TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 253, September 1979

σ-CONNECTEDNESS IN HEREDITARILY LOCALLY CONNECTED SPACES

BY

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ABSTRACT. B. Knaster, A. Lelek and J. Mycielski [Colloq. Math. **6** (1958), 227–246] had asked whether there exists a hereditarily locally connected planar set, which is the union of countably many disjoint arcs. They gave an example of a locally connected, connected planar set, which is the union of a countable sequence of disjoint arcs. Lelek proved in a paper in Fund. Math. in 1959, that connected subsets of planar hereditarily locally connected continua are weakly σ-connected (i.e., they cannot be written as unions of countably many disjoint, closed connected subsets). In this paper we generalize the notion of finitely Suslinian to noncompact spaces. We prove that there is a class of spaces, which includes the class of planar hereditarily locally connected spaces and the finitely Suslinian spaces, and which are weakly σ-connected, thus, answering the above question in the negative. We also prove that arcwise connected, hereditarily locally connected, planar spaces are locally arcwise connected. This answers in the affirmative a question of Lelek [Colloq. Math. **36** (1976), 87–96].

1. Introduction. A connected space is said to be *hereditarily locally connected* if each of its connected subsets is locally connected. A connected set is said to be *σ-connected* (resp., *weakly σ-connected*) if it cannot be decomposed into a countably infinite family of mutually separated (resp., mutually separated, connected) nonempty subsets. In [**5**, Problem 253], B. Knaster asked whether every hereditarily locally connected planar set is weakly σ-connected. A. Lelek proved in [**7**, Theorem 8] that every connected subset of a planar hereditarily locally connected continuum is σ-connected. H. M. Gehman proved in [**2**] that these continua are finitely Suslinian. In 1974, J. Grispolakis, Lelek and E. D. Tymchatyn proved that a hereditarily locally connected continuum is finitely Suslinian if and only if every connected subset is σ-connected if and only if every connected subset is weakly σ-connected (see [**3**, Theorem 2.2]). There exist examples of (nonplanar) hereditarily locally connected continua which contain connected subsets which are unions of countably many disjoint arcs (see [**6**, p. 270]).

Presented to the Society, August 23, 1977; received by the editors July 13, 1977.

AMS (MOS) subject classifications (1970). Primary 54D05; Secondary 54A05.

Key words and phrases. Hereditarily locally connected spaces, finitely Suslinian, weakly σ-connected spaces, σ-connected spaces.

[1]This research was supported in part by the National Research Council of Canada grant A5616.

0002-9947/79/0000-0413/$04.25

It was proved in [5, Theorem 1] that a connected metric space cannot be decomposed into a countable infinite null family of mutually separated nonempty sets. In §2, we give a very simple argument that extends this result to the class of normal T_1-spaces.

In §3, we prove that hereditarily locally connected, planar spaces are hereditarily weakly σ-connected. This answers Knaster's question in the negative. We have been unable, though, to answer Knaster's further question as to whether hereditarily locally connected, planar spaces are σ-connected.

In §4, we introduce the notion of finitely Suslinian spaces, which is a generalization of the notion of finitely Suslinian continua. We generalize Gehman's theorem by proving that hereditarily locally connected, planar spaces are finitely Suslinian if and only if they are semilocally connected. We obtain some rudimentary results concerning finitely Suslinian compactifications of finitely Suslinian spaces.

Finally, in §5, we answer a question of Lelek [8, Problem 987] in the affirmative by proving that planar hereditarily locally connected spaces that are arcwise connected are locally arcwise connected.

Let A be a subset of a topological space X. By $\mathrm{Cl}(A)$ and $\mathrm{Bd}(A)$ we denote the closure of A and the boundary of A in X, respectively.

Let X be a Tychonoff space. A family $\mathcal{A}$ of subsets of X is said to be *null* if for each $x \in X$ and for each convergent subnet $(A_\alpha)_{\alpha \in \Lambda}$ of $\mathcal{A}$ such that $x \in \operatorname{Lim\,sup}_{\alpha \in \Lambda} A_\alpha$ we have that $(A_\alpha)_{\alpha \in \Lambda}$ is eventually in every neighbourhood of x.

2. Decompositions of connected spaces. We shall need the following simple result.

2.1. THEOREM. *A nondegenerate, connected, normal, T_1-space does not admit an upper semicontinuous decomposition into countably many, pairwise disjoint, closed subsets.*

PROOF. Suppose, on the contrary, that X is a nondegenerate connected, normal, T_1-space, which admits a decomposition D as above. Let $Y = X/D$ be the quotient space, and let $\varphi: X \to Y$ be the natural projection. Then φ is a closed mapping and Y is a countable connected space. By [1, Theorem 5, p. 85], Y is a normal T_1-space. Thus, Y is degenerate. This contradiction completes the proof of Theorem 2.1.

