AN ALGEBRAIC CHARACTERIZATION OF CONNECTED SUM FACTORS OF CLOSED 3-MANIFOLDS

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Abstract. Let M and N be closed connected 3-manifolds. A knot group of M is the fundamental group of the complement of a tame simple closed curve in M. Denote the set of knot groups of M by K(M). A knot group G of M is realized in N if G is the fundamental group of a compact submanifold of N with connected boundary.

Theorem. Every knot group of M is realized in N iff N is a connected sum factor of M.

Corollary 1. K(M) = K(N) iff M is homeomorphic to N.

Given M, there exists a knot group G_M of M that serves to characterize M in the following sense.

Corollary 2. G_M is realized in N and G_N is realized in M iff M is homeomorphic to N.

Our proof depends heavily on the work of Bing, Feustal, Haken, and Waldhausen in the 1960s and early 1970s. A. C Conner announced Corollary 1 for orientable 3-manifolds in 1969 which Jaco and Myers have recently obtained using different techniques.

1. Preliminaries. We will work exclusively in the PL category. [Hem] is an excellent reference for definitions, notation and techniques. Manifolds are usually connected but not necessarily compact, orientable, or without boundary. \( \partial M \) denotes the boundary of a manifold M. A 2-manifold S properly embedded in a 3-manifold M or contained in \( \partial M \) is compressible provided (1) there exists a 2-cell D in M such that \( D \cap S = \partial D \) and \( \partial D \) does not bound a 2-cell in S, or (2) S bounds a 3-cell in M. We call a 2-cell D as in (1) a compressing 2-cell for S in M. If S is not compressible we say S is incompressible.

A 3-manifold M is \( P^2 \)-irreducible if every 2-sphere in M bounds a 3-cell in M and M contains no 2-sided projective planes. M is \( \delta \)-irreducible if every component of \( \partial M \) is incompressible. We usually follow Waldhausen [W1, p. 57] in using \( U(\cdot) \) for nice regular neighborhoods. (One exception is when \( U(J_{i+1}) \) is defined.) If N and M are compact manifolds with connected...
boundary and $N \subset \text{Int } M$, we use the notation $[N, M] = M - \text{Int } N$.

Projective planes, 2-spheres, and 2-cells share the property that any 2-sided simple closed curve they contain bounds a 2-cell, which is the key to proving Lemma A. (Proof omitted.)

**Lemma A.** Suppose $N_1$ and $N_2$ are $P^2$-irreducible, $\partial$-irreducible 3-manifolds such that $N_1 \cup N_2$ is a 3-manifold, $N_1 \cap N_2 = \partial N_1 \cap \partial N_2$ is a collection of 2-manifolds, $N_1 \cap N_2$ is incompressible in both $N_1$ and $N_2$, and no component of $N_1 \cap N_2$ is a 2-cell. Then $N_1 \cup N_2$ is $P^2$-irreducible and $\partial$-irreducible.

The next two lemmas are more general than we need but have other applications. First we give some definitions. Let $B$ be a 3-cell and $A$ an annulus on $\partial B$. Suppose $\alpha$ is an arc properly embedded in $B$ with an endpoint in each component of $\partial B - A$ and such that $C$, the closure of $B - U(\alpha)$, is not a solid torus. We call $C$ a cube-with-a-knotted-hole. If $M$ is a 3-manifold such that $C \cap M = \partial C \cap \partial M = A$. We say the 3-manifold $M \cup C$ is obtained from $M$ by attaching a cube-with-a-knotted-hole to $M$ along $A$.

**Lemma B.** Suppose $T$ is a torus or Klein bottle boundary component of a 3-manifold $M$. Let $A$ be an annulus in $T$ that is not contractible in $M$. If $N$ is obtained from $M$ by attaching a cube-with-a-knotted-hole to $M$ along $A$, then the boundary component of $N$ that intersects $T$ is incompressible in $N$.

**Proof.** Let $L$ be a cube-with-a-knotted-hole such that $L \cup M = N$ and $L \cap M = (\partial L) \cap (\partial M) = A$. Note $A$ is an incompressible surface in $N$. Let $S$ be the boundary component of $N$ that intersects $T$. Let $S \cap L = A'$, an annulus with $\partial A' = \partial A$.

