A connected space is said to be **hereditarily locally connected** if each of its connected subsets is locally connected. A connected set is said to be **σ-connected** (resp., **weakly σ-connected**) if it cannot be decomposed into a countably infinite family of mutually separated (resp., mutually separated, connected) nonempty subsets. In [5, Problem 253], B. Knaster asked whether every hereditarily locally connected planar set is weakly σ-connected. A. Lelek proved in [7, Theorem 8] that every connected subset of a planar hereditarily locally connected continuum is σ-connected. H. M. Gehman proved in [2] that these continua are finitely Suslinian. In 1974, J. Grispolakis, Lelek and E. D. Tymchatyn proved that a hereditarily locally connected continuum is finitely Suslinian if and only if every connected subset is σ-connected if and only if every connected subset is weakly σ-connected (see [3, Theorem 2.2]). There exist examples of (nonplanar) hereditarily locally connected continua which contain connected subsets which are unions of countably many disjoint arcs (see [6, p. 270]).

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It was proved in [5, Theorem 1] that a connected metric space cannot be decomposed into a countable infinite null family of mutually separated nonempty sets. In §2, we give a very simple argument that extends this result to the class of normal $T_1$-spaces.

In §3, we prove that hereditarily locally connected, planar spaces are hereditarily weakly $\sigma$-connected. This answers Knaster's question in the negative. We have been unable, though, to answer Knaster's further question as to whether hereditarily locally connected, planar spaces are $\sigma$-connected.

In §4, we introduce the notion of finitely Suslinian spaces, which is a generalization of the notion of finitely Suslinian continua. We generalize Gehman's theorem by proving that hereditarily locally connected, planar spaces are finitely Suslinian if and only if they are semilocally connected. We obtain some rudimentary results concerning finitely Suslinian compactifications of finitely Suslinian spaces.

Finally, in §5, we answer a question of Lelek [8, Problem 987] in the affirmative by proving that planar hereditarily locally connected spaces that are arcwise connected are locally arcwise connected.

Let $A$ be a subset of a topological space $X$. By $\text{Cl}(A)$ and $\text{Bd}(A)$ we denote the closure of $A$ and the boundary of $A$ in $X$, respectively.

Let $A'$ be a Tychonoff space. A family $\mathcal{A}$ of subsets of $X$ is said to be null if for each $x \in X$ and for each convergent subnet $(A_\alpha)_{\alpha \in \Lambda}$ of $\mathcal{A}$ such that $x \in \text{Lim sup}_{\alpha \in \Lambda} A_\alpha$ we have that $(A_\alpha)_{\alpha \in \Lambda}$ is eventually in every neighbourhood of $x$.

2. Decompositions of connected spaces. We shall need the following simple result.

2.1. THEOREM. A nondegenerate, connected, normal, $T_1$-space does not admit an upper semicontinuous decomposition into countably many, pairwise disjoint, closed subsets.

PROOF. Suppose, on the contrary, that $X$ is a nondegenerate connected, normal, $T_1$-space, which admits a decomposition $D$ as above. Let $Y = X/D$ be the quotient space, and let $\varphi: X \to Y$ be the natural projection. Then $\varphi$ is a closed mapping and $Y$ is a countable connected space. By [1, Theorem 5, p. 85], $Y$ is a normal $T_1$-space. Thus, $Y$ is degenerate. This contradiction completes the proof of Theorem 2.1.

The following result was first proved in [5].

2.2. COROLLARY [5, THEOREM 1]. A nondegenerate, connected, metric space cannot be decomposed into a null sequence of countably infinitely many, pairwise disjoint, closed subsets.

PROOF. Such a decomposition is upper semicontinuous.
3. Planar hereditarily locally connected spaces. We say that a topological space $X$ has property $(\ast)$ provided that for each connected subset $U$ of $X$ and for each sequence $A_1, A_2, \ldots$ of closed, connected subsets of $X$ each of which meets $U$ and such that $A_i \cap A_j \subseteq \text{Cl}(U)$ for each $i \neq j$, $i,j = 1, 2, \ldots$, we have

$$\limsup_{i \to \infty} A_i \subseteq \text{Cl}(U).$$

3.1. Theorem. If $X$ is a planar hereditarily locally connected space, then $X$ has property $(\ast)$.

Proof. Let $U, A_1, A_2, \ldots$ be connected closed sets in $X$ such that each $A_i$ meets $U$ and $A_i \cap A_j \subseteq U$ for each $i \neq j$. Just suppose that there exists some point $x \in \limsup_{i \to \infty} A_i \setminus U$.

