THE ANALYTIC CONTINUATION OF THE DISCRETE SERIES. I

BY

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Abstract. In this paper the analytic continuation of the holomorphic discrete series is defined. The most elementary properties of these representations are developed. The study of when these representations are unitary is begun.

1. Introduction. In Sally [8] (among other things), the analytic continuation of the holomorphic discrete series for the universal covering group of $SL(2, \mathbb{R})$ was studied. In this paper we generalize the results of Sally [8] to an arbitrary simply connected, semisimple, Lie group admitting (relative) holomorphic discrete series. We also show that the characters of the analytically continued representations are holomorphic functions of the parameter. This allows the use of the formula of Harish-Chandra [4] for the characters of discrete series to compute, in particular, the characters of the "limits of discrete series" in Knapp-Okamoto [6]. In Knapp-Okamoto [6] it is shown that these "limits of discrete series" are irreducible components of unitarily induced representations. Thus the results of this paper give a technique for computing the characters of irreducible components of certain unitarily induced representations. In particular for $SU(n, 1)$ it gives a technique for computing the characters of the irreducible components of an infinite class of reducible unitary principal series and, for $SU(1, 1)$ and $SU(2, 1)$, all of them.

In the course of our investigation we note that, in addition to the "limits of discrete series", there are unitary representations "past the limit" just as in the case of the universal covering group of $SL(2, \mathbb{R})$ (see Lemma 3.5 and §4).

Most of this paper is of an expository nature. We use some ideas of Murakami and Satake [9] to give the "bounded realization" of the holomorphic discrete series. In the proof of Proposition 2.6 and Lemma 2.7 this realization is shown to be equivalent to that of Harish-Chandra [3]. The critical observation in §2 is Lemma 2.5. §§3 and 4 contain whatever might be new in this paper.

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In a later paper in this series we will study the analytic continuation of the "nonholomorphic" discrete series for the universal covering group of SU(n, 1).

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2. The holomorphic discrete series. Let G be a connected, simply connected, simple Lie group. Let g = f ⊕ ρ be a Cartan decomposition of the Lie algebra, g, of G. We assume that f₁ = [f, f] ≠ f. Then f = ℜH₁ ⊕ f₁, [H₁, f₁] = 0 (here g ⊂ g, the complexification of g). Let h ⊂ f be a maximal abelian subalgebra of f and let h denote its complexification. Let Δ be the root system of g relative to h. Then f = f ⊗₀ R C = h + Σα∈Δ₀θα, ρ = Σα∈Δ₀θα (we use these formulas to define Δₖ and Δₜ). Then, as is well known, if α ∈ Δₜ, α(H₁) = ± c, c ∈ R, c ≠ 0. We normalize H₁ so that α(H₁) = ± 1 for α ∈ Δₜ. Let Δₜ be a positive system of roots relative to a lexicographic order starting with H₁. Since α(H₁) = 0 for α ∈ Δₖ, we see that if α ∈ Δₖ and β ∈ Δₜ = Δₜ ∩ Δₜ, then α + β ∈ Δ implies α + β ∈ Δₜ.

Let G_c be the connected, simply connected group with Lie algebra g_c. Then G_c is a complex, simple, Lie group. Let n⁺ = Σα∈Δ⁺θα, n⁻ = Σα∈Δ⁻θα. Set b₊ = {H ∈ b|α(H) ∈ R, α ∈ Δ}. Set A⁺ = exp b₊, N⁺ = exp n⁺, N⁻ = exp n⁻. We recall the following results of Harish-Chandra (cf. Helgason [5]).

(1) Set P⁺ = exp ρc⁺, P⁻ = exp ρc⁻ where ρc⁺ = ρc ∩ n⁺, ρc⁻ = ρc ∩ n⁻. Then P⁻K_cP⁺ is open in G_c, and the map P⁻ × K_c × P⁺ → G_c is a holomorphic diffeomorphism of P⁻ × K_c × P⁺ onto an open subset of G_c. Here K_c is the connected subgroup of G_c corresponding to f_c.

(2) If G₊ c G_c is the connected subgroup of G_c corresponding to g c g_c, then N⁻A⁺G_0 is an open subset of P⁻K_cP⁺. P⁻K_c ∩ G_0 = K_0 is the connected subgroup of G_0 corresponding to f.

If g ∈ P⁻K_cP⁺, let K(g) be defined by g ∈ P⁻K(g)P⁺. Then K: P⁻K_cP⁺ → K_c is a holomorphic map.

(3) Let γ₁ ∈ Δ⁺ be the smallest element. Let Φ₁ ⊂ Δ⁺ be the set of all α ∈ Δ⁺, α ≠ γ₁ such that γ₁ ± α ∉ Δ. If Φ₁ ≠ ∅ let γ₂ be the smallest element of Φ₁. Set Φ₂ = {α ∈ Φ₁ \ {γ₂}|γ₂ ± α ∉ Δ}. Continuing in this way we have Δ⁺ ⊃ Φ₁ ⊃ Φ₂ ⊃ ⋅⋅⋅ ⊃ Φ, and Φ₊₁ = ∅. We also have γ₁, ⋅⋅⋅, γ_r such that γᵢ ± γⱼ ∉ Δ. If α ∈ Δₜ, let X_α ∈ g be chosen so that X_α = X_α. (Here X = X₁ - iX₂ for X₁, X₂ ∈ g.) Then, if α = Σᵢ=₁R(Xᵢ, X₋ᵢ), α ∈ ρ is maximal abelian.

(4) Normalize X_α for α ∈ Δₜ so that [X_α, X₋α] = H_α, [H_α, X_α] = 2X_α and [H_α, X₋α] = -2X₋α. Then

\[ \exp \left( \sum t_i(X_α + X₋α) \right) = \exp \left( \sum (\tanh t_i)X_α \right) \cdot \exp \left( \sum (\log \cosh t_i)H_α \right) \exp \left( \sum (\tanh t_i)X_α \right). \]
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(5) Using the fact that \( G_0 = K_0(\exp a)K_\varphi \), we see that \( N^-A^+G_0 = P^-K_c\exp(\Omega) \) with \( \Omega = \{ \text{Ad}(k)(\sum \tanh t_iX_i^\prime)|k \in K, t_i \in R \} \).

Let \( \nu: G \to G_0 \) be the covering map. Then \( G \) acts on \( \Omega \) by \( P^-K_c\exp(z \cdot g) = P^-K_c(\exp z)\nu(g) \), \( z \in \Omega, g \in G \). This is the Harish-Chandra realization of \( G/K \) as a bounded homogeneous domain in \( C^n, n = \text{dim } \mathfrak{p}^+ = \frac{1}{2} \text{ dim } G/K \). Here, \( K \) is the connected subgroup of \( G \) corresponding to \( \mathfrak{p} \).

Let \( \tilde{K}_c \to K_c \) be the universal covering group of \( K_c \). Following Satake [9], we define \( \tilde{\mathcal{K}}: \Omega \times G \to \tilde{K}_c \) by the formula \( \tilde{\mathcal{K}}(z; g) = \tilde{\mathcal{K}}((\exp z)\nu(g)) \) for \( z \in \Omega, g \in G \). \( \tilde{\mathcal{K}} \) lifts to a holomorphic map \( \mathcal{K} \) of \( \Omega \times G \) to \( \tilde{K}_c \).