**Case 1.** $T - \text{Int } A$ is an annulus. Suppose $S$ is compressible. Then there exists a properly embedded 2-cell $E$ in $N$ such that $\partial E \subset S$, $\partial E$ does not bound a 2-cell in $S$, and $E$ is in general position with respect to $A$. If $(\partial E) \cap A = \emptyset$, there exists a properly embedded 2-cell $E'$ such that $\partial E = \partial E'$ and $E' \cap A = \emptyset$. But then $A$ is contractible in either $M$ or $L$, a contradiction. So assume $(\partial E) \cap A \neq \emptyset$. Since the closures of both components of $S - A$ are annuli, we can adjust $E$ by an ambient isotopy of $M$ so that the closure of each component of $\partial E - A$ is an arc with endpoints in different components of $\partial A$, and $E$ is in general position with respect to $A$. Note that if $\alpha$ is an arc that is a component of $E \cap A$, then the endpoints of $\alpha$ must also lie in different components of $\partial A$. Using the incompressibility of $A$ we can remove the simple closed curve components of $E \cap A$. Hence we can assume that $E \cap A$ consists of a finite number of arcs. Let $F$ be a component of $E \cap L$. $F$ is a 2-cell properly embedded in $L$ such that the algebraic intersection number of $\partial F$ with a component $J$ of $\partial A$ is a nonzero integer. Since $L$ is $\partial$-irreducible, $\partial F$ must bound a 2-cell in $\partial L$. Hence $\partial F$ must have zero
algebraic intersection number with $J$. So we must conclude that $S$ is incompressible.

**Case 2.** $T - \text{Int } A$ consists of two Möbius bands $Q_1$ and $Q_2$. Suppose $S$ is compressible. As in Case 1, there exists a properly embedded 2-cell $E$ in $N$ such that $\partial E \subseteq S$, $\partial E$ does not bound a 2-cell in $S$, $E$ is in general position with respect to $A$, and $\partial E \cap A \neq \emptyset$. We can assume each component of $(\partial E) \cap A'$ is an arc with an endpoint in each component of $\partial A'$, and each component of $\partial E \cap Q_i$ is an arc that does not separate $Q_i$. Since $A$ is incompressible in $N$ we can further assume that all the components of $E \cap A$ are arcs. Let $D$ be a 2-cell in $E$ such that $D \cap A = \alpha$ is an arc, $D \cap \partial E = \alpha'$ is an arc, and $\partial D = \alpha \cup \alpha'$. 

If $D \subseteq L$, then $\alpha' \subseteq \alpha'$. Recall that $\alpha'$ must have an endpoint in each component of $\partial A'$. Hence the existence of $D$ violates the fact that $L$ is not a solid torus.

If $D \subseteq M$, then $\partial D \subseteq T$. Suppose $\alpha' \subseteq Q_1$. Since $\alpha'$ does not separate $Q_1$, $\alpha \cup \alpha'$ must be 1-sided in $T$. But $\partial D = \alpha \cup \alpha'$ must be 2-sided in $T$.

We are forced to conclude again that $S$ is incompressible.

**Lemma C.** Suppose $M$ is a compact 3-manifold with no 2-sphere or projective plane boundary components. Then there exists a simple closed curve $J$ contained in $\text{Int } M$ such that $M - \text{Int } U(J)$ is $P^2$-irreducible and $\partial$-irreducible. Furthermore we may choose $J$ so that $U(J)$ is orientable.

**Proof.** Let $C$ be a compact collar on $\partial M$ in $M$ and let $N$ be the closure of $M - C$. Suppose $L$ is a triangulation of $M$ with subcomplex $K$ that triangulates $N$. Let $G$ be the 1-skeleton of $K''$, the second derived barycentric subdivision of $K$. Since $M - \text{Int } U(G)$ is homeomorphic to $C$ with a finite number of 1-handles attached, $M - G$ is $P^2$-irreducible and $\partial$-irreducible. We want to use one of Bing's techniques to find a simple closed curve $J_1$ that approximates $G$. Then we can tie a knot in $J_1$ to obtain $J$ to insure that $\partial U(J)$ is incompressible in $M - \text{Int } U(J)$.

Note $G$ has the following properties:

1. $G$ is a connected finite graph with no points of order one;
2. for all vertices $v$ of $G$, $G - \text{st}(v, G)$ is connected (st($v, G$) denotes the open star of $v$ in $G$);
3. for all vertices $v$ of $G$, $G - \text{st}(v, G)$ contains either $K^1$, the 1-skeleton of $K$, or $K_0$, the dual 1-skeleton of $K$.

