AN AXIOM FOR NONSEPARABLE BOREL THEORY

BY

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Abstract. Kuratowski asked whether the Lebesgue-Hausdorff theorem held for metrizable spaces. A. Stone asked whether a Borel isomorphism between metrizable spaces must be a generalized homeomorphism. The existence of a Q set refutes the generalized Lebesgue-Hausdorff theorem. In this paper we discuss the consequences of the axiom of the title, among which are "yes" answers to both Kuratowski's and Stone's questions. The axiom states that a point finite analytic additive family is σ discretely decomposable. We show that this axiom is valid in the model constructed by collapsing a supercompact cardinal to ω2 using Lévy forcing. Our proof displays relationships between σ discretely decomposable families, analytic additive families and d families.

1. Introduction. Several fields of mathematics have roots in the theory of a real variable, which flourished at the end of the nineteenth century and the beginning of the twentieth century. At this time, investigation centered on complete separable metric spaces and their metrizable continuous images. As mathematicians became interested in abstract topological spaces, it was natural to ask whether classical theorems in the theory of a real variable generalized to a wider class of spaces. Indeed, Kuratowski explicitly asked as early as 1935 [11], whether certain theorems of Baire, Lebesgue, Hausdorff, and Banach held for all metrizable spaces. More recently, A. Stone began an investigation of the Borel theory of metrizable spaces. Because this paper centers on such questions, we make the convention that all spaces considered are metrizable. (The exceptions, clear from context, are in remarks in §4.) Stone left open the important question of whether a Borel isomorphism must be a generalized homeomorphism. A one-to-one function f is a Borel isomorphism if both f and f⁻¹ take open sets to Borel sets; f is a generalized homeomorphism of type (α, β) if f takes open sets to Σβ sets and f⁻¹ takes open sets to Σα sets. (Σα is the family of Borel sets of additive class α; Πα is the family of Borel sets of multiplicative class α; α, β < ω1.) Significant progress on these questions was made by Stone's student...
Hansell [5], [6]. Hansell defined $\sigma$ discrete decompositions, and showed that much of the theory for separable spaces and complete spaces carries over to $\sigma$ discrete functions. That for absolutely analytic spaces the functions in question are $\sigma$ discrete follows from

1.1. Hansell’s Theorem [5]. A disjoint, absolutely analytic, additive family is $\sigma$ discretely decomposable.

A set is analytic in a space if it can be obtained from the open sets by the $\emptyset$ operation. A set is absolutely analytic if it is analytic in every superspace, or, equivalently, in one completion. Let $P$ be a property of sets, such as absolutely analytic. A family of sets is $P$ additive if the union of every subfamily has property $P$. A family of subsets in a space is discrete if every point of the space has a neighborhood meeting at most one element of the family. A family $S = \{S_i: i \in I\}$ is $\sigma$ discretely decomposable ($\sigma$ dd) if there is a family $S' = \{S_{i,n}: i \in I, n \in \omega\}$, a $\sigma$ discrete decomposition of $S$, satisfying

\begin{align*}
\text{for all } i \in I, \quad S_i &= \bigcup \{S_{i,n}: n \in \omega\}, \quad (1.1) \\
\text{for all } n \in \omega, \quad \{S_{i,n}: i \in I\} &\text{ is a discrete family.} \quad (1.2)
\end{align*}

Since each $x \in S_{i,n}$ has a neighborhood meeting only $S_{i,n}$ of $\{S_{j,n}: j \in I\}$,

\begin{equation}
\text{for all } i \in I, n \in \omega \text{ there is an open set } U_{i,n}^i, \quad (1.3)
\end{equation}

A function $f: X \to Y$ is $\sigma$ discrete if there is a base $\mathfrak{B}$ for $Y$ such that $\{f^{-1}B: B \in \mathfrak{B}\}$ is $\sigma$ dd, where

$$f^{-1}B = \{x \in X: f(x) \in B\}.$$

With the advent of set theoretic consistency results in the 1960's, it became clear that Kuratowski’s question could not be answered “yes” in ZFC, the usual axioms of set theory. Define a $Q$ set to be an uncountable subset $X$ of the real line such that every subset of $X$ is a relative $\mathcal{F}_a$ (i.e., $\mathfrak{P}(X) = \Sigma_a X$). Martin and Solovay [13] showed that the existence of a $Q$ set followed from Martin’s axiom and the negation of the Continuum Hypothesis, which Solovay and Tennenbaum [23] showed consistent with ZFC. We will show below that a $Q$ set gives a counterexample to the generalized Lebesgue-Hausdorff theorem.

In this paper we complement the $Q$ set result by showing consistent an axiom, Proposition P, which answers Kuratowski’s and Stone’s questions “yes” To be more precise, we show that Proposition P is consistent modulo a large cardinal axiom: by standard techniques our proof can by cast in the form $(\text{CON ZFC} + \exists \text{ a supercompact cardinal}) \to (\text{CON ZFC} + \text{Proposition P})$. Hansell [5], [6] implicitly proposed Proposition P (with “point finite” replaced by “disjoint”). Indeed, Hansell’s theorem is a special case.
Proposition P is simply a yes answer to Question 3 of Kaniewski and Pol [10]. Call a family *point finite* if every point is a member of at most finitely many elements of the family.

**Proposition P.** A point finite analytic additive family is σ discretely decomposable.

This paper is organized as follows. In the next section we state consequences of Proposition P. The generalized Baire-Lebesgue-Hausdorff, Banach, and Lavrentiev theorems are due to Hansell. One simply modifies Hansell’s proofs [5], [6] by replacing the application of Hansell’s theorem with the assumption of Proposition P. The selection theorems of Kaniewski and Pol [10] follow similarly. We do not know to whom to attribute the result that Proposition P implies that a Borel isomorphism is a generalized homeomorphism.

In §3 we discuss properties that a family of sets might have and discuss the implications between these properties with various extra axioms of set theory. We sketch the proof of Proposition P from certain other axioms. The proof is broken down into a number of lemmas, which we prove in §4.

In the final section we discuss the model constructed by using Lévy forcing to collapse a supercompact cardinal to ω2. We show that the axioms used in §4 are valid in this model.

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2. **Consequences of Proposition P.** The first part of this section lists consequences of Proposition P, which we mark with (P) to indicate the assumption of Proposition P. The second part gives counterexamples derived from other extra axioms of set theory and discusses open questions.

As we indicated in the Introduction, Theorems 2.1–2.4 are implicit in Hansell and Theorem 2.5 in Kaniewski and Pol. Theorems 2.1, 2.2 and 2.5 are generalizations in that the hypothesis that Y be separable is omitted. Theorems 2.3 and 2.4 are generalizations in that f is allowed to be a generalized homeomorphism rather than a homeomorphism.

2.1a. **Generalized Baire, Lebesgue-Hausdorff Theorem** [1], [9], [7], (P).

Let X be a space and Y a Banach space. There is a level-by-level correspondence between the hierarchy of Baire functions and the hierarchy of Borel measurable functions. Moreover these functions are continuous when restricted to a residual set.