Since $X$ is hereditarily locally connected we may suppose $A_i \subseteq U$ and $A_i \setminus \{a_i\}$ is connected for each $i = 1, 2, \ldots$. Let $U'$ denote the closure of $U$ in the plane. We may suppose without loss of generality that $x$ lies in the unbounded complementary component of $U'$ in the plane. We may also suppose that each $A_i \setminus \{a_i\}$ lies in the unbounded complementary component of $U'$ in the plane.

Let $W$ be a closed disk in the plane which contains $U'$ in its interior and which does not contain $x$. We may suppose that, for each $i$, $A_i$ is not contained in $W$. For each $i$ let $z_i$ be a point in the boundary of $W$ such that $z_i$ is in the component of $A_i \cap W$ which contains $a_i$. Such a point exists since $A_i$ is connected and locally connected. We may suppose that there exist a point $z_0$ in the boundary of $W$ and an arc $[z_i, z_0]$ in the boundary of $W$ such that, for each $i > 2$, $z_i$ separates $z_{i-1}$ from $z_0$ in $[z_i, z_0]$.

Let $Y = U \cup A_1 \cup A_2 \cup \ldots$. We may suppose $x \notin Y$. If for each $i$, $A_i \setminus W$ were open in $Y$ then $Y \cup \{x\}$ would be a connected subset of $X$ which is not locally connected at $x$. We suppose, therefore, that there exist $y \in A_i \setminus W$ and $z \in A_i \setminus W$ such that $y, z \in \lim_{i \to \infty} A_i$.

We may suppose (by picking a subsequence if necessary) that $\varepsilon_3 > \varepsilon_4 > \ldots$ is a sequence of real numbers converging to zero such that, for each $i > 3$, $A_i$ meets the $\varepsilon_j$ neighbourhood $S(z, \varepsilon_j)$ of $z$ in the plane and $A_j$ does not meet $S(z, \varepsilon_j)$ for $3 < j < i$. For each $i > 3$ let $Z_i$ denote the closure in the plane of $W \cup A_2 \cup S(z, \varepsilon_j) \cup A_i$. It is easy to see by using standard arguments for the plane that $Z_i$ separates $y$ from $A_i \setminus W$ for $3 < j < i$.

Let $Z = U \cup A_1 \cup A_3 \cup A_5 \cup \ldots$. Then $Z$ is a connected subset of $X$ which is not locally connected at $y$. To see this let $V$ be a neighbourhood of $y$ in the plane. If $i > 1$ and $A_{2i+1}$ meets $V$ then $Z_{2i+1}$ separates $A_{2i+1} \cap V$ from $y$ in $V$ if $V \cap W = \emptyset$. Also, if $i$ is sufficiently large then $A_{2i+1}$ meets $V$, thus $V \cap Z$ is not connected.

Example 4.4 shows that the converse of Theorem 3.1 need not be true.
3.2. Theorem. Every locally connected, normal, $T_1$-space with property (*)& is weakly $\sigma$-connected.

Proof. Suppose, on the contrary, that $X$ is a locally connected, normal, $T_1$-space with property (*), which is not weakly $\sigma$-connected. Then $X = \bigcup_{i=1}^{\infty} A_i$, where $A_1, A_2, \ldots$ are mutually disjoint, closed, connected, non-empty subsets of $X$.

Let $U_1$ be a connected neighbourhood of $A_1$ such that $\text{Cl}(U_1) \subset X \setminus A_2$. Let $D_2^1$ be the component of $A_2$ in $X \setminus \text{Cl}(U_1)$, and consider the set

$$A_2^1 = \bigcup \{ A_i | A_i \cap \text{Cl}(D_2^1) \neq \emptyset \}.$$

Then $D_2^1$ is an open connected set, since $X \setminus \text{Cl}(U_1)$ is locally connected, and $A_2 \subset D_2^1 \subset \text{Int}(A_2^1)$. By property (*), we have that $A_2^1$ is a closed, connected subset of $X$, disjoint from $A_1$.

Let $V_1$ be a connected neighbourhood of $A_1$ such that $\text{Cl}(V_1) \subset X \setminus A_2$. Let $D_1^1$ be the component of $A_2$ in $X \setminus \text{Cl}(V_1)$, and consider the set

$$A_1^1 = (X \setminus D_1^1) \cup \bigcup \{ A_i | A_i \cap \text{Bd}(V_1) \neq \emptyset \}.$$

Then $X \setminus D_1^1$ is a closed, connected subset of $X$ and $A_1 \subset V_1 \subset \text{Int}(A_1^1)$. By property (*), we have that $A_1^1$ is a connected, closed subset of $X$, disjoint from $A_2^1$.