The following result was first proved in [5].

2.2. COROLLARY [5, THEOREM 1]. *A nondegenerate, connected, metric space cannot be decomposed into a null sequence of countably infinitely many, pairwise disjoint, closed subsets.*

PROOF. Such a decomposition is upper semicontinuous.

3. Planar hereditarily locally connected spaces. We say that a topological space X has *property* (*) provided that for each connected subset U of X and for each sequence $A_1, A_2, \ldots$ of closed, connected subsets of X each of which meets U and such that $A_i \cap A_j \subset \mathrm{Cl}(U)$ for each $i \neq j$, $i,j = 1, 2, \ldots,$ we have

$$\operatorname*{Lim\,sup}_{i\to\infty} A_i \subset \mathrm{Cl}(U).$$

3.1. THEOREM. *If X is a planar hereditarily locally connected space, then X has property* (*).

PROOF. Let $U, A_1, A_2, \ldots$ be connected closed sets in X such that each A_i meets U and $A_i \cap A_j \subset U$ for each $i \neq j$. Just suppose that there exists some point $x \in \mathrm{Lim\,sup}_{i\to\infty} A_i \setminus U$.

Since X is hereditarily locally connected we may suppose $A_i \cap U = \{a_i\}$ and $A_i \setminus \{a_i\}$ is connected for each $i = 1, 2, \ldots$. Let U' denote the closure of U in the plane. We may suppose without loss of generality that x lies in the unbounded complementary component of U' in the plane. We may also suppose that each $A_i \setminus \{a_i\}$ lies in the unbounded complementary component of U' in the plane.

Let W be a closed disk in the plane which contains U' in its interior and which does not contain x. We may suppose that, for each i, A_i is not contained in W. For each i let z_i be a point in the boundary of W such that z_i is also in the component of $A_i \cap W$ which contains a_i. Such a point exists since A_i is connected and locally connected. We may suppose that there exist a point z_0 in the boundary of W and an arc $[z_1, z_0]$ in the boundary of W such that, for each $i \geqslant 2$, z_i separates z_{i-1} from z_0 in $[z_1, z_0]$.

Let $Y = U \cup A_1 \cup A_2 \cup \ldots$. We may suppose $x \notin Y$. If for each i, $A_i \setminus W$ were open in Y then $Y \cup \{x\}$ would be a connected subset of X which is not locally connected at x. We suppose, therefore, that there exist $y \in A_1 \setminus W$ and $z \in A_2 \setminus W$ such that $y, z \in \mathrm{Lim}_{i\to\infty} A_i$.

We may suppose (by picking a subsequence if necessary) that $\varepsilon_3 > \varepsilon_4 > \ldots$ is a sequence of real numbers converging to zero such that, for each $i \geqslant 3$, A_i meets the ε_i neighbourhood $S(z, \varepsilon_i)$ of z in the plane and A_j does not meet $S(z, \varepsilon_i)$ for $3 \leqslant j < i$. For each $i \geqslant 3$ let Z_i denote the closure in the plane of $W \cup A_2 \cup S(z, \varepsilon_i) \cup A_i$. It is easy to see by using standard arguments for the plane that Z_i separates y from $A_j \setminus W$ for $3 \leqslant j < i$.

Let $Z = U \cup A_1 \cup A_3 \cup A_5 \cup \ldots$. Then Z is a connected subset of X which is not locally connected at y. To see this let V be a neighbourhood of y in the plane. If $i > 1$ and A_{2i+1} meets V then Z_{2i+2} separates $A_{2i+1} \cap V$ from y in V if $V \cap W = \varnothing$. Also, if i is sufficiently large then A_{2i+1} meets V, thus $V \cap Z$ is not connected.

Example 4.4 shows that the converse of Theorem 3.1 need not be true.

3.2. THEOREM. *Every locally connected, normal, T_1-space with property* (*) *is weakly σ-connected.*

PROOF. Suppose, on the contrary, that X is a locally connected, normal, T_1-space with property (*), which is not weakly σ-connected. Then $X = \bigcup_{i=1}^{\infty} A_i$, where $A_1, A_2, \ldots$ are mutually disjoint, closed, connected, nonempty subsets of X.

