EXPANSIVE HOMEOMORPHISMS AND TOPOLOGICAL DIMENSION

BY

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ABSTRACT. Let $K$ be a compact metric space. A homeomorphism $f: K \to K$ is expansive if there exists $e > 0$ such that if $x, y \in K$ satisfy $d(f^n(x), f^n(y)) < e$ for all $n \in \mathbb{Z}$ (where $d(\cdot, \cdot)$ denotes the metric on $K$) then $x = y$. We prove that a compact metric space that admits an expansive homeomorphism is finite dimensional and that every minimal set of an expansive homeomorphism is 0-dimensional.

A homeomorphism $f$ of a compact metric space $K$ is expansive if there exists $c > 0$ (called an expansivity constant for $f$) such that $d(f^n(x), f^n(y)) < c$ for all $n$ implies $x = y$. This property has frequent applications in stability theory, symbolic dynamics and ergodic theory. Specially interesting are expansive homeomorphisms of 0-dimensional spaces because they can be embedded in shifts, or, in other words, they are equivalent to subshifts. In [1] Bowen proved that hyperbolic minimal sets of diffeomorphisms are 0-dimensional. Since the restriction of a diffeomorphism to a hyperbolic set is always expansive, it is natural to ask whether minimal sets of expansive homeomorphisms are 0-dimensional. The purpose of this paper is to prove this property.

THEOREM. If $f: K \to K$ is an expansive homeomorphism of the compact metric space $K$ then $\dim K < \infty$ and every minimal set of $f$ is 0-dimensional.

Recall that a compact metric space has dimension $< n$ if for all $r > 0$ there exists a covering $\mathcal{U}$ of $K$ by open sets with diameter $< r$ such that every point belongs to at most $n + 1$ sets of $\mathcal{U}$ [2]. Moreover it is known [2] that $K$ is 0-dimensional if and only if it is totally disconnected i.e. if the connected component of every point $x$ is $\{x\}$.

Let us see an application of the theorem to the symbolic dynamics of an expansive homeomorphism $f: K \to K$. Let $\mathcal{U} = \{U_1, \ldots, U_k\}$ be a covering of $K$ by open sets with diameter smaller than an expansivity constant $c$ of $f$. Let $\Sigma(f, \mathcal{U})$ be the subshift associated to $f$ and $\mathcal{U}$ i.e. the set of sequences $\theta: \mathbb{Z} \to \mathcal{U}$ such that $\bigcap_{n=-\infty}^{+\infty} f^{-n}(\theta_n) \neq \emptyset$. Endow $\Sigma(f, \mathcal{U})$ with the topology induced by the space $\mathcal{U}^\mathbb{Z}$. Then $\Sigma(f, \mathcal{U})$ is compact and since the diameter of the sets in $\mathcal{U}$ is smaller than $c$ we have a continuous map $\pi: \Sigma(f, \mathcal{U}) \to K$.
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defined by \( \sigma(\theta) = \cap_{n \geq 0} f^{-n}(\tilde{\theta}_n) \). Moreover if \( \sigma: \Sigma(f, \mathcal{U}) \leftrightarrow \) is the shift homeomorphism we have \( f\sigma = \sigma f \).

**Corollary.** If \( \Delta \subset K \) is a minimal set for \( f \) then \( \pi^{-1}(\Delta) \) contains a minimal set \( \Lambda_0 \) that is mapped homeomorphically onto \( \Delta \) by \( \pi \).

