CR SUBMANIFOLDS OF A KAehler MANIFOLD. II

BY

AUREL BEJANCU

ABSTRACT. The differential geometry of CR submanifolds of a Kähler manifold is studied. Theorems on parallel normal sections and on a special type of flatness of the normal connection on a CR submanifold are obtained. Also, the nonexistence of totally umbilical proper CR submanifolds in an elliptic or hyperbolic complex space is proven.

1. Introduction and basic formulas. A study of the differential geometry of CR submanifolds of a Kähler manifold has been initiated in [2]. Some problems, mostly concerning integrability conditions on such submanifolds have been investigated by S. Ishihara and K. Yano [6]. Also, results on the general theory of Cauchy-Riemann manifolds have been obtained by A. Andreotti and C. D. Hill [1], R. Nirenberg and R. O. Wells [8], V. Oproiu [7], and others.

The purpose of the present paper is to study further CR submanifolds by the method which has been used in [2]. First, in §2 we state theorems on parallel normal sections on a CR submanifold and some lemmas for later use. A special type of flatness for the normal connection of a CR submanifold is studied in §3. Finally, in §4 we give some characterizations for a class of CR submanifolds, and the nonexistence of totally umbilical CR submanifolds in an elliptic or hyperbolic complex space is proven. In this paragraph we give the basic definitions and formulas.

Let \( \tilde{M} \) be a Kähler manifold of complex dimension \( n \) and \( M \) be a submanifold of \( \tilde{M} \) of real dimension \( m \). The submanifold \( M \) is supposed to be endowed with two complementary orthogonal distributions \( D \) and \( D^\perp \) of real dimensions \( 2p \) and \( q \) respectively. The first one is called the horizontal distribution and it is invariant by the almost complex structure \( J \) on \( \tilde{M} \): that is, \( J(D_x) = D_x \) for each \( x \in M \). The second one is called the vertical distribution and it is anti-invariant by \( J \); that is, \( J(D_x) \subset v_x \) where \( v_x \) is the normal space to \( M \) at \( x \).

The submanifold \( M \) endowed with the pair of distributions \( (D, D^\perp) \) is called a CR submanifold of \( \tilde{M} \) [2]. It is easily seen that each real hypersurface of \( \tilde{M} \) is a CR submanifold.
Remark. Of course, the definition of a CR submanifold might be considered for submanifolds of an almost Hermitian manifold [3], but we are concerned, in this paper, only with problems on CR submanifolds of a Kaehler manifold.

The Kaehlerian metric on $\tilde{M}$ and the Riemannian metric induced by it on $M$ will be denoted by the same symbol $g$. The Kaehlerian connection on $\tilde{M}$ is denoted by $\tilde{\nabla}$, the Levi-Civita connection on $M$ is denoted by $\nabla$ and by $\nabla^\perp$ is denoted the linear connection induced by $\tilde{\nabla}$ on the normal bundle $\nu$. Then the equations of Gauss and Weingarten are

\[
\tilde{\nabla}_X Y = \nabla_X Y + h(X, Y),
\]

\[
\tilde{\nabla}_X N = -A_N X + \nabla^\perp_X N,
\]

for any vector fields $X, Y$ tangent to $M$ and any vector field $N$ in the normal bundle $\nu$. We put

\[
JN = BN + CN,
\]

where $BN$ and $CN$ are the vertical and the normal component of $JN$ respectively.

The second fundamental form $h$ of $M$ verifies

\[
g(h(X, Y), N) = g(A_N X, Y).
\]

Our study is made on the domain of a local chart of coordinates on $M$. For any vector bundle $S \rightarrow M$ on $M$, the module of its local sections is denoted by $\mathcal{S}(S)$, without specifying the local chart. Also, throughout the paper, all the manifolds and mappings are supposed to be differentiable of class $C^\infty$.

The curvature tensor of type $(1, 3)$ of $M$ is given by

\[
R(X, Y) = \left[\nabla_X, \nabla_Y\right] - \nabla_{[X, Y]}.
\]

Also, we shall use the curvature tensor of type $(0, 4)$ given by

\[
R(X, Y; Z, W) = g(R(X, Y)Z, W).
\]

Let $\{E_1, \ldots, E_m\}$ be a local orthonormal frame on $M$. Then, the Ricci tensor $S$ of $M$ is given by

\[
S(X, Y) = \sum_{i=1}^{m} \{R(E_i, X; Y, E_i)\}
\]

where $X, Y$ are vector fields tangent to $M$.