Let U_1 be a connected neighbourhood of A_1 such that $\mathrm{Cl}(U_1) \subset X \setminus A_2$. Let D_2^1 be the component of A_2 in $X \setminus \mathrm{Cl}(U_1)$, and consider the set

$$A_2^1 = \bigcup \{A_i | A_i \cap \mathrm{Cl}(D_2^1) \neq \varnothing\}.$$

Then D_2^1 is an open connected set, since $X \setminus \mathrm{Cl}(U_1)$ is locally connected, and $A_2 \subset D_2^1 \subset \mathrm{Int}(A_2^1)$. By property (*), we have that A_2^1 is a closed, connected subset of X, disjoint from A_1.

Let V_1 be a connected neighbourhood of A_1 such that $\mathrm{Cl}(V_1) \subset X \setminus A_2^1$. Let D_1^1 be the component of A_2 in $X \setminus \mathrm{Cl}(V_1)$, and consider the set

$$A_1^1 = (X \setminus D_1^1) \cup \bigcup \{A_i | A_i \cap \mathrm{Bd}(V_1) \neq \varnothing\}.$$

Then $X \setminus D_1^1$ is a closed, connected subset of X and $A_1 \subset V_1 \subset \mathrm{Int}(A_1^1)$. By property (*), we have that A_1^1 is a connected, closed subset of X, disjoint from A_2^1.

Let n_1 be the smallest integer such that $A_{n_1} \subset X \setminus (A_1^1 \cup A_2^1)$. Let $\mathfrak{U}$ be an open cover of X consisting of connected sets whose closures refine the cover $\{X \setminus A_1^1, X \setminus A_2^1\}$. Let $\{C_1, \ldots, C_m\}$ be an irreducible chain in $\mathfrak{U}$ joining A_{n_1} and $A_1^1 \cup A_2^1$. Without loss of generality, assume that C_1 meets A_{n_1} and C_m meets A_1^1, and put

$$B_1^1 = \bigcup \{A_i | A_i \cap C_j \neq \varnothing \text{ for some } j \text{ with } 1 \leqslant j \leqslant m\} \cup A_1^1$$

and $B_2^1 = A_2^1$. By property (*), B_1^1 and B_2^1 are closed, connected, nonempty sets.

Let U_2 be a connected neighbourhood of B_1^1 such that $\mathrm{Cl}(U_2) \subset X \setminus B_2^1$. Let D_2^2 be the component of B_2^1 in $X \setminus \mathrm{Cl}(U_2)$, and consider the set

$$A_2^2 = \bigcup \{A_i | A_i \cap \mathrm{Cl}(D_2^2) \neq \varnothing\}.$$

Then D_2^2 is an open connected set, since $X \setminus \mathrm{Cl}(U_2)$ is locally connected, and $B_2^1 \subset \mathrm{Int}(A_2^2)$. By property (*), we have that A_2^2 is a closed, connected subset of X disjoint from B_1^1.

Let V_2 be a connected neighbourhood of B_1^1 such that $\mathrm{Cl}(V_2) \subset X \setminus A_2^2$. Let D_1^2 be the component of B_2^1 in $X \setminus \mathrm{Cl}(V_2)$, and consider the set

$$A_1^2 = (X \setminus D_1^2) \cup \bigcup \{A_i | A_i \cap \mathrm{Bd}(V_2) \neq \varnothing\}.$$

Then $X \setminus D_1^2$ is a closed, connected subset of X and $B_1^1 \subset \mathrm{Int}(A_1^2)$. By

property (*), we have that A_1^2 is a closed, connected subset of X, disjoint from A_2^2.

Inductively, we construct two sequences of closed connected sets B_1^0, B_1^1, $B_1^2, \ldots$ and B_2^0, B_2^1, $B_2^2, \ldots$ where $B_1^0 = A_1$ and $B_2^0 = A_2$, such that, for each $i \geqslant 0$, $B_1^i \subset \operatorname{Int}(B_1^{i+1})$ and $B_2^i \subset \operatorname{Int}(B_2^{i+1})$ and $A_{i+1} \subset B_1^i \cup B_2^i$, and such that if $A_j \cap B_k^i \neq \varnothing$, then $A_j \subset B_k^i$.

Consider the sets $A = \bigcup_{i=0}^{\infty} B_1^i$ and $B = \bigcup_{i=0}^{\infty} B_2^i$. Then A and B are open, disjoint, nonempty subsets of X such that $X = A \cup B$. The latter statement is obvious, since for each $m > 1$ there exists some k such that either $A_m \subset B_1^k$ or $A_m \subset B_2^k$. This contradicts the connectedness of X and the proof is complete.

The following corollary answers in the negative Knaster's question (see [**5**, Problem 253]).

3.3. COROLLARY. *Every planar, hereditarily locally connected space is weakly σ-connected.*

The corollary follows from Theorem 3.1 and Theorem 3.2.

