SIMPLE PERIODIC ORBITS OF MAPPINGS
OF THE INTERVAL

BY

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Abstract. Let $f$ be a continuous map of a closed, bounded interval into itself. A criterion is given to determine whether or not $f$ has a periodic point whose period is not a power of 2, which just depends on the periodic orbits of $f$ whose period is a power of 2. Also, a lower bound for the topological entropy of $f$ is obtained.

1. Introduction. Let $I$ denote a closed and bounded interval on the real line and let $C^0(I, I)$ denote the space of continuous maps from $I$ into itself. This paper is concerned with periodic orbits of mappings $f \in C^0(I, I)$. Such mappings (sometimes called first order difference equations) arise as mathematical models for phenomena in the natural sciences (see [4] and [5] for some discussion and further references).

Let $f \in C^0(I, I)$. Consider the following ordering of the positive integers:

\[
1, 2, 4, 8, \ldots \ldots, 7 \cdot 8, 5 \cdot 8, 3 \cdot 8, \ldots, 7 \cdot 4, 5 \cdot 4, 3 \cdot 4, \ldots,
\]

\[
7 \cdot 2, 5 \cdot 2, 3 \cdot 2, \ldots, 7, 5, 3.
\]

A. N. Šarkovskii has proven that if $m$ is to the left of $n$ (in the above ordering) and $f$ has a periodic point of period $n$, then $f$ has a periodic point of period $m$ (see [6] or [7]). This theorem suggests that the following property implies a rich orbit structure:

(1) $f$ has a periodic point whose period is not a power of 2.

This suggestion is supported by the fact that (1) implies the following:

(2) $f$ has a homoclinic point (see [1]);

(3) $f$ has positive topological entropy (see [1], [2], or [7]).

Also, (2) is equivalent to (1) (see [1]) and it has been conjectured (and proved for a special case in [3]) that (3) is equivalent to (1).

In this paper we give a criterion for determining whether or not $f$ satisfies (1) which just depends on the periodic orbits of $f$ whose period is a power of 2. In the process we obtain a lower bound for topological entropy. The criterion we give is based on the following definition.

Definition. Let $P$ be a periodic orbit of $f \in C^0(I, I)$ of period $m$, where $m$
is a power of 2 and \( m > 2 \). We say \( P \) is simple if for any subset \( \{q_1, \ldots, q_n\} \) of \( P \) where \( n \) divides \( m \) and \( n > 2 \), and any positive integer \( r \) which divides \( m \), such that \( \{q_1, \ldots, q_n\} \) is periodic orbit of \( f' \) with \( q_1 < q_2 < \cdots < q_n \), we have

\[
f'(\{q_1, \ldots, q_{n/2}\}) = \{q_{n/2+1}, \ldots, q_n\}.
\]

The reader may wish to see §4 where the definition of "simple" is discussed for a periodic orbit of period 8, and some examples are given.

Our main results are the following: (In this paper we include \( 1 = 2^0 \) as a power of 2.)

**Theorem A.** Let \( f \in C^0(I, I) \), \( f \) has a periodic point whose period is not a power of 2 if and only if \( f \) has a periodic orbit of period a power of 2 which is not simple.

**Theorem B.** Let \( f \in C^0(I, I) \). Suppose \( f \) has a periodic orbit \( P \) of period \( m \) (where \( m = 2^k \) for some \( k \geq 2 \)) which is not simple. Then \( f \) has a periodic point of period \( 3 \cdot 2^{k-2} \).

The proof of Theorems A and B uses some results of [6] and [7] which will be stated in §2. Štefan in [7] also obtains the following result which improves a theorem of Bowen and Franks (see [2]).

**Theorem C.** Let \( f \in C^0(I, I) \) and suppose \( f \) has a periodic point of period \( n \), where \( n = 2^d \cdot m \) and \( m \geq 3 \) is odd. Then the topological entropy of \( f \) is greater than \((1/2^d) \log \sqrt{2} \).

Thus (using Theorem C) the following is an immediate corollary of Theorem B.

**Corollary D.** Let \( f \in C^0(I, I) \). Suppose \( f \) has a periodic orbit of period \( m \) (where \( m = 2^k \) for some \( k \geq 2 \)) which is not simple. Then the topological entropy of \( f \) is greater than \((1/2^{k-2}) \log \sqrt{2} \).