Of course, similar formulas are given for the curvature tensor $\tilde{R}$ and the Ricci tensor $\tilde{S}$ of $\tilde{M}$.

Denote by $R^\perp$ the curvature tensor of the normal connection $\nabla^\perp$. A local normal vector field $N \neq 0$ is said to be a $D$-parallel normal section if $\nabla^\perp_X N = 0$ for each $X \in \mathcal{S}(\nu)$.
The equations of Gauss, Codazzi and Ricci are given respectively by [4, p. 46]

\[ \tilde{R}(X, Y; Z, W) = R(X, Y; Z, W) + g(h(X, Z), h(Y, W)) \]
\[ - g(h(Y, Z), h(X, W)) \]  
\[ (1.5) \]
\[ [\tilde{R}(X, Y)Z]_X - (V_Xh)(Y, Z) - (V_Yh)(X, Z), \]  
\[ (1.6) \]
\[ \tilde{R}(X, Y; N, \bar{N}) = R(X, Y; N, \bar{N}) - g([A_N, A_{\bar{N}}](X), Y) \]  
\[ (1.7) \]
for any vector fields \( X, Y, Z, W \) tangent to \( M \) and normal vector fields \( N, \bar{N} \).

The left-hand side of (1.6) is the normal component of \( \tilde{R}(X, Y)Z \) and \( \nabla_Xh \)
from the right-hand side is given by

\[ (\nabla_Xh)(Y, Z) = \nabla_X^h(h(Y, Z)) - h(V_XY, Z) - h(Y, WXZ). \]  
\[ (1.8) \]

Now, suppose the distributions \( D \) and \( D^\perp \) are given by the projectors \( P \) and \( Q \) respectively. Then, since \( \tilde{V} \) is a Kaehlerian connection, from the equations of Gauss and Weingarten, by comparing horizontal, vertical and normal parts, we obtain respectively

\[ P(\nabla_XJPY) - P(A_{JQY}X) = JP(\nabla_XY), \]  
\[ (1.9) \]
\[ Q(\nabla_XJPY) - Q(A_{JQY}X) = Bh(X, Y), \]  
\[ (1.10) \]
\[ h(X, JPY) + V_pQY = JQ(V_XY) + Ch(X, Y) \]  
\[ (1.11) \]
for any vector fields \( X, Y \) tangent to \( M \).

2. **D-parallel normal sections on a CR submanifold.** Let \( M \) be a CR submanifold of a Kaehler manifold \( \tilde{M} \). As it was supposed in §1, \( JD^\perp \) is a vector subbundle of the normal bundle \( \nu \). The complementary orthogonal vector bundle of \( JD^\perp \) in \( \nu \) will be denoted by \( \mu \). It is easily seen that \( \mu \) is invariant by the almost complex structure \( J \).

The CR submanifold \( M \) is said to be **mixed totally geodesic** if \( h(X, Y) = 0 \) for any vector fields \( X \in \mathcal{S}(D) \) and \( Y \in \mathcal{S}(D^\perp) \).

**Lemma 2.1.** A CR submanifold \( M \) of a Kaehler manifold \( \tilde{M} \) is mixed totally geodesic if and only if \( A_NX \in \mathcal{S}(D) \) for each \( X \in \mathcal{S}(D) \) and \( N \in \mathcal{S}(\nu) \).

**Proof.** If \( M \) is mixed totally geodesic, then by using (1.4) we have \( g(A_NX, Y) = 0 \) for each \( X \in \mathcal{S}(D), \ Y \in \mathcal{S}(D^\perp) \) and \( N \in \mathcal{S}(\nu) \). This implies \( A_NX \in \mathcal{S}(D) \).

Conversely, suppose \( A_NX \in \mathcal{S}(D) \) for any \( X \in \mathcal{S}(D) \) and \( N \in \mathcal{S}(\nu) \). Let \( \{N_1, \ldots, N_{2n-m}\} \) be a local orthonormal field of frames on \( \nu \). Then we have

\[ 0 = g(A_NX, Y) = g(h(X, Y), N_\alpha), \quad 1 < \alpha < 2n - m. \]

Since \( h(X, Y) \in \mathcal{S}(\nu) \), from the relations above we have \( h(X, Y) = 0 \). Therefore \( M \) is mixed totally geodesic.
Lemma 2.2. Let $M$ be a mixed totally geodesic CR submanifold of a Kaehler manifold $\tilde{M}$. Then we have

$$A_{JN}X = JAX$$

(2.1)

for any $X \in \mathcal{S}(D)$ and $N \in \mathcal{S}(v)$.