1 and (3) clearly hold. (2) holds since st($v, G$) is one component of $G - \text{lk}(v, K'')$ and lk($v, K''$) is connected (lk($v, K''$) denotes the link of $v$ in $K''$). Recall that lk($v, K''$) is the 1-skeleton of lk($v, K''$), a 2-sphere or 2-cell.) Note any subdivision of a graph with properties (1)--(3) also has properties (1)--(3).
There are an even number of vertices of \( G \) with odd order. Hence we can pair these vertices and connect them by polygonal arcs that intersect \( G \) only in their endpoints. So we obtain a finite graph \( G_1 \) that contains \( G \), has properties (1)-(3), and such that each point of \( G_1 \) has even order. Note that \( M - G_1 \) is \( P^2 \)-irreducible and \( \partial \)-irreducible.

We now wish to modify \( G_1 \) to obtain a graph \( G_2 \) that satisfies (1) and (2) such that each point of \( G_2 \) has order 2 or 4, and \( M - G_2 \) is homeomorphic to \( M - G_1 \).

Let \( w_1, \ldots, w_k \) be the points of \( G_1 \) that have order greater than 4. Let \( D_1, \ldots, D_k \) be small regular neighborhoods of \( w_1, \ldots, w_k \), respectively, in \( M \) such that \( D_i \) collapses to \( G_1 \cap D_i \). There exist 2-cells \( B_i \) properly embedded in \( D_i \) such that \( G_1 \cap D_i \subseteq B_i \). There exist trees \( F_i \subseteq B_i \) such that \( F_i \cap D_i = G_1 \cap \partial D_i \) consists of the endpoints of \( F_i \), and each point of \( F_i \) in \( Int B_i \) has order 2 or 4. Let \( D = \bigcup_{i=1}^{k} D_i \) and \( G_2 = (G_1 - D) \cup \left( \bigcup_{i=1}^{k} F_i \right) \).

Let \( z_1, \ldots, z_s \) be the points of \( G_2 \) that have order 4, and let \( H_1, \ldots, H_s \) be small regular neighborhoods of \( z_1, \ldots, z_s \), respectively, in \( Int D \) such that \( G_2 \cap \partial H_j \) consists of 4 points. Let \( H = \bigcup_{j=1}^{s} H_j \). Replace \( G_2 \cap (H \cup \text{Int } H) \) by a pair of arcs \( a_j, a'_j \) properly embedded in \( H_j \) such that \( a_j \cup a'_j \cap \partial H_j = G_2 \cap \partial H_j \), and \( a_j \) and \( a'_j \) cannot be separated in \( H_j \) by a properly embedded 2-cell, and such that

\[
J_1 = (G_2 - H) \cup \left( \bigcup_{j=1}^{s} \left( a_j \cup a'_j \right) \right)
\]

is a simple closed curve. See [B, Figure 3 and Lemma 6] for one way to choose \( a_j \) and \( a'_j \). In the next four paragraphs we will show \( M - J_1 \) is \( P^2 \)-irreducible and \( \partial \)-irreducible.

\( H_j - J_1 \) is clearly \( P^2 \)-irreducible. Using linking arguments it is easy to verify that \( H_j - J_1 \) is \( \partial \)-irreducible.

We claim \( D_i - (J_1 \cup \text{Int } H) \) is \( P^2 \)-irreducible and \( \partial \)-irreducible. \( D_i - (J_1 \cup H) \), \( D_i - G_2 \), \( D_i - G_1 \) and \( (\partial D_i - J_1) \times [0, 1) \) are homeomorphic. So it is sufficient to show that if \( E \) is a properly embedded 2-cell in \( D_i - (J_1 \cup \text{Int } H) \) with \( \partial E \subseteq \partial H_j - J_1 \), then \( \partial E \) bounds a 2-cell in \( \partial H_j - J_1 \). By property (2), \( G_2 - \text{Int } H_j \) is connected. Note \( E \) separates \( M - \text{Int } H_j \). Hence \( J_1 \cap \partial H_j = G_2 \cap \partial H_j \) must be contained in one component of \( \partial H_j - \partial E \). Therefore \( \partial E \) bounds a 2-cell in \( \partial H_j - J_1 \).

