number of open neighborhoods of $x$. Let $C(X)$ be the Banach algebra of complex valued continuous functions on $X$ (with sup norm and pointwise multiplication).

**Theorem 8.1.** Let $C(X)$ be as above. If $F: B \to C(X)$ is a convex biholomorphic function in $B$ such that $DF(0) = I$, then $F$ is $L$-analytic in $B$ and hence bianalytic in $B$.

Without loss of generality, we assume $F(0) = 0$. The proof will be given in the following six lemmas.

**Lemma 8.2.** If $k, u \in C(X), k \equiv 0$ on an open neighborhood of $x_0 \in X$, $u(x_0) = 1$ and $|u(x)| < 1$ if $x \neq x_0$ (such a peaking function $u$ exists since every point of $X$ is $G_\delta$), then when $|\alpha| < 1$, we have

$$\left[ DF(\alpha u) \right]^{-1} (D^n F(\alpha u)(k^n))(x_0) = 0 \quad \text{for } n = 2, 3, \ldots .$$

**Proof.** Assume $k \equiv 0$ on the open neighborhood $N$ of $x_0$. Then $N^c$, complement of $N$, is compact, so that for fixed $\alpha, 0 < |\alpha| < 1$, we can choose $r > 0$ (say $r = |\alpha(1 - m)/(||k|| + 1)$ where $m = \sup\{|u(x)| | x \in N^c\} < 1)$ so that

$$||\alpha u + \beta k|| = |(\alpha u + \beta k)(x_0)| = |\alpha| \quad \text{for all } \beta \in \mathbb{C}, |\beta| < r.$$

Define $l \in C(X)^*$ by $l(f) = |\alpha| f(x_0)/\alpha$ for all $f \in C(X)$. Then $l(\alpha u) = |\alpha| = ||\alpha u||$ and $||l|| = 1$. Since $F$ is convex biholomorphic in $B$, we know by Theorem 4 \cite[p. 583]{16} there is a function $w: B \times B \to C(X)$ such that $w$ is $F$-holomorphic in each variable, $w(\alpha u, \alpha u) = 0$, $Re \, l(w(\alpha u, \alpha u + \beta k)) > 0$ if $|\beta| < r$, and

$$F(\alpha u) - F(\alpha u + \beta k) = DF(\alpha u)(w(\alpha u, \alpha u + \beta k)).$$
Expanding $F(\alpha u + \beta k)$ about $\alpha u$, we have

$$F(\alpha u + \beta k) = F(\alpha u) + \sum_{n=1}^{\infty} \frac{\beta^n}{n!} D^n F(\alpha u)(\alpha u)^n$$

so that

$$w(\alpha u, \alpha u + \beta k) - \beta k - 2 \sum_{n=1}^{\infty} \frac{\beta^n}{n!} [DF(\alpha u)]^{-1} D^n F(\alpha u)(\alpha u)^n.$$ 

Applying $l$, we have

$$v(\beta) \equiv lw(\alpha u, \alpha u + \beta k) = -\sum_{n=1}^{\infty} \frac{\beta^n}{n!} l\left([DF(\alpha u)]^{-1} D^n F(\alpha u)(\alpha u)^n\right)$$

is a holomorphic function of $\beta$ for $|\beta| < r$ and Re $v(\beta) > 0$. Since $v(0) = 0$, $v(\beta) \equiv 0$ and Lemma 8.2 follows.

**Lemma 8.3.** If $k, u \in C(X)$, $k(x_0) = 0$ and $u$ is as in Lemma 8.2, then $[DF(\alpha u)]^{-1}(D^n F(\alpha u)(\alpha u)^n)(x_0) = 0$ for $n = 2, 3, \ldots$.

We will prove this without utilizing the $G_\delta$ property. Let $A$ be an index set of the open neighborhoods of $x_0$ and let $A_\alpha$ be the set of functions $u_\alpha k$ where $u_\alpha \in C(X)$, $u_\alpha = 1$ on $U_\alpha^c$, complement of $U_\alpha$, support of $u_\alpha \subseteq \overline{U_\alpha}^c$, $0 < u_\alpha \leq 1$, where $U_\delta$ is an open neighborhood of $x_0$ and $\overline{U_\alpha} \subseteq U_\alpha$. $A_\alpha \neq \emptyset$ by Urysohn’s Lemma. Let $\mathcal{U} = \{A_\alpha | \alpha \in A\}$. It is routine to show that $\mathcal{U}$ is a filterbase in $C(X)$ which converges (in the norm topology) to $k$ [2, p. 211]. Apply Lemma 8.2 to each $u_\alpha k$ and the result follows.