To prove the Corollary take a covering of \( \Delta \) by disjoint open sets in \( \Delta \), \( \tilde{\mathcal{U}} = \{ V_1, \ldots, V_i \} \) with diameters so small that there exists a function \( \varphi: \tilde{\mathcal{U}} \to \mathcal{U} \) with the property \( \varphi(V_i) \supseteq V_i \) for all \( i \). Consider the subshift \( \Sigma_1 = \Sigma(f, \Sigma) \) associated to \( f/\Lambda \) and \( \tilde{\mathcal{U}} \) and the maps \( \tilde{\pi}: \Sigma_1 \to \Lambda \) defined by \( \tilde{\pi}(\theta) = \cap_{n \geq 0} f^{-n}(\tilde{\theta}_n) \) and \( \tilde{\varphi}: \Sigma_1 \to \Sigma(f, \mathcal{U}) \) defined by \( \tilde{\varphi}(\theta) = \varphi \circ \theta \). Now \( \tilde{\pi} \) is a homeomorphism because the sets in \( \tilde{\mathcal{U}} \) are disjoint and we have \( f\tilde{\pi} = \tilde{\pi} \sigma, \) \( \tilde{\sigma} = \tilde{\varphi} \sigma \), where \( \sigma \) also denotes the shift homeomorphism of \( \Sigma_1 \). Since \( \tilde{\pi} \) is a homeomorphism then \( \sigma: \Sigma_1 \leftrightarrow \) is a minimal homeomorphism (because \( f\tilde{\pi} = \tilde{\pi} \sigma \)). Hence \( \tilde{\varphi}(\Sigma_1) \) is a minimal set for \( \sigma: \Sigma(f, \mathcal{U}) \leftrightarrow \) and \( \tilde{\pi}(\Sigma_1) = \tilde{\pi}\Sigma_1 = \Lambda_1 \). Finally \( \pi/\tilde{\varphi}(\Sigma_1) \) is one-to-one because \( \pi\tilde{\varphi}(\theta) = \pi\tilde{\varphi}(\theta') \) implies \( \tilde{\pi}(\theta) = \tilde{\pi}(\theta') \) and then \( \theta = \theta' \).

1. **The dimension of minimal sets.** Let \( K \) be a compact metric space, with metric \( d(\cdot, \cdot) \) and \( f: K \leftrightarrow \) an expansive homeomorphism with expansivity constant \( c > 0 \). In this section we shall assume that \( \dim K > 0 \) and we shall prove that \( f \) cannot be a minimal homeomorphism.

If \( \varepsilon > 0 \) and \( x \in K \), let \( W^s(\varepsilon, x) \), \( W^u(\varepsilon, x) \) be the local stable and unstable sets defined by:

\[
W^s(\varepsilon, x) = \{ y \in K | d(f^n(x), f^n(y)) < \varepsilon, \forall n > 0 \},
\]

\[
W^u(\varepsilon, x) = \{ y \in K | d(f^{-n}(x), f^{-n}(y)) < \varepsilon, \forall n > 0 \}.
\]

Fix \( 0 < \varepsilon < c/2 \).

The idea of the proof is the following: using the expansiveness we show that for some \( x \in K \) there exists a compact connected set \( \Lambda_0 \subset W^s(\varepsilon, x) \) with \( \dim(\Lambda_0) = c > 0 \). Then we prove that some power \( f^{-m} \) of \( f \) expands every compact connected set \( \Lambda \) with \( \dim(\Lambda) = c \) contained in a local stable set. More precisely \( \dim f^{-m}(\Lambda) > 3c \). Using this we show that \( f^{-m}(\Lambda) \) contains two compact connected sets \( \Lambda', \Lambda'' \), contained in local stable sets, with \( \dim(\Lambda') = \dim(\Lambda'') = c \) and satisfying \( \inf(d(x, y) | x \in \Lambda', y \in \Lambda'') > c/2 \). This property contradicts the minimality of \( f^m \) because if we take an open set \( U \) with \( \dim(U) < c/2 \) then either \( \Lambda_0' \) or \( \Lambda_0'' \) (where \( \Lambda_0', \Lambda_0'' \) are related to \( \Lambda_0 \) as \( \Lambda', \Lambda'' \) to \( \Lambda \) in the previous explanation) does not intersect \( U \). Suppose \( \Lambda_0' \cap U = \emptyset \). Define \( \Lambda_1 = \Lambda_0' \). Again \( \Lambda_1 \) or \( \Lambda_1'' \) does not intersect \( U \). Suppose \( \Lambda_1 \cap U = \emptyset \) and define \( \Lambda_2 = \Lambda_1 \). Using this method we find \( \Lambda_0, \Lambda_1, \ldots \) such that \( f^{-m}(\Lambda_j) \subset \Lambda_{j+1} \) and \( \Lambda_j \cap U = \emptyset \). Let \( x \in \cap_{j \geq 0} f^m(\Lambda_j) \). Then the backwards orbit of \( x \) under \( f^m \) does not intersect \( U \); hence \( K \) is not minimal for \( f^m \). In order to prove that \( K \) is not minimal for \( f \) we shall follow the same
idea being more careful in the choice of $U$ in order to make possible the construction of the sequence $\Lambda_i$ satisfying $f^{-i}(\Lambda_i) \cap U = \emptyset$ for all $0 < i < m$. The existence of the initial set $\Lambda_0$ is proved in Lemma III and the expanding property in Lemma IV.