Since \( G_c \) is simply connected, there is a Lie group isomorphism, \( g \to \tilde{g} \) such that \( (\exp \tilde{X}) = \exp \tilde{X} \). We define for \( z_1, z_2 \in \Omega,

\[
D(z_1 : z_2) = \tilde{\mathcal{K}}((\exp z_2)(\exp \tilde{z}_1))^{-1}.
\]

Then \( D: \Omega \times \Omega \to K_c \). Hence \( D \) lifts to a map \( D: \Omega \times \Omega \to \tilde{K}_c \).

(6) For \( z_1 \cdot g \cdot z_2 \cdot g \) \( = \tilde{\mathcal{K}}(z_1 : g^{-1})\tilde{D}(z_1 : z_2)\mathcal{K}(z_2 : g) \).

**Lemma 2.1.**

\[
D \left( \text{Ad}(k) \left( \sum_{i=1}^{r} t_iX_i^\prime \right) \right) = \text{Ad}(k) \left( \sum_{i=1}^{r} t_iX_i^\prime \right) \left( \sum_{i=1}^{r} t_iX_i^\prime \right) \left( \sum_{i=1}^{r} t_iX_i^\prime \right) = \text{Ad}(k) \left( \sum_{i=1}^{r} t_iX_i^\prime \right) \left( \sum_{i=1}^{r} t_iX_i^\prime \right)
\]

for \( -1 < t_i < 1, i = 1, \ldots, r, k \in K \).

**Proof.**

\[
D \left( \text{Ad}(k) \left( \sum_{i=1}^{r} t_iX_i^\prime \right) \right) = k^{-1}D \left( \sum_{i=1}^{r} t_iX_i^\prime \right) k
\]

by (6).

\[
\exp \left( \sum_{i=1}^{r} t_iX_i^\prime \right) \exp \left( \sum_{i=1}^{r} t_iX_i^\prime \right) = \exp \left( \sum_{i=1}^{r} t_iX_i^\prime \right) \exp \left( - \sum_{i=1}^{r} t_iX_i^\prime \right).
\]

Now \( [X_i, X_j] = [X_i, X_j] = 0 \) if \( i \neq j \). Note that \( CX_i + CH_i + CX_i \) is isomorphic to the Lie algebra of \( SL(2, \mathbb{C}) \) with

\[
X_i \leftrightarrow \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \; X_i \leftrightarrow \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \; H_i \leftrightarrow \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.
\]

Under this identification, we see that

\[
\exp(tX_i)\exp(-tX_i) \leftrightarrow \begin{bmatrix} 1 - t^2 & t \\ -t & 1 \end{bmatrix}\exp \left( \frac{t}{1 - t^2} X_i \right)\exp \left( \frac{-t}{1 - t^2} X_i \right).
\]

The lemma follows from these observations.
Lemma 2.2. If $f \in C_0(\Omega)$ (continuous with compact support) then, for any $g \in G$,

$$
\int_{\Omega} f(z \cdot g) \, dz = \int_{\Omega} f(z) |\det(\text{Ad}(K(z; g^{-1}))_{(g^{-1})})|^{-2} \, dz
$$

where $dz$ is Lebesgue measure on $p^+$.

Proof. We note that $\exp: p^+ \to P^+$ is a holomorphic diffeomorphism and since $[p^+, p^+] = 0$ we see that, if $f \in C_0(P^+)$, then

$$
\int_{p^+} f(\exp z) \, dz = \int_{p^+} f(p^+) \, dp^+,
$$

where $dp^+$ is a Haar measure on $P^+$.

Thus, if $f \in C_0(\Omega)$, we may identify $f$ with an element of $C_0(P^+)$. We note that if all measures are properly normalized, and if $F \in C_0(G)$ then

$$
\int_{P^{-} \times K \times P^+} F(g) \, dg = \int_{P^{-} \times K \times P^+} F(p^{-} k p^+) |\det(\text{Ad}(k))_{(p^+)}|^2 \, dp^+ \, dk \, dp^+.
$$

Let $F \in C_0(G)$ be such that

$$
\int_{P^{-} \times K \times P^+} F(p^{-} k \exp(z)) \, dp^{-} \, dk = f(z) \quad \text{for all } z \in \Omega.
$$

Then

$$
\int_{\Omega} f(z \cdot g) \, dz = \int_{P^{-} \times K \times P^+} F(p^{-} k \exp(z)\nu(g)) \, dp^{-} \, dk \, dz
$$

$$
= \int_{P^{-} \times K \times P^+} F(\exp(z)) |\det(\text{Ad}(\mathcal{K}(z; g^{-1}))_{(g^{-1})})|^{-2} \, dx
$$

$$
= \int_{P^{-} \times K \times P^+} F(z) |\det(\text{Ad}(\mathcal{K}(z; g^{-1}))_{(g^{-1})})|^{-2} \, dx
$$

$$
= \int_{P^{-} \times K \times P^+} F(p^{-} k \exp(z)) |\det(\text{Ad}(k))_{(p^+)}|^2
$$

$$
\cdot |\det(\text{Ad}(\mathcal{K}(p^{-} k \exp(z)))_{(g^{-1})})|^{-2} \, dp^{-} \, dk \, dz
$$

$$
= \int_{\Omega} f(z) |\det(\text{Ad}(\mathcal{K}(z; g^{-1}))_{(g^{-1})})|^{-2} \, dz. \quad \text{Q.E.D.}
$$

Lemma 2.3. The $G$-invariant measure on $\Omega$ is

$$
d\mu(z) = |\det(\text{Ad}(D(z : z)_{(g^{-1})}))| \, dz.
$$

Proof. Let $f \in C_0(\Omega)$. Then

$$
\int_{\Omega} f(z \cdot g) d\mu(z) = \int_{\Omega} f(z \cdot g) \det(\text{Ad}(D(z : z))_{(g^{-1})}) \, dz
$$

$$
= \int_{\Omega} f(z) |\det(\text{Ad}(\mathcal{K}(z; g^{-1}))_{(g^{-1})})|^{-2} \det(\text{Ad}(D(z : g^{-1}) : z \cdot g^{-1}))_{(g^{-1})}) \, dz.
$$
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Now \( D(z \cdot g^{-1} : z \cdot g^{-1}) = \mathcal{K}(z : g^{-1})^{-1}D(z : z)\mathcal{K}(z : g^{-1}) \). If \( k \in K_c \), then 
\[ k = \exp iH_1 \cdot k_1 \text{ with } k_1 \in [K_0, K_c] \text{ and } k = \exp i\tilde{H}_1 \cdot \tilde{k}_1. \]
Now 
\[ \det(\text{Ad}(K_0))_{\rho^+} = \det(\text{Ad}(\tilde{k}_1))_{\rho^+} = 1 \text{ and } \det(\text{Ad}(k))_{\rho^+} = e^{int} (n = \dim_C \rho^+). \]
Hence
\[ |\det \text{Ad}(k)|_{\rho^+}^{-2\dim \rho^+} \det(\text{Ad}(\tilde{k}_1))_{\rho^+}^{-1} \det(\text{Ad}(k))_{\rho^+} = e^{2n \Im t} e^{-2n \Im t} = 1. \]
This proves the lemma.

**Lemma 2.4.** If \( (\pi, \rho, \langle , \rangle) \) is a finite dimensional unitary representation of \( K \) extended to \( \tilde{K}_c \) as a holomorphic representation then \( \pi(D(z : z)) \) is a positive definite operator for \( z \in \Omega \).