A residual set is the complement of a first category set. To express more
precisely the correspondence between the Borel measurable hierarchy, \(\{\Psi_\alpha XY: \alpha < \omega_1\}\), and the Baire hierarchy, \(\{\Phi_\alpha XY: \alpha < \omega_1\}\), we define
\[
\Psi_\alpha XY = \{f: (\forall U \in \Sigma_0 Y)(f^{-1} U \in \Sigma_\alpha X)\},
\]
or more concisely,
\[
\Psi_\alpha XY = \{f: f^{-1} \Sigma_0 Y \subseteq \Sigma_\alpha X\}, \quad (2.1)
\]
\[
\Phi_0 XY = \Psi_0 XY, \quad (2.2)
\]
\[
\Phi_\alpha XY = \bigcup \{\Phi_\beta: \beta < \alpha\} \quad \text{for } \alpha > 0, \quad (2.3)
\]
where in general \(\Delta_\alpha\) denotes the set of all functions which are limits of convergent sequences of functions belonging to the class \(\Delta\). To obtain an analogue of the Lebesgue-Hausdorff theorem for arbitrary \(Y\), Banach [2] defined.
\[
\Phi_0^*XY = \Psi_0 XY, \quad \Phi_1^*XY = \Psi_1 XY, \quad (2.4)
\]
\[
\Phi_\alpha^*XY = \bigcup \{\Phi_\beta: \beta < \alpha\} \quad \text{for } \alpha > 1. \quad (2.5)
\]

2.1b. Generalized Baire, Lebesgue-Hausdorff Theorem. (P). Let \(X\) be a space and \(Y\) a Banach space. Then (a) and (b) are equivalent and imply (c):
(a) \(f \in \Psi_\alpha XY\),
(b) \(f \in \Phi_\alpha XY\) if \(\alpha\) is finite, \(f \in \Phi_\alpha+1 XY\) if \(\alpha\) is infinite.
(c) \(f: X \to Y\) and there is a residual set \(T\) such that \(f\vert T\) is continuous. Further, if \(\alpha = 1\), \(T\) may be chosen so that \(f\) is continuous on \(T\).

2.2. Generalized Banach Theorem [2]. (P). As in Theorem 2.1b with \(Y\) any space and \(\Phi^*\) replacing \(\Phi\).

2.3. Generalized Lavrentiev Theorem [8]. (P). Let \(\alpha\) and \(\beta\) be two given nonlimit ordinals. Let \(f\) be a generalized homeomorphism of type \((\alpha, \beta)\) between arbitrary subsets of two complete spaces \(X, Y\). Then \(f\) can be extended to a generalized homeomorphism of type \((\alpha, \beta)\) from a member of \(\Pi_{\alpha+\beta+1} X\) to a member of \(\Pi_{\beta+\alpha+1} Y\).

2.4. Theorem. (P). If \(X\) is absolutely \(\Pi_\alpha\) and \(f: X \to Y\) is a generalized homeomorphism of type \((\alpha, \beta)\), then \(f^{-1} X = \{f(x): x \in X\}\) is absolutely \(\Pi_{\beta+\alpha}\).

2.5. Generalized Kuratowski and Ryll-Nardzewski Selection Theorem. (P). Every compact valued map \(F: X \to 2^Y\) of lower class \(\alpha\), \(0 < \alpha < \omega_1\), admits a selector of class \(\alpha\). If \(F\) is lower Borel, \(F\) admits a Borel selector.

We explain the first sentence of Theorem 2.5. The hypothesis is that if \(F\) is a function with domain \(X\), \(F(x)\) is a compact subset of \(Y\) for all \(x \in X\), and \(\{x: F(x) \cap U \neq \emptyset\} \in \Sigma_\alpha X\) for all \(U\) open in \(Y\). The conclusion is that there
is an \( f \in \Psi_\alpha XY \) with \( f(x) \in F(x) \) for all \( x \in X \). The last statement of Theorem 2.5 uses Lemma 2.8, below.

Several applications use Lemma 2.8, a simple consequence of Preiss's theorem, Hansell's converse, and Proposition P.

2.6. **Preiss's Theorem** [20]. If \( S \) is a disjoint Borel additive family, then there is an \( \alpha < \omega_1 \), such that \( S \subseteq \Sigma_\alpha \).

**Proof.** We assume that for all \( \alpha < \omega_1 \) \( S \not\subseteq \Sigma_\alpha \), and aim for a contradiction. We can find \( \omega_1 \) disjoint subsets \( S_\beta \) of \( S \) such that, for all \( \alpha, \beta < \omega_1 \), \( S_\beta \not\subseteq \Sigma_\alpha \). Because \( S \) is Borel additive, there is a function \( \delta: \omega_1 \to \omega_1 \) so that \( \bigcup S_\beta \subseteq \Sigma_\delta(\beta) \) and \( \delta(\beta) > \beta \). Now choose \( S_\beta \subseteq S_\beta \not\subseteq \Sigma_\delta(\beta) \). Let \( A = \bigcup \{ S_\beta: \beta < \omega_1 \} \). Then for some \( \alpha, A \in \Sigma_\alpha \). Now \( S_\alpha = A \cap \bigcup S_\alpha \subseteq \Sigma_\delta(\alpha) \). This contradicts our choice of \( S_\alpha \). \( \square \)

2.7. **Hansell's Converse** [5]. Suppose \( S = \{ S_i: i \in I \} \) is a \( \sigma \) discretely decomposable subset of \( \Sigma_\alpha X \). Then \( S \subseteq \Sigma_\alpha \). Similarly with \( S_\alpha \) replaced by analytic.

**Proof.** First we note that the union of a discrete family of \( \Sigma_\alpha \) (respectively, \( \Pi_\alpha \)) sets is \( \Sigma_\alpha \) (respectively, \( \Pi_\alpha \)). (This is proved straightforwardly by induction on \( \alpha \).)

Let \( \{ S_{i,n}: i \in I, n \in \omega \} \) be a \( \sigma \) discrete decomposition as in (1.1)-(1.3). \( S_{i,n} \), the intersection of a \( \Sigma_\alpha \) set and an open set, is \( \Sigma_\alpha \). Let \( J \) be an arbitrary subset of \( I \). \( S^J_n = \bigcup \{ S_{i,n}: i \in J \} \), the discrete union of \( \Sigma_\alpha \) sets, is \( \Sigma_\alpha \). Finally

\[
\bigcup \{ S_i: i \in J \} = \bigcup \{ S^J_n: n \in \omega \},
\]

the countable union of \( \Sigma_\alpha \) sets, is \( \Sigma_\alpha \). \( \square \)

2.8. **Lemma.** (P). If \( S \) is a disjoint Borel additive family, then there is an \( \alpha < \omega_1 \) such that \( S \subseteq \Sigma_\alpha \).

2.9. **Corollary.** (P). Borel isomorphisms are generalized homeomorphisms.

**Proof.** Let \( f: X \to Y \) be a Borel isomorphism. Let \( B = \bigcup \{ B_m: m \in \omega \} \) be a base for \( Y \), \( B_m \) discrete. Then \( f^{-1}B = \{ f^{-1}B: B \in B_m \} \) is a disjoint Borel additive family. By Lemma 2.8, there is an \( \alpha_m < \omega_1 \) such that \( f^{-1}B_m \) is \( \Sigma_{\alpha_m} \) additive. Let \( \alpha = \sup \{ \alpha_m: m \in \omega \} \). Then \( f^{-1}B \) is \( \Sigma_\alpha \) additive, hence \( f \in \Psi_\alpha XY \). Similarly there is \( \beta < \omega_1 \) such that \( f^{-1} \in \Psi_\beta YX \). So \( f \) is a generalized homeomorphism of type \( (\alpha, \beta) \). \( \square \)

We now turn to examining whether any of the above results can be improved by strengthening the conclusion, or by weakening the hypothesis, in particular, by omitting the assumption of Proposition P.

We begin by ruling out some improvements on the Baire, Lebesgue-Hausdorff theorem, even in the case \( X = R, Y = R \). First, (c) cannot imply (b)
because there are $2^c$ functions satisfying (c) and only $c$ satisfying (c). Second, we cannot replace "residual" with "almost everywhere" because there are closed nowhere dense sets of positive measure.