That the local connectedness is essential in Theorem 3.2 is apparent from the following:

3.4. EXAMPLE. Let X be the Knaster indecomposable continuum (see [**6**, p. 205]). Then every proper connected subset of X has property (*), since every connected subset of X which is not dense in X is contained in an arc. On the other hand, there exist connected subsets of X which are not weakly σ-connected (see [**4**, Proposition 2.1]).

4. Finitely Suslinian spaces. A Tychonoff space is said to be *finitely Suslinian* provided it is locally connected and each net $(A_\alpha)_{\alpha \in I}$ of distinct, closed, connected, pairwise disjoint subsets of it is null (i.e., if $x \in \operatorname{Lim\,sup}_{\alpha \in I} A_\alpha$ and V is a neighbourhood of x, then there is a subnet of $(A_\alpha)_{\alpha \in I}$ which is eventually in V). It is clear that this class contains the class of finitely Suslinian continua (see [**14**]). It was proved that a hereditarily locally connected metric continuum is finitely Suslinian if and only if every connected subset is σ-connected if and only if every connected subset is weakly σ-connected (see [**3**, Theorem 2.2]). There are simple examples of hereditarily locally connected spaces, which are not finitely Suslinian, but which have the property that every connected subset is σ-connected.

We say that a space X *is semilocally connected* provided that for each point $x \in X$ and for each neighbourhood U of x, there exists a neighbourhood V of x such that $V \subset U$ and $X \setminus V$ has finitely many components. The space X is said to be *hereditarily semilocally connected* (h.s.l.c.) provided each connected subset of X is semilocally connected.

4.1. THEOREM. *Let X be a connected, semilocally connected first countable Tychonoff space with property* (*). *Then X is hereditarily locally connected.*

PROOF. Let C be a connected subset of X. We shall prove that C is locally connected. Suppose, on the contrary, that C is not locally connected at the point $x_0 \in C$, and let U be a neighbourhood of x_0 in X such that each neighbourhood of x_0 contained in $U \cap C$ is not connected, and such that $X \setminus U$ has finitely many components. Let $C_1, \ldots, C_n$ be the components of $X \setminus U$. Since x_0 is not in the interior of its component in $U \cap C$, we can find a sequence $F_1, F_2, \ldots$ of nonempty closed-open subsets of $U \cap C$, which are mutually separated and such that $x_0 \in \operatorname{Lim\,sup}_{i\to\infty} F_i$. For each i let $x_i \in F_i$ be such that $\operatorname{Lim}_{i\to\infty} x_i = x_0$. Since C is connected, we have that

$$\operatorname{Cl}(F_i) \cap \operatorname{Bd}(U) \neq \varnothing$$

for each i (the closure of F_i is taken in X). If $n = 1$, then $C_1 \cup F_i$ is connected for each i. Then $C_1 \cup \bigcup_{i=1}^{\infty} F_i \cup \{x_0\}$ is a connected set, which does not have property (*), which is a contradiction.

Suppose $n > 1$ and let $1 \leqslant m < n$ be such that x_0 is in the interior of the component of $C \setminus (C_1 \cup \cdots \cup C_m)$ which contains x_0. Let K be the component of $C \setminus (C_1 \cup \cdots \cup C_m)$, which contains x_0. Suppose x_0 is not in the interior of the component of $K \setminus C_{m+1}$ which contains x_0. As in the case $n = 1$ replacing C_1 by C_{m+1} and C by K, we again get a contradiction. Thus, C is locally connected and the proof of the theorem is complete.

4.2. THEOREM. *A connected, first countable Tychonoff space is finitely Suslinian if and only if it is semilocally connected and has property* (*).

PROOF. The necessity of the condition is obvious by the definition of finitely Suslinian space. To prove the sufficiency of the condition, suppose, on the contrary, that there exists a space X which satisfies the conditions of the theorem but is not finitely Suslinian. Then there exists a sequence $A_1, A_2, \ldots$ of disjoint, closed, connected subsets of X which is not null, that is, there exist $x_0 \in \operatorname{Lim\,sup}_{i\to\infty} A_i$ and a neighbourhood V_0 of x_0 such that for each i there exists $n_i > i$ such that

$$A_{n_i} \cap (X \setminus \operatorname{Cl}(V_0)) \neq \varnothing.$$

Assume, without loss of generality, that for each i we have $A_i \cap V_0 \neq \varnothing$ and

$$A_i \cap (X \setminus \operatorname{Cl}(V_0)) \neq \varnothing.$$

Since X is semilocally connected we can take V_0 to be such that $X \setminus V_0$ has finitely many components. Since $A_i \cap (X \setminus V_0) \neq \varnothing$, there exist a component C of $X \setminus V_0$ and a convergent subsequence $(A_{n_i})_{i \in N}$ of $(A_i)_{i \in N}$ such that $C \cap A_{n_i} \neq \varnothing$ for each $i \in N$ and $x_0 \in \operatorname{Lim\,sup}_{i\to\infty} A_{n_i}$. But then $x_0 \notin \operatorname{Cl}(C)$. This contradicts the fact that X has property (*). This contradiction combined with Theorem 4.1 implies that X is finitely Suslinian.