Proof. Since $\tilde{\nabla}$ is a Kaehlerian connection, from the equation of Weingarten we get

$$JAX - J\nabla^\perp_X N = A_{JN}X - \nabla^\perp_X JN.$$  

(2.2)

From Lemma 2.1 the first terms from both sides of (2.2) belong to $\mathcal{S}(D)$. On the other hand, $\nabla^\perp_X JN \in \mathcal{S}(v)$ and $J\nabla^\perp_X N \in \mathcal{S}(D \perp \mu)$. Hence (2.1) follows from (2.2) by comparing the horizontal parts of both sides.

Also, from (2.2) we obtain

Corollary 2.1. If $M$ is a mixed totally geodesic CR submanifold of a Kaehler manifold $M$, then we have

$$J\nabla^\perp_X N = \nabla^\perp_X JN$$

(2.3)

and

$$\nabla^\perp_X N \in \mathcal{S}(\mu)$$

(2.4)

for any vector fields $X \in \mathcal{S}(D)$ and $N \in \mathcal{S}(v)$.

If the horizontal distribution is involutive, then $M$ will be called a foliate CR submanifold of $M$. We have proved [2]

Proposition 2.1. A CR submanifold $M$ of a Kaehler manifold $\tilde{M}$ is foliate if and only if the second fundamental form $h$ satisfies

$$h(X, JY) = h(JX, Y)$$

(2.5)

for each $X, Y \in \mathcal{S}(D)$.

Now we prove

Lemma 2.3. Let $M$ be a foliate CR submanifold of a Kaehler manifold $\tilde{M}$. If $M$ is mixed totally geodesic, then we have

$$JAX = -A_NJX$$

(2.6)

for any $X \in \mathcal{S}(D)$ and $N \in \mathcal{S}(v)$.

Proof. From (1.4) and (2.5) we have

$$g(JAX, Y) = -g(h(X, JY), N) = -g(h(JX, Y), N) = -g(A_NJX, Y)$$

(2.7)

for any vector fields $X, Y \in \mathcal{S}(D)$ and $N \in \mathcal{S}(v)$. Thus (2.6) follows from (2.7) by using Lemma 2.1.
The holomorphic bisectional curvature for the pair of vector fields \((X, Y)\) on \(\tilde{M}\) is given by

\[
H(X, Y) = \tilde{R}(X, JX; JY, Y)/g(X, X)g(Y, Y).
\]

Now we can state the following theorem which has been proved by B. Y. Chen and H. S. Lue for complex submanifolds of a Kaehler manifold [5].

**Theorem 2.1.** Let \(M\) be a mixed totally geodesic foliate CR submanifold of a Kaehler manifold \(\tilde{M}\). If there exists a unit vector field \(X \in \mathcal{S}(D)\) such that for all normal sections \(N \in \mathcal{S}(\mu)\), the holomorphic bisectional curvatures \(H(X, N)\) are positive, then the normal subbundle \(\mu\) does not admit \(D\)-parallel section.

**Proof.** Suppose \(N\) is a parallel section of \(\mu\). Then \(R^\perp(X, Y)N = 0\) for each \(X, Y \in \mathcal{S}(D)\). Hence, by using the equation of Ricci we have

\[
\tilde{R}(X, Y; N, JN) = -g\left([A_N, A_JN](X), Y\right). \tag{2.8}
\]

Next, by (2.1) and (2.6) the second-hand side of (2.8) becomes \(2g(JA_N^2X, Y)\).

On the other hand, the hypothesis on the holomorphic bisectional curvatures and (2.8) imply

\[
0 > \tilde{R}(X, JX; N, JN) = 2g(A_N^2X, X).
\]

This is clearly impossible, since \(g\) is positive definite.

Now we give a characterization for the parallel normal sections which belong to the normal subbundle \(JD^\perp\).

**Theorem 2.2.** Let \(M\) be a mixed totally geodesic CR submanifold of a Kaehler manifold \(\tilde{M}\). Then the normal section \(N \in \mathcal{S}(JD^\perp)\) is \(D\)-parallel if and only if \(\nabla_X N \in \mathcal{S}(D)\) for each vector field \(X \in \mathcal{S}(D)\).