Next, to see why $Y$ must be a Banach space, consider $X = \mathbb{R}$, $Y = \{0, 1\} \subseteq \mathbb{R}$. Every continuous function from $X$ to $Y$ is constant, so $\Phi_\alpha XY = \Phi_0 XY$ for all $\alpha$. On the other hand, the characteristic function of an open, not closed set is in $\Psi_1XY - \Psi_0 XY$. The definition of $\Phi^*$ eliminates this example, and, as Banach showed, allows proving a Lebesgue-Hausdorff theorem for arbitrary separable $Y$. Kuratowski [11] made a different suggestion which, he points out, reduces to the case where $Y$ is a Banach space.

Let us note that the inclusion $\Phi_\alpha XY \subset \Psi_\alpha XY$ can be proved in ZFC for all $X$, $Y$, and $\alpha$.

Now we construct a counterexample to Theorem 2.1 from a $Q$ set $X \subseteq \mathbb{R}$. Let $f$ be a one-to-one map from $X$ to a sufficiently large Banach space $Y$. Clearly $f \in \Psi_1 XY$ as every subset of $X$ is $\Sigma_1 X$, and it is easy to show that every element of $\Phi_\alpha XY$ has separable range. Thus $f \in \Psi_1 XY - \Phi_1 XY$.

Unfortunately, the above is not a counterexample to Theorem 2.2, as $\Phi^* XY = \Psi_1 XY$ by definition. It is plausible to try to construct a counterexample from a $Q_\alpha$ set. We define a $Q_\alpha$ set to be an uncountable subset $X$ of $\mathbb{R}$ such that $\alpha$ is least such that $\mathcal{P}(X) = \Sigma_\alpha X$. For every $\alpha < \omega_1$, a model of set theory with a $Q_{\alpha+1}$ set can be constructed by A. Miller's techniques [15]. Of course, if $X$ is a $Q_\alpha$ set, then $2^{\text{card } X} = c$.

It follows from a result of Kuratowski [12, p. 517] that if $X$ is a $Q$ set then (c) is vacuously true. Kunen has extended this to show that $Q_\alpha$ sets are first category in themselves. Condition (c) is not the conclusion in Baire's formulation of his theorem; rather it is an intermediate conclusion. Baire proved his category theorem to conclude from condition (c) that $T$ was dense in $X$.

We append a list of some of the many questions left open by this discussion.

Question 1. Is it consistent that $\Phi^* \neq \Psi_2$? Can a counterexample be constructed from any $Q_2$ set? From Miller's $Q_2$ sets?

Question 2. Can Theorem 2.1 be strengthened by replacing "residual" with "dense"?

The next question, recently raised by Michael and Namioka [14], and answered affirmatively by Fremlin, seems closely related.

Question 3. Suppose $f|H$ has at least one point of continuity for every closed subset $H$ of $X$. Is $f$ continuous on a dense subset of $X$?

The author knows of no counterexample to the other consequences of Proposition P.

Question 4. Can Theorems 2.3, 2.4, 2.5, 2.8, and 2.9 be proved in ZFC? (Miller [15] has the following partial result: if $X \subseteq \mathbb{R}$ and every subset of $X$ is
Borel in $X$, then there is $\alpha < \omega_1$ such that every subset of $X$ is $\Sigma_\alpha X$.

**Question 5.** Can “disjoint” be replaced by “point finite” in Theorems 2.6 and 2.8 (with or without Proposition P)?

**Question 6.** Can the first sentence be omitted in the hypothesis of Theorem 2.3?

3. **Outline of the proof of Proposition P.** We begin by discussing six properties a family of subsets of a space can have. The analytic additive (aa) and $\sigma$ discretely decomposable (odd) properties have already been defined in the Introduction. (When we say that $S$ is aa, we mean that $S$ is aa relative to the subspace $\bigcup S$.)

Following Pol, let us call $S = \{S_i: i \in I\}$ a $d$ family if whenever $x_i \in S_i$ then $\{x_i: i \in I\}$ is odd. Clearly a odd family is a $d$ family.

It will be convenient to define subfamily to be something different from a subset. Suppose $S = \{S_i: i \in I\}$ and $\mathcal{K} = \{H_j: j \in J\}$. We say $\mathcal{K}$ is a subfamily of $S$, and write $\mathcal{K} \subset * S$, if $J \subset I$ and for all $j \in J$, $H_j = S_j \cap \bigcup \mathcal{K}$. The point of this definition is that if $S$ is aa and $\mathcal{K} \subset * S$ then $\mathcal{K}$ is aa. Similarly subfamilies inherit properties odd and $d$.

![Diagram 1. Trivial implications](https://example.com/diagram.png)

0. Elements analytic
1. In general
2. $\forall S \subseteq S$, point-finite
3. ZFC
4. $SC\omega_2$, ultrametric
5. $V = L$, point-finite
6. $\exists \kappa E(\kappa)$

![Diagram 2. Nontrivial implications](https://example.com/diagram.png)
Let \( \nu \) be a cardinal. We say that \( S \) is aa|\( \nu \) if whenever \( \mathcal{K} \subset \ast S \) and \( \cup \mathcal{K} \subset \nu \), then \( \mathcal{K} \) is aa. We define odd|\( \nu \) and d|\( \nu \) analogously.

Now we consider the implications between the six properties: aa, odd, d, aa|\( \omega_2 \), odd|\( \omega_2 \), and d|\( \omega_2 \). There are six trivial implications, and these are displayed in Diagram 1. The downward arrows are because the properties are hereditary, the rightward arrows are because odd families are d families, and the leftward arrows are from Hansell’s converse with the additional hypothesis that the members of the family are analytic.

Diagram 2 displays some nontrivial implications. Of course, our main concern is the arrow from aa to odd. We start with some examples showing that aa does not imply even odd|\( \omega_2 \) without additional hypotheses (arrow 1).

Let \( X = \{x_\alpha: \alpha < \omega_1\} \) be a subset of \( \mathbb{R} \). Let \( S_\beta = \{x_\alpha: \beta < \alpha\} \). It is easy to verify that \( S = \{S_\beta: \beta < \omega_1\} \) is point countable, \( \Pi_1 \) additive, and not odd. By assuming the Continuum Hypothesis, we can make \( S \) absolutely \( \Pi_1 \) additive by letting \( X = \mathbb{R} \setminus \{10\} \).

To get an absolutely \( \Pi_1 \) additive not odd example without the Continuum Hypothesis, let \( X \) be the space of functions from \( \omega \) to \( \omega_1 \), with the distance between two points \( x, y \) defined to be \( 2^{-n} \), where \( n \) is the least integer such that \( x|n \neq y|n \). \( X \) is complete. Define \( S_\beta = \{x: \exists n x(n) > \beta\} \). Again it is easy to see that \( S = \{S_\beta: \beta < \omega_1\} \) is point countable and absolutely \( \Pi_1 \) additive. That \( S \) is not odd follows from Lemma 4.6.

The above example shows that there is no hope of proving \( \text{aa} \to \text{odd} \) without the additional hypothesis that the set is point finite. Moreover, a \( Q \) set shows that an additional set theoretic hypothesis is needed, even for disjoint \( \Pi_1 \) additive decompositions.

Now we start on our journey from aa to odd, adding hypotheses along the way. The downward arrow is trivial. Lemma 4.8 shows that arrow 2 holds for disjoint families assuming \( \diamond \) for stationary sets (abbreviated \( \forall S \diamond \mathcal{S} \)). Recall that a subset of \( \omega_1 \) is cub if it is closed and unbounded, and that a subset of \( \omega_1 \) is stationary if it meets every cub set. \( \forall S \diamond \mathcal{S} \) is the assertion that for every stationary \( S \subset \omega_1 \) there is a function \( \Gamma_S \) with domain \( S \), such that for all \( Y \subset \omega_1 \), \( \{\alpha: \Gamma_S(\alpha) = Y \cap \alpha\} \) is stationary.