Let $n_1$ be the smallest integer such that $A_1 \subset X \setminus (A_1^1 \cup A_2^1)$. Let $\mathcal{U}$ be an open cover of $X$ consisting of connected sets whose closures refine the cover $\{X \setminus A_1^1, X \setminus A_2^1\}$. Let $\{C_1, \ldots, C_m\}$ be an irreducible chain in $\mathcal{U}$ joining $A_n$ and $A_1^1 \cup A_2^1$. Without loss of generality, assume that $C_1$ meets $A_1$, and $C_m$ meets $A_1^1$, and put

$$B_1^1 = \bigcup \{ A_i | A_i \cap C_j \neq \emptyset \text{ for some } j \text{ with } 1 \leq j \leq m \} \cup A_1^1$$

and $B_2^1 = A_2$. By property (*), $B_1^1$ and $B_2^1$ are closed, connected, nonempty sets.

Let $U_2$ be a connected neighbourhood of $B_1^1$ such that $\text{Cl}(U_2) \subset X \setminus B_2^1$. Let $D_2^2$ be the component of $B_2^1$ in $X \setminus \text{Cl}(U_2)$, and consider the set

$$A_2^2 = \bigcup \{ A_i | A_i \cap \text{Cl}(D_2^2) \neq \emptyset \}.$$

Then $D_2^2$ is an open connected set, since $X \setminus \text{Cl}(U_2)$ is locally connected, and $B_2^1 \subset \text{Int}(A_2^2)$. By property (*), we have that $A_2^2$ is a closed, connected subset of $X$ disjoint from $B_1^1$.

Let $V_2$ be a connected neighbourhood of $B_1^1$ such that $\text{Cl}(V_2) \subset X \setminus A_2^2$. Let $D_2^1$ be the component of $B_1^1$ in $X \setminus \text{Cl}(V_2)$, and consider the set

$$A_1^2 = (X \setminus D_2^1) \cup \bigcup \{ A_i | A_i \cap \text{Bd}(V_2) \neq \emptyset \}.$$

Then $X \setminus D_2^1$ is a closed, connected subset of $X$ and $B_1^1 \subset \text{Int}(A_1^2)$. By
property (\(*\)), we have that $A_1^2$ is a closed, connected subset of $X$, disjoint from $A_2^2$.

Inductively, we construct two sequences of closed connected sets $B_1^0, B_1^1, B_1^2, \ldots$ and $B_2^0, B_2^1, B_2^2, \ldots$ where $B_1^0 = A_1$ and $B_2^0 = A_2$, such that, for each $i > 0$, $B_1^i \subset \text{Int}(B_1^{i+1})$ and $B_2^i \subset \text{Int}(B_2^{i+1})$ and $A_{i+1} \subset B_1^i \cup B_2^i$, and such that if $A_j \cap B_k^i \neq \emptyset$, then $A_j \subset B_k^i$.

Consider the sets $A = \bigcup_{i=0}^{\infty} B_1^i$ and $B = \bigcup_{i=0}^{\infty} B_2^i$. Then $A$ and $B$ are open, disjoint, nonempty subsets of $X$ such that $X = A \cup B$. The latter statement is obvious, since for each $m > 1$ there exists some $k$ such that either $A_m \subset B_1^k$ or $B_m \subset B_2^k$. This contradicts the connectedness of $X$ and the proof is complete.

The following corollary answers in the negative Knaster’s question (see [5, Problem 253]).

3.3. Corollary. Every planar, hereditarily locally connected space is weakly $\sigma$-connected.

The corollary follows from Theorem 3.1 and Theorem 3.2.

That the local connectedness is essential in Theorem 3.2 is apparent from the following:

3.4. Example. Let $X$ be the Knaster indecomposable continuum (see [6, p. 205]). Then every proper connected subset of $X$ has property (\(*\)), since every connected subset of $X$ which is not dense in $X$ is contained in an arc. On the other hand, there exist connected subsets of $X$ which are not weakly $\sigma$-connected (see [4, Proposition 2.1]).

4. Finitely Suslinian spaces. A Tychonoff space is said to be finitely Suslinian provided it is locally connected and each net $(A_a)_{a \in I}$ of distinct, closed, connected, pairwise disjoint subsets of it is null (i.e., if $x \in \limsup_{a \in I} A_a$ and $V$ is a neighbourhood of $x$, then there is a subnet of $(A_a)_{a \in I}$ which is eventually in $V$). It is clear that this class contains the class of finitely Suslinian continua (see [14]). It was proved that a hereditarily locally connected metric continuum is finitely Suslinian if and only if every connected subset is $\sigma$-connected if and only if every connected subset is weakly $\sigma$-connected (see [3, Theorem 2.2]). There are simple examples of hereditarily locally connected spaces, which are not finitely Suslinian, but which have the property that every connected subset is $\sigma$-connected.