**Proof.** If \( X \in \mathfrak{k}_c \) then \( \langle \pi(x)\rho, \omega \rangle = -\langle \rho, \pi(\bar{x})\omega \rangle \), \( \rho, \omega \in \rho \). Thus, if \( H \in i\mathfrak{h}_a \), then \( \langle \pi(H)\rho, \omega \rangle = \langle \rho, \pi(H)\omega \rangle \). The result now follows from Lemma 2.1.

Let \( \pi = \{\alpha_1, \ldots, \alpha_l\} \) be the set of simple roots for \( \Delta^+ \). We may assume that \( \Delta^+_K \cap \pi = \{\alpha_2, \ldots, \alpha_l\} \). Let \( \Lambda_0 \in \mathfrak{h}^* \) be such that

\[ \langle \ast, \ast \rangle \Lambda_0, \alpha_j \rangle \langle \alpha_j, \alpha_j \rangle \text{ is a nonnegative integer } j = 2, \ldots, l \text{ and equal to } 0 \text{ if } j = 1. \]

Let \( (\pi_0, V^{\Lambda_0}) \) be the irreducible unitary representation of \( K \) with highest weight \( \Lambda_0 \). Let \( \langle \Lambda_1, \alpha_j \rangle = 0, j = 2, \ldots, l, 2\langle \Lambda_1, \alpha_1 \rangle / \langle \alpha_1, \alpha_1 \rangle = 1 \). Then \( \Lambda_1 \) is the differential of a character of \( K \), which we denote \( e^{\Lambda_1} \). Furthermore \( e^{\Lambda_1} \) is defined for all \( \lambda \in \mathbb{C} \), since \( K \) is simply connected. Thus \( e^{\Lambda_1} \otimes \pi_0 \) defines an irreducible holomorphic representation of \( \tilde{K}_c \) for all \( z \in \mathbb{C} \).

Let \( \mathcal{K}^{\rho_0} \) be the space of all \( f : \Omega \to V^{\rho_0} \) such that \( f \) is holomorphic. If \( g \in G \) and \( f \in \mathcal{K}^{\rho_0} \), define for \( \lambda \in \mathbb{C} \)

\[ (T_{\rho_0, \lambda}(g)f)(z) = (e^{\lambda\Lambda_1} \otimes \pi_0)(\mathcal{K}(x : g))f(x \cdot g). \]

Then it is easily seen that
\[ T_{\rho_0, \lambda}(g_1 g_2) = T_{\rho_0, \lambda}(g_1)T_{\rho_0, \lambda}(g_2). \]

Furthermore we have

**Lemma 2.5.** \( T_{\rho_0, \lambda}|_K = (e^{\lambda\Lambda_1} \otimes \pi_0)|_K \). Also \( T_{\rho_0, \lambda} = T_{\rho_0, \lambda}(g') \) if \( v(g) = v(g') \). That is \( T_{\rho_0, \lambda}|_K \) is actually a representation of \( K_0 \).

If \( f \in \mathcal{K}^{\rho_0} \) define, for \( \lambda \in \mathbb{R} \),

\[ ||f||^2_{\rho_0, \lambda} = \int_{\Omega} \langle (e^{\lambda\Lambda_1} \otimes \pi_0)(D(z : z))f(z), f(z) \rangle d\mu(z). \]

Lemma 2.4 implies that the right-hand side of the above formula is positive if it converges. Let \( H^{\rho_0, \lambda} \) be the space of all \( f \in H^{\rho_0} \) such that \( ||f||^2_{\rho_0, \lambda} < \infty \). Let \( \langle , \rangle_{\rho_0, \lambda} \) denote the associated inner product on \( H^{\rho_0, \lambda} \).
**Proposition 2.6.** If $f \in H^{\pi_\alpha}$ then $\|T_{\pi_\alpha}(g)f\|_{\pi_\alpha} = \|f\|_{\pi_\alpha}$ for all $g \in G$. If $H^{\pi_\alpha} \neq (0)$ then the constant functions are in $H^{\pi_\alpha}$. Furthermore $H^{\pi_\alpha}$ is complete and if $H^{\pi_\alpha} \neq (0)$ then the polynomial functions $f : \Omega \to V^{\pi_0}$ are dense in $H^{\pi_\alpha}$. Finally $(T_{\pi_\alpha}, H^{\pi_\alpha})$ is a unitary representation of $G$.

**Proof.**

\[
\|T_{\pi_\alpha}(g)f\|_{\pi_\alpha}^2 = \int_{\Omega} \langle (e^{\lambda_1} \otimes \pi_0)(D(z : z)) \cdot (e^{\lambda_1} \otimes \pi_0)(\mathbb{C}(z : g)f(z \cdot g)), f(z \cdot g) \rangle \, d\mu(z),
\]

\[
= \int_{\Omega} \langle (e^{\lambda_1} \otimes \pi_0)(\overline{\mathbb{C}(z : g)})^{-1}(e^{\lambda_1} \otimes \pi_0)(D(z : z)) \cdot (e^{\lambda_1} \otimes \pi_0)(\mathbb{C}(z : g))f(z \cdot g), f(z \cdot g) \rangle \, d\mu(z)
\]

\[
= \int_{\Omega} \langle (e^{\lambda_1} \otimes \pi_0)(D(z \cdot g : z \cdot g))f(z \cdot g), f(z \cdot g) \rangle \, d\mu(z)
\]

\[
= \int_{\Omega} \langle (e^{\lambda_1} \otimes \pi_0)(D(z : z))f(z), f(z) \rangle \, d\mu(z) = \|f\|_{\pi_\alpha}^2.
\]

This proves the first assertion.

If $f \in \mathcal{C}_c$ and $\gamma \in \hat{K}_0$ let

\[
f_\gamma(z) = d(\gamma)\int_{K_0} \overline{\chi_\gamma(k)} \pi_0(k)f(z \cdot k) \, dk
\]

\[
= d(\gamma)\int_{K_0} \overline{\chi_\gamma(k)} \pi_0(k)f(Ad(k)^{-1}z) \, dk.
\]

Here $\chi_\gamma$ is the character of $\gamma$ and $d(\gamma)$ is the dimension. The integral defining $f_\gamma$ clearly converges uniformly on compact subsets of $\Omega$. Thus $f_\gamma \in \mathcal{C}_c$. Furthermore $f = \sum f_\gamma$ with uniform and absolute convergence on compact subsets of $\Omega$. Using the Stone-Weierstrass theorem on an arbitrary $Ad(K_0)$ invariant open subset $\omega$ of $\Omega$ so that $\overline{\omega} \subset \Omega$, we see that $f_\gamma$ is a polynomial mapping of $\Omega$ to $V^{\lambda_0}$.