We now show \( M - (J_1 \cup \text{Int } D) \) is \( P^2 \)-irreducible and \( \partial \)-irreducible. Recall \( N - G_1 \), which is homeomorphic to \( M - (J_1 \cup D) \), is \( P^2 \)-irreducible and \( \partial \)-irreducible. So we just need to show \( \partial D_i - J_1 \) is incompressible in \( M - (J_1 \cup \text{Int } D) \). Let \( E \) be a properly embedded 2-cell in \( M - (J_1 \cup \text{Int } D) \) with \( \partial E \subseteq \partial D_i - J_1 \). We claim \( E \) separates \( M - (J_1 \cup \text{Int } D) \). Let \( E' \) be a 2-cell
in \( \partial D_i \) with \( \partial E' = \partial E \). The 2-sphere \( E \cup E' \) is contained in either \( M - K_1 \) or \( M - K^1 \) by property (3) for \( G_1 \). Since both \( M - K_1 \) and \( M - K^1 \) are irreducible, \( E \cup E' \) separates \( M \). Hence \( E \) separates \( M - (J_1 \cup \text{Int } D) \). Using property (2) for \( G_1 \), we see \( G_1 - \text{Int } D_i \) is contained in one component of \( M - (J_1 \cup \text{Int } D \cup E) \). Hence \( G_1 \cap \partial D_i = J_1 \cap \partial D_i \) cannot meet both components of \( \partial D_i - \partial E \). Therefore \( \partial E \) bounds a 2-cell in \( \partial D_i - J_1 \).

The results of the preceding three paragraphs imply that each component of \( (\partial H \cup \partial D) - J_1 \) is a separating incompressible surface in \( M - J_1 \), and the closures of the components of \( (M - J_1) - ((\partial H \cap \partial D) - J_1) \) are \( P^2 \)-irreducible and \( \partial \)-irreducible. By Lemma A, \( M - J_1 \) is \( P^2 \)-irreducible and \( \partial \)-irreducible.

We claim \( J_1 \) does not pierce any 2-sphere in \( M \). Suppose \( S \) is a 2-sphere in \( M \) with \( S \cap J_1 = \{ p \} \). We can assume \( p \notin D \) and that \( S \cap \partial D = \emptyset \). So \( S \cap (J_1 \cup D) = \{ p \} = S \cap G_1 \). Hence \( S \) is contained in an irreducible submanifold of \( M \), either \( M - K_1 \) or \( M - K^1 \). We conclude \( J_1 \) does not pierce \( S \).

\( \partial U(J_1) \) may not be incompressible in \( M - \text{Int } U(J_1) \). So we need to "tie a knot" in \( J_1 \). More precisely, let \( P \) be a 3-cell in \( U(J_1) \) such that \( P \cap \partial U(J_1) = A \) is an annulus, \( \text{cl}(U(J_1) - P) = P' \) is a 3-cell, and \( P \cap J_1 = \alpha \) is an arc with an endpoint in each component of \( \partial P - A \). Let \( \alpha' \) be the arc \( P' \cap J_1 \). Let \( \beta \) be a properly embedded arc in \( P \) such that \( \partial \beta = \partial \alpha \) and \( L = \text{cl}(P - U(\beta)) \) is a cube-with-a-knotted-hole. \( U(\beta) \) is a standard regular neighborhood of \( \beta \) in \( P \).) Let \( J_2 = \beta \cup \alpha' \). We say \( J_2 \) is obtained from \( J_1 \) by tying a knot in \( J_1 \). Note \( U(\beta) \cup P' \) is a regular neighborhood of \( J_2 \) in \( M \). Hence \( M - \text{Int } U(J_2) \) is homeomorphic to \( M - \text{Int } U(J_1) \) with the cube-with-a-knotted-hole \( L \) attached along \( A \). Since \( J_1 \) does not pierce any 2-sphere in \( M \), \( A \) must not be contractible in \( M - \text{Int } U(J_1) \). Applying Lemma B we see \( M - \text{Int } U(J_2) \) is \( \partial \)-irreducible. \( M - \text{Int } U(J_2) \) is clearly \( P^2 \)-irreducible. Hence \( J_2 \) is our required \( J \).