We say that a space $X$ is semilocally connected provided that for each point $x \in X$ and for each neighbourhood $U$ of $x$, there exists a neighbourhood $V$ of $x$ such that $V \subset U$ and $X \setminus V$ has finitely many components. The space $X$ is said to be hereditarily semilocally connected (h.s.l.c.) provided each connected subset of $X$ is semilocally connected.
4.1. THEOREM. Let \( X \) be a connected, semilocally connected first countable Tychonoff space with property (\( \ast \)). Then \( X \) is hereditarily locally connected.

PROOF. Let \( C \) be a connected subset of \( X \). We shall prove that \( C \) is locally connected. Suppose, on the contrary, that \( C \) is not locally connected at the point \( x_0 \in C \), and let \( U \) be a neighbourhood of \( x_0 \) in \( X \) such that each neighbourhood of \( x_0 \) contained in \( U \cap C \) is not connected, and such that \( X \setminus U \) has finitely many components. Let \( C_1, \ldots, C_n \) be the components of \( X \setminus U \). Since \( x_0 \) is not in the interior of its component in \( U \cap C \), we can find a sequence \( F_1, F_2, \ldots \) of nonempty closed-open subsets of \( U \cap C \), which are mutually separated and such that \( x_0 \in \text{Lim sup}_i F_i \). For each \( i \) let \( x_i \in F_i \) be such that \( \text{Lim}_{i \to \infty} x_i = x_0 \). Since \( C \) is connected, we have that
\[
\text{Cl}(F_i) \cap \text{Bd}(U) \neq \emptyset
\]
for each \( i \) (the closure of \( F_i \) is taken in \( X \)). If \( n = 1 \), then \( C_1 \cup F_i \) is connected for each \( i \). Then \( C_1 \cup \bigcup_{i=1}^\infty F_i \cup \{x_0\} \) is a connected set, which does not have property (\( \ast \)), which is a contradiction.

Suppose \( n > 1 \) and let \( 1 < m < n \) be such that \( x_0 \) is in the interior of the component of \( C \setminus (C_1 \cup \cdots \cup C_m) \) which contains \( x_0 \). Let \( K \) be the component of \( C \setminus (C_1 \cup \cdots \cup C_m) \), which contains \( x_0 \). Suppose \( x_0 \) is not in the interior of the component of \( K \setminus C_{m+1} \) which contains \( x_0 \). As in the case \( n = 1 \) replacing \( C_1 \) by \( C_{m+1} \) and \( C \) by \( K \), we again get a contradiction. Thus, \( C \) is locally connected and the proof of the theorem is complete.

4.2. THEOREM. A connected, first countable Tychonoff space is finitely Suslinian if and only if it is semilocally connected and has property (\( \ast \)).

PROOF. The necessity of the condition is obvious by the definition of finitely Suslinian space. To prove the sufficiency of the condition, suppose, on the contrary, that there exists a space \( X \) which satisfies the conditions of the theorem but is not finitely Suslinian. Then there exists a sequence \( A_1, A_2, \ldots \) of disjoint, closed, connected subsets of \( X \) which is not null, that is, there exist \( x_0 \in \text{Lim sup}_{i \to \infty} A_i \) and a neighbourhood \( V_0 \) of \( x_0 \) such that for each \( i \) there exists \( n_i > i \) such that
\[
A_{n_i} \cap (X \setminus \text{Cl}(V_0)) \neq \emptyset.
\]
Assume, without loss of generality, that for each \( i \) we have \( A_i \cap V_0 \neq \emptyset \) and \( A_i \cap (X \setminus \text{Cl}(V_0)) \neq \emptyset \).

Since \( X \) is semilocally connected we can take \( V_0 \) to be such that \( X \setminus V_0 \) has finitely many components. Since \( A_i \cap (X \setminus V_0) \neq \emptyset \), there exist a component \( C \) of \( X \setminus V_0 \) and a convergent subsequence \( (A_{n_i})_{i \in N} \) of \( (A_i)_{i \in N} \) such that \( C \cap A_{n_i} \neq \emptyset \) for each \( i \in N \) and \( x_0 \in \text{Lim sup}_{i \to \infty} A_{n_i} \). But then \( x_0 \in \text{Cl}(C) \). This contradicts the fact that \( X \) has property (\( \ast \)). This contradiction combined with Theorem 4.1 implies that \( X \) is finitely Suslinian.
4.3. Corollary. Every connected subset of a finitely Suslinian first countable space is finitely Suslinian. Therefore, first countable finitely Suslinian spaces are hereditarily semilocally connected, and hereditarily locally connected.

The proof of Corollary 4.3 follows from the fact that property (•) is hereditary, and from Theorems 4.1 and 4.2.

Problem I. Are finitely Suslinian spaces hereditarily locally connected?

The following example shows that semilocal connectedness is essential in the hypothesis of Theorem 4.2.