Let us now choose, for each $j \in \mathbb{Z}, j > 0$, $\Omega_j \subset \Omega$, $\Omega_{j+1} \supset \Omega_j$, and $\bigcup_{j=0}^{\infty} \Omega_j = \Omega$. (For example take $\Omega_j = \{Ad(k_0)(\Sigma_{i=1}^j X_{i,j})|k_0 \in K, |t_i| < j/(j + 1)\}$.) Then

\[
\int_{\Omega_j} \langle (e^{\lambda_1} \otimes \pi_0)(D(z : z))f(z), f(z) \rangle \, d\mu(z)
\]

\[
= \int_{\Omega_j} \sum_{\gamma \in \hat{K}} \langle (e^{\lambda_1} \otimes \pi_0)(D(z : z))f_\gamma(z), f_\gamma(z) \rangle \, d\mu(z)
\]

by Schur orthogonality.
The Lebesgue monotone convergence theorem implies that
\[ \|f\|^2_{\pi_\omega} = \lim_{f \to \infty} \int_{\Omega} \langle (e^{\lambda_1} \otimes \pi_0)(D(z : z))f(z), f(z) \rangle \, d\mu(z). \]
Suppose that \( f \in H^\pi_{\omega,\alpha} \). Then, setting \( \pi = e^{\lambda_1} \otimes \pi_0 \), we have
\[ \int_{\Omega} \langle \pi(D(z : z))f(z), f(z) \rangle \, d\mu(z) > \int_{\Omega} \langle \pi(D(z : z))f_j(z), f_j(z) \rangle \, d\mu(z) \]
for all \( j \). Hence \( f_j \in H^\pi_{\omega,\alpha} \). This also implies that if \( H^\pi_{\omega,\alpha} \neq (0) \) then there are polynomials in \( H^\pi_{\omega,\alpha} \) and the set of polynomials in \( H^\pi_{\omega,\alpha} \) is dense.

The fact that \( (T_{\pi_\omega}, H^\pi_{\omega,\alpha}) \) is a (continuous) unitary representation is standard once we have proved that \( H^\pi_{\omega,\alpha} \) is complete. To see that \( H^\pi_{\omega,\alpha} \) is complete, we note that if \( \omega \subset \Omega \) is any compact subset then there is \( C_\omega > 0 \) so that
\[ \|f\|^2_{\pi_\omega} > C_\omega \int_\omega \|f(z)\|^2 \, dz. \]
The completeness follows from
\[ \int_\omega \|f(z)\|^2 \, dz > C'_\omega \sup_{z \in \omega} \|f(z)\| \text{ with } C'_\omega > 0 \]
for \( \omega \) the closure of an open subset of \( \Omega \), \( \omega \) compact (cf. Helgason [4, Chapter 8]). We have therefore shown that \( (T_{\pi_\omega}, H^\pi_{\omega,\alpha}) \) is a unitary representation of \( G \).

Using the fact that the polynomial functions in \( H^\pi_{\omega,\alpha} \) are dense, we see that, if \( H^\pi_{\gamma,\alpha} = \{f_\gamma| f \in H^\pi_{\omega,\alpha}\} \), then \( H^\pi_{\gamma,\alpha} \) is finite dimensional. Using results of Harish-Chandra [1], we see that, if \( H^\pi_{\gamma,\alpha} = \sum_{\gamma \in K_\alpha} H^\pi_{\gamma,\alpha} \) (algebraic direct sum), then
\[ T_{\pi_\omega}(X)f = \frac{d}{dt} (T_{\pi_\omega}(\exp tX)f)|_{t=0} \]
defines a representation of \( g \) on \( H^\pi_{\gamma,\alpha} \).

We need the following lemma.

**Lemma 2.7.** Let for \( \gamma \in K_\alpha, \mathcal{H}_{\gamma} = \{f_\gamma| f \in \mathcal{H}\} \) (see the second part of the proof of Proposition 2.5). Let for \( X \in g \),
\[ (T_{\pi_\omega}(X)f)(z) = \frac{d}{dt} (T_{\pi_\omega}(\exp tX)f)(z)|_{t=0} \text{ for } f \in \mathcal{H}. \]
Then \( T_{\pi_\omega}(X)f \in \sum_{\gamma \in K_\alpha} \mathcal{H}_{\gamma} = \mathcal{H}_{\gamma} \) and \( (T_{\pi_\omega}, \mathcal{H}_{\gamma}) \) is a representation of \( g \). Furthermore, if \( W \subset \mathcal{H}_{\gamma} \) is a nonzero invariant subspace of \( \mathcal{H}_{\gamma} \), then \( W \) contains the constant functions.
Proof. Let \( f \in \mathcal{C}_f \). Then \( f \) extends to \( P - K_c \exp \Omega \) (looked upon as the universal covering space of \( P - K_c \exp \Omega \) by defining \( f(p \cdot k \exp z) = (e^{\lambda_1} \otimes \pi_0)(D(p \cdot k)(z)) \) for \( p^{-1} \in P^-, k \in K_c, z \in \Omega \). Let \( \mu \) be a nonzero element in the lowest weight space of \((V^{pe})^*\). Let for \( f \in \mathcal{C}_f\), \( A(f)(g) = \mu(f(g)) \) for \( g \in P - K_c \exp \Omega = N A^G \). If \( g \in U(\mathfrak{g}_0) \), the universal enveloping algebra of \( \mathfrak{g}_0 \), let \( \tilde{f}(g) = (g \cdot A(f))(e) \). Then \( \tilde{f}(zg) = (\lambda \Lambda_1 + \Lambda_0)(z)(f)(g) \) for \( z \in \mathfrak{n}^- \oplus \mathfrak{h} \) (here \( (\lambda \Lambda_1 + \Lambda_0)(\mathfrak{n}^-) = 0 \)). Since the map \( f \rightarrow \tilde{f} \) is injective (\( f \) is holomorphic), the Poincaré-Birkhoff-Witt theorem implies that \( \mathcal{C}_{f^A} = \Sigma_{\Lambda \in \mathfrak{h}^*} \mathcal{C}_{\Lambda} \) with \( \Lambda = (\lambda \Lambda_1 + \Lambda_0) - \Sigma n_i \alpha_i \) with \( n_i > 0, n_i \in \mathbb{Z} \). Here for \( \Lambda \in \mathfrak{h}^* \), \( \mathcal{C}_{\Lambda} = \{ f \in \mathcal{C}_{f^A} | T_{\varpi_\Lambda}(h)f = \Lambda(h)f \) for \( h \in \mathfrak{h} \} \). Furthermore, if \( f \in \mathcal{C}_{f^A} \) and \( T_{\varpi_\Lambda}(n^+)^{f^A} = 0 \), then \( f \in \mathcal{C}_{\Lambda + \Lambda_0} \). We observe that \( \mathcal{C}_{\Lambda + \Lambda_0} \) consists of the functions \( f(z) = v \) with \( v \) in the highest weight space of \( V^{\Lambda_0} \).

Now \( W \subset \mathcal{C}_{f^A}, W \neq 0 \) implies \( W = \Sigma(W \cap \mathcal{C}_{\Lambda}) \). Let \( \Lambda' \) be such that \( W \cap \mathcal{C}_{\Lambda} \neq 0 \) and \( \Lambda' = \lambda \Lambda_1 + \Lambda_0 - \Sigma n_i \alpha_i \) with \( \Sigma n_i \) minimal. Then \( T_{\varpi_\Lambda}(n^+)(W \cap \mathcal{C}_{\Lambda}) = 0 \). Hence \( \Lambda = \lambda \Lambda_1 + \Lambda_0 \). Thus the constants are in \( W \).