2. Preliminary definitions and results. Let \( f \in C^0(I, I) \) and let \( N \) denote the set of positive integers. For any \( n \in N \), we define \( f^n \) inductively by \( f^1 = f \) and \( f^n = f \circ f^{n-1} \). Let \( f^0 \) denote the identity map of \( I \).

Let \( x \in I \). \( x \) is said to be a periodic point of \( f \) if \( f^n(x) = x \) for some \( n \in N \). In this case the smallest element of \( \{n \in N: f^n(x) = x\} \) is called the period of \( x \).

We define the orbit of \( x \) to be \( \{f^n(x): n = 0,1,2, \ldots \} \). If \( x \) is a periodic point we say the orbit of \( x \) is a periodic orbit, and we define the period of the orbit to be the period of \( x \). Clearly, if \( x \) is a periodic point of period \( n \), then the orbit of \( x \) contains \( n \) points and each of these points is a periodic point of \( f \) of period \( n \).
Note that a periodic point of $f$ is always a periodic point of $f^n$ (for any $n \in \mathbb{N}$), but the periods may be different. The following proposition (which follows immediately from the definitions) gives an example of this.

**Proposition 1.** Let $f \in C^0(I, I)$. Suppose $P$ is a periodic orbit of $f$ of period $n$ where $n$ is even. Then there are disjoint subsets $P_1$ and $P_2$ of $P$ which are periodic orbits of $f^2$ of period $n/2$.

Now, let $P$ be a periodic orbit of $f$ containing at least two points. Let $P_{\text{min}}(f)$ denote the smallest element of $P$ and $P_{\text{max}}(f)$ denote the largest element of $P$. Let

$$U(f) = \{ x \in I : f(x) > x \} \quad \text{and} \quad D(f) = \{ x \in I : f(x) < x \}.$$  

Let $P_U(f)$ denote the largest element of $P \cap U(f)$ and $P_D(f)$ denote the smallest element of $P \cap D(f)$.

We will use the following lemma, proved by Štefan in [7] (see (9) in §B of [7]).

**Lemma 2.** Let $f \in C^0(I, I)$ and let $P$ be a periodic orbit of $f$. If $f$ has a fixed point between $P_{\text{min}}(f)$ and $P_{\text{max}}(f)$ (or between $P_D(f)$ and $P_U(f)$), then $f$ has periodic orbits of every period.

The following corollary to Lemma 2 also appears in [7].

**Lemma 3.** Let $f \in C^0(I, I)$ and let $P$ be a periodic orbit of $f$. If $P_D(f) < P_U(f)$, then $f$ has periodic orbits of every period.

**Proof.** Suppose $P_D(f) < P_U(f)$. Since $f(P_D(f)) < P_D(f)$ and $f(P_U(f)) > P_U(f)$, $f$ has a fixed point between $P_D(f)$ and $P_U(f)$. Thus, the hypothesis of Lemma 2 is satisfied. Q.E.D.

**Lemma 4.** Suppose $f \in C^0(I, I)$. Let $J \subset I$ and $K \subset I$ be closed intervals with $f(J) \supset K$. There is a closed interval $H \subset J$ with $f(H) = K$.

**Proof.** Let $K = [a, b]$ and let $A = f^{-1}(a) \cap J$ and $B = f^{-1}(b) \cap J$. Let $d$ denote the usual metric on the real line. Since $A$ and $B$ are nonempty disjoint compact sets, there are points $a_1 \in A$ and $b_1 \in B$ such that $d(a_1, b_1) = d(A, B)$. Let $H$ be the closed interval with endpoints $a_1$ and $b_1$. Then $H \cap A = \{a_1\}$ and $H \cap B = \{b_1\}$. Hence $f(H) = K$. Q.E.D.

**Lemma 5.** Let $f \in C^0(I, I)$. Suppose $H$ and $K$ are closed intervals with $H \subset K \subset I$ and $f(H) = K$. Then $f$ has a fixed point in $H$.