4.4. Example. This is a planar, connected, locally connected space with property (•), which is not hereditarily locally connected.

By (x, y) we denote the point of the plane \( \mathbb{R}^2 \) with cartesian coordinates \( x \) and \( y \).

Let \( C_n = \{(x, y) | x^2 + (y - (1/2^n))^2 = ((2^n - 1)/2^n)^2\} \) for each \( n = 1, 2, \ldots \). Consider the set \( X = \bigcup_{n=1}^{\infty} C_n \). Since \( C_n \cap C_m = \{(0,1)\} \) for each \( n \neq m, n, m = 1, 2, \ldots \), we have that \( X \) is a connected subspace of \( \mathbb{R}^2 \).

Now it is easy to check that \( X \) is locally connected space with property (•), but \( X \) is not hereditarily locally connected.

A connected space \( X \) is said to be hereditarily \( \sigma \)-connected (resp., hereditarily weakly \( \sigma \)-connected) provided that each connected subset of \( X \) is \( \sigma \)-connected (resp., weakly \( \sigma \)-connected). For a study of these spaces see [3] and [4].

4.5. Theorem. Every finitely Suslinian, normal, \( T_1 \)-space is weakly \( \sigma \)-connected.

Proof. Let \( X \) be a finitely Suslinian space. Just suppose \( X = \bigcup_{i=1}^{\infty} A_i \), where \( A_1, A_2, \ldots \) are pairwise disjoint, closed, connected, nonempty subsets of \( X \). Since \( X \) is finitely Suslinian, the sequence \( A_1, A_2, \ldots \) is null. Therefore, \( D = \{A_1, A_2, \ldots \} \) is an upper semicontinuous decomposition of the connected set \( X \). But \( X \) is a normal \( T_1 \)-space. This contradicts Theorem 2.1.

4.6. Corollary. Every completely normal, \( T_1 \), finitely Suslinian space is hereditarily weakly \( \sigma \)-connected.

In [2], Gehman proved that a planar hereditarily locally connected continuum is finitely Suslinian. It is clear that every locally connected continuum is semilocally connected. Therefore, the following theorem generalizes Gehman's result.

4.7. Theorem. A planar, hereditarily locally connected space is finitely Suslinian if and only if it is semilocally connected.

Proof. The necessity follows from Theorem 4.2, and the sufficiency follows from Theorems 3.1 and 4.2.
A space $X$ is said to be rational (resp., regular) provided each point of $X$ has arbitrarily small open neighbourhoods with countable (resp., finite) boundaries. In [10, p. 593], an example of a hereditarily locally connected, planar space is given which is not rational. In the same paper, it is proved that semilocally connected hereditarily locally connected separable metric spaces are rational (see [10, Theorem 2.4]). This together with our Theorems 4.1 and 4.2 imply the following:

4.8. Theorem. Every finitely Suslinian, separable metric space is rational.

The above mentioned example [10, p. 593] contains a totally disconnected set which is one dimensional.

4.9. Theorem. Let $X$ be a finitely Suslinian separable metric space, and let $Y$ be a nonempty totally disconnected subset of $X$. Then $\dim Y = 0$.

Proof. The theorem follows from Corollary 4.3 and [10, Theorem 1.7].

We present now one theorem on compactifications of finitely Suslinian spaces. By a compactification of a space $X$ we mean a compact space $Y$ such that $X$ can be embedded densely in $Y$.

It is known that a regular space can be compactified to a regular space [11] (see also J. R. Isbell [Uniform spaces, Math. Surveys, no. 12, Amer. Math. Soc., Providence, R. I., 1964, p. 111]). In [10] and [13], the problem of the compactification of hereditarily locally connected spaces has been investigated extensively. The following problems concerning compactifications of finitely Suslinian spaces are still open.

Problem II. Do finitely Suslinian (metric) spaces have finitely Suslinian compactifications?

Problem III. Do finitely Suslinian (metric) spaces have hereditarily locally connected compactifications?

Problem IV. Do finitely Suslinian (metric) spaces have rational compactifications?

Since finitely Suslinian continua are hereditarily locally connected the question raised in Problem III is strictly weaker than that in Problem II. It is well known that hereditarily locally connected metric continua are rational (see [15]) hence the question raised in Problem IV is at least in the metric case strictly weaker than that in Problem III. Spaces which admit a rational metric compactification have been characterized by Tymchatyn [Houston J. Math. 3 (1977), 131–139]. The following theorem now gives a partial answer to Problem II.

4.10. Theorem. Let $X$ be a cyclic finitely Suslinian planar space which admits a hereditarily locally connected compactification. Then $X$ admits a planar finitely Suslinian compactification.
PROOF. Let \( \tilde{X} \) be a hereditarily locally connected compactification of \( X \). By [13, Theorem 7], \( \tilde{X} \) can be considered to be a perfect, metric compactification with punctiforme remainder. By Gehman's theorem it suffices to prove that \( \tilde{X} \) is planar. To prove that \( \tilde{X} \) is planar, it suffices to prove that every graph in \( \tilde{X} \) is planar (see Claytor [0]).