**Proof.** Let \(Y \in \mathcal{S}(D^\perp)\) be such that \(JY = N\). Then from (1.9) we have

\[
P A_N X = -JP(\nabla_X Y)
\]

for each \(X \in \mathcal{S}(D)\). From Lemma 2.1 we have \(A_N X \in \mathcal{S}(D)\). Hence the relation above becomes

\[
A_N X = -JP(\nabla_X Y). \tag{2.9}
\]

Now, from the equations of Gauss and Weingarten, by using \(\tilde{\nabla} J = 0\) and (2.9) we obtain

\[
\nabla^\perp_X N = JQ(\nabla_X Y) + Ch(X, Y). \tag{2.10}
\]

Since \(M\) is mixed geodesic, from (2.10) we have

\[
\nabla^\perp_X N = -JQ(\nabla_X JN), \tag{2.11}
\]

which clearly proves the theorem.

3. **The normal connection of a CR submanifold.** The normal connection \(\nabla^\perp\) of the CR submanifold \(M\) is called \((D, \mu)\)-flat, if the restriction of the
curvature tensor $R^\perp$ of $\nabla^\perp$ to $\mathbb{S}(D) \times \mathbb{S}(D) \times \mathbb{S}(\mu)$ vanishes, i.e.

$$R^\perp(X, Y)N = [\nabla^\perp_X, \nabla^\perp_Y](N) - \nabla^\perp_{[X, Y]}N = 0 \quad (3.1)$$

for each $X, Y \in \mathbb{S}(D)$ and $N \in \mathbb{S}(\mu)$.

The theorem which follows establishes a link between the $(D, \mu)$-flatness of the normal connection and the existence of $D$-parallel normal sections from $\mu$. Suppose the fibre of the vector bundle $\mu$ at each point of $M$ is of complex dimension $s > 1$.

**Theorem 3.1.** Let $M$ be a mixed totally geodesic foliate CR submanifold of a Kaehler manifold $M$. Then, the normal connection is $(D, \mu)$-flat if and only if there exist locally $2s$ mutually orthogonal unit normal vector fields $N_a \in \mathbb{S}(\mu)$ such that each of the $N_a$ is $D$-parallel in the normal subbundle $\mu$.

**Proof.** Suppose the normal connection $\nabla^\perp$ is $(D, \mu)$-flat. Take a point $x \in M$ and denote by $M_x$ the maximal integral manifold of $D$ passing through $x$. For each vector field $X$ on $M_x$ put

$$\nabla^\perp_x N_a = \omega^b_a(X)N_b.$$  

This is possible by virtue of (2.4). Then, from (3.1) we get

$$d\omega^b_a = -\omega^c_b \wedge \omega^b_c; \quad \omega^b_a + \omega^b_b = 0.$$  

Then we can find an $(s \times s)$-nonsingular matrix $A = (A^b_a)$ of functions such that $dA^b_a = -A^c_b \omega^b_c$. Now, we take $N'_a = A^b_a N_b$. Of course, $N'_a$ are also $2s$ mutually orthogonal unit normal vector fields from $\mu$. Moreover, if we put

$$\nabla^\perp_x N'_a = \omega^b_a(X)N'_b,$$

then we have

$$\omega^c_a A^b_c = dA^b_a + A^c_a \omega^b_c = 0,$$

which implies $\omega^c_a = 0$ for all indices $a$ and $c$. Hence $N'_a$ are $2s$ $D$-parallel orthonormal vector fields from the normal subbundle $\mu$.

Conversely, suppose that there exist locally $2s$ mutually orthogonal unit normal vector fields $N_a$ such that each $N_a$ is $D$-parallel in the normal subbundle $\mu$. Let $N$ be a certain normal vector field from $\mu$. Since $R^\perp(X, Y)N_a = 0$, by a simple computation we get $R^\perp(X, Y)N = 0$ for any $X, Y \in \mathbb{S}(D)$. Thus the normal connection is $(D, \mu)$-flat and the proof is complete.

Let $\{F_1, \ldots, F_q\}$ be a local field of frames on the vertical distribution. The CR submanifold $M$ is said to be $D^\perp$-minimal, if the second fundamental form $h$ of $M$ satisfies