We now conclude the proof of Proposition 2.6. \( H^{\varpi_\Lambda} \subset \mathcal{C}_{f^A} \) is \( T_{\varpi_\Lambda} \) invariant and nonzero if \( H^{\varpi_\Lambda} \neq 0 \). Thus, by the above, \( H^{\varpi_\Lambda} \neq 0 \) implies \( H^{\varpi_\Lambda} \) contains the constants. Thus, if \( H^{\varpi_\Lambda} \neq 0 \), then

\[
\int_{\Omega} \langle \pi(D(z : z))v, v \rangle d\mu(z) < \infty \quad \text{for all } v \in V^{\Lambda_0}. \tag{1}
\]

If \( f \) is a polynomial function from \( \Omega \) to \( V_0 \), then \( f \) is a linear combination of elements of the form \( \varphi \cdot v \) with \( \varphi: \Omega \rightarrow C \) a polynomial and \( v \in V^{pe} \). But

\[
\int_{\Omega} \langle \pi(D(z : z))\varphi(z)v, \varphi(z)v \rangle d\mu(z) = \int_{\Omega} |\varphi(z)|^2 \langle \pi(D(z : z))v, v \rangle d\mu(z) < \infty.
\]

Corollary 2.8. If \( \lambda \in \mathbb{R} \), then \( H^{\varpi_\Lambda} \neq 0 \) if and only if \( \int_{\Omega} \langle \pi(\lambda \Lambda_1 \otimes \pi_0)(D(z : z))v, v \rangle d\mu(z) < \infty \) for all \( v \in V^{\Lambda_0} \).

Corollary 2.9. If \( (T_{\varpi_\Lambda}, \mathcal{C}_f) \) admits an invariant inner product then \( (T_{\varpi_\Lambda}, \mathcal{C}_f) \) is irreducible.

Proof. Let \( W \subset \mathcal{C}_f \) be invariant, \( W \neq 0 \). Then \( W^\perp \subset \mathcal{C}_f \) is nonzero if and only if \( W \) contains the constants (see Lemma 2.7). Hence \( W^\perp = 0 \) since \( W \) contains the constants.

Lemma 2.10. Let \( z \in C \). If

\[
\int_{\Omega} (e^{z \Lambda_1} \otimes \pi_0)(D(z : z)) d\mu(z)
\]

converges, then it is equal to \( d(\pi_0, z)I \) with \( d(\pi_0, z) \in C \).
Proof. $\int_{\Omega} (e^{zA_1} \otimes \pi_0) (D(z : u : z \cdot u)) \, d\mu(z) = \int_{\Omega} (e^{zA_1} \otimes \pi_0) (D(z : z)) \, d\mu(z)$ for all $u \in K$. The result now follows from the irreducibility of $(\pi_0, V^{A_0})$ and (6) (preceding Lemma 2.1).

We observe that more is true.

**Corollary 2.11.**

$$d(\pi_0, z) = \frac{1}{d_{\pi_0}} \int_{\Omega} e^{zA_1} (D(z : z)) \chi_{\pi_0} (D(z : z)) \, d\mu(z)$$

with $\chi_{\pi_0}$ the character of $(\pi_0, V^{A_0})$ and $d_{\pi_0} = \dim V^{A_0}$.

3. The analytic continuation of the holomorphic discrete series. We retain the notation of §2. Let $\mathcal{P} (p^+)$ be the space of holomorphic polynomial functions from $p^+$ to $\mathbb{C}$. On $p^+$ put the inner product $\langle X, Y \rangle = - B(X, \tau Y)$ where $B$ is the Killing form of $g_c$ and $\tau$ is the conjugation of $g_u = f \otimes ip$. Extend $\langle \cdot, \cdot \rangle$ to an inner product on $\mathcal{P} (p^+) = S((p^+)^*)$ in the usual fashion ($S(V)$ denotes the symmetric algebra on the vector space $V$). Let $\langle \cdot, \cdot \rangle$ also denote a $K$-invariant inner product on $V^{A_0}$. On $\mathcal{P} (p^+) \otimes V^{A_0}$ put the tensor product the inner product which we also denote $\langle \cdot, \cdot \rangle$. In the following, $\mathcal{P}^j (V)$ will denote the polynomial functions on $V$ homogeneous of degree $j$.

For the rest of this section $(\pi_0, V^{A_0})$ will be fixed.

**Lemma 3.1.** If $f \in \mathcal{P}^j (p^+) \otimes V^{A_0}$ then

$$\int_{\Omega} \langle (e^{zA_1} \otimes \pi_0) (D(z : z)) f(z), f(z) \rangle \, d\mu(z) = \langle A_j(\lambda) f, f \rangle,$$

converges absolutely for $((\Re \lambda) A_1 + A_0 + \rho)(H_\beta) < 0$ for all $\beta \in \Delta^+$. ($\rho = \frac{1}{2} \sum_{a \in \Delta_+} a.)$ Moreover, $\lambda \mapsto A_j(\lambda)$ extends to a rational function from $\mathbb{C}$ to $\text{End} (\mathcal{P} (p^+) \otimes V^{A_0})$. Furthermore the singularities of $A_j(\lambda)$ are only at the points $\lambda \in \mathbb{Z}/2$ (half integers). Finally, if $\lambda \in \mathbb{R}$ and if $A_j(\lambda)$ is defined then $A_j(\lambda)$ is Hermitian.

**Proof.** Let $b_+ = \sum_{i=1} r_i C H_{n_i}$. Let $b^- = \{ H \in b | \gamma_i (H) = 0 \, \text{for} \, i = 1, \ldots, r \}$. Let $\Delta_0 = \{ a \in \Delta | a|_{b^+} = 0 \}$. Then (up to constants of normalization) if $f \in C_0 (\Omega)$

$$\int_{\Omega} f(z) \, dz = \int_{\Delta_0} \int_{1 > t_1 > t_2 > \cdots > t_r > 0} f \left( \frac{\text{Ad}(k) \left( \sum_{i=1}^r t_i X_{n_i} \right)}{\prod_{a \in \Delta^+ - \Delta_0} |a| \left( \sum_{i=1}^r t_i H_{n_i} \right)} \right) \, dk \, dt_1 \cdots \, dt_r. \quad (1)$$
To prove (1) we note that if $A: \mathfrak{p}^+ \to \mathfrak{p}$ is defined by $A(X) = \frac{1}{2}(X + \bar{X}) \quad ((X_1 + iX_2) = X_1 - iX_2$ for $X_1, X_2 \in \mathfrak{g})$, then $A$ is a real linear isomorphism such that $A \circ \text{Ad}(k) = \text{Ad}(k) \circ A$ for $k \in K_0$. This implies that $\mathfrak{p}^+ \to \text{Ad}(K_0)\mathfrak{g}$. In fact, if $a = \sum \mathbb{R} X_{\gamma^+} + X_{\gamma^-}$, then $a$ is maximal abelian in $\mathfrak{p}$ and $\text{Ad}(K_0)a = \mathfrak{p}$ (cf. Helgason [5]). On the other hand,

$$\int_{\mathfrak{p}} f(x) \, dx = \int_{\mathfrak{a}^+} \left( \prod_{\lambda \in \Lambda^+} |\lambda(H)|^{m_\lambda} \right) \left( \int_{K_0} f(\text{Ad}(k)H) \, dk \right) \, dH \quad (\ast)$$

where $\Lambda^+$ is the set of positive restricted roots of $(\mathfrak{g}, \mathfrak{a})$ relative to some order, $m_\lambda$ is the dimension of the restricted weight space and $\alpha^+$ is a positive Weyl chamber ($\lambda(H) > 0$ for $\lambda \in \Lambda^+$). $(\ast)$ can be found in say Helgason [5]. Now (1) follows by applying the Cayley transform (cf. Harish-Chandra [3]).

We now compute

$$\prod_{\alpha \in \Delta^+ - \Delta_0} |\alpha(\sum t_i H_i)|, \quad t_1, \ldots, t_r \in \mathbb{R}, \quad t_i > 0.$$ 

To do this, we recall some results of Harish-Chandra [3].