Arrow 3, due to Pol [18], holds in ZFC (cf. Lemma 4.6). The author independently proved an analogous lemma while seeking a nonmetrizable normal Moore space.

Lemma 4.9 shows that arrow 4 holds for certain metrizable subspaces, ultrametric spaces, under a new axiom of set theory \( SC\omega_2 \). \( SC\omega_2 \), whose complicated statement we defer, roughly says that \( \omega_2 \) retains a few properties of a supercompact cardinal.

One drawback to our proof of (the consistency of) Proposition P is that it requires a supercompact cardinal, a rather large cardinal. The author has no
evidence to affirm or deny that $V = L$ implies Proposition P. So we have labelled arrow 5 with a question mark.

The situation with respect to arrow 6 and large cardinals is different. Let $E(\kappa)$ be an axiom stating that $\kappa$ is a regular cardinal greater than $\omega_1$ and that there is a stationary $E \subseteq \kappa$ of $\omega$ limits such that, for all $\alpha < \kappa$, $E \cap \alpha$ is not stationary in $\alpha$. Jensen showed that $V = L$ implies $E(\kappa)$ for all regular, not weakly compact cardinals greater than $\omega_1$. It seems (at least to the author) that $\neg E(\omega_2)$ requires a weakly compact cardinal and that $\forall \kappa \neg E(\kappa)$ requires a strongly compact cardinal. Returning to metrizable spaces, we note that $\exists \kappa E(\kappa)$ implies that there is a $d$ family that is not odd [19]. Thus we have marked arrow 6 a nonimplication.

We conclude this section with a sketch of the proof of Proposition P from $\mathcal{S}C\omega_2 + \forall \mathcal{S}^\omega_2$. Of course the basic idea is to follow the arrows. But because of the extra hypotheses the arrows require, it is easier to prove the contrapositive. First, by Lemma 4.4, if there is a point finite $aa$ not odd family there is such a family $\mathcal{S}$ of subsets of an ultrametric space.

Now we follow the arrows backwards. By Lemma 4.9 $\mathcal{S}$ is not $\sigma$odd$\omega_2$; by Lemma 4.6, $\mathcal{S}$ is not $d\omega_2$. Lemma 4.7 gives us a disjoint, not $d\omega_2$ subfamily $\mathcal{S}'$ of $\mathcal{S}$. Then by Lemma 4.8 $\mathcal{S}'$ is not $aa\omega_2$, a fortiori, not $aa$, and hence $\mathcal{S}$ is not $aa$. This completes the proof of the contrapositive of Proposition P.

4. Many lemmas. In this section we prove the lemmas used in the proof of Proposition P. The lemmas are presented roughly in order of set theoretic sophistication asked of the reader.

Our first goal is to show that if there is a counterexample to Proposition P, then there is a counterexample in an ultrametric space. $X$ is an ultrametric space if $X$ has a base $\mathcal{B} = \bigcup \{ \mathcal{B}_m : m \in \omega \}$ where each $\mathcal{B}_{m+1}$ is a clopen cover of $X$ refining $\mathcal{B}_m$. A continuous function $f$ from $X$ onto $Y$ is perfect if $f^{-1}K$ is closed in $Y$ for all $K$ closed in $X$ and $f^{-1}(y)$ is compact for all $y \in Y$. A map $f : X \to Y$ is irreducible if for all closed $K \subset X$, $f^{-1}K \neq f^{-1}X$.

4.1. Lemma. Every space is the irreducible perfect image of an ultrametric space.

Proof. This is Corollary 2 to Theorem VI.12 of Nagata [17]. As van Douwen pointed out to us, there is a proof along the following lines. A hedgehog is a space composed of unit intervals attached at 0; a Cantor set hedgehog is a space composed of Cantor sets attached at 0. Every space $X$ with metric $\rho$ can be embedded in the product of hedgehogs. $X$ has a $\sigma$ discrete base $\mathcal{B} = \bigcup \{ \mathcal{B}_m : m \in \omega \}$ where the members of $\mathcal{B}_m = \{ B^m_j : j \in J_m \}$ are farther than $2/N_m$ apart. Let $Y_m$ be the hedgehog $\bigcup \{ [0, 1]_j : j \in J_m \}$. For $n > N_m$, define $e_{m,n} : X \to Y_m$ by
\[
e_{m,n}(x) = \max\{0, (1 - n \cdot \rho(x, B_j^n)); j \in J\}.
\]

Define \( e: X \to \prod \{ Y^n_m; n > N_m, m \in \omega \} \) by
\[
e(x) = \langle e_{m,n}(x); n > N_m, m \in \omega \rangle.
\]

Finally, the perfect map from the Cantor set to the unit interval induces a perfect map from products of Cantor set hedgehogs onto products of hedgehogs. It is easy to use Zorn’s lemma to show that if \( f: X \to Y \) is perfect, there is a closed subset \( K \) of \( X \) so that \( f|K \) is perfect and irreducible. □

4.2. **Lemma.** Let \( \mathcal{K} \) be a locally finite family in \( X \), and \( f \) a perfect map from \( X \) onto \( Y \). Then \( f^{-1}\mathcal{K} = \{ f^{-1}H; H \in \mathcal{K} \} \) is locally finite.

**Proof.** Fix \( y \in Y \). For each \( x \in f^{-1}\{y\} \) there is an open set \( U_x \) containing \( x \) and meeting at most finitely elements of \( \mathcal{K} \). Because \( f^{-1}\{y\} \) is compact, finitely many \( U_x \)'s cover \( f^{-1}\{y\} \). Now \( V \), the union of these finitely many \( U_x \)'s, is open, contains \( f^{-1}\{y\} \), and meets only finitely elements of \( \mathcal{K} \). Because \( X - V \) is closed, \( Y - f^{-1}(X - V) \) is an open set containing \( y \) and meeting at most finitely many elements of \( f^{-1}\mathcal{K} \). □

4.3. **Lemma.** Let \( f \) be a perfect map from \( X \) onto \( Y \), and \( \mathcal{S} \) a family of subsets of \( Y \):

(a) If \( \mathcal{S} \) is \( \Sigma_\alpha \) additive, analytic additive, \( \sigma \) discretely decomposable, or a \( d \) family, then so is \( f^{-1}\mathcal{S} \);

(b) If \( \mathcal{S} \) is not a \( \sigma \) discretely decomposable, then neither is \( f^{-1}\mathcal{S} \);

(c) If \( f \) is irreducible and \( \mathcal{S} \) is Baire property additive, then so is \( f^{-1}\mathcal{S} \).

**Proof.** (a) follows routinely from the fact that the inverse image of an open set is open. (b) follows from the converse of Lemma 4.2 and the fact that because of the \( \sigma \) discrete base, a locally finite family is \( \sigma \)d. (c) follows from the fact that the irreducible preimage of a nowhere dense set is nowhere dense. (A set \( \mathcal{S} \) has the Baire property if there is an open set \( U \) such that \( (U - \mathcal{S}) \cup (\mathcal{S} - U) \) is first category.) □

4.4. **Corollary.** If there is a counterexample to Proposition \( P \) there is a counterexample in an ultrametric space.