Suppose \( U(J_2) \) is nonorientable. We wish to find a simple closed curve \( J \) such that \( U(J) \) is orientable and \( M - \text{Int } U(J) \) is \( P^2 \)-irreducible and \( \partial \)-irreducible. Let \( Q \) be a Möbius band, with center line \( J_2 \), that is properly embedded as a 2-sided subset of \( U(J_2) \). Let \( J_3 = \partial Q \). Note \( U(J_3) \) is orientable and \( J_3 \) is not contractible in \( M - J_2 \).

We claim \( \text{Int } Q \) is incompressible in \( M - J_3 \). Let \( E \) be a 2-cell in \( M - J_3 \) such that \( E \cap Q = \partial E \). \( \partial E \) must be 2-sided in \( Q \). So we can assume \( \partial E \cap J_2 = \emptyset \). Now \( \partial E \) is not parallel to \( J_3 \) in \( Q \) since \( \partial E \) is contractible in \( M - J_2 \). Hence \( \partial E \) must bound a 2-cell in \( Q \).

Since \( M - Q \) and \( M - J_2 \) are homeomorphic, \( M - Q \) is \( P^2 \)-irreducible and \( \partial \)-irreducible. It follows that \( M - J_3 \) is \( P^2 \)-irreducible and \( \partial \)-irreducible. \( J_3 \) cannot pierce a 2-sphere in \( M \) since \( J_3 \) bounds \( Q \). Let \( J \) be obtained from
$J_i$ by tying a knot in $J_i$. $U(J)$ is orientable. As before, $M - \text{Int } U(J)$ is $P^2$-irreducible and $\partial$-irreducible.

This completes the proof of Lemma C.

2. Proof of the main theorem and corollaries. Let $M$ be a closed 3-manifold. Recall that $K(M)$ is the set of fundamental groups of complements of $PL$ simple closed curves in $M$. $S(M)$ is the set of fundamental groups of compact sub-3-manifolds of $M$ that have connected boundary.

**Main Theorem.** Let $M, N$ be closed 3-manifolds. $S(M)$ contains $K(N)$ if and only if $N$ is a connected sum factor of $M$.

**Proof.** Suppose $N$ is a connected sum factor of $M$. Let $\alpha$ be a simple closed curve in $N$, $U(\alpha)$ a regular neighborhood of $\alpha$ in $N$, and $B$ a 3-cell in $\text{Int } U(\alpha)$. Since $N - \text{Int } B$ is homeomorphic to a subset of $M$, $\pi_1(N - \text{Int } U(\alpha))$ belongs to $S(M)$.

Now suppose $S(M)$ contains $K(N)$. By Lemma C there exists an orientable simple closed curve $J_0$ contained in $N$ such that $L_0 = N - \text{Int } U(J_0)$ is $P^2$-irreducible and $\partial$-irreducible. Let $h$ be a positive integer such that $h - 1$ is the maximal number of disjoint, 2-sided, nonparallel, incompressible tori contained in $M$ [Ha]. We wish to add structure to $L_0$ by doubling $J_0$ and then tying a knot in the result, repeating the process $h + 1$ times. Note that $h$ is the only information about $M$ used in this construction. More precisely assume $J_i$, $U(J_i)$, and $L_i$ have been defined.

Let $J_i^+$ be a simple closed curve in $\text{Int } U(J_i)$ with winding number two. Let $A_i$ be an annulus on $\partial U(J_i^+)$ that is contractible in $U(J_i^+)$ and does not separate $\partial U(J_i^+)$. Let $L_i^+ = N - \text{Int } U(J_i^+)$. Let $C_i$ be a cube-with-a-knotted-hole in $U(J_i^+)$ such that $C_i \cap \partial U(J_i^+) = A_i$ and $L_{i+1} = L_i^+ \cup C_i$ is obtained from $L_i^+$ by attaching $C_i$ along $A_i$. Now $N - \text{Int } L_{i+1}$, being a solid torus, is a regular neighborhood of a simple closed curve $J_{i+1}$. We depart from our $U(\cdot)$ convention and define $U(J_{i+1}) = N - \text{Int } L_{i+1}$. Recall $[L_i, L_i^+] = L_i^+ - \text{Int } L_i$. We collect some useful facts in the following lemma. Figure 1 should help picture the construction.