For this, let \( K \) be an arbitrary graph in \( \tilde{X} \). Let \( \mathcal{U} = \{ U_1, \ldots, U_n \} \) be a family of open connected sets in \( \tilde{X} \) such that \( U_i \cap K \) is connected for each \( i \), \( K \subseteq \cup \mathcal{U} \), and the nerve of \( \mathcal{U} \) is homeomorphic to \( K \). Since \( \tilde{X} \) is a perfect compactification of \( X \), \( \mathcal{V} = \{ \bigcap U_i \cap X \mid i = 1, \ldots, n \} \) is a family of connected open sets in \( X \), whose nerve is homeomorphic to \( K \). Let \( W_1, \ldots, W_n \) be connected open sets in the plane such that \( W_i \cap X = U_i \cap X \) for each \( i \), and such that the nerve of \( \mathcal{W} = \{ W_1, \ldots, W_n \} \) is homeomorphic to \( K \). Then, it is easy to verify that the nerve of \( \mathcal{W} \) is embeddable in \( \cup \mathcal{U} \cap X \), and hence \( K \) is a planar graph.

4.11. Theorem. Let \( X \) be a finitely Suslinian, separable metric space, which admits a hereditarily locally connected compactification. Then \( X \) is hereditarily \( \sigma \)-connected.

PROOF. Let \( \tilde{X} \) be a hereditarily locally connected compactification of \( X \). By [13, Theorem 9], \( \tilde{X} \) may be considered to be metric. It suffices to show that \( X \) is \( \sigma \)-connected. Suppose, on the contrary, that \( X = \bigcup_{i=1}^{\infty} A_i \) is a decomposition of \( X \) into countably many mutually disjoint, closed, nonempty subsets of \( X \). By [3, Lemma 1.2], there exists a null sequence \( D_{11}, D_{12}, \ldots \) of mutually disjoint, closed-open subsets of \( X \). By [3, Lemma 1.2], there exists a null sequence \( D_{11}, D_{12}, \ldots \) of mutually disjoint, closed-open subsets \( D_{ij} \) of \( A_i \), such that

\[
\text{Dis}(A_i) \subseteq U_i = \bigcup_{j=1}^{\infty} D_{ij} \quad (i = 1, 2, \ldots) \tag{1}
\]

and

\[
\text{diam} D_{ij} < 1/i \quad (i, j = 1, 2, \ldots), \tag{2}
\]

where \( \text{Dis}(A_i) \) denotes the union of all the degenerate components of \( A_i \) \((i = 1, 2, \ldots)\).

Let \( \mathcal{C}_i \) denote the collection of all components of the set \( A_i \cap U_i \) \((i = 1, 2, \ldots)\), and let \( \mathcal{C} = \mathcal{C}_1 \cup \mathcal{C}_2 \cup \ldots \). Clearly, the members of \( \mathcal{C} \) are mutually disjoint, closed nondegenerate subsets of \( X \). Since \( \tilde{X} \) is metric and hereditarily locally connected, it is rational (see [15, p. 94]). Therefore, each family of mutually separated nondegenerate connected subsets of \( \tilde{X} \) is countable. Thus, \( \mathcal{C} \) is a countable family. By Corollary 4.6, \( X \) is hereditarily weakly \( \sigma \)-connected, therefore, \( \cup \mathcal{C} \) is not connected and each member of \( \mathcal{C} \) is a component of \( \cup \mathcal{C} \). By [6, Theorem 9, p. 272], \( \mathcal{C} \) is a null sequence. It follows from \( (2) \) that the members of \( \mathcal{C} \) together with the sets \( D_{ij} \) form a null sequence whose union is \( X \). By Corollary 2.2, \( X \) cannot be connected. This
contradiction shows that $X$ is $\sigma$-connected. Finally, since each connected subset of $X$ is finitely Suslinian, we conclude that $X$ is hereditarily $\sigma$-connected.

5. Local arcwise connectedness. In [8], Lelek established a relationship between local arcwise connectedness and the class of finitely Suslinian metric continua. He proved that each arcwise connected subset of a finitely Suslinian metric continuum is locally arcwise connected [8, Theorem 2.1], and he asked whether arcwise connected, hereditarily locally connected, planar spaces are locally arcwise connected [8, Problem 987]. In this problem the hereditary local connectedness is essential, since, in 1959, M. Shimrat constructed an arcwise connected, locally connected, planar space which is not locally arcwise connected [12, pp. 184–185].