Our second goal is to relate \( \sigma \text{dd}|\omega_2 \) and \( d|\omega_2 \) to stationary subsets of \( \omega_1 \). For \( I \) a set, \( \nu \) a cardinal, let
\[
[I]^{<\nu} = \{ J \subset I; \text{card } J < \nu \}.
\]

4.5. **Lemma.** Let \( \mathcal{S} = \{ S_i; i \in I \} \) be a family of subsets of a space \( X \) with \( \sigma \) discrete base \( \mathcal{B} = \bigcup \{ \mathcal{B}_m; m \in \omega \} \). \( \mathcal{S} \) is \( \sigma \) discretely decomposable iff there is a function \( \psi: \mathcal{B} \to [I]^{<\omega_1} \) satisfying
\[
\forall x \forall i, \ x \in S_i \text{ implies } i \in \bigcup \{ \psi(B); x \in B \}.
\]
Proof. First we suppose there is such a function $\psi$. Let $\psi(B) = \{i(B, n) : n \in \omega\}$. Then

$$S_{i,m,n} = \{x \in S_i : (\exists B \in B_m)(x \in B \text{ and } i(B, n) = i)\}$$

is a $\sigma$ discrete decomposition of $S$.

Conversely, let $\{S_{i,m} : i \in I, n \in \omega\}$ be a $\sigma$ discrete decomposition of $S$. Define

$$\psi(B) = \{i : \exists n \{i : B \cap S_{i,n} \neq \emptyset\} = \{i\}\}. \quad \square$$

The density $d(X)$ of a space $X$ is the least cardinal $\nu$ such that $X$ has a dense subset of cardinality $\nu$. Following Pol [18], [19], we call a disjoint family $\mathcal{K} = \{H_{a} : \alpha < d(X)\}$ of subsets of $X$ a natural decomposition if, setting $X_{\beta} = \bigcup \{H_{a} : a < \beta\}$, $X_{\beta}$ is closed, and $d(X_{\beta}) < d(X)$, (4.1)

$$X = \bigcup \{H_{a} : \alpha < d(X)\} = \bigcup \{X_{\beta} : \beta < d(X)\}, \quad \ (4.2)$$

$$\forall \text{ limit } \lambda < d(X), \ X_{\lambda} = \text{closure } \bigcup \{H_{a} : \alpha < \lambda\}. \quad (4.3)$$

4.6. Lemma. Let $\mathcal{S} = \{S_{i} : i \in I\}$ be a family of subsets of a space $X$ of density $\omega_1$. Let $\mathcal{K} = \{H_{a} : \alpha < \omega_1\}$ be a natural decomposition of $X$. Define $g : I \to \omega_1$ by

$$g(i) = \min\{\alpha : H_{a} \cap S_{i} \neq \emptyset\}.$$ 

Then the following are equivalent:

(a) $\mathcal{S}$ is a discretely decomposable,

(b) $\mathcal{S}$ is a $d$ family,

(c) $g$ is countable-to-one and range $g$ is not stationary.

Proof. (a) $\to$ (b). Trivial.

(b) $\to$ (c). Let $x_{i} \in S_{i} \cap H_{g(i)}$. By (b), we may express $I = \bigcup \{I_{n} : n \in \omega\}$ so that each $Y_{n} = \{x_{i} : i \in I_{n}\}$ is discrete. Let $\mathcal{B}_{m} = \bigcup \{B_{m} : m \in \omega\}$ be a $\sigma$ discrete base for $X$. Define $h : \mathcal{B}_{m} \to \omega_1$ by

$$h(B) = \min\{\alpha : H_{a} \cap B \neq \emptyset\}.$$ 

Let $\{B_{i} : i \in I_{n}\} \subset \mathcal{B}_{m}$ be a disjoint family with $x_{i} \in B_{i}$. Define $k : I_{n} \to \omega_1$ by $k(i) = h(B_{i})$. By (4.3) if $g(i) = \lambda$ is a limit ordinal, then $k(i) < g(i)$. Then the function defined by taking $g(i)$ to $k(i)$ is countable-to-one and presses down on limit ordinals. By the contrapositive of the pressing down lemma, range $g|I_{n}$ is not stationary. As the countable union of nonstationary sets is nonstationary, we conclude (c).

(c) $\to$ (a). By (c) there is a cub $C$, $C \cap \text{range } g = \emptyset$. Let $\gamma(\alpha) = \sup\{C \cap \alpha\} \subset C$. Define $\psi(B) = \{i : \gamma(g(i)) < h(B)\}$. We verify that $\psi$ is as in Lemma 4.5. Both $g$ and $\gamma$ are countable-to-one, so $\psi(B)$ is countable. Note that replacing $H_{a}$ by $X_{\alpha}$ in the definition does not change $g$ and $h$. Because
C ∩ range g = ∅, γ(g(i)) < g(i), and hence S_i ∩ X_{γ(g(i))} = ∅. As X_{γ(g(i))} is closed, for all x ∈ S_i there is B_x ∈ ☰ with x ∈ B_x and B_x ∩ X_{γ(g(i))} = ∅, so i ∈ ψ(B_x).

Remark. The above lemma is due to Pol [18, Theorem 1 and Remark 2]. The author independently proved an analogous lemma about Moore spaces M(X, S) (Moore spaces may be nonmetrizable) constructed from X and S as in [4, §3]. This lemma is that 4.6(c) is equivalent to both

(d) M(X, S) is metrizable,
(e) M(X, S) is collectionwise Hausdorff.

One may further prove (a) ↔ (d) and (b) ↔ (e) without the assumption d(X) < ω₁. The author is grateful for a letter from Roman Pol expressing this parallel and interesting the author in the subject of this paper.

4.7. Lemma. Suppose that S = {S_i: i ∈ I} is a point finite not d family of subsets of X, d(X) < ω₁. Then there is a disjoint, not d family S' of singletons, S' c S.

Proof. Let H = {H_α: α < ω₁} be a natural decomposition of X. Let g: I → ω₁ be as in Lemma 4.6. Then there are two cases.

Case 1. For some δ < ω₁, g ~ (δ) is uncountable. For each i ∈ g ~ (δ) choose x_i ∈ S_i ∩ H_δ.

Let F_i = {x_j: j ∈ g ~ (δ)} and x_i ∈ S_j. Because each F_i is finite, we may apply the Δ-system lemma to get an uncountable I' ⊆ g ~ (δ) such that i, i' ∈ I', i ≠ i implies F_i ∩ F_i' = ∅.

Let S' = {{x_i}: i ∈ I'}. And x_i ∉ ∩ {F_i: i ∈ I'}. Let S' = {{x_i}: i ∈ I'}.

Case 2. Range g is stationary. By restricting I, we may assume that g is one-to-one onto S, a stationary set of limit ordinals. Choose x_i ∈ S_i ∩ H_{g(i)}.

Let {F_α: α < ω₁} list the finite subsets of ω₁ in such a manner that for every limit ordinal λ < ω₁, {F_α: α < λ} lists the finite subsets of λ. For β = g(i) ∈ S, we define h(β) so that

F_{h(β)} = {g(j): x_j ∈ S_j, g(j) < β}.

Since h presses down on a stationary set, there is δ such that h ~ (δ) is stationary. Let S' = {{x_i}: h(g(i)) = δ}.

We now aim towards showing arrow 2 of Diagram 2. We begin by reviewing operation @. Let FSω be the set of finite sequences of integers. Given a function U: FSω → ☰(X), we define

@ (U) = ∪_{f ∈ ω} ∩_{n ∈ ω} U_{f;n}.