**Lemma 8.4.** If $k, u$ are as in Lemma 8.3, we have

$$[DF(\alpha u)]^{-1} D^2 F(\alpha u)(u, k)(x_0) = 0.$$

**Proof.** First assume $k$ is as in Lemma 8.2. Set $(1 + t^2)^{1/2}g = \alpha(1 + it)u + \beta k$ where $|\beta| = t^{1/2}$, and $t$ is sufficiently small and positive so that

$$\|g\| = \left|\frac{\alpha(1 + it)u(x_0) + \beta k(x_0)}{\sqrt{1 + t^2}}\right| < |\alpha| < 1.$$ 

Let $l$ be the same as in the proof of Lemma 8.2. Since $F$ is convex biholomorphic in $B$, we know again by Theorem 4 [16] that $F(\alpha u) - F(g) = DF(\alpha u)(w(\alpha u, g))$ where Re $l(w(\alpha u, g)) > 0$. Expanding $F(g)$ about $\alpha u$, we have

$$F(g) = F(\alpha u) + \sum_{n=1}^{\infty} \frac{1}{n!} D^n(\alpha u)(g - \alpha u)^n$$

$$= F(\alpha u) + DF(\alpha u)(g - \alpha u)$$

$$+ D^2 F(\alpha u)\left(\frac{i + \alpha u}{\sqrt{1 + t^2}}, \frac{\beta k}{\sqrt{1 + t^2}}\right) + O(t^2)$$

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since \((1 + t^2)^{-1/2} - 1 = O(t^2)\) and \(|\beta| = t^{1/2}\). By Lemma 8.2 we then have
\[
w(au, g) = -(g - au) - \frac{i\alpha u f \beta}{1 + t^2} \left[ DF(au) \right]^{-1} D^2 F(au)(u, k) + O(t^2)
\]
so that
\[
\left| \frac{\alpha}{\alpha} \right| w(au, g)(x_0) = \frac{-i|\alpha|}{\sqrt{1 + t^2}} \sum_{l=0}^{n} \frac{-i|\alpha| \beta}{1 + t^2} \left[ DF(au) \right]^{-1} D^2 F(au)(u, k)(x_0) + O(t^2).
\]
Since \(\Re I(w(au, g)) > 0\), we conclude that
\[
\Re \frac{-i|\alpha| \beta}{1 + t^2} \left[ DF(au) \right]^{-1} D^2 F(au)(u, k)(x_0) > 0.
\]
Since \(\arg \beta\) is arbitrary, we have
\[
\left[ DF(au) \right]^{-1} D^2 F(au)(u, k)(x_0) = 0 \quad \text{if } k \equiv 0
\]
in an open neighborhood of \(x_0\). Now assume \(k \in C(X)\) and \(k(x_0) = 0\). Apply the argument in Lemma 8.3 to obtain the conclusion.

**Lemma 8.5.** If \(k, \alpha\) are as in Lemma 8.3, then
\[
\left[ DF(au) \right]^{-1} D^n F(au)(u^l, k^{n-l})(x_0) = 0
\]
when \(0 < l < n, n = 2, 3, 4, \ldots\).

**Proof.** When \(l = 0\) and \(n > 2\), the result is Lemma 8.3. When \(l = 1\) and \(n = 2\), the result is Lemma 8.4. Assume the result is true for fixed \(l\) and \(n\) and prove it for \(l + 1\) and \(n + 1\). Let \(G = F^{-1}\). Then, for all \(f \in B\), we have \(G \circ F(f) = f\), \(DG(F(f)) \circ DF(f) = I\), identity map from \(C(X)\) into \(C(X)\), and
\[
D^2 G(F(f))(DF(f)(g), DF(f)(h)) + DG(F(f))(D^2 F(f)(g, h)) = 0,
\]
zero map from \(C(X)\) into \(C(X)\), for all \(f \in B\) and all \(g, h \in C(X)\). Hence
\[
D^2 G(F(f))(g, h) = -DG(F(f))\left[ D^2 F(f)\left[ DF(f) \right]^{-1}(g), DF(f)^{-1}(h) \right] = -\left[ DF(f) \right]^{-1}\left[ D^2 F(f)\left[ DF(f) \right]^{-1}(g), DF(f)^{-1}(h) \right].
\]
Define \(H: B \to C_n(C(X), C(X))\), space of all continuous symmetric \(n\)-linear maps of \(C(X)^n\) into \(C(X)\), by
\[
H(f) = \left[ DF(f) \right]^{-1} D^n F(f) = DG(F(f)) D^n F(f) \quad \text{for all } f \in B.
\]
By the induction assumption $H(\alpha u)(u'^l, k^{n-l})(x_0) = 0$ and $H(\alpha u + \epsilon u)(u'^l, k^{n-l})(x_0) = 0$ for small $|\epsilon| > 0$. Hence