$$\sum_{k=1}^q \{h(F_k, F_k)\} = 0. \quad (3.2)$$

---

The indices $a, b, c, \ldots$ run over the range $1, 2, \ldots, 2s$, and we use Einstein’s convention on summing indices.
The definition above does not depend on the local field of frames \( \{ F_1, \ldots, F_q \} \). In fact, the second fundamental form \( h \) can be written as
\[
h(X, Y) = h^\alpha(X, Y)N_\alpha, \quad \alpha = 1, \ldots, 2n - m,
\]
where \( \{ N_1, \ldots, N_{2n-m} \} \) is a local field of orthonormal frames in the normal bundle. Denote by \( h^\alpha \) the restriction of \( h^\alpha \) to \( S(D^+) \times S(D^+) \). Then, the condition (3.2) is equivalent to
\[
\text{trace } h^\alpha = 0, \quad \text{for each } \alpha = 1, \ldots, 2n - m. \quad (3.3)
\]
Now it is easily seen from (3.3) that the definition of a \( D^\perp \)-minimal \( CR \) submanifold does not depend on the basis. In the same way, we say that \( M \) is \( D\)-minimal, if the second fundamental form satisfies
\[
\sum_{i=1}^{2p} \{ h(E_i, E_i) \} = 0, \quad (3.4)
\]
where \( \{ E_1, \ldots, E_{2p} \} \) is a local field of orthonormal frames on the horizontal distribution.

**Theorem 3.2.** Let \( M \) be a mixed totally geodesic foliate \( CR \) submanifold of a Kaehler manifold \( \tilde{M} \). If the normal connection is \( (D, \mu) \)-flat and \( M \) is \( D^\perp \)-minimal, then the Ricci tensors \( S \) and \( \tilde{S} \) of \( M \) and respectively \( \tilde{M} \) satisfy the relation
\[
S(X, Y) = \tilde{S}(X, Y) - \sum_{k=1}^{q} \left\{ g(P\nabla_x F_k, P\nabla_y F_k) + \tilde{R}(JF_k, X; Y, JF_k) \right\},
\]
for each \( X, Y \in S(D) \) and for any local field of orthonormal frames \( \{ F_1, \ldots, F_q \} \) on the vertical distribution.

**Proof.** Let
\[
\{ E_1, \ldots, E_{2p}, F_1, \ldots, F_q, JF_1, \ldots, JF_q, N_1, \ldots, N_s, JN_1, \ldots, JN_s \}
\]
be a local field of orthonormal frames of \( \tilde{M} \) such that \( \{ E_1, \ldots, E_{2p} \} \), \( \{ F_1, \ldots, F_q \} \) and \( \{ N_1, \ldots, N_s, JN_1, \ldots, JN_s \} \) are local fields of frames of the horizontal distribution, vertical distribution and of the normal subbundle \( \mu \) respectively. Moreover, the vector fields \( \{ N_1, \ldots, N_s \} \) are taken as \( D \)-parallel vector fields from those whose existence has been proven by Theorem 3.1. From (2.3) we see that \( \{ JN_1, \ldots, JN_s \} \) are also \( D \)-parallel normal sections. Next, by using the definition of Ricci tensor and the equation of Gauss, we get
\[ S(X, Y) = \tilde{S}(X, Y) + g\left(h(X, Y), \sum_{i=1}^{2p} \{h(E_i, E_i)\} + \sum_{k=1}^{q} \{h(F_k, F_k)\}\right) \]

\[ - \sum_{i=1}^{2p} \{g(h(E_i, X), h(E_i, Y))\} \]

\[ - \sum_{k=1}^{q} \{g(h(F_k, X), h(F_k, Y)) + \tilde{R}(JF_k, X; Y, JF_k)\} \]

\[ - \sum_{\alpha=1}^{s} \{\tilde{R}(N_{\alpha}, X, Y, N_{\alpha}) + R(JN_{\alpha}, X; Y, JN_{\alpha})\} \quad (3.6) \]

for each \( X, Y \in \mathcal{S}(D) \).

Since the horizontal distribution is involutive, \( M \) is \( D \)-minimal [2]. On the other hand, \( M \) is supposed to be \( D^\perp \)-minimal. Hence the second term from the right-hand side of (3.6) vanishes.

Now, from (1.4) and by using Lemmas 2.1 and 2.2, we get

\[ \sum_{k=1}^{q} \{g(h(E_i, X), h(E_i, Y))\} = 2 \sum_{\alpha=1}^{s} \{g(A_{N_{\alpha}}X, A_{N_{\alpha}}Y)\}. \]

We make use again of the equations of Gauss and Weingarten and obtain

\[ A_{JF_k}X = -JF_k \nabla_X F_k, \] which transforms the relation above into

\[ \sum_{i=1}^{2p} \{g(h(E_i, X), h(E_i, Y))\} \]

\[ = \sum_{k=1}^{q} \{g(\nabla_X F_k, P \nabla_X F_k)\} + 2 \sum_{\alpha=1}^{s} \{g(A_{N_{\alpha}}X, A_{N_{\alpha}} Y)\}. \quad (3.7) \]