**Lemma I.** For all $r > 0$ there exists $N > 0$ such that

\[ f^n(W^s_\epsilon(x)) \subset W^s_\epsilon(f^n(x)), \quad f^{-n}(W^u_\epsilon(x)) \subset W^u_\epsilon(f^{-n}(x)) \]

for all $x \in K$, $n > N$.

**Proof.** If the lemma is false we can find sequences $x_n, y_n \in K$, $m_n > 0$ such that $y_n \in W^s_\epsilon(x_n)$, $\lim m_n = +\infty$ and $d(f^{m_n}(x_n), f^{m_n}(y_n)) > r$. Since $y_n \in W^s_\epsilon(x_n)$ we have $d(f^n(f^{m_n}(x_n)), f^n(f^{m_n}(y_n))) < \epsilon$ for all $-m_n < n$. Then if $x_n \to x$, $y_n \to y$ when $n \to +\infty$ we obtain that $d(f^n(x), f^n(y)) < \epsilon$ for all $n \in \mathbb{Z}$. Moreover $d(x, y) = \lim d(f^{m_n}(x_n), f^{m_n}(y_n)) > r$ thus contradicting the expansivity.

Now define the stable and unstable sets $W^s(x), W^u(x)$ as

\[ W^s(x) = \bigcup_{n>0} f^{-n}(W^s_\epsilon(f^n(x))), \quad W^u(x) = \bigcup_{n>0} f^n(W^u_\epsilon(f^{-n}(x))). \]

By Lemma I we have:

\[ W^s(x) = \{ y \in K | \lim_{n \to +\infty} d(f^n(x), f^n(y)) = 0 \}, \]
\[ W^u(x) = \{ y \in K | \lim_{n \to +\infty} d(f^{-n}(x), f^{-n}(y)) = 0 \}. \]

**Lemma II.** If for some $x \in K$ and $m > 0$ we have $f^m(W^s(x)) \cap W^u(x) \neq \emptyset$ then $K$ contains a periodic point.

**Proof.** Suppose $f^m(W^s(x)) \cap W^u(x) \neq \emptyset$. Take $y \in W^s(x) \cap f^m(W^s(x))$. Let $z = f^{-m}(y)$. Then $f^m(z) \in W^s(x) = W^s(z)$. Therefore $\lim_{n \to +\infty} d(f^n(f^m(z)), f^n(z)) = 0$. Suppose that for some subsequence $m_n$ we have that $f^{m_n}(z)$ converges to some $w \in K$. Then

\[ d(w, f^{m_n}(w)) = \lim d(f^{m_n}(f^m(z)), f^{m_n}(z)) = \lim d(f^{m_n}(f^m(z)), f^{m_n}(z)) = 0. \]

Define $\Sigma^s_\delta(x), \Sigma^u_\delta(x)$ as the connected components of $x$ in $W^s_\epsilon(x) \cap B_\delta(x)$ and $W^u_\epsilon(x) \cap B_\delta(x)$ respectively, where $B_\delta(x) = \{ y/d(y, x) < \delta \}$. Let $S_\delta(x) = \{ y/d(y, x) = \delta \}.$

**Lemma III.** There exists $\epsilon > r > 0$ such that if $0 < \delta < r$ there exists $a \in K$ such that $\Sigma^s_\delta(a) \cap S_\delta(a) \neq \emptyset$ or $\Sigma^u_\delta(a) \cap S_\delta(a) \neq \emptyset$.