In this section, we generalize Lelek's theorem and we resolve Problem 987 in [8] in the affirmative.

A space is said to be a local dendron if it has a basis of open sets which are connected subsets of finite graphs.

5.1. Lemma. Let $X$ be a hereditarily locally connected, separable metric space with property ($\ast$). If $A$ is an arcwise connected subset of $X$ and $x \in \text{Cl}(A)$, then $A \cup \{x\}$ is arcwise connected.

Proof. Suppose, on the contrary, that $A \cup \{x\}$ is not arcwise connected. Then, there exist an $\varepsilon > 0$ and a sequence of points $x_1, x_2, \ldots$ in $A$ converging to $x$ such that each arc in $A$ from $x_i$ to $x_j$ has diameter $> \varepsilon$ for each $i \neq j$. For each $i$, let $[x_i, x_{i+1}]$ be an arc from $x_i$ to $x_{i+1}$ in $A$ such that $[x_1, x_{i+1}] \cap [x_1, x_i]$ is connected for each $j < i$. Without loss of generality, assume that $A = \bigcup_{i=1}^{\infty} [x_i, x_{i+1}]$. If $A$ is not a local dendron, let $y \in A$ such that no neighbourhood of $y$ is contained in the union of finitely many of the arcs $[x_i, x_j]$. Since $A \cup \{x\}$ is a connected subset of $X$, it is locally connected. Therefore, if $U$ is a connected neighbourhood of $y$ in $A \cup \{x\}$ such that $x \not\in \text{Cl}(U)$, then there exists a subsequence $x_{i_1}, x_{i_2}, \ldots$ of $x_1, x_2, \ldots$, and points $y_j \in [x_1, x_j] \cap U$ such that for each $j \neq k$ we have

$$[y_j, x_j] \cap [y_k, x_k] \subset U,$$

where $[y_j, x_j]$ is the arc in $[x_1, x_j]$ with endpoints $y_j$ and $x_j$. But since $x \in \text{Lim sup}_{j \to \infty} [y_j, x_j]$, it is obvious that $A$ does not satisfy property ($\ast$). Hence, $A$ is a local dendron. Thus, each arc in $A$ has at most finitely many ramification points and each of these points is of finite order. Let each arc $[x_1, p]$ in $A$ have its natural order with initial point $x_1$. Let $a_1$ be the maximum point in $[x_1, x_j]$ such that $a_1 \in [x_1, x_j]$ for infinitely many $j > 2$. Then $a_1$ exists, and either $a_1 = x_1$, $a_1 = x_2$, or $a_1$ is a ramification point of $A$. Without loss of generality, assume that $a_1 \in [x_1, x_j]$ for each $j > 2$ and $x_k \not\in [x_1, x_j]$. 

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for $j < k$. Let $a_2$ be the maximum point in $[x_1, x_3]$ such that $a_2 \in [x_1, x_j]$ for infinitely many $j$. Then $a_2 > a_1$ in $[x_1, a_2]$. By induction, we define for each $i$ a point $a_i \in [x_1, x_j]$ for infinitely many $j$. Without loss of generality, we can assume that

$$[x_1, x_{i+1}] \cap [x_1, x_j] = [x_1, a_i]$$

for each $j > i + 2$.

Let $B = \cup_{i=1}^{\infty} [x_1, a_i]$. Then, $B$ is a ray (i.e., a homeomorphic image of $[0, 1]$). By property (•) and (1), $x \in \text{Cl}(B)$. Therefore, $B \cup \{x\}$ is a connected, and hence a locally connected subset of $X$. Let $U$ be a connected neighbourhood of $x$ in $B \cup \{x\}$. Then $U \cap B$ is an open arc in $B$, since $B$ is a ray. Hence, $B \cup \{x\}$ is an arc from $x_1$ to $x$ in $A \cup \{x\}$.

Let $X$ be a topological space and $x$ be a point of $X$. By the arc-component of $x$ in $X$ is meant the union of $x$ and of all arcs in $X$, which contain the point $x$. Whyburn had proved in [16] that finitely Suslinian metric continua have the property that each connected subset has closed arc-components. The following theorem generalizes the above result.

5.2. Theorem. Let $X$ be a hereditarily locally connected, separable metric space with property (•). Then every arc-component of $X$ is a closed subset of $X$.

5.3. Theorem. Let $X$ be a hereditarily locally connected, separable metric space with property (•). Then every arcwise connected subset of $X$ is locally arcwise connected.