For the last sum of the right-hand side of (3.6) we have the following evaluations:

\[ \sum_{\alpha=1}^{s} \{\tilde{R}(N_{\alpha}, X, Y, N_{\alpha}) + \tilde{R}(JN_{\alpha}, X; Y, JN_{\alpha})\} \]

\[ = - \sum_{\alpha=1}^{s} \{\tilde{R}(X, JY; N_{\alpha}, JN_{\alpha})\} = -2 \sum_{\alpha=1}^{s} \{g(A_{N_{\alpha}}X, A_{N_{\alpha}} Y)\}. \quad (3.8) \]

Finally, we see that (3.5) follows from (3.6) by using (3.7), (3.8) and the fact that \( M \) is mixed totally geodesic. Thus the proof is done.

The vertical distribution is said to be parallel along the horizontal distribution if we have \( \nabla_X Y \in \mathcal{S}(D) \perp \) for each \( X \in \mathcal{S}(D) \) and \( Y \in \mathcal{S}(D^\perp) \).
Corollary 3.1. Let $M$ be a CR submanifold of the Kaehler manifold $\tilde{M}$ under the conditions of Theorem 3.2. Suppose the following two conditions are satisfied:

1. The vertical distribution is parallel along the horizontal distribution.
2. The curvature tensor $R$ satisfies $R(Z, X)Y \in \mathfrak{S}(D)$, for any $X, Y \in \mathfrak{S}(D)$ and $Z \in \mathfrak{S}(D^\perp)$.

Then, the Ricci tensors $S$ and $\tilde{S}$ satisfy

$$S(X, Y) = \tilde{S}(X, Y), \quad (3.9)$$

for any $X, Y \in \mathfrak{S}(D)$.

Proof. From condition (1) of the corollary we see that $PV_XF_k = 0$ for each $X \in \mathfrak{S}(D)$ and $1 \leq k \leq q$. Hence the first $q$ terms from the sum of the right-hand side of (3.5) vanish. It is known that the curvature tensor $\tilde{R}$ of the Kaehler manifold $\tilde{M}$ satisfies

$$\tilde{R}(JX, JY) = \tilde{R}(X, Y) \quad (3.10)$$

and

$$\tilde{R}(X, Y)Z = J\tilde{R}(X, Y)Z, \quad (3.11)$$

for any vector fields $X, Y, Z$ tangent to $\tilde{M}$. Then, by using (3.10) and (3.11) we obtain

$$\tilde{R}(JF_k, X; Y, JF_k) = \tilde{R}(F_k, JX; JY, F_k), \quad (3.12)$$

for any vector fields $X, Y \in \mathfrak{S}(D)$. Now, from (1.5), taking into account that $M$ is $D^\perp$-minimal and mixed totally geodesic, we get

$$\sum_{k=1}^{q} \{ \tilde{R}(F_k, JX; JY, F_k) \} = \sum_{k=1}^{q} \{ R(F_k, JX; JY, F_k) \}. \quad (3.13)$$

Condition (2) of the corollary implies

$$R(F_k, JX; JY, F_k) = 0, \quad k = 1, \ldots, q. \quad (3.14)$$

Finally, from (3.12), (3.13) and (3.14) we conclude that

$$\sum_{k=1}^{q} \{ \tilde{R}(JF_k, X; Y, JF_k) \} = 0, \quad (3.15)$$

and the corollary follows from (3.5).

The Kaehler manifold $\tilde{M}$ is said to be an Einstein space if there exists a constant $\rho$ such that $\tilde{S}(X, Y) = \rho g(X, Y)$ for all $X, Y$ tangent to $\tilde{M}$.

Now we can prove

Theorem 3.3. Let $\tilde{M}$ be an Einstein-Kaehler manifold. If $M$ is a CR submanifold of $\tilde{M}$ and satisfies the conditions from Corollary 3.1 then each maximal integral manifold of $D$ is totally geodesic immersed in $M$ if and only if it is an Einstein space with the same scalar curvature as $\tilde{M}$.
Proof. Let \( M^* \) be the maximal integral manifold of \( D \) passing through the point \( x \in M \). Denote by \( S^* \) the Ricci tensor of \( M^* \) and by \( h^* \) the second fundamental form of \( M^* \) considered as a submanifold of \( M \). Then, for each pair of local vector fields \( X, Y \) tangent to \( M^* \) we obtain