**Lemma 1.** $[L_i, L_i^+]$ is $\partial$-irreducible. $[L_i^+, L_i^{+\ast}]$ is not a parallelity component. $L_i$ is $P^2$-irreducible and $\partial$-irreducible. No nontrivial loop on $A_i$ is freely homotopic in $[L_i, L_i^+]$ to a loop on $\partial L_i$.

**Proof.** Suppose $D$ is a compressing 2-cell for $[L_i, L_i^+]$. Let $U(D)$ be a regular neighborhood of $D$ in $[L_i, L_i^+]$. If $\partial D \subset \partial L_i$, the closure of $U(J_i) - U(D)$ is a 3-cell in $U(J_i)$ that contains $J_i^+$, If $\partial D \subset \partial L_i^+$, then $U(J_i^+) \cup U(D)$ is a 3-cell in $U(J_i)$ containing $J_i^+$. In either case the winding number of $J_i^+$ in $U(J_i)$ would be zero, contrary to construction.
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$[(L^+_{i+1}, L^+_i)] = U(J_i) - \text{Int } U(J_i^+)$

Figure 1

\[ \text{Int } [L^+, L^+_{i+1}] \text{ contains an incompressible torus (a parallel copy of } \partial C_i \text{ in Int } C_i \text{) that does not separate } \partial L^+_i \text{ from } \partial L^+_{i+1} \text{ in } [L^+_i, L^+_{i+1}]. \text{ So by the appendix to } [\text{Ha}], [L^+_i, L^+_{i+1}] \text{ cannot be a parallelity component.} \]

Using Lemmas A and B with the above facts we see $L_i$ is $P^2$-irreducible and $\partial$-irreducible.

Suppose $l^+$ is a nontrivial loop on $A_i$ that is freely homotopic in $[L_i, L^+_i]$ to a loop $l$ on $\partial L_i$. Since $A_i$ is incompressible $l$ must be nontrivial on $\partial L_i$. By [W2] there exists an annulus $A$ in $[L_i, L^+_i]$ with one boundary component $\alpha^+$ on $A_i$, the other $\alpha$ on $\partial L_i$, and both are nontrivial. Note $\alpha$ bounds a 2-cell in $U(J_i)$. So if we consider $U(J_i)$ as embedded in $E^3$, $\alpha$ does not link $J_{i+1}$ mod 2 but $\alpha^+$ does link $J_{i+1}$ mod 2. This contradiction completes the proof of Lemma 1.

**Lemma 2.** $M$ contains a compact $P^2$-irreducible, $\partial$-irreducible 3-manifold $K$ such that $\partial K$ is a torus or a Klein bottle and $K$ is homotopy equivalent to $L_{n+1}$. 

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Proof. By hypothesis there is a compact 3-manifold $K'$ in $M$ such that $\pi_1(K') \cong \pi_1(L_{h+1})$ and $\partial K'$ is connected. Note $\pi_1(L_{h+1})$ has no elements of finite order [E, Theorem 3.2], is not a nontrivial free product [Hem, Theorem 7.1], and is not free abelian. $\partial K'$ is incompressible since otherwise $\pi_1(K')$ would be free abelian or a nontrivial free product. If $K'$ contained a 2-sided projective plane, $\pi_1(K')$ would have elements of order two. Hence $K' = K \# H$ where $H$ is a homotopy 3-sphere and $K$ is $P^2$-irreducible with $\partial K$ connected and incompressible. We can assume $K$ is contained in $M$. Since $K$ and $L_{h+1}$ are $K(\pi, 1)$'s, $K$ is homotopy equivalent to $L_{h+1}$. Hence
\[
0 = X(\partial L_{h+1}) = 2X(L_{h+1}) = 2X(K) = X(\partial K)
\]
which implies $\partial K$ is a torus on a Klein bottle. ($X(\cdot)$ denotes the Euler characteristic.) Lemma 2 is completed.

Suppose $f: K \to \text{Int } L_{h+1}$ is a homotopy equivalence. We can assume $f^{-1}(A_h)$ and $f^{-1}(\partial L_h)$ are collections of 2-sided properly embedded incompressible surfaces in $K$ [Hel] with a minimum number of components. Both collections are nonempty since $f_\ast: \pi_1(K) \to \pi_1(L_{h+1})$ is an isomorphism and $\pi_1(L_{h+1})$ splits as nontrivial free products with amalgamation along both $\pi_1(A_h)$ and $\pi_1(\partial L_h)$. Suppose $S$ is a component of either $f^{-1}(A_h)$ or $f^{-1}(\partial L_h)$ that has boundary.