Proof. Suppose, on the contrary, that $A$ is an arcwise connected subset of $X$, which is not locally arcwise connected at some point $x \in A$. Then, there exists a connected neighbourhood $V$ of $x$ in $A$, which is not arcwise connected. Let $\mathcal{C}$ be the family of the arc-components of the set $V$. Then the closure in $A$ of each arc-component of $V$ meets the boundary of $V$; therefore, each member of $\mathcal{C}$ is nondegenerate. By [10, Theorem 2.3], $\mathcal{C}$ is a countable family of connected sets. By Theorem 5.2, each member of $\mathcal{C}$ is a closed subset of $V$. Thus, $V$ is a hereditarily locally connected metric space with property (•) which is not weakly $\sigma$-connected. This contradicts Theorem 3.2, and the proof of the theorem is complete.

Remarks. In both Theorems 5.2 and 5.3 the assumption that the space has property (•) is essential. As a matter of fact, there exists a hereditarily locally connected space in the Euclidean 3-space $\mathbb{R}^3$, which is arcwise connected but not locally arcwise connected, and which does not satisfy property (•). This space consists of a one-to-one continuous image $C$ of the real line, which is not locally compact (see [9, p. 322]), and of a point of the boundary of $C$ in the continuum where $C$ is embedded.

The following corollary resolves in the affirmative Problem 987 in [8].
5.4. Corollary. Let $X$ be a hereditarily locally connected, planar space. Then each arcwise connected subset of $X$ is locally arcwise connected.

The proof of Corollary 5.4 follows from Theorems 3.1 and 5.3.

We have proved (see Theorem 4.2 and Corollary 4.3) that finitely Suslinian metric spaces are hereditarily locally connected and have property (*). Therefore, the following corollary is a generalization of [8, Theorem 2.1]:

5.5. Corollary. Let $X$ be a finitely Suslinian, separable metric space. Then, each arcwise connected subset of $X$ is locally arcwise connected.

In [8, Corollary 2.4], Lelek gives a characterization of locally connected metric continua which have the property that each arcwise connected subset is locally arcwise connected. We give a similar characterization for complete, separable metric hereditarily connected spaces by proving the following:

5.6. Theorem. A hereditarily locally connected, complete, separable metric space $X$ has property (*) if and only if each arcwise connected subset of $X$ is locally arcwise connected.

Proof. If $X$ is a hereditarily locally connected, complete, separable metric space with property (*), then by Theorem 5.3, each arcwise connected subset of $X$ is locally arcwise connected.

Conversely, let $X$ be a complete, metric, hereditarily locally connected space such that each arcwise connected subset of $X$ is locally arcwise connected. Just suppose that $U$ is a connected subset of $X$, $A_1, A_2, \ldots$ is a sequence of connected, closed subsets of $X$ such that $A_i \cap A_j \subset \text{Cl}(U)$ for each $i \neq j$, and $x \in X \setminus \text{Cl}(U)$ is a point in $\text{Lim sup}_{i \to \infty} A_i$. Let $V$ be a connected neighbourhood of $U$ in $X$ such that $x \notin \text{Cl}(V)$. Then $V$ is an open connected subset of a complete metric space, so $V$ is a locally connected, complete metric space, and by Moore's theorem, $V$ is arcwise connected. Also, $A_1, A_2, \ldots$ are arcwise connected subsets of $X$ since they are closed, connected subsets of a hereditarily locally connected, complete metric space.

Consider the set

$$B = V \cup \bigcup_{i=1}^{\infty} A_i \cup \{x\}.$$ 

Then $B$ is a connected subset of $X$, and so locally connected. That means that $A_i \setminus \text{Cl}(V)$ is not open in $B$ for all but finitely many $i$'s. Let $y \in A_n \setminus \text{Cl}(V)$ such that $y \in \text{Lim sup}_{i \to \infty} A_i$, and consider the set $C = V \cup \bigcup_{i=1}^{\infty} A_i$. Then $C$ is an arcwise connected subset of $X$, which is not locally arcwise connected at $y$, since

$$[A_i \setminus \text{Cl}(V)] \cap [A_j \setminus \text{Cl}(V)] = \emptyset$$

for each $i \neq j$. This contradiction completes the proof of Theorem 5.6.
As the following example shows, the characterization in Theorem 5.6 of complete metric spaces with property (\(\ast\)) is not true if we do not assume that the space is hereditarily locally connected.

5.7. Example. This is a locally connected, complete metric space \(X\) such that each arcwise connected subset of \(X\) is locally arcwise connected, but \(X\) does not have property \(\ast\). Let

\[
X = \bigcup_{i=0}^{1} \{(x, i)|0 < x < 1\} \cup \bigcup_{n=1}^{\infty} \{(1/n, y)|0 < y < 1\}.
\]

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Department of Mathematics, University of Saskatchewan, Saskatoon, Saskatchewan, Canada S7N 0W0 (Current address of E. D. Tymchatyn)

Current address (J. Grispolakis): Department of Mathematics, University of Crete, Iraklion, Crete, Greece