4.3. COROLLARY. *Every connected subset of a finitely Suslinian first countable space is finitely Suslinian. Therefore, first countable finitely Suslinian spaces are hereditarily semilocally connected, and hereditarily locally connected.*

The proof of Corollary 4.3 follows from the fact that property (*) is hereditary, and from Theorems 4.1 and 4.2.

Problem I. Are finitely Suslinian spaces hereditarily locally connected?

The following example shows that semilocal connectedness is essential in the hypothesis of Theorem 4.2.

4.4. EXAMPLE. This is a planar, connected, locally connected space with property (*), which is not hereditarily locally connected.

By (x, y) we denote the point of the plane R^2 with cartesian coordinates x and y.

Let $C_n = \{(x, y) | x^2 + (y - (1/2^n))^2 = ((2^n - 1)/2^n)^2\}$ for each $n = 1, 2, \ldots$. Consider the set $X = \bigcup_{n=1}^{\infty} C_n$. Since $C_n \cap C_m = \{(0, 1)\}$ for each $n \neq m$, $n, m = 1, 2, \ldots$, we have that X is a connected subspace of R^2. Now it is easy to check that X is locally connected space with property (*), but X is not hereditarily locally connected.

A connected space X is said to be hereditarily σ-connected (resp., *hereditarily weakly σ-connected*) provided that each connected subset of X is σ-connected (resp., weakly σ-connected). For a study of these spaces see [3] and [4].

4.5. THEOREM. *Every finitely Suslinian, normal, T_1-space is weakly σ-connected.*

PROOF. Let X be a finitely Suslinian space. Just suppose $X = \bigcup_{i=1}^{\infty} A_i$, where $A_1, A_2, \ldots$ are pairwise disjoint, closed, connected, nonempty subsets of X. Since X is finitely Suslinian, the sequence $A_1, A_2, \ldots$ is null. Therefore, $D = \{A_1, A_2, \ldots\}$ is an upper semicontinuous decomposition of the connected set X. But X is a normal T_1-space. This contradicts Theorem 2.1.

4.6. COROLLARY. *Every completely normal, T_1, finitely Suslinian space is hereditarily weakly σ-connected.*

In [2], Gehman proved that a planar hereditarily locally connected continuum is finitely Suslinian. It is clear that every locally connected continuum is semilocally connected. Therefore, the following theorem generalizes Gehman's result.

4.7. THEOREM. *A planar, hereditarily locally connected space is finitely Suslinian if and only if it is semilocally connected.*

PROOF. The necessity follows from Theorem 4.2, and the sufficiency follows from Theorems 3.1 and 4.2.

A space X is said to be rational (resp., regular) provided each point of X has arbitrarily small open neighbourhoods with countable (resp., finite) boundaries. In [**10**, p. 593], an example of a hereditarily locally connected, planar space is given which is not rational. In the same paper, it is proved that semilocally connected hereditarily locally connected separable metric spaces are rational (see [**10**, Theorem 2.4]). This together with our Theorems 4.1 and 4.2 imply the following:

4.8. THEOREM. *Every finitely Suslinian, separable metric space is rational.*

The above mentioned example [**10**, p. 593] contains a totally disconnected set which is one dimensional.

4.9. THEOREM. *Let X be a finitely Suslinian separable metric space, and let Y be a nonempty totally disconnected subset of X. Then* $\dim Y = 0$.

PROOF. The theorem follows from Corollary 4.3 and [**10**, Theorem 1.7].

We present now one theorem on compactifications of finitely Suslinian spaces. By a *compactification* of a space X we mean a compact space Y such that X can be embedded densely in Y.

It is known that a regular space can be compactified to a regular space [**11**] (see also J. R. Isbell [*Uniform spaces*, Math. Surveys, no. 12, Amer. Math. Soc., Providence, R. I., 1964, p. 111]). In [**10**] and [**13**], the problem of the compactification of hereditarily locally connected spaces has been investigated extensively. The following problems concerning compactifications of finitely Suslinian spaces are still open.

Problem II. Do finitely Suslinian (metric) spaces have finitely Suslinian compactifications?

Problem III. Do finitely Suslinian (metric) spaces have hereditarily locally connected compactifications?