Each component of $\partial S$ is a nontrivial simple closed curve on $\partial K$. In order to see this note $(f|_S)^\ast: \pi_1(S) \to G$ is a monomorphism where $G$ is either $\pi_1(A_h)$ or $\pi_1(\partial L_h)$. Hence $\pi_1(S)$ is free abelian. If $S$ is an annulus the boundary components must be nontrivial. If $S$ were a Möbius band with trivial boundary we could find a 2-sided projective plane in $K$. If $S$ were a 2-cell with trivial boundary, one component of $K - \text{Int } U(S)$ would be a 3-cell and we could reduce the number of components of either $f^{-1}(A_h)$ or $f^{-1}(\partial L_h)$ that has boundary.

Note $f^{-1}(A_h) \cap \partial K$ is nonempty.

Lemma 3. $f^{-1}(\partial L_h) \cap \partial K$ is empty.

Proof. Suppose not. Then there exist nontrivial 2-sided simple closed curves $\alpha$ and $\beta$ on $\partial K$ such that $f(\alpha) \subseteq A_h$ and $f(\beta) \subseteq \partial L_h$. Let $U(\alpha \cup \beta)$ be a regular neighborhood of $\alpha \cup \beta$ in $\partial K$. Each component of $\partial K - \text{Int } U(\alpha \cup \beta)$ must have Euler characteristic zero. Since at least one such component must have two boundary simple closed curves, $\alpha \cup \beta$ must bound an annulus $A'$ in $\partial K$. We can assume $\text{Int } A'$ misses $f^{-1}(A_h)$ and $f^{-1}(\partial L_h)$. Hence $f(\alpha)$ is a nontrivial loop in $A_h$ that is freely homotopic in $[L_h, L_h^+]$ to a loop in $\partial L_h$, which violates Lemma 1. So Lemma 3 holds.

Hence $f(\partial K) \subseteq \text{Int } [L_h, L_{h+1}]$. We can modify $f$ so that $f(\partial K) \subseteq [L_h, L_{h+1}]$ but $f(\partial K) \cap \partial L_h$ is nonempty. Let $x$ be a point in $\partial K$ such that $f(x)$ belongs to $\partial L_h$. We now wish to apply the main result of [F]. Consider the geometric
splitting of $L_{n+1} = L_n \cup \partial L_n [L_n, L_{n+1}]$ and $f_a: \pi_1(K, x) \to \pi_1(L_{n+1}, f(x))$. Under $(f_a)^{-1}$ we obtain an algebraic splitting of $\pi_1(K, x)$ that respects the peripheral structure of $\pi_1(K, x)$, i.e., $\pi_1(\partial K, x)$ under inclusion is a subgroup of $(f_a)^{-1}(\pi_1([L_n, L_{n+1}], f(x)))$. So applying [F] there exists a separating incompressible torus $T \subset \text{Int} K$ (denote the closure of the component of $K - T$ that has boundary $T$ as $K_h$; then the closure of the other component is $K - \text{Int} K_h = [K_h, K]$; let $y$ belong to $T$) and an isomorphism $d: \pi_1(K, y) \to \pi_1(K, x)$ such that

$$f_a d(\pi_1(T, y)) = \pi_1(\partial L_n, f(x))$$

and

$$f_a d(\pi_1(K_h, y)) = \pi_1([L_n, L_{n+1}], f(x)).$$

Since $\pi_1(\partial L_n)$ is a peripheral subgroup of both $\pi_1(L_n)$ and $\pi_1([L_n, L_{n+1}])$ and neither $L_n$ nor $[L_n, L_{n+1}]$ is a product I-bundle, [W] and [Hei] apply to conclude $K_h$ is homeomorphic to either $L_n$ or $[L_n, L_{n+1}]$. The only possibility is $L_n$.

Let $g: L_n \to K_h$ be a homeomorphism. First some notation is needed. Let $K_i = g(L_i)$ and $K_i^+ = g(L_i^+)$. Now $\partial K_0^+, \ldots, \partial K_{h-1}^+$ are nonparallel disjoint 2-sided tori in $M$. At least one must be compressible, say $\partial K_j^+$, $0 < j < h - 1$. Let $J$ be a boundary component of the annulus $A_j$.