\[
S(X, Y) = S^*(X, Y) + \sum_{i=1}^{2p} \left\{ g(h^*(E_i, X), h^*(E_i, Y)) \right\}
- g\left( h^*(X, Y), \sum_{i=1}^{2p} \{ h^*(E_i, E_i) \} \right) + \sum_{k=1}^{q} \{ g\left( R(F_k, X) Y, F_k \right) \},
\]

where \( \{ E_1, \ldots, E_{2p}, F_1, \ldots, F_q \} \) is a local orthonormal field of frames on \( M \) such that \( E_i, 1 \leq i \leq 2p, \) are tangent to \( M \) and \( F_k \in S(D^\perp) \). Denote by \( \tilde{h} \) the second fundamental form of \( M^* \) as submanifold of \( M \). Then we have

\[
\tilde{h}(X, Y) = h^*(X, Y) + h(X, Y).
\]

(3.17)

\( M^* \) is a Kaehler submanifold of \( \tilde{M} \); hence it is minimal. On the other hand, \( M \) is \( D \)-minimal since it is foliate. Therefore, from (3.17) \( M^* \) is minimal as submanifold of \( M \). Then by using Corollary 3.1, (3.16) becomes

\[
\tilde{S}(X, Y) = S^*(X, Y) + \sum_{i=1}^{2p} \left\{ g(h^*(E_i, X), h^*(E_i, Y)) \right\}.
\]

(3.18)

Now it is easily seen that (3.18) proves the theorem.

4. CR submanifolds of a complex space form. Let \( \tilde{M} \) be a complex space form of constant holomorphic sectional curvature \( c \). The curvature tensor of \( \tilde{M} \) is given by

\[
\tilde{R}(X, Y)Z = \frac{1}{4} \left\{ g(Y, Z)X - g(X, Z)Y + g(Z, JY)JX
- g(Z, JX)JY + 2g(X, JY)JZ \right\},
\]

(4.1)

where \( X, Y, Z \) are vector fields on \( \tilde{M} \).

A CR submanifold \( M \) of \( \tilde{M} \) is called \( (D, \mu) \)-totally geodesic, if we have \( A_N X = 0 \) for each \( X \in S(D) \) and \( N \in S(\mu) \).

Theorem 4.1. Let \( M \) be a mixed totally geodesic foliate CR submanifold of a complex space form \( \tilde{M}(c) \). If the normal connection is \( (D, \mu) \)-flat, then \( c < 0 \). The equality holds good if and only if \( M \) is \( (D, \mu) \)-totally geodesic.

Proof. From the equation of Ricci, taking into account the \( (D, \mu) \)-flatness of the normal connection we obtain

\[
g\left( \left[ \tilde{R}(X, Y)N \right]^\perp, JN \right) = g(A_{N} Y, A_{N} X) - g(A_{N} X, A_{N} Y),
\]

for each \( N \in S(\mu) \) and \( X, Y \in S(D) \). Next, by using (2.1) and (2.6) the relation above becomes
Kähler manifold. II

\[ g\left( [\tilde{R}(X, Y)N]^\perp, JN \right) = -2g(A_N X, A_N Y). \]  

(4.2)

On the other hand, from (4.1) we have

\[ g\left( [\tilde{R}(X, Y)N]^\perp, JN \right) = \frac{\varepsilon}{2} g(X, JY) g(N, N). \]  

(4.3)

Take \( N \) as a unit section of the normal subbundle \( \mu \). Then from (4.2) and (4.3) we obtain

\[ cg(X, Y) + 4g(A_N X, A_N Y) = 0. \]  

(4.4)

Since \( g \) is a positive definite metric, from (4.4) we get \( c < 0 \). The last part of the theorem follows from (4.4).

The vertical distribution \( D^\perp \) is said to be flat with respect to the Levi-Civita connection on \( M \) if we have \( \nabla_X Y \in \mathfrak{S}(D^\perp) \) for any vector fields \( X, Y \in \mathfrak{S}(D^\perp) \). Now, we can state

**Theorem 4.2.** Let \( M \) be a CR submanifold of real codimension \( q > 2 \) of a complex space form \( \tilde{M}(c) \) of complex dimension \( p + q \). If the vertical distribution of \( M \) is flat with respect to the Levi-Civita connection on \( M \), and \( [A_N, A_{N'}] = 0 \) for any normal vectors \( N \) and \( N' \), then the sectional curvature \( K_M \) of \( M \) satisfies

\[ K_M (X \wedge Y) = \frac{\varepsilon}{4}, \]  

(4.5)

for each pair of orthonormal vectors \((X, Y)\) from \( D^\perp \).