**Proof.** Let $K = [a, b]$. For some $x \in H$ and $y \in H$, $f(x) = a$ and $f(y) = b$. Hence $f(x) < x$ and $f(y) > y$. Thus, $f$ has a fixed point between $x$ and $y$. Q.E.D.
Lemma 6. Let $f \in C^0(I, I)$. Let $g = f^r$ for some positive integer $r$ which is a power of 2. Suppose there is a periodic orbit $P_0 = \{q_1, \ldots, q_n\}$ of $g$ of period $n$, where $n$ is a power of 2 and $n \geq 2$. Suppose $q_1 < q_2 < \cdots < q_n$ and for some $i < n/2$ and $j < n/2$, $g(q_i) = q_j$. Then there is a periodic orbit $P$ of $f$ of period a power of 2 which is not simple.

Proof. Let $P$ be the orbit of $q_1$ with respect to $f$. Then $P$ is a periodic orbit and $P_0 \subset P$. Let $m$ be the period of $P$. We will show that $m = n \cdot r$.

We claim that, for any positive integer $s < r$ and any $q_1 \in P_0$, $f^s(q_1) \notin P_0$. To prove this, suppose that, for some positive integer $s < r$ and some $q_1 \in P_0$, $f^s(q_1) \in P_0$. We may assume (by choosing $s$ smaller if necessary) that, for any positive integer $t < s$, $f^t(q_1) \notin P_0$ for $i = 1, \ldots, n$. Note that for $k = 0, \ldots, n - 1$,

$$f^k(f^{kr}(q_1)) = f^{kr}(f^k(q_1)) \in P_0.$$ 

Hence $f^s(P_0) \subset P_0$. Since $f$ restricted to $P$ is one-to-one, $f^s(P_0) = P_0$. Since $f^r(P_0) = P_0$ and $s < r$, it follows from the choice of $s$ that $s$ divides $r$.

Now, $f^s(P_0) = P_0$ implies that some subset $P_i$ of $P_0$ is a periodic orbit of $f^s$. Hence $f^r(P_i) = P_i$. Since $s$ divides $r$, $f^s(P_i) = P_i$. Hence $P_i = P_0$. Thus $P_0$ is a periodic orbit of $f$ and a periodic orbit of $f'$. Since $r$ is a power of 2, $s$ divides $r$, $s < r$, and $P_0$ has at least two elements, we obtain a contradiction by repeated application of Proposition 1. This contradiction establishes our claim.

Next, we will show that the points

$$q_1, \ldots, q_n, f(q_1), \ldots, f(q_n), \ldots, f^{r-1}(q_1), \ldots, f^{r-1}(q_n)$$

are all distinct. Suppose $f^a(q_1) = f^b(q_1)$ where $0 < a < r - 1$, $0 < b < r - 1$, $1 < i < n$, and $1 < j < n$. We may assume that $a < b$. By applying $f^{r-b}$ to the points $f^a(q_1)$ and $f^b(q_1)$, we see that $f^{r-b+a}(q_1) \in P_0$. Since $a < b$, $r - b + a < r$. By our claim above, $r - b + a = r$. Hence $a = b$. Again, applying $f^{r-b}$ to the points $f^a(q_1)$ and $f^b(q_1)$ we see that $f^r(q_1) = f^r(q_1)$. Since $f^r$ restricted to $P_0$ is one-to-one, $q_i = q_j$. Hence $i = j$.

Clearly,

$$P = \{q_1, \ldots, q_n, f(q_1), \ldots, f(q_n), \ldots, f^{r-1}(q_1), \ldots, f^{r-1}(q_n)\}.$$ 

Hence $m = n \cdot r$. Thus $m$ is a power of 2 and $r$ divides $m$. It follows from this and our hypothesis that $P$ is not simple. Q.E.D.

3. Proof of Theorems A and B.

Lemma 7. Let $f \in C^0(I, I)$. Let $n > 3$ be an odd integer and suppose that $f^n$ does not have any periodic orbits of period 3. Let $P$ be a periodic orbit of $f$ of period $k$, where $k$ is a power of 2 and $k > 2$. Then $P_U(f^n) = P_U(f)$ and $P_D(f^n) = P_D(f)$.
PROOF. Note that $P_U(f^n)$ and $P_D(f^n)$ are well defined because $P$ is a periodic orbit of $f^n$.

Our hypothesis implies that $f$ does not have any periodic orbits of period $3 \cdot n$. By Lemma 3, $P_U(f) < P_D(f)$. It follows from this (and the definitions of $P_U(f)$ and $P_D(f)$) that there are no elements of $P$ between $P_U(f)$ and $P_D(f)$. Also, by Lemma 3, $P_U(f^n) < P_D(f^n)$ and there are no elements of $P$ between $P_U(f^n)$ and $P_D(f^n)$.