Let for $1 < i < r$, $C_i = \{\alpha \in \Delta^+_a | |\alpha|_{\mathfrak{h}^+} = -\frac{1}{2} \gamma_i\}$, for $1 < i < j < r$, $C_{ij} = \{\alpha \in \Delta^+_a | |\alpha|_{\mathfrak{h}^+} = \frac{1}{2}(\gamma_i - \gamma_j)\}$. Then

$$\Delta^+_a = \Delta^+_a \cup \bigcup_{i=1}^{r} C_i \cup \bigcup_{1 < i < j < r} C_{ij}.$$ 

Let $P_i = \{\alpha \in \Delta^+_a | |\alpha|_{\mathfrak{h}^+} = \frac{1}{2} \gamma_i\}$, $P_{ij} = \{\alpha \in \Delta^+_a | |\alpha|_{\mathfrak{h}^+} = \frac{1}{2}(\gamma_i + \gamma_j)\}$, $1 < i < j < r$. Then

$$\Delta^+_a = \{\gamma_1, \ldots, \gamma_r\} \cup \bigcup_{i=1}^{r} P_i \cup \bigcup_{1 < i < j < r} P_{ij}.$$ 

Let $r_i$ (resp. $r_{ij}$) be the order of $C_i$ (resp. $C_{ij}$). Then $r_i$ (resp. $r_{ij}$) is the order of $P_i$ (resp. $P_{ij}$). This says that

$$\prod_{\alpha \in \Delta^+ - \Delta_0} |\alpha(\sum t_i H_i)| = C \prod_{i=1}^{r} |t_i^{2r_i+1}| \prod_{1 < i < j < r} |(t_i^2 - t_j^2)|^{\gamma_{ij}}.$$ 

Furthermore, by the above, $\alpha^+$ corresponds to $\{\sum t_i (X_{\gamma^+} + X_{\gamma^-}), \quad t_1 > t_2 > \ldots > t_r > 0\}$. This implies that for $t_1 > t_2 > \ldots > t_r > 0$

$$\prod_{\alpha \in \Delta^+ - \Delta_0} |\alpha(\sum t_i H_i)| = C \prod_{i=1}^{r} t_i^{2r_i+1} \prod_{1 < i < j < r} t_i^2 - t_j^2 = P(t_1, \ldots, t_r).$$
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Now (from (1), Lemmas 2.2, 2.3 and \( \rho_\rho = \frac{1}{2}(\text{tr ad } H|_{1_\rho}) \))

\[
\left( e^{\lambda A_i} \otimes \pi_0 \right) (D(z; z)) \langle f(z), f(z) \rangle \, d\mu(z)
\]

\[
= \int_{1 > t_1 > t_2 > \cdots > t_r > 0} \prod_{i=1}^{r} \prod_{1 < j < r} (t_i^2 - t_j^2)
\]

\[
\cdot \prod_{i=1}^{r} (1 - t_i^2)^{-(\lambda A_i + 2\rho)(H_n)}
\]

\[
\cdot \left( \int_{K_0} \left( \pi_0(k)^{-1} \pi_0(\exp(-\sum (\log |1 - t_i|^2 H_n)) \pi_0(k)
\right)
\]

\[
\cdot f(\text{Ad}(k)\left( \sum t_i X_n \right)), f(\text{Ad}(k)\left( \sum t_i X_n \right)) \right) \, dk_0 \right) \, dt_1 \cdots \, dt_r.
\]

The integral over \( K_0 \) above is a polynomial \( q(t_1, \ldots, t_r) \) in \( t_1, \ldots, t_r \). Thus we have

\[
\left( e^{\lambda A_i} \otimes \pi_0 \right) (D(z; z)) \langle f(z), f(z) \rangle \, d\mu(z)
\]

\[
= \int_{1 > t_1 > t_2 > \cdots > t_r > 0} \prod_{i=1}^{r} (1 - t_i^2)^{-(\lambda A_i + 2\rho)(H_n)}
\]

\[
\cdot \rho(t_1, \ldots, t_r) q(t_1, \ldots, t_r) \, dt_1 \cdots \, dt_r.
\]

As observed in Harish-Chandra [3] such an integral is a rational function of \( \lambda \) (it also clearly converges for \( \lambda < 0 \) and \( \lambda \) large since \( \Lambda_i(H_n) = 1, i = 1, \ldots, r \). The assertion on the pole structure follows from the fact that \( 2\rho_\rho(H_n) \) is an integer for \( i = 1, \ldots, r \).

The absolute convergence statement follows from the case \( f \equiv v \in V^{\Lambda_0} \) in Corollary 2.11 and the work of Harish-Chandra [3]. The last assertion is clear (see Lemma 2.4).

**Lemma 3.2 (Harish-Chandra [3]).**

\[
d(\pi_0, z)^{-1} = \prod_{\alpha \in \Delta_\rho^+} \left\{ \frac{-(\Lambda_0 + z\Lambda_1 + \rho)(H_\alpha)}{\rho(H_\alpha)} \right\}
\]

up to constants of normalization.

Now let \( c(\pi_0) = \text{Sup}\{ z \in \mathbb{R}|(z\Lambda_1 + \Lambda_0 + \rho)(H_\alpha) < 0 \text{ for } \alpha \in \Delta_\rho^+ \} \).

Lemma 2.4 combined with Lemma 3.1 and Corollary 2.7 imply that \( (T_{\pi_0}, H^{\rho^+}) \) is an irreducible representation of \( G \) for \( z < c(\pi_0) \). This implies that \( (T_{\pi_0}, \mathcal{X}_\rho^+) \) is irreducible as a representation of \( U(\mathfrak{g}_C) \). However, more is true.

**Lemma 3.3 (Harish-Chandra [2]).** \( (T_{\pi_0}, \mathcal{X}_\rho^+) \) is irreducible for \( z < c(\pi_0) \).

Combining this with results of Harish-Chandra [2], [3], we have the following lemma.
Lemma 3.4. If \( f \in \mathcal{H}_p \) then
\[
\lim_{\lambda \to c(\pi_0)} d_{\sigma_0}(\lambda)^{-1} \| f \|_{\pi_0, \lambda}^2 = \| f \|_{\pi_0}^2
\]
exists and induces a positive definite invariant inner product on \( \mathcal{H}_p \). This representation extends to \( G \) on the Hilbert space completion of \( \mathcal{H}_p \).

Note. This gives another realization of the “limits of holomorphic discrete series” in Knapp-Okamoto [5].

We also note that Lemmas 3.1, 3.4 have the following consequence.

Lemma 3.5. Let for \( f \in \mathcal{H}_p \),
\[
| f |_{\sigma_0, \lambda}^2 = d(\sigma_0, \lambda)^{-1} \| f \|_{\pi_0, \lambda}^2.
\]
Then \( \lambda \to | f |_{\sigma_0, \lambda}^2 \) is a rational function of \( \lambda \) and there is a constant \( \tilde{c}(\pi_0) > c(\pi_0) \) so that \( \cdot \cdot \cdot |_{\sigma_0, \lambda}^2 \) defines a positive definite invariant inner product on \( \mathcal{H}_p \) for \( \lambda \in R \) and \( \lambda < \tilde{c}(\pi_0) \).

In §4 we will determine \( \tilde{c}(1) \) for \( G_0 = SU(n, 1) \).