**Lemma 4.** $g(J)$ bounds a 2-cell in $M - \text{Int} K_{j+1}$.

**Proof.** We adapt the Bing-Martin proof that composite knots have property P [B, M]. Recall $L_{j+1} = C_j \cup L_j^+$. Let $K_j'$ be a concentric copy of $K_j^+$ in $\text{Int} K_j^+$. Since $\partial K_j^+$ is compressible in $M$, there exists a compressing 2-cell $D$ for $\partial K_j'$ in $M - \text{Int} K_j'$. We can assume $D$ misses $g(C_j)$ (put $D$ in general position with respect to $g(C_j)$). If all the simple closed curves in $D \cap g(C_j)$ are trivial on $g(C_j)$ we can find the desired compressing 2-cell. Otherwise we can find a compact 3-manifold $Q'$ that contains $g(C_j)$, is contained in $M - K_j'$ and has a 2-sphere boundary. Once again we can find the desired 2-cell.) Now put $D$ in general position with respect to $\partial K_j^+$. We can assume all simple closed curves in $D \cap \partial K_j^+$ are nontrivial on $\partial K_j^+$. $D \cap \partial K_j^+$ is nonempty since $K_j^+ - \text{Int} K_j'$ has incompressible boundary. Let $E \subset D$ be a 2-cell such that $E \cap \partial K_j^+ = \partial E$. $E$ is contained in $M - \text{Int} K_{j+1}$ since $D$ misses $g(C_j)$ and $K_j^+ - K_j'$ has incompressible boundary. Since $\partial E$ misses $g(A_j)$ and is nontrivial in $\partial K_j^+$, $\partial E$ is parallel to $g(J)$ on $\partial K_j^+$. The required 2-cell exists and the proof of Lemma 4 is complete.

Let $D_1$ be a 2-cell in $N$ with $D_1 \cap L_{j+1} = \partial D_1 = J$. Let $D_2$ be a 2-cell in $M$ with $D_2 \cap K_{j+1} = \partial D_2 = g(J)$. Extend $g|_{L_{j+1}: L_{j+1} \to K_{j+1}}$ first to $g^*: L_{j+1} \cup D_1 \to K_{j+1} \cup D_2$ and then to regular neighborhoods $g^*: (L_{j+1} \cup D_1) \to (K_{j+1} \cup D_2)$. Since $N - \text{Int} U(L_{j+1} \cup D_1)$ is a 3-cell, there exists a closed 3-manifold $Q$ such that $M$ is a connected sum of $N$ and $Q$. The proof of the main theorem is complete.
**Corollary 1.** Suppose $M$ and $N$ are closed 3-manifolds. $K(M) = K(N)$ if and only if $M$ is homeomorphic to $N$.

**Proof.** By the main theorem, $M$ is a connected sum of $N$ and a closed 3-manifold $Q_1$. Again $N$ is a connected sum of $M$ and a closed 3-manifold $Q_2$. So $M$ is a connected sum of $M$, $Q_2$, and $Q_1$. [Hem, Theorem 3.21] applies to allow us to conclude $Q_1$ and $Q_2$ are trivial. Hence $M$ and $N$ are homeomorphic.

Suppose $N$ is a closed 3-manifold. Let $h_N = h$ be the positive integer such that $h - 1$ is the maximal number of disjoint, 2-sided, nonparallel, incompressible tori in $N$ [Ha]. Choose $L_{h+1} \subseteq N$ as in the proof of the main theorem. Denote $\pi_1(L_{h+1})$ by $G_N$. If $M$ is a closed 3-manifold and $G_N$ belongs to $S(M)$, we say $G_N$ is realized in $M$.

**Corollary 2.** Let $M$ and $N$ be closed 3-manifolds. $G_M$ is realized in $N$ and $G_N$ is realized in $M$ if and only if $M$ is homeomorphic to $N$.

**Proof.** Consider the positive integers $h_N$ and $h_M$ as defined above. Suppose $h_M < h_N$. By the proof of the main theorem, $N$ is a connected sum factor of $M$. So $h_N < h_M$. Apply the proof of the main theorem again to conclude $M$ is a connected sum factor of $N$. As in Corollary 1, $M$ is homeomorphic to $N$.

We conclude with the following question.

**Question.** Does the main theorem hold for compact 3-manifolds with no 2-sphere boundary components?

**References**


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