We say a set A is analytic in X if there is a function U from FSω to the open sets of X (equivalently, the closed sets of X) such that A = @ (U). We
will find it convenient to define the analytic sets from basic open sets. So let \( \mathcal{B} \) be a \( \sigma \)-discrete base for \( X \). As every open set is the union of basic open sets, we may define the class of sets analytic in \( X \) to be \( \{ \mathcal{G}'(u): u: \mathcal{B} \to \mathcal{P}(FS\omega) \} \), where

\[
\mathcal{G}'(u) = \bigcup_{f \in \omega} \bigcap_{\omega \in \omega} \bigcup_{f \in \omega(B)} B.
\]

Recall that \( \diamond \) for stationary sets, abbreviated \( \forall S \diamond \) is the assumption that whenever \( S \subset \omega_1 \) is stationary there is a function \( \Gamma_S \) with domain \( S \) such that for all \( Y \subset \omega_1 \), \( \{ \alpha: \Gamma_S(\alpha) = Y \cap \alpha \} \) is stationary. Assuming that \( d(X) < \omega_1 \), letting \( \{ B_\alpha: \alpha < \omega_1 \} = \mathcal{B} \) be a \( \sigma \)-discrete base, recalling that \( \forall S \diamond \) implies the Continuum Hypothesis, and doing some bookkeeping, we can and do use the following variation of \( \forall S \diamond \), convenient for dealing with operation \( \mathcal{G} \): whenever \( S \subset \omega_1 \) is stationary there is a function \( \Gamma_S \) with domain \( S \) such that for all \( u: \mathcal{B} \to \mathcal{P}(FS\omega) \), \( \{ \alpha: \Gamma_S(\alpha) = u(\alpha) \} \) is stationary.

4.8. Lemma (\( \forall S \diamond \)). If \( d(X) < \omega_1 \) and \( S \) is an analytic additive disjoint family of subsets of \( X \), then \( S \) is a \( d \) family; equivalently, \( S \) is \( \sigma \) discretely decomposable.

Proof. We prove the contrapositive. So let \( S \) be not a \( d \) family. Let \( \mathcal{K} = \{ H_\alpha: \alpha < \omega_1 \} \) be the natural decomposition induced by setting

\[
X_\alpha = \{ x \in X: \forall \gamma x \in B_\gamma \rightarrow \gamma < \alpha \}.
\]

By Lemma 4.6, the function \( g \) is either uncountable-to-one or range \( g \) is stationary. In the former case \( \mathcal{S} \) must be uncountable, and as a separable space has only \( 2^\omega \) analytic subsets, \( 2^\omega < 2^\omega \) yields that \( S \) is not \( \sigma \)-discrete.

In the latter case, we may assume that \( S \) is of the form \( \{ \{ x_\alpha \}: \alpha \in S \} \) where \( x_\alpha \in H_\alpha \) and \( S \) is stationary. The key fact that allows us to use \( \forall S \diamond \) is that

\[
\text{if } u(\alpha) = u'(\alpha) \text{ then } x_\alpha \in \mathcal{G}'(u) \iff x_\alpha \in \mathcal{G}'(u').
\]

Thus we may define \( x_\alpha \in \mathcal{G}'(v) \), for \( v \) from \( \{ B_\beta: \beta < \alpha \} \) to \( \mathcal{P}(FS\omega) \) to mean \( x_\alpha \in \mathcal{G}'(u) \) for one, or equivalently, for all \( u \) extending \( v \).

Let \( \Gamma_S \) be the function \( \forall S \diamond \) asserts to exist. Define \( Y \subset \omega_1 \) by \( Y = \{ \alpha: x_\alpha \in \mathcal{G}'(\Gamma_S(\alpha)) \} \).

We check that \( Z = \{ x_\alpha: \alpha \in Y \} \) is not analytic, i.e., \( Z \neq \mathcal{G}'(u) \) for all \( u \). Fix \( u \). By our choice of \( \Gamma_S \), there is an \( \alpha \in S \) such that \( u(\alpha) = \Gamma_S(\alpha) \).

Thus \( x_\alpha \in \mathcal{G}'(u) \) just in case \( x_\alpha \in Z \). □

Our final goal is to demonstrate arrow 4 of Diagram 2. Axiom \( SC\omega_2 \) is rather technical, so for expository purposes, we precede its formulation with two applications.

Application 1. Let \( \mathcal{S} = \{ S_i: i \in I \} \) be a family of subsets of an ultrametric space \( X \), with base \( \mathcal{B} = \bigcup \{ \mathcal{B}_m: m \in \omega \} \) where each \( \mathcal{B}_{m+1} \) is a clopen cover
of $X$ refining $\mathcal{B}_m$. The reason we have considered ultrametric spaces is that we need (at least, the author needs) to define a tree $T$, $<$. Let $T = \mathcal{B} \cup \{\{x\}: x \in X\}$; define $t' < t$ iff $t' \supset t$. Define $t = \{t': t' < t\}$.

Now Lemma 4.5 says that $\mathcal{S}$ is odd iff there is a $\psi: \mathcal{B} \rightarrow [I]^{<\omega_1}$ satisfying

$$\forall x \forall i, \quad x \in S_i \text{ implies } i \in \bigcup \{\psi(B): x \in B\}.$$ 

Translating this into tree terminology

$$\forall t \in X, \quad \forall i \in I, \quad t \in S_i \text{ implies } i \in \bigcup \{\psi(t'): t' \in i\}.$$ 

Next, we note that whether $\psi$ meets the condition at $t$ depends only on part of the information contained in $\psi$. Only the restriction of $\psi$ to $i$ and the set $\psi(t') \cap \{i: t \in S_i\}$ are relevant. We indicate this by rephrasing our expression:

$$\forall t \in X, \quad \{i: t \in S_i\} \subset \bigcup \{\psi(t') \cap \{i: t \in S_i\}: t' \in i\}.$$ 

Now $SC\omega_2$ will say that if for all $T' \in [T]^{<\omega_2}$ there is a $\psi_{T'}$ meeting such a condition at all $t \in T'$, there is a $\psi$ which meets the condition at all $t \in T$. Thus, we prove Lemma 4.9 (below).

Application 2. Let $T$, $<$ be a tree of height $< \omega_1$. That is, for all $t \in T$, $t = \{t': t < t\}$ is countable and well ordered by $<$. We topologize $T$ by declaring sets of the form $\{t \in T: a < t < b\}$ and $\{\{t\}: (-\exists t')(t' < t)\}$ to be basic open. $T$ is a regular, first countable, locally countable, locally compact Moore, possibly nonmetrizable space.

A space is collectionwise Hausdorff (respectively $\omega_2$ collectionwise Hausdorff) if every closed discrete set (respectively, of cardinality $< \omega_2$) can be screened (i.e., simultaneously separated by disjoint open sets).

Let $Y$ be a closed discrete set of nonisolated points of $T$. $Y$ can be screened iff there is a function $F$ from $T$ to $[Y]^{<\omega_1}$ such that for all $y$ there is a $t < y$ such that for all $t'$, $t < t' < y$ implies $F(t') = \{y\}$.

As in Application 1, whether $F$ meets this condition at $y$ depends only on part of the information contained in $F$. We may rephrase the condition as: for all $y \in Y$ there is a $t < y$ such that for all $t'$, $t < t' < y$ implies $F|\bar{y}(t') \cap \{y\} = \{y\}$. Thus, we prove Lemma 4.10 (below).

(It happens that in this particular case we can improve a “point countable almost screening” to a screening. But for convenience in this case, and perhaps necessity in some other case, we include in $SC\omega_2$ a provision to make $F(t)$ a singleton.)

$SC\omega_2$ will say that if for all $Y' \in [Y]^{<\omega_1}$ there is an $F_{Y'}$ meeting such a condition at all $y \in Y'$ there is an $F$ meeting the condition at all $y \in Y$.