It suffices to prove that $P_U(f^n) = P_U(f)$. Suppose $P_U(f^n) \neq P_U(f)$. We have two cases.

**Case 1.** $P_U(f) < P_U(f^n)$. Then $P_U(f) < P_D(f) < P_U(f^n) < P_D(f^n)$. Since $f(P_{\min}(f)) > P_{\min}(f)$ and $f(P_D(f)) < P_D(f)$, $f$ has a fixed point between $P_{\min}(f)$ and $P_D(f)$. Hence, $f^n$ has a fixed point between $P_{\min}(f^n) = P_{\min}(f)$ and $P_U(f^n)$. By Lemma 2, $f^n$ has periodic orbits of every period, a contradiction.

**Case 2.** $P_U(f^n) < P_U(f)$. Then $P_U(f^n) < P_D(f^n) < P_U(f) < P_D(f)$. It follows that $f$ has a fixed point between $P_U(f)$ and $P_{\max}(f)$, so $f^n$ has a fixed point between $P_D(f^n)$ and $P_{\max}(f^n)$. By Lemma 2, $f^n$ has periodic orbits of every period, a contradiction. Q.E.D.

**LEMMA 8.** Let $f \in C^0(I, I)$. Let $P = \{p_1, \ldots, p_n\}$ be a periodic orbit of $f$ of period $n$, where $n$ is a power of 2 and $n > 2$, and $p_1 < p_2 < \cdots < p_n$. Suppose that, for every odd positive integer $m < n$, $f^m$ does not have any periodic orbits of period 3. Then

$$f(\{p_1, \ldots, p_{n/2}\}) = \{p_{n/2+1}, \ldots, p_n\}$$

and

$$f(\{p_{n/2+1}, \ldots, p_n\}) = \{p_1, \ldots, p_{n/2}\}.$$  

**PROOF.** We claim that $f(P \cap U(f)) \subset P \cap D(f)$. Suppose the claim is false. Then for some $p_0 \in P \cap U(f)$, $f(p_0) \in P \cap U(f)$. Let $k$ be the smallest nonnegative integer with $f^k((f(p_0)) = p_1$. Note that $1 < k < n$.

If $k$ is odd then $p_1 < f(p_0)$ and $f^k$ has a fixed point between $p_1$ and $f(p_0)$. Now $f(p_0) \leq P_U(f)$ and, by Lemma 7, $P_U(f) = P_U(f^k)$. Hence, $f^k$ has a fixed point between $P_{\min}(f^k)$ and $P_U(f^k)$. By Lemma 2, $f^k$ has periodic orbits of every period. This contradicts our hypothesis.

If $k$ is even then $k + 1$ is odd, $k + 1 < n$, and $f^{k+1}(p_0) = p_1$. Hence, $p_1 < p_0$ and $f^{k+1}$ has a fixed point between $p_1$ and $p_0$. By Lemma 7, $f^{k+1}$ has a fixed point between $P_{\min}(f^{k+1})$ and $P_U(f^{k+1})$. Again, using Lemma 2, we obtain a contradiction.

This establishes our claim that $f(P \cap U(f)) \subset P \cap D(f)$. By a similar proof, it follows that $f(P \cap D(f)) \subset P \cap U(f)$. Since the restriction of $f$ to $P$ is a bijection it follows that
\[ f(P \cap U(f)) = P \cap D(f) \quad \text{and} \quad f(P \cap D(f)) = P \cap U(f). \]

Hence \( P \cap U(f) \) and \( P \cap D(f) \) have an equal number of points. Since \( P_0(f) < P_D(f) \) by Lemma 3, this proves Lemma 8. Q.E.D.

**Proposition 9.** Let \( f \in C^0(I, I) \). Suppose \( \{p_1, \ldots, p_n\} \) is a periodic orbit of \( f \) of period \( n \), where \( n \) is a power of 2 and \( n > 2 \). Suppose \( p_1 < p_2 < \cdots < p_n \) and \( f(\{p_{n/2}, \ldots, p_n\}) \neq \{p_{n/2+1}, \ldots, p_n\} \). Then \( f \) has a periodic point of period \( s \), where \( s \) is odd and \( 3 \leq s \leq 3(n - 1) \).