$$DH(\alpha u)(u)(u'^l, k^{n-l})(x_0) = \lim_{\epsilon \to 0} \frac{H(\alpha u + \epsilon u)(u'^l, k^{n-l})(x_0) - H(\alpha u)(u'^l, k^{n-l})(x_0)}{\epsilon} = 0.$$ 

On the other hand,

$$DH(f)(u) = D^2G(F(f))(D^nF(f), DF(f)(u)) + DG(F(f))(D(D^nF(f))(u))$$

$$= -[DF(f)]^{-1}[D^2F(f)(H(f), u)] + [DF(f)]^{-1}(D(D^nF(f))(u)).$$

Since $H(\alpha u)(u'^l, k^{n-l}) \in C(X)$ and vanishes at $x_0$, the first term is zero when evaluated at $f = \alpha u$ and $(u'^l, k^{n-l})$ and $x_0$ by Lemma 8.4. Hence

$$0 = DH(\alpha u)(u)(u'^l, k^{n-l})(x_0)$$

$$= [DF(\alpha u)]^{-1}(D(D^nF(\alpha u)(u'^l, k^{n+1-(l+1)}))(u))(x_0)$$

which is the result for $l + 1$ and $n + 1$.

**Lemma 8.6.** Let $u \in C(X)$ such that $u(x_0) = 1$ and $0 < u(x) < 1$ if $x \neq x_0$. If $f \in B$, then $F(f)(x_0) = F(uf)(x_0)$.

**Proof.** If $f(x_0) = 0$, then $(uf)(x_0) = 0$ and

$$F(f)(x_0) = \sum_{n=1}^{\infty} \frac{1}{n!} D^nF(0)(f^n)(x_0) = f(x_0)$$

since $f$ plays the role of $k$ in Lemma 8.3. Similarly, $F(uf)(x_0) = (uf)(x_0)$.

Assume $f(x_0) \neq 0$. Let $N = \{x \in X|u(x) < |f(x_0)|/2||f||\}$ and set $v(x) = u(x)$ if $x \in N$ and $v(x) = \min(u(x), |f(x_0)|/|f(x)|)$ if $x \in N^c$. Then $v \in C(X)$ and $|(vf)(x)| < \frac{1}{2}|f(x_0)|$ if $x \in N$ and $|(vf)(x)| = \min(u(x), |f(x)|)$ if $x \in N^c$. Therefore $vf/(f(x_0))$ plays the role of $u$ in Lemma 8.5. Setting $\alpha = 0$, in Lemma 8.5, we have, for all nonnegative integers $l < n$,

$$D^nF(0)((vf/f(v_0))', ((1 - v)f)^{n-l})(x_0) = 0$$

and hence

$$D^nF(0)((vf)'', ((1 - v)f)^{n-l})(x_0) = 0.$$
Then
\[ F(f)(x_0) = F(\psi f + (1 - \nu)f)(x_0) \]
\[ = \sum_{n=1}^{\infty} \frac{1}{n!} D^n F(0)((\psi f + (1 - \nu)f)^n)(x_0) \]
\[ = \sum_{n=1}^{\infty} \frac{1}{n} \sum_{i=0}^{n} \frac{1}{i!(n-i)!} D^n F(0)((\psi f)'((1 - \nu)f)^n-i)(x_0) \]
\[ = \sum_{n=1}^{\infty} \frac{1}{n!} D^n F(0)((\psi f)^n)(x_0) = F(\psi f)(x_0). \]

If we use \( uf \), instead of \( f \), the same \( \nu \) works for \( uf \) and we have \( F(uf)(x_0) = F(vuf)(x_0). \) Since \( uu \) can be used instead of \( \nu \) for \( f \), we have \( F(f)(x_0) = F(vuuf)(x_0) \) and hence \( F(uf)(x_0) = F(uf)(x_0). \)