We introduce notation and terminology to express $SC\omega_2$. We call a hextuple $\langle T, <, A, \delta, z, \theta \rangle$ acceptable if

(a) $\langle T, < \rangle$ is a tree of height $< \omega_1$, 

(b) $A$ is a set,
(c) $\delta$, $z$, and $\theta$ are functions with domain $T$ such that for all $t \in T$:

\begin{align*}
& (1) \delta(t) \in [A]^{<\omega_1}, \\
& (2) z(t) \text{ is a set of functions from } \tilde{t} \text{ to } [A]^{<\omega_1}, \\
& (3) \theta(t) \text{ is a function from } \tilde{t} \text{ to } \omega + 1. \\
\end{align*} 

(4.4)

If $T' \subseteq T$, set $T'' = \bigcup \{\tilde{t} : t \in T'\}$ and

$$\mathcal{F}' = \langle T'', < |T'' \times T'', A, \delta|T'', z|T'', \theta|T''\rangle.$$ 

Set

$$[f, t, \delta] = \{ \langle t', f(t') \cap \delta(t) \rangle : t' \in \tilde{t} \}.$$ 

We call an acceptable hextuple $\mathcal{F}$ doable if there is a function $f : T \to [A]^{<\omega_1}$ such that for all $t \in T$

\begin{align*}
& (a) [f, t, \delta] \subseteq z(t), \\
& (b) \text{ for all } t' \in t, \text{ card } f(t') < \theta(t)(t'). \\
\end{align*} 

(4.5)

In such case we say $f$ does $\mathcal{F}$. We say that $\mathcal{F}$ is $\omega_2$ doable if for all $T' \subseteq [T]^{<\omega_1}$, $\mathcal{F}'$ is doable.

Axiom $SC_{\omega_2}$ is the assertion that an acceptable hextuple $\mathcal{F}$ is doable iff $\mathcal{F}$ is $\omega_2$ doable.

We remark that one can modify $SC_{\omega_2}$ to $SC_{\nu^+}$ by changing $\omega$ to $\nu$, $\omega_1$ to $\nu^+$, and $\omega_2$ to $\nu^{++}$ throughout the above discussion and in the proof that $SC_{\omega_2}$ holds in a model obtained by Levy collapsing a supercompact cardinal.

4.9. Lemma. ($SC_{\omega_2}$) Let $\mathcal{S}$ be a family of subsets of an ultrametric space $X$. $\mathcal{S}$ is $\sigma$ discretely decomposable if $\mathcal{S}$ is odd$|\omega_2$.

**Proof.** Set $T = \mathcal{B} \cup \{ \{ x \} : x \in X \}$, $\subseteq = \supseteq$, $A = I$,

$$\delta(x) = \{ i : x \in S_i \}, \quad \delta(B) = \emptyset,$$

$$z(x) = \{ f : f \text{ is a function from } \bar{x} \text{ to } [I]^{<\omega_i}, \delta(x) \subseteq \bigcup \text{ range } f \},$$

$$z(B) = \{ f : f \text{ is a function from } \bar{B} \text{ to } [I]^{<\omega_i}, \theta(t)(t') = \omega \}.$$ 

4.10. Lemma. ($SC_{\omega_2}$) A tree space $T$, $<$ of height $< \omega_1$ is collectionwise Hausdorff if $\omega_2$ collectionwise Hausdorff.

**Proof.** Let $Y$ be a closed discrete subset of $T$. Set $A = Y$; for $y \in Y$,

$$\delta(y) = \{ y \},$$

$$z(y) = \{ f : f \text{ is a function from } \bar{y} \text{ to } [Y]^{<\omega_1},$$

$$\exists t < y, \forall t', t < t' < y \rightarrow f(t') = \{ y \},$$
\[ (y)(t') = 1; \text{ for } t \not\in Y, \delta(t) = \emptyset, \]

\[ z(t) = \{ f : f \text{ is a function from } t \text{ to } [Y]^\omega \}, \theta(t)(t') = 1. \quad \square \]

We remark that there are counterexamples to Lemmas 4.9 and 4.10 assuming \( \exists \kappa E(\kappa) \).

It would be nice to have other applications of \( SC\omega_2 \) or, more likely, strengthen \( SC\omega_2 \) to have other applications. Some candidates are

1. A graph has chromatic number \( < \omega \) iff every subgraph of cardinality \( < \omega_2 \) has chromatic number \( < \omega \);
2. A first countable space is metrizable iff every subspace of cardinality \( < \omega_2 \) is metrizable;
3. Application 2 with "\( T \) is a tree space of height \( \omega_1 \)" replaced with "\( T \) is first countable".

Shelah [22, p. 322] has claimed (1) with "chromatic" replaced by "coloring". In seeking applications of this type it is prudent to note that there is a family of \( \omega_2 \) countable sets without a transversal, yet every subfamily of cardinality \( < \omega_2 \) has a transversal [16].

5. Collapsing a supercompact cardinal to \( \omega_2 \). We begin by defining the partial order of Lévy forcing. For ordinals \( \lambda < \nu \), set \([\lambda, \nu) = \{ \rho : \lambda < \rho < \nu \} \). For \( \omega_2 < \lambda < \nu \), let \( P(\lambda, \nu) \) be the set of \( p \) satisfying (5.1)--(5.4) below, with the partial order given by (5.5):

\[ p \text{ is countable,} \quad (5.1) \]
\[ p \text{ is a function,} \quad (5.2) \]
\[ \text{domain } p \subset [\lambda, \nu) \times \omega_1, \quad (5.3) \]
\[ p(\rho, \alpha) \in p, \quad (5.4) \]
\[ p' < p \text{ iff } p' \supseteq p. \quad (5.5) \]

It is obvious that \( P(\lambda, \nu) \) is countably closed. So \( \omega_1 \) is preserved and we omit superscripts. If \( \nu \) is inaccessible, \( P(\lambda, \nu) \) has the \( \nu \) chain condition (apply the \( \Delta \)-system lemma to the domains of the \( p \)'s). In this case \( \nu \) becomes \( \omega_2 \) in the extension. Also note that when \( \lambda < \kappa < \nu \), \( P(\lambda, \nu) = P(\lambda, \kappa) \oplus P(\kappa, \nu) \).

5.1. Lemma. Let \( \kappa \) be inaccessible and let \( G \) be \( M \) generic over \( P(\omega_2, \kappa) \). Then \( \diamond \) for stationary sets holds in \( M[G] \).

Proof. Because \( \kappa \) is inaccessible \( 2^\omega < \kappa \). List \( \{ \mathfrak{A}(\alpha) : \alpha < \omega_1 \} \) as \( \{ a_\beta : \beta < 2^\omega \} \).

Suppose that \( M[G] \models S \subset \omega_1 \) is stationary. By the \( \kappa \) chain condition, there is a \( \nu, 2^\omega < \nu < \kappa \), such that \( M[G \cap P(\omega_2, \nu)] \models S \subset \omega_1 \) is stationary. Define, in \( M[G] \), a function \( \Gamma \) with domain \( S \) by \( \Gamma(\gamma) = a_{\bigcup G(\alpha, \gamma)} \).