**Proof.** By hypothesis and Lemma 8, for some odd integer \( m < n \), \( f^m \) has a periodic point of period 3. The conclusion follows easily from this. Q.E.D.

**Lemma 10.** Let \( f \in C^0(I, I) \). Suppose \( f \) has a periodic orbit \( \{p_1, p_2, p_3, p_4\} \) with \( p_1 < p_2 < p_3 < p_4 \) and \( f(\{p_1, p_2\}) \neq \{p_3, p_4\} \). Then \( f \) has a periodic point of period 3.

**Proof.** Let \( I_1 = [p_1, p_2] \), \( I_2 = [p_2, p_3] \), and \( I_3 = [p_3, p_4] \). Our hypothesis implies that either \( f(p_1) = p_2 \) or \( f(p_2) = p_1 \).

**Case 1.** \( f(p_1) = p_2 \). Since \( f(p_2) = p_3 \) or \( f(p_2) = p_4 \), we have \( f(I_1) \supset I_2 \). Also, since \( f(p_2) = p_4 \) or \( f(p_3) = p_4 \), we have \( f(I_2) \supset I_3 \). Finally, since \( f(p_3) = p_1 \) or \( f(p_4) = p_1 \), we have \( f(I_3) \supset I_1 \).

By Lemma 4, there are closed intervals \( J_1 \subset I_1, J_2 \subset I_2, \) and \( J_3 \subset I_3 \) such that \( f(J_1) = I_1, f(J_2) = J_3, \) and \( f(J_3) = J_2 \). It follows that \( f^3(J_1) = I_1 \). By Lemma 5, \( f^3 \) has a fixed point \( x \in J_1 \). Since \( f(x) \in I_2 \), \( x \) is a periodic point of \( f \) of period 3.

**Case 2.** \( f(p_2) = p_1 \). Then \( f(p_1) = p_3 \) or \( f(p_1) = p_4 \). Thus, \( f(I_1) \supset I_1 \), and \( f(I_1) \supset I_2 \). Also, \( f(p_2) = p_1 \) implies that \( f(I_2) \supset I_1 \). By Lemma 4, there are closed intervals \( J_1 \subset I_1, J_2 \subset I_2, \) and \( J_3 \subset I_3 \) such that \( f(J_1) = I_1, f(J_2) = J_3, \) and \( f(J_3) = J_2 \). It follows that \( f^3(J_1) = I_1 \). By Lemma 5, \( f^3 \) has a fixed point \( x \in J_1 \). Since \( f^2(x) \in I_2 \), \( x \) is a periodic point of \( f \) of period 3. Q.E.D.

**Theorem B.** Let \( f \in C^0(I, I) \). Suppose \( f \) has a periodic orbit \( P \) of period \( m \) (where \( m = 2^k \) for some \( k > 2 \)) which is not simple. Then \( f \) has a periodic point of period \( 3 \cdot 2^{k-2} \).

**Proof.** By hypothesis there is a subset \( \{q_1, \ldots, q_n\} \) of \( P \) and a positive integer \( r \) which divides \( m \) such that \( \{q_1, \ldots, q_n\} \) is a periodic orbit of \( f^r \) with \( q_1 < q_2 < \cdots < q_n \) and

\[ f^r(\{q_1, \ldots, q_{n/2}\}) \neq \{q_{n/2+1}, \ldots, q_n\}. \]

This implies \( n > 2 \). It follows from the proof of Lemma 6 that \( m = n \cdot r \). Hence \( r < 2^{k-2} \).

First suppose \( r = 2^{k-2} \). Then \( n = 4 \), so \( \{q_1, q_2, q_3, q_4\} \) is a periodic orbit of \( f^r \) of period 4 with \( f^r(\{q_1, q_2\}) \neq \{q_3, q_4\} \). By Lemma 10, \( f^r \) has a periodic
point of period 3. By the theorem of Šarkovskii (stated in §1), \( f \) has a periodic point of period \( 3 \cdot r = 3 \cdot 2^k - 2 \).

Now suppose \( r < 2^{k-2} \). Then \( r < 2^{k-3} \). By Proposition 9, \( f' \) has a periodic point of period \( s \), where \( s \) is odd and \( s > 3 \). By the theorem of Šarkovskii, \( f \) has a periodic point of period \( 3 \cdot 2^k - 2 \). Q.E.D.