Let \( \mathcal{H}_p \) be as in Lemma 2.7. Then \( \mathcal{H}_p = \Sigma_{\gamma \in K_0} \mathcal{H}_\gamma \) where \( \mathcal{H}_\gamma = \{ f : f \in \mathcal{H}_p \} \) and \( f_\gamma \) is as in the proof of Proposition 2.6. Let \( E_\gamma : \mathcal{H}_\gamma \to \mathcal{H}_\gamma \) be defined (as in the proof of Proposition 2.6) by
\[
(E_\gamma f)(z) = d(\gamma) \int_{K_0} \chi_\gamma(k) \pi_0(k) f(Ad(k)^{-1}z) \, dk.
\]

If \( g \in G, \gamma \in K_0 \) define
\[
\phi_{\gamma, \sigma_0}(g) = tr(E_\gamma T_{\sigma_0}(g)|_{\mathcal{H}_p}).
\]

Lemma 3.6. The function \( C \times G \to C \) given by \( (\lambda, g) \mapsto \phi_{\gamma, \sigma_0}(g) \) is continuous and holomorphic in \( \lambda \).

Proof.
\[
(E_\gamma T_{\sigma_0}(g)f)(z) = d(\gamma) \int_{K_0} \chi_\gamma(k) (T_{\sigma_0}(g)f)(Ad(k)^{-1}z) \, dk
\]

\[
= d(\gamma) \int_{K_0} \chi_\gamma(k) (e^{\lambda h_1} \otimes \pi_0(3k \cdot k : g))f(z \cdot kg) \, dk
\]

\[
= d(\gamma) \int_{K_0} \chi_\gamma(k) e^{\lambda h_1}(3k \cdot g)(\pi_0(3k \cdot k : g))f(z \cdot kg) \, dk.
\]

If \( \mu \in (V^{\lambda_0})^* \) and \( z \in \Omega \) are fixed, then the above computation clearly implies that \( (\lambda, g) \mapsto \mu((E_\gamma T_{\sigma_0}(g)f)(z)) \) satisfies the continuity and holomorphy properties asserted for \( \phi_{\gamma, \sigma_0} \). It is easily seen that, if \( (\mu \otimes \varepsilon_2)f = \mu(f(z)) \), then the set \( \{ \mu \otimes \varepsilon_2 | \mu \in (V^{\lambda_0})^*, z \in \Omega \} \) spans \( (\mathcal{H}_p)^* \). Thus \( \phi_{\gamma, \sigma_0} \) is a linear combination of functions of the form \( g \mapsto (\mu \otimes \varepsilon_2)(E_\gamma T_{\sigma_0}(g)f) \).

This proves the lemma.
Lemma 3.7. Let \( \mathcal{D}'(G) \) be the space of distributions on \( C_0^\infty(G) \) with the weak topology. If \( f \in C_0^\infty(G) \) define

\[
\theta_{\pi, \lambda}(f) = \sum_{\gamma \in K_0} \int_G \phi_{\gamma, \lambda}(g) f(g) \, dg.
\]

Then the series defining \( \theta_{\pi, \lambda} \) converges absolutely and uniformly on compact subsets of \( C \). Furthermore the function \( \lambda \to \theta_{\pi, \lambda} \) defines a holomorphic function from \( C \) to \( \mathcal{D}'(G) \).

Proof. Let \( a = \sum_{i=1}^r R(X_{\gamma_i} + X_{-\gamma_i}) \). Let \( A = \exp a \). Let \( g = t \oplus a \oplus n \) be an Iwasawa decomposition of \( g \) corresponding to \((t, a)\). Then \( G = KAN \) is an Iwasawa decomposition of \( G \). Let \( \tilde{\rho}(H) = \frac{1}{2} \text{tr}(\text{ad} \, H a) \) for \( H \in a \). Then, if \( f \) is integrable on \( G \), we have

\[
\int_G f(g) \, dg = \int_{K \times A \times N} f(kan) e^{2\pi i \text{log}(a)} \, dk \, da \, dn
\]

where \( \text{log} : A \to \mathbb{R} \) is the inverse map to \( \exp : \mathbb{R} \to A \).

Now \( t = R(t_1 H_1) \oplus t_1, t_1 = [k, k] \). Let \( K_1 \) be the connected subgroup of \( K \) corresponding to \( t_1 \). Then \( K_1 \) is compact and simply connected and the map \( R \times K_1 \to K \) given by \( (t, k_1) \mapsto \exp(t_i H_1)k_1 \) is a Lie isomorphism of \( R \times K_1 \) with \( K \). We therefore see that

\[
\int_G f(g) \, dg = \int_{R} \left( \int_{K_1 \times A \times N} f(\exp(t_i H_1)k_1 an) e^{2\pi i \text{log}(a)} \, dk_1 \, da \, dn \right) \, dt
\]

for \( f \) absolutely integrable on \( G \). Now let \( f \in C_0^\infty(G) \). Then

\[
\int_G \phi_{\gamma, \lambda}(g) f(g) \, dg = \int_R \left( \int_{K_1 \times A \times N} \phi_{\gamma, \lambda}(\exp t_i H_1) k_1 an \right)
\]

\[
\cdot f(\exp(t(i H_1))k_1 an) e^{2\pi i \text{log}(a)} \, dk_1 \, da \, dn \right) \, dt.
\]

Now

\[
T_{\pi, \lambda}(\exp(t_i H_1)k_1 an) |_{\mathcal{D}'_0} = e^{it(\lambda_1(H_1) + \lambda)} T_{\pi, \lambda}(k_1 an),
\]

where \( \pi_1(\exp t_i H_1) = e^{it \lambda} \) for \( t \in \mathbb{R} \) (see Lemma 2.5). Hence we have

\[
\int_G \phi_{\gamma, \lambda}(g) f(g) \, dg = \int_R e^{it(\lambda_1(H_1) + \lambda)} \int_{K_1 \times A \times N} \phi_{\gamma, \lambda}(k_1 an)
\]

\[
\cdot f(\exp(t(i H_1))k_1 an) e^{2\pi i \text{log}(a)} \, dk_1 \, da \, dn.
\]

If \( \mu \in C \) and \( \delta \in \hat{K}_1 \) define

\[
\hat{f}(\mu : \delta : k_1 an) = d(\delta) \int_R e^{i\mu t} \left( \int_{K_1} \chi_{\delta}(k) f((\exp(t_i H_1))k^{-1}k_1 an) dk \right) \, dt.
\]
Let \( \| \delta \| \) be the norm of the highest weight of \( \delta \). Now \( f(\mu : \delta : an) \) has compact support on \( AN \). If \( \omega \subset C \) is a compact subset of \( C \) and \( l_1, l_2 > 0 \) then

\[
|\phi_{\gamma}^{\pi^\omega}(k_1 an) f(\lambda \Lambda_1 (H_1) + \lambda_\gamma : \gamma_0 : k_1 an)| \\
< C_{l_1, l_2, 0}(f)(1 + |\lambda + \lambda_\gamma|^2)^{-l_1}(1 + |\gamma_0|^2)^{-l_2}\psi(an)
\]

with \( \psi \in C_0^\infty(AN) \) and \( f \to C_{l_1, l_2, 0}(f)\psi(an) \) a continuous function from \( C_0^\infty(G) \to C \). Here \( \gamma_0 \) is the restriction of \( \gamma \) to \( K_1 \). These estimates follow the Paley-Wiener theorem for \( R \) and \( K_1 \). This implies that

\[
\sum_{\gamma \in K_0} \left| \int_G \phi_{\gamma}^{\pi^\omega}(g) f(g) \, dg \right|
< C_{l_1, l_2, 0}(f) \sum_{\gamma \in K} \left( 1 + |\lambda \Lambda_1 (H_1) + \lambda_\gamma|^2 \right)^{-l_1}(1 + |\gamma_0|^2)^{-l_2}
\]

for \( \lambda \in \omega \) with \( f \to C_{l_1, l_2, 0}(f) \) continuous on \( C_0^\infty(G) \). This clearly implies the lemma (cf. Wallach [10, Chapter 5]).