Problem IV. Do finitely Suslinian (metric) spaces have rational compactifications?

Since finitely Suslinian continua are hereditarily locally connected the question raised in Problem III is strictly weaker than that in Problem II. It is well known that hereditarily locally connected metric continua are rational (see [**15**]) hence the question raised in Problem IV is at least in the metric case strictly weaker than that in Problem III. Spaces which admit a rational metric compactification have been characterized by Tymchatyn [Houston J. Math. **3** (1977), 131–139]. The following theorem now gives a partial answer to Problem II.

4.10. THEOREM. *Let X be a cyclic finitely Suslinian planar space which admits a hereditarily locally connected compactification. Then X admits a planar finitely Suslinian compactification.*

PROOF. Let $\tilde{X}$ be a hereditarily locally connected compactification of X. By [**13**, Theorem 7], $\tilde{X}$ can be considered to be a perfect, metric compactification with ponctiforme remainder. By Gehman's theorem it suffices to prove that $\tilde{X}$ is planar. To prove that $\tilde{X}$ is planar, it suffices to prove that every graph in X is planar (see Claytor [**0**]).

For this, let K be an arbitrary graph in $\tilde{X}$. Let $\mathcal{U} = \{U_1, \ldots, U_n\}$ be a family of open connected sets in $\tilde{X}$ such that $U_i \cap K$ is connected for each i, $K \subset \bigcup \mathcal{U}$, and the nerve of $\mathcal{U}$ is homeomorphic to K. Since $\tilde{X}$ is a perfect compactification of X, $\mathcal{V} = \{U_i \cap X | i = 1, \ldots, n\}$ is a family of connected open sets in X, whose nerve is homeomorphic to K. Let $W_1, \ldots, W_n$ be connected open sets in the plane such that $W_i \cap X = U_i \cap X$ for each i, and such that the nerve of $\mathcal{W} = \{W_1, \ldots, W_n\}$ is homeomorphic to K. Then, it is easy to verify that the nerve of $\mathcal{W}$ is embeddable in $\bigcup \mathcal{W}$, and hence K is a planar graph.

4.11. THEOREM. *Let X be a finitely Suslinian, separable metric space, which admits a hereditarily locally connected compactification. Then X is hereditarily σ-connected.*

PROOF. Let $\tilde{X}$ be a hereditarily locally connected compactification of X. By [**13**, Theorem 9], $\tilde{X}$ may be considered to be metric. It suffices to show that X is σ-connected. Suppose, on the contrary, that $X = \bigcup_{i=1}^{\infty} A_i$ is a decomposition of X into countably many mutually disjoint, closed, nonempty subsets of X. By [**3**, Lemma 1.2], there exists a null sequence $D_{i1}, D_{i2}, \ldots$ of mutually disjoint, closed-open subsets D_{ij} of A_i, such that

$$\mathrm{Dis}(A_i) \subset U_i = \bigcup_{j=1}^{\infty} D_{ij} \qquad (i = 1, 2, \ldots) \tag{1}$$

and

$$\operatorname{diam} D_{ij} < 1/i \qquad (i, j = 1, 2, \ldots), \tag{2}$$

where $\mathrm{Dis}(A_i)$ denotes the union of all the degenerate components of A_i $(i = 1, 2, \ldots)$.

Let $\mathcal{C}_i$ denote the collection of all components of the set $A_i \setminus U_i$ $(i = 1, 2, \ldots)$, and let $\mathcal{C} = \mathcal{C}_1 \cup \mathcal{C}_2 \cup \ldots$. Clearly, the members of $\mathcal{C}$ are mutually disjoint, closed nondegenerate subsets of X. Since $\tilde{X}$ is metric and hereditarily locally connected, it is rational (see [**15**, p. 94]). Therefore, each family of mutually separated nondegenerate connected subsets of $\tilde{X}$ is countable. Thus, $\mathcal{C}$ is a countable family. By Corollary 4.6, X is hereditarily weakly σ-connected, therefore, $\bigcup \mathcal{C}$ is not connected and each member of $\mathcal{C}$ is a component of $\bigcup \mathcal{C}$. By [**6**, Theorem 9, p. 272], $\mathcal{C}$ is a null sequence. It follows from (2) that the members of $\mathcal{C}$ together with the sets D_{ij} form a null sequence whose union is X. By Corollary 2.2, X cannot be connected. This

contradiction shows that X is σ-connected. Finally, since each connected subset of X is finitely Suslinian, we conclude that X is hereditarily σ-connected.