Working in \( M[G \cap P(\omega_2, \nu)] \), we verify that \( \Gamma \) witnesses \( \diamond \) for \( S \). Suppose
$p \in P(\nu, \kappa), p \Vdash C^0, Y^0 \subset \omega_1, C^0$ cub. We will find $p' < p$ and $\gamma < \omega_1$ so that $p' \Vdash \gamma \in C^0 \cap S, \Gamma(\gamma) = Y^0 \cap \gamma$. Using the facts that $\omega_1$ is regular and that $P(\nu, \kappa)$ is countable closed, we can define by induction on $\alpha$, $\alpha < \omega_1$, conditions $p_\alpha \in P(\nu, \kappa)$, ordinals $\beta_\alpha < \omega_1$, and sets $Y_\alpha \subset \beta_\alpha$ satisfying for all $\alpha' < \alpha$ and all limit ordinals $\lambda$,

$$
p > p_\alpha > p_{\alpha'}, \quad \beta_\alpha < \beta_\alpha',
p_\lambda = \bigcup \{p_\alpha : \alpha < \lambda\}, \quad \beta_\lambda = \bigcup \{\beta_\alpha : \alpha < \lambda\},$$

$$p_\alpha \Vdash C^0 \cap [\beta_\alpha, \beta_\alpha) \neq \emptyset,$$

$$p_\alpha \Vdash Y^0 \cap \beta_\alpha = Y_\alpha,$$

$$\langle \nu, \delta \rangle \in \text{dom} \, p_\alpha \rightarrow \delta < \beta_\alpha.$$

Because $\{\beta_\alpha : \alpha < \omega_1\}$ is cub and $S$ is stationary, there is a limit ordinal $\lambda$ so that $\beta_\lambda \in S$. Note that because $p \Vdash C^0$ cub, $p_\lambda \Vdash \beta_\lambda \in C^0$. Suppose $Y_\lambda = \alpha_\delta$. Then

$$p_\lambda \cup \{\langle \nu, \lambda, \delta \rangle\} \Vdash \beta_\lambda \in C^0 \cap S, \quad \Gamma(\beta_\lambda) = Y^0 \cap \beta_\lambda. \quad \square$$

A cardinal $\kappa$ is supercompact iff for all $\nu > \kappa$ there is a transitive class $N$ and an elementary embedding $j : V \rightarrow N$ satisfying (we will write $j_x$ instead of $j(x)$)

$$j|\kappa = \text{id}|\kappa, \quad (5.6)$$

$$j\kappa > \lambda, \quad (5.7)$$

$$\lambda N \subset N. \quad (5.8)$$

The standard reference on supercompact cardinals is [21].

Let $\mathcal{B}$ be the regular open algebra of $P(\omega_2, \kappa)$. Then $j$, as above, is a complete Boolean algebra embedding of $\mathcal{B}$ into $j^\mathcal{B}$, which is the regular open algebra of $P(\omega_2, j\kappa)$. This induces an elementary embedding, also denoted $j$, of the Boolean valued model $M^\mathcal{B}$ into $N^j$. An important fact about this embedding follows from (5.6).

5.2. LEMMA. If $x \in M^\mathcal{B}$ and card $x < \kappa$, then $j_x = \{j_y : y \in x\}$.

5.3. THEOREM. Let $\kappa$ be a supercompact cardinal in a model $M$. Then $SC\omega_2$ is valid in $M^\mathcal{B}$.

PROOF. We outline the proof, supplying the reasons later. Suppose

$$M \Vdash \text{\|#} \mathcal{F} \text{ is } \omega_2 \text{ doable} \| \mathcal{B} = b. \quad (5.9)$$

Choose an elementary embedding $j$ with $\lambda$ large enough that (5.8) gives $\mathcal{F} \in N$ and (5.7) gives card $T < j\kappa$.

$$N \Vdash \text{\|#} j^\mathcal{F} \text{ is } j \kappa \text{ doable} \| j^\mathcal{B} > j b, \quad (5.10)$$

$$N \Vdash \text{\|#} \mathcal{F} \text{ is doable} \| j^\mathcal{B} > j b, \quad (5.11)$$
First, (5.10) follows from (5.9) because \( j \) is elementary, recalling that \( \kappa = \omega_2 \) in \( M^\mathcal{G} \). We chose \( j \) so that \( \mathcal{F} \in N \), thus (5.11) is meaningful. If \( T \subset jT \) and \( A = jA \), then (5.11) follows from (5.10) because we chose \( j \) so that card \( T < j \kappa \). While \( T \subset jT \) and \( A = jA \) need not hold, we will show in Lemma 5.4 below that this basic idea works. That (5.12) follows from (5.11) boils down to Lemma 5.5, that "not doable" is preserved by countably closed forcing, in particular \( P(\kappa, j\kappa) \). Since \( N^\mathcal{G} \subset M^\mathcal{G} \) and \( \mathcal{F} \) is doable is a \( \Sigma_1 \) formula, (5.12) gives (5.13). This derivation of (5.13) from (5.9) shows that \( SC\omega_2 \) is valid in \( M^\mathcal{G} \).

5.4. Lemma. \( N \models j\mathcal{F} \) is doable\( \| j\mathcal{F} \| < \| \mathcal{F} \| \) is doable\( \| j\mathcal{F} \| \).

Proof. As pointed out above, if \( T \subset jT \) and \( jA = A \), the lemma is obvious. The idea of this proof is to note that there is a subset of \( jT \) which is a copy of \( T \), and that because of card \( \delta(t) < \omega_2 \), \( jA - A \) is irrelevant.

Define \( T^* \) by (omitting the superscript \( j\mathcal{G} \))

\[ \| x \in T^* \| = \| (\exists t \in T)(x = jt) \|. \]

(We check that \( \| (\exists t \in T)(x = jt) \| \cdot \| x = y \| < \| (\exists t \in T)(y = jt) \| \) so the definition of \( T^* \) is valid.) Now card \( T^* < j\kappa \) and \( T^* \subset jT \), so there is \( g: \mathcal{T} \rightarrow [jA]^<\omega_1 \) with

\[ \| g \text{ does } T^* \| > \| j\mathcal{F} \text{ is doable} \|. \]

Define \( A^* \subset jA \), \( h: T^* \rightarrow [A^*]^<\omega_1 \), and \( f: T \rightarrow [A]^<\omega_1 \) by

\[ \| x \in A^* \| = \| (\exists a \in A)(x = ja) \|, \]

\[ \| x \in h(t) \| = \| x \in g(t) \| \cdot \| x \in A^* \|, \]

\[ \| a \in f(t) \| = \| ja \in g(jt) \|. \]

We claim that

\[ \| g \text{ does } \mathcal{F}^* \| = \| h \text{ does } \mathcal{F}^* \| < \| f \text{ does } \mathcal{F} \|. \]

First we note that for all \( t \in T \), card \( g(jt) > \text{card} h(jt) = \text{card} f(t) \). Thus we verify that range \( h \subset [A^*]^<\omega_1 \), range \( f \subset [A]^<\omega_1 \), \( h \) satisfies (4.5b) and \( f \) satisfies (4.5b). Next we see that by Lemma 5.2, for all \( t \in T \), \( j\delta(jt) = \{ ja: a \in \delta(t) \} \subset A^* \). So \( [g, jt, j\delta] = [h, jt, j\delta] = [f, t, \delta] \), hence

\[ \| [g, jt, j\delta] \in jz(jt) \| = \| [h, jt, j\delta] \in jz(jt) \| \]

\[ = \| [f, t, \delta] \in z(t) \|. \]

5.5. Lemma. Suppose that in \( M \), \( P \) is countably closed and \( \mathcal{F} \) is an acceptable tuple. Let \( G \) be \( M \) generic over \( P \). If \( \mathcal{F} \) is doable in \( M[G] \), then \( \mathcal{F} \) is doable in \( M \).
Proof. By the basic forcing lemmas, if $\mathcal{T}$ is doable in $M[G]$, there is a condition $p \in G$ and a term $f^0$ such that $p \vdash f^0$ does $\mathcal{T}$. Note that because $P$ is countably closed, $[A]^{<\omega}$ is the same in both $M$ and $M[G]$. Thus in $M$ we may define by induction on $<_\mathcal{T}$ conditions $p_t \in P$ and elements $f(t)$ of $M$ for all $t \in T$ such that

$$p_t \leq p, t' <_\mathcal{T} t \text{ implies } p_{t'} \geq p_t,$$

and

$$p_t \vdash f^0(t) = f(t).$$

For all $t \in T$, $p_t \vdash f^0(t) = f(t)$ does $\mathcal{T}$. Thus $f \in M$ does $\mathcal{T}$ and $\mathcal{J}$ is doable in $M$. □

Bibliography


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