**Proof.** Let $\Sigma_\epsilon(x)$ be the connected component of $x$ in $B_\epsilon(x)$. Since $\dim K > 0$ we can find $x \in K$ and $r > 0$ such that $\Sigma_\epsilon(x) \cap S_\delta(x) \neq \emptyset$. If $0 < \delta < r$ it follows that $\Sigma_\delta(x) \cap S_\delta(x) \neq \emptyset$. Suppose that for some $0 < \delta < r$ we have $\Sigma^s_\delta(y) \cap S_\delta(y) = \emptyset$ for all $y$. We shall prove that there exists
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$a \in K$ such that $\Sigma_g^L(a) \cap S_g(a) \neq \emptyset$. To find $a$ we shall construct a family of compact connected sets $\Lambda_n$, $n > 0$, and a sequence of points $x_n \in \Lambda_n$ such that for some sequence of integers $m_n > 0$ they satisfy the following conditions:

(1) $f^{-m_n}(x_n) = x_{n+1};$
(2) $f^{-m_n}(\Lambda_n) \supset \Lambda_{n+1};$
(3) $\Lambda_n \cap S_g(x_n) \neq \emptyset;$
(4) $f^n(\Lambda_n) \subset B_g(f^n(x_n))$ if $0 < n < m_n - 1$.

Once the sets $\Lambda_n$ are constructed the lemma follows easily taking $a = \lim x_n$ and defining $\Lambda$ as the set of points $y \in K$ such that $y = \lim y_n$ for some sequence $y_n \in \Lambda_n$. Then $\Lambda$ is connected and from (4) follows that $\Lambda \subset W_g^s(x)$. By (3) $\Lambda \cap S_g(a) \neq \emptyset$. Hence the connected component of $a$ in $\Lambda$ intersects $S_g(a)$. Therefore the same thing is true for $\Sigma_g^L(a)$ because $\Lambda \subset W_g^s(a)$. To construct the sets $\Lambda_n$ start taking $\Lambda_0 = \Sigma_g^L(x)$. Suppose $\Lambda_0, \Lambda_1, \ldots, \Lambda_{n-1}$ is constructed. If $\Lambda_{n-1}$ is the connected component of $x_{n-1}$ in $B_g(x_{n-1}) \cap \Lambda_{n-1}$ then

$$\Lambda_{n-1} \cap S_g(x_{n-1}) \neq \emptyset$$

(because $\Lambda_{n-1}$ is connected) and $\Lambda_{n-1}$ cannot be contained in $W_g^s(x_{n-1})$. Hence there exists $y \in \Lambda_{n-1} \setminus W_g^s(x_{n-1})$. Then for some $m > 0$ we have

$$\sup\{d(f^{-m}(z), f^{-m}(x_{n-1})) | z \in \Lambda_{n-1}\} = d(f^{-m}(y), f^{-m}(x_{n-1})) > \epsilon$$

and we can suppose that:

$$\sup\{d(f^{-j}(z), f^{-j}(x_{n-1})) | z \in \Lambda_{n-1}, 0 < j < m\} < \epsilon.$$  (0)

Let $\Lambda_n$ be the connected component of $x_n = f^{-m_n}(x_{n-1})$ in $B_g(x_n) \cap f^{-m}(\Lambda_{n-1})$. Since $f^{-m}(\Lambda_{n-1})$ is connected we obtain that $\Lambda_n \cap S_g(x_n) \neq \emptyset$ thus proving (3). From (0) follows that $\Lambda_n$ satisfies (4).

**Lemma IV.** For all $0 < \delta < \epsilon$ there exists $N > 0$ such that for all $x \in K$ and $y \in W_g^s(x)$ with $d(y, x) = \delta$ there exists $0 < n < N$ satisfying

$$d(f^{-n}(y), f^{-n}(x)) > \epsilon.$$

**Proof.** If the lemma were false there would exist sequences $x_n \in K$, $y_n \in W_g^s(x_n)$ such that $d(x_n, y_n) = \delta$ and if $j < n$, $d(f^{-j}(x_n), f^{-j}(y_n)) < \epsilon$. Then if $x_n \to x, y_n \to y$ it is easy to check that $x \neq y$ and $d(f^n(x), f^n(y)) < \epsilon$ for all $n \in \mathbb{Z}$.

**Lemma V.** There exists $\delta_0 > 0$ such that $W_g^s(x) \cap B_g(x) = W_g^s(x) \cap B_g(x)$ for all $x \in K, 0 < \delta < \delta_0$.