5. Local arcwise connectedness. In [**8**], Lelek established a relationship between local arcwise connectedness and the class of finitely Suslinian metric continua. He proved that each arcwise connected subset of a finitely Suslinian metric continuum is locally arcwise connected [**8**, Theorem 2.1], and he asked whether arcwise connected, hereditarily locally connected, planar spaces are locally arcwise connected [**8**, Problem 987]. In this problem the hereditary local connectedness is essential, since, in 1959, M. Shimrat constructed an arcwise connected, locally connected, planar space which is not locally arcwise connected [**12**, pp. 184–185].

In this section, we generalize Lelek's theorem and we resolve Problem 987 in [**8**] in the affirmative.

A space is said to be a *local dendron* if it has a basis of open sets which are connected subsets of finite graphs.

5.1. LEMMA. *Let X be a hereditarily locally connected, separable metric space with property* (*). *If A is an arcwise connected subset of X and $x \in \mathrm{Cl}(A)$, then $A \cup \{x\}$ is arcwise connected.*

PROOF. Suppose, on the contrary, that $A \cup \{x\}$ is not arcwise connected. Then, there exist an $\varepsilon > 0$ and a sequence of points $x_1, x_2, \ldots$ in A converging to x such that each arc in A from x_i to x_j has diameter $\geqslant \varepsilon$ for each $i \neq j$. For each i, let $[x_1, x_{i+1}]$ be an arc from x_1 to x_{i+1} in A such that $[x_1, x_{i+1}] \cap [x_1, x_j]$ is connected for each $j < i$. Without loss of generality, assume that $A = \bigcup_{i=1}^{\infty} [x_1, x_{i+1}]$. If A is not a local dendron, let $y \in A$ such that no neighbourhood of y is contained in the union of finitely many of the arcs $[x_1, x_i]$. Since $A \cup \{x\}$ is a connected subset of X, it is locally connected. Therefore, if U is a connected neighbourhood of y in $A \cup \{x\}$ such that $x \notin \mathrm{Cl}(U)$, then there exists a subsequence $x_{i_1}, x_{i_2}, \ldots$ of $x_1, x_2, \ldots$, and points $y_j \in [x_1, x_{i_j}] \cap U$ such that for each $j \neq k$ we have

$$[y_j, x_{i_j}] \cap [y_k, x_{i_k}] \subset U,$$

where $[y_j, x_{i_j}]$ is the arc in $[x_1, x_{i_j}]$ with endpoints y_j and x_{i_j}. But since $x \in \mathrm{Lim\,sup}_{j\to\infty} [y_j, x_{i_j}]$, it is obvious that A does not satisfy property (*). Hence, A is a local dendron. Thus, each arc in A has at most finitely many ramification points and each of these points is of finite order. Let each arc $[x_1, p]$ in A have its natural order with initial point x_1. Let a_1 be the maximum point in $[x_1, x_2]$ such that $a_1 \in [x_1, x_j]$ for infinitely many $j \geqslant 2$. Then a_1 exists, and either $a_1 = x_1$, $a_1 = x_2$, or a_1 is a ramification point of A. Without loss of generality, assume that $a_1 \in [x_1, x_j]$ for each $j \geqslant 2$ and $x_k \notin [x_1, x_j]$

for $j < k$. Let a_2 be the maximum point in $[x_1, x_3]$ such that $a_2 \in [x_1, x_j]$ for infinitely many j. Then $a_2 \geqslant a_1$ in $[x_1, a_2]$. By induction, we define for each i a point $a_i \in [x_1, x_j]$ for infinitely many j. Without loss of generality, we can assume that

$$[x_1, x_{i+1}] \cap [x_1, x_j] = [x_1, a_i] \tag{1}$$

for each $j \geqslant i + 2$.

Let $B = \bigcup_{i=1}^{\infty}[x_1, a_i]$. Then, B is a ray (i.e., a homeomorphic image of $[0, 1)$). By property $(*)$ and (1), $x \in \mathrm{Cl}(B)$. Therefore, $B \cup \{x\}$ is a connected, and hence a locally connected subset of X. Let U be a connected neighbourhood of x in $B \cup \{x\}$. Then $U \cap B$ is an open arc in B, since B is a ray. Hence, $B \cup \{x\}$ is an arc from x_1 to x in $A \cup \{x\}$.

Let X be a topological space and x be a point of X. By the *arc-component* of x in X is meant the union of x and of all arcs in X, which contain the point x. Whyburn had proved in [**16**] that finitely Suslinian metric continua have the property that each connected subset has closed arc-components. The following theorem generalizes the above result.