**Corollary 3.8.** The character of a "limit of holomorphic discrete series" is the limit of the characters of holomorphic discrete series. That is

\[
\lim_{\lambda \to C(\pi_0)} \theta_{\pi_{0, \lambda}} = \theta_{\pi_{0, C(\Lambda_0)}}
\]

and, since the representations \((T_{\pi_{0, \lambda}}, H_{\pi_{0, \lambda}}), \lambda < C(\Lambda_0), \) and \((T_{\pi_{0, C(\Lambda_0)}}, H_{\pi_{0}})\) are trace class this implies the statement.

**Note.** This implies that the Harish-Chandra formula [4] for the character of holomorphic discrete series is also true for limits of discrete series. Lemma 3.7 also says that the "signs" involved in the formula can be chosen so that the formula is holomorphic in \( \lambda \) where \( \Lambda = \Lambda_0 + \lambda \Lambda_1 \). This also allows one to compute characters of unitary representations which are "past the limit of holomorphic discrete series" (see Lemma 3.5).

**4. Example:** The universal covering group of \( SU(n, 1) \). In this case we may identify \( p^+ \) with \( C^n \) and \( \Omega \) with the unit ball in \( C^n \): \( \Omega = \{ z \in C^n | \sum |z_i|^2 < 1 \} \). If \( g \in SU(n, 1) = G_0 \) then

\[
g = \begin{bmatrix} A & b \\ c^* & d \end{bmatrix}
\]

with \( A, n \times n; c, b, n \times 1; d \in C \). The condition that \( g \in G_0 \) is

\[
gJ^Tg = J, \quad \det g = 1
\]

with

\[
J = \begin{bmatrix} I & 0 \\ 0 & -1 \end{bmatrix}
\]
The action of $G_0$ on $\Omega$ (which we write on the left) is

$$g \cdot z = (\langle z, c \rangle + d)^{-1}(Az + b).$$

Actually $G_0$ acts on $\Omega$. $K_0$ is the subgroup of $G_0$ consisting of the matrices

$$\begin{bmatrix} u & 0 \\ 0 & (\det u)^{-1} \end{bmatrix},$$

$u \in U(n)$. In particular $U(n)$ acts on $\mathfrak{p}^+ = \mathbb{C}^n$ by

$$u \cdot z = (\det u)uz$$

where $Uz$ is the usual action of $U(n)$ on $\mathbb{C}^n$. It is classical that the representation of $U(n)$ on $\mathfrak{p}^/(\mathbb{C}^n)$ given by

$$(u \cdot f)(z) = f(u^{-1} \cdot z)$$

is irreducible.

We consider the case $\pi_0 = 1$. For $f \in \mathfrak{p}^/(\mathbb{C}^n)$

$$\int_{\Omega} e^{\lambda \Lambda_1(D(z : z))} f(z) \overline{f(z)} \, d\mu(z) = c_\lambda \langle f, f \rangle$$

since $\mathfrak{p}^/(\mathbb{C}^n)$ is irreducible. It is therefore convenient to take $X_{\gamma_1} = e_1 \in \mathbb{C}^n (e_1, \ldots, e_n$ the standard basis of $\mathbb{C}^n)$ and $f(z) = z^t$. From this we see that

$$\int_{\Omega} e^{\lambda \Lambda_1(D(z : z))} f(z) \overline{f(z)} \, d\mu(z)$$

$$= \int_0^1 t^{2n-1} \int_{U(n)} e^{\lambda \Lambda_1(D(tX_{\gamma_1} : tX_{\gamma_1}))} |f(Ad(k)tX_{\gamma_1})|^2 \, dk \, \frac{dt}{(1 - t^2)^{n+1}}$$

$$= \int_0^1 t^{2n-1}(1 - t^2)^{-\lambda \Lambda_1(H_{\gamma_1})-n-1} \left( \int_{U(n)} |B(Ad(k)tX_{\gamma_1}, X_{-\gamma_1})|^2 \, dk \right) \, dt$$

$$= \frac{1}{d_j} \int_0^1 t^{2n-1}(1 - t^2)^{-\lambda \Lambda_1(H_{\gamma_1})-n-1} \, dt\, dt.$$

(Here we have used the orthogonality relations for $U(n)$ and $d_j = \dim S^j(\mathfrak{p}^+)$.)

$$= \frac{1}{2d_j} B(n + j, -\lambda \Lambda_1(H_{\gamma_1}) - n)$$

where $B(z, \omega)$ is the classical beta function (cf. Whittaker and Watson [11]). Now $\Lambda_1(H_{\gamma_1}) = 1$. We therefore have the following lemma.

**Lemma 4.1.** If $G_0 = SU(n, 1)$ and $\pi_0 = 1$ then $C(1) = -n$. That is $T_{1,\lambda}$ is holomorphic discrete series for $\lambda < -n$. Furthermore
\[ A_j(\lambda) = \frac{1}{2d_j} B(n + j, -\lambda - n) I = \left( \frac{1}{2d_j} \right) \frac{(n + j - 1)!}{\prod_{s=0}^{n+j-1} (-\lambda + j - s)} I. \]

In particular

\[ d(1, \lambda) = \frac{(-1)^{n-1}}{2 \prod_{s=0}^{n-1} (\lambda + n - s)} \]

(see Lemma 3.2).

**Corollary 4.1.**

\[ d(1, \lambda)^{-1} A_j(\lambda) = \frac{(-1)^{j}(n + j - 1)!}{d_j} \left( \prod_{s=0}^{j-1} (\lambda - s) \right)^{-1} I. \]

**Corollary 4.2.** If \( f_1, f_2 \in \mathcal{B}(C^n) \) then \( d(1, \lambda)^{-1} \langle f_1, f_2 \rangle_{1, \lambda} \) defines a positive definite \( T_{1, \lambda} \) invariant inner product on \( \mathcal{B}(C^n) = \mathcal{H}_F \) for \( \lambda \in \mathbb{R}, \lambda < 0 \). These representations extend to unitary representations of \( G \) on the Hilbert space completion \( H^{1,\lambda} \) of \( \mathcal{B}(C^n) \).

**Note.** The representations \( -n < \lambda < 0 \) are the direct generalization of the "extra representations" of Sally [8] that go "past" the limit holomorphic relative discrete series of the universal covering group of \( SL(2, \mathbb{R}) \).

We also note that \( (T_{1, -n}, H^{1, -n}) \) is the Hardy space for \( \Omega \).

**Lemma 4.2.** There is a constant \( C > 0 \) so that if \( f \in \mathcal{B}(C^n) \) then

\[ \lim_{\lambda \to -n} d(1, \lambda)^{-1} ||f||_{1, \lambda}^2 = \lim_{t \to 1} C \int_{S^{2n-1}} |f(t\omega)|^2 d\omega, \]

where \( d\omega \) is invariant measure on \( S^{2n-1} \).

**Proof.** This result follows directly from Corollary 4.2, Corollary 2.7 and the results of Knapp-Okamoto [6].

**References**


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