**Proof.** If the lemma is false there exist sequences $x_n, y_n \in K$ such that $d(x_n, y_n) \to 0$, and $y_n \in W_g^s(x_n)$. Hence for some $m_n > 0$ we must have $d(f^{m_n}(x_n), f^{m_n}(y_n)) > \epsilon$ and $m_n \to +\infty$. We also have $d(f^n(f^{m_n}(x_n)))$,
\( f^n(y_n) < 2\varepsilon \) for all \(-m < m \leq m\), because \( y_n \in W^l_z(x_0) \). Then if \( f^n(x_n) \to x \) and \( f^n(y_n) \to y \) when \( n \to +\infty \) we conclude that \( d(x, y) > \varepsilon \) and \( d(f^n(x), f^n(y)) < 2\varepsilon \) for all \( m \in \mathbb{Z} \) thus contradicting the expansivity of \( f \).

Now define the constant \( \tilde{\varepsilon} = \inf\{d(x, y)\mid d(f^{-1}(x), f^{-1}(y)) > \varepsilon\} \).

**Lemma VI.** For all \( 0 < \delta < \min(\delta_0, \tilde{\varepsilon}/3) \) (where \( \delta_0 \) is given by Lemma V) there exists \( N = N(\delta) > 0 \) such that if \( x \in K \) and \( \Lambda \subset W^s(x) \) is a compact connected set containing \( x \) and intersecting \( S_\delta(x) \) then there exist \( 0 < m < N \), points \( \alpha, \beta \in f^{-m}(\Lambda) \) and compact connected sets \( \Lambda_\alpha, \Lambda_\beta \) satisfying:

(a) \( \alpha \in \Lambda_\alpha, \beta \in \Lambda_\beta, \alpha \in W^s_z(\beta) \);
(b) \( \Lambda_\alpha \cap S_\delta(\alpha) \neq \emptyset, \Lambda_\beta \cap S_\delta(\beta) \neq \emptyset \);
(c) \( \inf\{d(z, w)\mid z \in B_\delta(\alpha), w \in B_\delta(\beta)\} > \delta \);
(d) \( \Lambda_\alpha \subset W^s_\delta(\alpha) \cap B_\delta(\alpha), \Lambda_\beta \subset W^s_\delta(\beta) \cap B_\delta(\beta) \).

**Proof.** Take \( N = N(\delta) \) given by Lemma IV. Since \( S_\delta(x) \cap \Lambda \neq \emptyset \), by Lemma IV there exists \( 0 < m < N \) such that

\[
\sup\{d(f^{-m+1}(z), f^{-m+1}(x))\mid z \in \Lambda\} > \varepsilon
\]

and we can suppose:

\[
\sup\{d(f^{-j}(z), f^{-j}(x))\mid x \in \Lambda, 0 < j < m\} < \varepsilon.
\]

Hence \( f^{-m}(\Lambda) \subset W^s_z(f^{-m}(x)) \) and then:

\[
f^{-m}(\Lambda) \subset W^s_z(w)
\]

for all \( w \in f^{-m}(\Lambda) \). Moreover by the definition of \( \tilde{\varepsilon} \) we have \( \text{diam } f^{-m}(\Lambda) < \tilde{\varepsilon} \). Then we can find points \( \alpha, \beta \in f^{-m}(\Lambda) \) such that \( B_\delta(\alpha), B_\delta(\beta) \) satisfy (c) (here is used the property \( 3\delta < \tilde{\varepsilon} \)). Let \( \Lambda_\alpha, \Lambda_\beta \) be the connected components of \( \alpha \) and \( \beta \) in \( f^{-m}(\Lambda) \cap B_\delta(\alpha), f^{-m}(\Lambda) \cap B_\delta(\beta) \) respectively. Since \( f^{-m}(\Lambda) \) is connected, contains \( \alpha \) and \( \beta \), and \( \alpha \notin B_\delta(\beta), \beta \notin B_\delta(\alpha) \), it follows that \( \Lambda_\alpha, \Lambda_\beta \) satisfy (b). By (1) \( \alpha \) and \( \beta \) satisfy (a) and \( \Lambda_\alpha \cap B_\delta(\alpha) \cap W^s_z(\alpha), \Lambda_\beta \subset B_\delta(\beta) \cap W^s_z(\beta) \). Hence, by Lemma V, \( \Lambda_\alpha, \Lambda_\beta \) satisfy (d).