5.2. THEOREM. *Let X be a hereditarily locally connected, separable metric space with property $(*)$. Then every arc-component of X is a closed subset of X.*

5.3. THEOREM. *Let X be a hereditarily locally connected, separable metric space with property $(*)$. Then every arcwise connected subset of X is locally arcwise connected.*

PROOF. Suppose, on the contrary, that A is an arcwise connected subset of X, which is not locally arcwise connected at some point $x \in A$. Then, there exists a connected neighbourhood V of x in A, which is not arcwise connected. Let $\mathcal{C}$ be the family of the arc-components of the set V. Then the closure in A of each arc-component of V meets the boundary of V; therefore, each member of $\mathcal{C}$ is nondegenerate. By [**10**, Theorem 2.3], $\mathcal{C}$ is a countable family of connected sets. By Theorem 5.2, each member of $\mathcal{C}$ is a closed subset of V. Thus, V is a hereditarily locally connected metric space with property $(*)$ which is not weakly σ-connected. This contradicts Theorem 3.2, and the proof of the theorem is complete.

REMARKS. In both Theorems 5.2 and 5.3 the assumption that the space has property $(*)$ is essential. As a matter of fact, there exists a hereditarily locally connected space in the Euclidean 3-space R^3, which is arcwise connected but not locally arcwise connected, and which does not satisfy property $(*)$. This space consists of a one-to-one continuous image C of the real line, which is not locally compact (see [**9**, p. 322]), and of a point of the boundary of C in the continuum where C is embedded.

The following corollary resolves in the affirmative Problem 987 in [**8**].

5.4. COROLLARY. *Let X be a hereditarily locally connected, planar space. Then each arcwise connected subset of X is locally arcwise connected.*

The proof of Corollary 5.4 follows from Theorems 3.1 and 5.3.

We have proved (see Theorem 4.2 and Corollary 4.3) that finitely Suslinian metric spaces are hereditarily locally connected and have property (*). Therefore, the following corollary is a generalization of [**8**, Theorem 2.1]:

5.5. COROLLARY. *Let X be a finitely Suslinian, separable metric space. Then, each arcwise connected subset of X is locally arcwise connected.*

In [**8**, Corollary 2.4], Lelek gives a characterization of locally connected metric continua which have the property that each arcwise connected subset is locally arcwise connected. We give a similar characterization for complete, separable metric hereditarily connected spaces by proving the following:

5.6. THEOREM. *A hereditarily locally connected, complete, separable metric space X has property (*) if and only if each arcwise connected subset of X is locally arcwise connected.*

PROOF. If X is a hereditarily locally connected, complete, separable metric space with property (*), then by Theorem 5.3, each arcwise connected subset of X is locally arcwise connected.

Conversely, let X be a complete, metric, hereditarily locally connected space such that each arcwise connected subset of X is locally arcwise connected. Just suppose that U is a connected subset of X, $A_1, A_2, \ldots$ is a sequence of connected, closed subsets of X such that $A_i \cap A_j \subset \mathrm{Cl}(U)$ for each $i \neq j$, and $x \in X \setminus \mathrm{Cl}(U)$ is a point in $\mathrm{Lim\,sup}_{i\to\infty} A_i$. Let V be a connected neighbourhood of U in X such that $x \notin \mathrm{Cl}(V)$. Then V is an open connected subset of a complete metric space, so V is a locally connected, complete metric space, and by Moore's theorem, V is arcwise connected. Also, $A_1, A_2, \ldots$ are arcwise connected subsets of X since they are closed, connected subsets of a hereditarily locally connected, complete metric space. Consider the set

$$B = V \cup \bigcup_{i=1}^{\infty} A_i \cup \{x\}.$$

Then B is a connected subset of X, and so locally connected. That means that $A_i \setminus \mathrm{Cl}(V)$ is not open in B for all but finitely many i's. Let $y \in A_n \setminus \mathrm{Cl}(V)$ such that $y \in \mathrm{Lim\,sup}_{i\to\infty} A_i$, and consider the set $C = V \cup \bigcup_{i=1}^{\infty} A_i$. Then C is an arcwise connected subset of X, which is not locally arcwise connected at y, since

$$[A_i \setminus \mathrm{Cl}(V)] \cap [A_j \setminus \mathrm{Cl}(V)] = \varnothing$$

for each $i \neq j$. This contradiction completes the proof of Theorem 5.6.

As the following example shows, the characterization in Theorem 5.6 of complete metric spaces with property (*) is not true if we do not assume that the space is hereditarily locally connected.

5.7. EXAMPLE. This is a locally connected, complete metric space X such that each arcwise connected subset of X is locally arcwise connected, but X does not have property (*). Let

$$X = \bigcup_{i=0}^{1} \{(x, i) | 0 \leqslant x \leqslant 1\} \cup \bigcup_{n=1}^{\infty} \{(1/n, y) | 0 < y \leqslant 1\}.$$

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