**Lemma 8.7.** If \( f, g \in B \) such that \( f(x_0) = g(x_0) \), then \( F(f)(x_0) = F(g)(x_0). \)

**Proof.** Let \( \epsilon > 0 \) be given. Since \( F \) is continuous at \( f \), there is \( \delta > 0 \) such that \( \|F(f) - F(h)\| < \epsilon \) if \( \|f - h\| < \delta \). Let \( N_1 = \{x \in X | f(x) - g(x) < \delta \} \) and let \( N \) be an open neighborhood of \( x_0 \) such that \( \overline{N} \subset N_1 \). Since \( x_0 \) is a \( G_\delta \), there is \( u \in C(X) \) such that \( u(x_0) = 1, 0 < u(x) < 1, \) if \( x \neq x_0 \), and \( u \equiv 0 \) on \( N_1 \). By Urysohn's Lemma, there is \( v \in C(X) \) such that \( 0 < v(x) < 1 \) for all \( x \in X \), \( v \equiv 1 \) on \( N \) and \( v \equiv 0 \) on \( N_1 \). Set \( h = vg + (1 - \nu)f \). Then \( \|f - h\| < \delta \) and \( ug = uh \). Therefore, by Lemma 8.6,
\[ |F(f)(x_0) - F(g)(x_0)| = |F(f)(x_0) - F(ug)(x_0)| \]
\[ = |F(f)(x_0) - F(ug)(x_0)| \]
\[ = |F(f)(x_0) - F(h)(x_0)| < \|F(f) - F(h)\| < \epsilon. \]

Since \( \epsilon \) can be made arbitrarily small, \( F(f)(x_0) = F(g)(x_0). \)

We can now prove Theorem 8.1.

**Proof.** Let \( f, g \in B \) and \( x_0 \in X \). For small \( |\alpha| \), we have that \( f + \alpha g \), \( f + \alpha g(x_0)l \in C(X) \) and agree at \( x_0 \); therefore, by Lemma 8.7, \( F(\alpha f + \alpha g)(x_0) = F(f + \alpha g(x_0)l). \) Therefore,
\[ DF(f)(g)(x_0) = \lim_{\alpha \to 0} \frac{F(f + \alpha g)(x_0) - F(f)(x_0)}{\alpha} \]
\[ = \lim_{\alpha \to 0} \frac{F(f + \alpha g(x_0)l)(x_0) - F(f)(x_0)}{\alpha} \]
\[ = DF(f)(g(x_0)l)(x_0) = g(x_0)DF(f)(1)(x_0). \]

Since \( x_0 \) is arbitrary, we have \( DF(f)(g) = gDF(f)(1); \) i.e., \( F \) is \( L \)-analytic in \( B \) and \( D'(f) = DF(f)(1). \)
Remark 1. The normalization $DF(0) = I$ is necessary in Theorem 8.1 as is seen in the following example. Define $F: C[0, 1] \to C[0, 1]$ by $F(f)(x) = (x + \frac{1}{2})f(x) + (\frac{1}{2} - x)f(x + \frac{1}{2})$ if $x \in [0, \frac{1}{2})$ and $F(f)(x) = f(x)$ if $x \in [\frac{1}{2}, 1]$. Then $F$ is a continuous linear map of $C[0, 1]$ onto $C[0, 1]$ and so $F$ is a convex biholomorphic function in $B$. But $F$ is not $L$-analytic and $DF(0) = F \neq I$.

Without the normalization, we have

**Corollary.** If $F$ is biholomorphic in $B$, and $F(B)$ is convex then $F = L \circ G$ where $L$ is a univalent affine map of $C(X)$ onto $C(X)$ and $G$ is bianalytic in $B$.

**Proof.** Define $L(f) = F(0) + DF(0)(f)$ for all $f \in B$ and $G = L^{-1} \circ F$ satisfies Theorem 8.1.

Remark 2. The proof of Theorem 8.1 depends on the existence of a peaking function $u$ at each point $x \in X$. In general, an arbitrary compact $T_\alpha$-space $X$ does not have peaking functions at each point; for example, if $X$ is the set of all ordinals which are less than or equal to the first uncountable ordinal with the order topology, then the first uncountable ordinal is not a $G_\delta$ point. See [10, Exercises 1.1, 5.C, and 4.J]. It would be interesting to know if Theorem 8.1 is true for an arbitrary compact $T_\alpha$-space.