Now we are ready to prove that \( f \) cannot be a minimal homeomorphism. Take \( 0 < \delta < \min(\delta_0, \tilde{\varepsilon}/3, r) \) (\( r \) given by Lemma III) and define:

\[
r_1 = \inf\{d(f^j(x), f^j(y))\mid x \in W^s_{a_1}(y), d(x, y) > \delta, 0 < j < N(\delta), 0 < i < N(\delta)\}
\]

where \( N(\delta) \) is as in Lemma VI. Using Lemma II it is easy to see that this number is positive, otherwise we should have sequences \( x_n, y_n \in K, n > 0, y_n \in W^s_{a_1}(x_n), d(x_n, y_n) > \delta, 0 < j_n < i_n < N(\delta) \) such that \( d(f^n(x_n), f^n(y_n)) \to 0 \) when \( n \to +\infty \) and that \( j_n = j, i_n = i \) for all \( n > 0 \). Then \( y \in W^s_{a_1}(x), d(y, x) > \delta \) and \( f^n(x) = f^n(y) \). Hence \( j \neq i \) and \( f^{j-i}(W^s(x)) \cap W^s(y) \neq \emptyset \). This, by Lemma II, implies that \( K \) contains a periodic point. So we can assume \( r_1 > 0 \). We shall show that \( r_1 > 0 \) also contradicts the minimality of \( K \) by showing the
existence of an open set $U$ and a point $p$ such that $f^n(p) \notin U$ for all $n > 0$.

First we construct a family of compact connected sets $\Lambda_n$, $n > 0$, a sequence of points $x_n \in \Lambda_n$ and an open set $U \subset K$ such that:

(a) $\Lambda_n \cap S_\delta(x_n) \neq \emptyset$;
(b) $\Lambda_n \subset W^s_\varepsilon(x_n)$;
(c) For some $0 < m_n < N(\delta)$, $f^{-m_n}(\Lambda_n) = \Lambda_{n+1}$;
(d) $f^{-j}(\Lambda_n) \cap U = \emptyset$ for all $0 < j < m_n$.

By Lemma III we can suppose that $S_\delta(a) \cap S_\delta(b)$ for some $a \in K$ (because $\delta < r$). Define $x_0 = a$, $\Lambda_0 = S_\delta(a)$ and take an open set $U$ with diameter $< r/2$, and $U \cap \Lambda_0 = \emptyset$. Suppose that we constructed $\Lambda_0, \Lambda_1, \ldots, \Lambda_{n-1}$. To find $\Lambda_n$ we apply Lemma VI obtaining two compact connected sets $\Lambda_a, \Lambda_b$ such that defining $\Lambda_n$ as one of them, properties (a), (b), (c) would be satisfied. In order to satisfy condition (d) observe that by the definition of $r_1$ and the fact that the diameter of $U$ is $< r_1/2$ then if $U \cap (\bigcup_{j=0}^\infty f^{-j}(\Lambda_n)) = \emptyset$ ($m$ being the number given in Lemma VI) then $U \cap (\bigcup_{j=0}^\infty f^{-j}(\Lambda_n)) = \emptyset$ and we chose $\Lambda_n = \Lambda_b$. Finally define $p = \cap N_{m_n}$ where $N_n = \Sigma_{j=0}^\infty m_j$. Clearly $f^n(p) \notin U$ for all $n > 0$, thus proving that $K$ is not minimal.

2. The dimension of $K$. Let $f, K$ be as in §1. Here we shall prove that $\dim K < \infty$. Let, as in §1, $c > 0$ be an expansivity constant for $f$. Fix $0 < \varepsilon < c/2$.

**Lemma.** There exists $\delta > 0$ such that if $x, y \in K, d(x, y) < \delta$, and for some $n > 0$ satisfy $\varepsilon < \sup \{d(f^j(x), f^j(y))|0 < j < n\} < 2\varepsilon$, then $d(f^n(x), f^n(y)) > \delta$.

**Proof.** If this property is false we can find sequences $x_n, y_n \in K, m_n > l_n > 0$ such that $d(x_n, y_n) \to 0$, $d(f^{m_n}(x_n), f^{m_n}(y_n)) \to 0$, $d(f^j(x_n), f^j(y_n)) > \varepsilon$ and $\sup \{d(f^m(x_n), f^m(y_n))|0 < m < m_n\} < 2\varepsilon$. Suppose that $f^{m_n}(x_n) \to x$ and $f^{m_n}(y_n) \to y$. Then $d(x, y) > \varepsilon$ and $d(f^n(x), f^n(y)) < 2\varepsilon$ for all $n \in \mathbb{Z}$.