**Proof.** From the Ricci equation, by using (4.1) and the assumption \([A_N, A_{N'}] = 0\), we get

\[ \frac{\varepsilon}{4} \left\{ g(BN, QY) g(B\bar{N}, QX) - g(BN, QX) g(B\bar{N}, QY) \right\} 
+ 2g(PX, JPY) g(CN, \bar{N}) \right\} \]  

\[ = g\left( R^\perp(X, Y)N, \bar{N} \right), \]  

for any vector fields \( X, Y \) tangent to \( M \) and \( N, \bar{N} \) normal to \( M \). Since \( \dim \mu = 0 \), we have \( C = 0 \). Thus, for the particular case when \( X, Y \in \mathfrak{S}(D^\perp) \), the relation above becomes

\[ \frac{\varepsilon}{4} \left\{ g(JN, Y) g(J\bar{N}, X) - g(JN, X) g(J\bar{N}, Y) \right\} = g\left( R^\perp(X, Y)N, \bar{N} \right). \]  

(4.6)

Now let \( Z \in \mathfrak{S}(D^\perp) \) be such that \( JZ = N \). The assumption on the codimension of \( M \) assures the existence of the vector field \( Z \). Then from (1.1) and by using the fact that \( D^\perp \) is flat with respect to \( \nabla \) we have \( \nabla_X^\perp N = J\nabla_X Z \). This implies \( R^\perp(X, Y)N = JR(X, Y)Z \). If we put \( \bar{N} = JW, W \in \mathfrak{S}(D^\perp) \), then from (4.6) we get
A CR submanifold $M$ of a Kaehler manifold $\tilde{M}$ is called $D^\perp$-totally geodesic if $h(X, Y) = 0$ for any $X, Y \in \mathcal{S}(D^\perp)$. We have proved the following theorem [2]:

**Theorem 4.3.** If $M$ is a $D^\perp$-minimal CR submanifold of a complex space form $\tilde{M}(c)$, then $M$ is $D^\perp$-totally geodesic if and only if $K_M(X \wedge Y) = \frac{\xi}{\gamma}$ for each pair of vector fields $X, Y \in \mathcal{S}(D^\perp)$.

Now, combining Theorems 4.2 and 4.3 we find

**Theorem 4.4.** If a CR submanifold $M$ of a Kaehler manifold $\tilde{M}$ satisfies the conditions of Theorem 4.2, then $M$ is $D^\perp$-totally geodesic.

In what follows we shall study the existence of a certain class of CR submanifolds in a complex space form of non-null holomorphic sectional curvature.

A proper CR submanifold $M$ of a Kaehler manifold $\tilde{M}$ is a CR submanifold with both distributions $D$ and $D^\perp$ of non-null dimensions. Also, $M$ is said to be totally umbilical if there exists a normal vector field $L$, such that the second fundamental form $h$ satisfies

$$h(X, Y) = g(X, Y)L,$$

(4.8)

for any vector fields $X, Y$ tangent to $M$.

Now we can state

**Theorem 4.5.** There exist no totally umbilical proper CR submanifolds of an elliptic or hyperbolic complex space.

**Proof.** Suppose there exists a totally umbilical proper CR submanifold $M$ of a complex space form $\tilde{M}$ ($c \neq 0$). Let $X$ and $Y$ be two non-null vector fields from $D$ and $D^\perp$ respectively. Then, for the normal part of $\tilde{R}(X, JX)Y$ we get

$$[\tilde{R}(X, JX)Y]_N = -\frac{\xi}{\gamma} g(X, X)JY \neq 0.$$

(4.9)

On the other hand, since $M$ is totally umbilical, from the Codazzi equation we have

$$[\tilde{R}(X, JX)Y]_N = g(JX, Y)\nabla^\perp_X L - g(X, Y)\nabla^\perp_J X L = 0,$$

(4.10)

which contradicts (4.9). Thus the proof is complete.

From this theorem we obtain

**Corollary 4.1.** There exist no totally geodesic proper CR submanifolds of an elliptic or hyperbolic complex space.
From this corollary the following question arises: Since $\|h\| \neq 0$ for any proper $CR$ submanifold of an elliptic or hyperbolic complex space, then how far from zero is $\|h\|$? Thus, instead of pinching theorems for $CR$ submanifolds of elliptic or hyperbolic complex spaces, one should consider theorems concerning different evaluations of $\|h\|$ for classes of $CR$ submanifolds.

REFERENCES