Note that Theorem 8.1 contains Theorem 3 of [15]. To see this, take $X = \{1, 2, \ldots, n\}$ with the discrete topology. Also compare Theorem 8 of [16].

9. Example and a remark. We now give an example of a function which is univalent and $F$-holomorphic in $B$ such that $F^{-1}$ is not $F$-holomorphic in $F(B)$, $F(B)$ contains an open set, but $F(B)$ is not open. The example is in the Banach algebra $H^\infty = \{f|f: \Delta \to C\text{ is holomorphic in }\Delta \text{ and } \sup\{|f(z)| | z \in \Delta\} < \infty\}$ [3], [8]. Define $F: B \to H^\infty$ by $F(f) = f + af^2$ for nonconstant $a \in H^\infty$ such that for some $z_0 \in \Delta$, $|a(z_0)| = \frac{1}{2}$. To show that $F$ is univalent in $B$, suppose that $f, g \in B$ such that $F(f) = F(g)$. Then $(f - g)(1 + a(f + g)) = 0$ which implies that $f(z) = g(z)$ or $f(z) + g(z) = -1/a(z)$ for each $z \in \Delta$. We claim the first equation always holds. By hypothesis, there is $z_0 \in \Delta$ such that $|a(z_0)| = \frac{1}{2}$. Since $f, g \in B$, there is $\delta > 0$ such that $|f(z)| < 1 - \delta$ and $|g(z)| < 1 - \delta$ in some open neighborhood of $z_0$. The second equation implies $1/|a(z)| < 2 - 2\delta$ in this neighborhood, hence $|a(z_0)| > 1/(2 - 2\delta) > \frac{1}{2}$, which is a contradiction. Hence $f(z) = g(z)$ in this open neighborhood of $z_0$ and therefore $f = g$.

To prove that $F^{-1}$ is not $F$-holomorphic in $F(B)$, observe that $F'(f) = 1 + 2af$ and so $F'(f)$ is nonsingular as long as $1 + 2a(z)f(z) \neq 0$ for $z \in \Delta$. If $|a(z_0)| > \frac{1}{2}$, let $f$ be the constant function $-1/2a(z_0)$. Then $\|f\| < 1$ and $F'(f)$ is singular. It follows that $F^{-1}$ is not $F$-holomorphic in $F(B)$.

$F(B)$ contains an open set since the equation $\epsilon = f + af^2$ has the solution
\[ f = \frac{-1 + \sqrt{1 + 4ae}}{2a} = \frac{2e}{(1 + \sqrt{1 + 4ae})} \text{ where } e \in H^\infty \text{ and } ||e|| \text{ is small.} \]

To prove that \( F(B) \) is not open, suppose \( |a(z_1)| > \frac{1}{2} \) and let \( f \) be the constant function \( -1/2a(z_1) \). Then \( F(f) = a/4a^2(z_1) - 1/2a(z_1) \), and if \( e \in H^\infty, ||e|| \text{ small,} \) we have \( a/4a^2(z_1) - 1/2a(z_1) + e = f + af^2 \) for some \( f \in B \) if and only if \( f = -1/2a(z_1) + \delta \) where \( \delta \in H^\infty \) and \( e(z) = \delta(z)(1 - a(z)/a(z_1)) + a(z)\delta^2(z) \). Let \( e \) have the property that \( e(z_1) = 0 \). Then \( 0 = e(z_1) = a(z_1)\delta^2(z_1) \) so we conclude \( \delta(z_1) = 0 \). Since \( e(z) = \delta(z)(1 - a(z)/a(z_1)) + a(z)\delta^2(z) \), it follows that \( e \) has at least a double zero at \( z_1 \). This means that any function of the form \( a/4a^2(z_1) - 1/2a(z_1) + e \) such that \( e(z_1) = 0 \) and \( e'(z_1) \neq 0 \) cannot lie in \( F(B) \).

This example shows that Theorem 5 [5] is false. \( F \) is univalent and \( F \)-holomorphic in \( B \) with \( F(0) = 0 \) and \( DF(0) = I \), but \( F(B) \) is not open. Hence \( F \) is not locally biholomorphic in \( B \) and therefore \( F \) is not \( \Phi \)-like for any function \( \Phi \).

**References**


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