**Lemma II.** For all $\rho > 0$ there exists $N = N(\rho)$ such that $d(x, y) > \rho$ implies that $\sup \{d(f^n(x), f^n(y))| |n| < N\} > \varepsilon$.

**Proof.** If the property is false there exist sequences $x_n, y_n \in K$ with $d(x_n, y_n) > \rho$ and such that $\sup \{d(f^j(x), f^j(y))| |j| < n\} < \varepsilon$. Then if $x_n \to x$, $y_n \to y$, we obtain $d(x, y) > \rho$ and $d(f^n(x), f^n(y)) < \varepsilon$ for all $n \in \mathbb{Z}$.

To prove that $\dim K < \infty$ take a covering $\{U_i|1 < i < l\}$ of $K$ by open sets with diameter $< \delta, \delta$ as in Lemma I. We claim that $\dim(K) < l^2 - 1$. To prove this for each $n > 0$ choose $\delta_n > 0$ such that $d(x, y) < \delta_n$ implies $d(f^j(x), f^j(y)) < \varepsilon$ for all $|j| < n$. Let $U_{ij}^n = f^n(U_i) \cap f^{-n}(U_j)$ and let $U_{ij}^{nk}$, $1 < k < k(i, j, n)$, be the $\delta_n$-components of $U_{ij}^n$, i.e. the equivalence classes of $U_{ij}^n$ under the relation $x \sim y$ if there exists a sequence $x = x_0, x_1, \ldots, x_p = y$.  

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such that $d(x_r, x_{r+1}) < \delta_n$ for all $0 < r < p - 1$ and $x_r \in U_{ij}^{n,k}$ for all $0 < r < p$. Observe that the sets $U_{ij}^{n,k}$ are open and cover $K$. We have that:

$$\lim_{n \to +\infty} \left( \sup_{k,i,j} \text{diam } U_{ij}^{n,k} \right) = 0 \quad (2)$$

because otherwise we could find $\rho > 0$ and large values of $n$, say $n > 2N(\rho)$, $N(\rho)$ given by Lemma II, such that in some $U_{ij}^{n,k}$ there exist $x, y$ with $d(x, y) > \rho$. Let $x = x_0, x_1, \ldots, x_p = y$ a sequence in $U_{ij}^n$ such that $d(x_r, x_{r+1}) < \delta_n$ for all $0 < r < p$. Define $S_r = \sup\{d(f^m(x_r), f^m(x_0)) | m < n\}$. By Lemma II $S_p > \varepsilon$ and by the choice of $\delta_n S_1 < \varepsilon$ and $|S_{r+1} - S_r| < \varepsilon$ for $1 < r < p$. Take $r$ such that $S_r < \varepsilon$ if $r' < r$ and $S_r > \varepsilon$. Then $S_r < 2\varepsilon$. Therefore the points $x = x_0$ and $x_r$ satisfy $d(f^{-n}(x), f^{-n}(x_r)) < \delta$ because $f^{-n}(x), f^{-n}(x_r)$ belong to $U_i$, and $d(f^m(x), f^m(x_r)) < \delta$ because $f^m(x), f^m(x_r)$ belong to $U_j$, and $\varepsilon < s_r = \sup\{d(f^m(x), f^m(x_r)) | m < n\} < 2\varepsilon$ thus contradicting Lemma I. Then (2) is proved and it remains to show only that for each $n$, every point of $K$ belongs at most to $l^2$ sets of the covering $\{U_{ij}^{n,k} | 1 < i < l, 1 < j < l, 1 < k < k(i,j)\}$. Suppose that $\nbigcup_{m} \{U_{m,i,j}^{n,k} | 1 < m < s\} \neq \emptyset$. Then $(i_m, j_m) = (i_m, j_m)$ implies that $U_{m,i,j}^{n,k} = U_{m,i,j}^{n,k}$ because they are both $\delta_n$-components of $U_{m,i,j}^{n,k}$ and have nonempty intersection.

This means that to different values of $m$ correspond different values of the couple $(i_m, j_m)$. Therefore $s < m^2$.

REFERENCES


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