**Theorem A.** Let \( f \in C^0(I, I) \). \( f \) has a periodic point whose period is not a power of 2 if and only if \( f \) has periodic orbit of period a power of 2 which is not simple.

**Proof.** The “if” part of the theorem follows from Theorem B.

Suppose \( f \) has a periodic point whose period is not a power of 2. By the theorem of Šarkovskii, stated in §1, for some positive integer \( r \) which is a power of 2, \( f' \) has a periodic orbit \( P \) of period 3. Let \( P = \{ p_1, p_2, p_3 \} \) with \( p_1 < p_2 < p_3 \).

Let \( g = f' \). Then \( g(p_1) = p_2 \) or \( g(p_3) = p_2 \). We may assume without loss of generality that \( g(p_1) = p_2 \). This implies that \( g(p_2) = p_3 \) and \( g(p_3) = p_1 \).

Since \( g(p_2) > p_2 \) and \( g(p_3) < p_3 \), \( g \) has a fixed point \( e \in (p_2, p_3) \). Let \( I_1 = [p_1, p_2] \), \( I_2 = [p_2, e] \), and \( I_3 = [e, p_3] \). Then \( g(I_1) \supset I_2 \), \( g(I_1) \supset I_3 \), \( g(I_2) \supset I_3 \), \( g(I_3) \supset I_1 \), and \( g(I_3) \supset I_2 \). By Lemma 4, there are closed intervals \( J_8 \subseteq I_3 \) with \( g(J_8) = I_1 \), \( J_7 \subseteq I_2 \) with \( g(J_7) = J_8 \), \( J_6 \subseteq I_3 \) with \( g(J_6) = J_7 \), \( J_5 \subseteq I_2 \) with \( g(J_5) = J_6 \), \( J_4 \subseteq I_1 \) with \( g(J_4) = J_5 \), \( J_3 \subseteq I_3 \) with \( g(J_3) = J_4 \), \( J_2 \subseteq I_2 \) with \( g(J_2) = J_3 \), and \( J_1 \subseteq I_1 \) with \( g(J_1) = J_2 \).

It follows that \( g^8(J_1) = I_1 \). By Lemma 5, \( g^8 \) has a fixed point \( c \in J_1 \). Hence \( c \) is a periodic point of \( g \) of period 1, 2, 4, or 8. Since \( g(c) \in I_2 \), \( g^2(c) \in I_1 \), and \( g^4(c) \in I_2 \), \( c \) is a periodic point of \( g \) of period 8.

Let \( \{ q_1, \ldots, q_8 \} \) denote the orbit of \( c \) where \( q_1 < q_2 < \cdots < q_8 \). We claim that, for some \( i \leq 4 \) and \( j \leq 4 \), \( g(q_i) = q_j \). Note that \( c \in I_1 \), \( g^4(c) \in I_1 \), \( g(c) \in I_2 \), \( g^4(c) \in I_2 \), \( g^6(c) \in I_2 \), \( g^2(c) \in I_3 \), \( g^2(c) \in I_3 \), and \( g^6(c) \in I_3 \). Hence, \( \{ q_1, q_2, q_3, q_4 \} \) contains \( c \), \( g^4(c) \), and two of the points \( g(c), g^4(c), \) and \( g^6(c) \).

First, suppose that \( g(c) \in \{ q_1, q_2, q_3, q_4 \} \). Then the claim is true with \( q_i = c \) and \( q_j = g(c) \). Now, suppose that \( g(c) \not\in \{ q_1, q_2, q_3, q_4 \} \). Then \( g^4(c) \in \{ q_1, q_2, q_3, q_4 \} \). So the claim holds with \( q_i = g^3(c) \) and \( q_j = g^4(c) \).

Thus, our claim holds in either case. By Lemma 6, \( f \) has a periodic orbit of period a power of 2 which is not simple. Q.E.D.

**4. Some examples.** Let \( f \in C^0(I, I) \) and let \( P = \{ p_1, \ldots, p_8 \} \) be a periodic orbit of \( f \) of period 8 with \( p_1 < p_2 < \cdots < p_8 \). Then \( P \) is simple if and only if the following two conditions hold:

1. \( f(P) = \{ p_2, p_6, p_7, p_3 \} \).
2. \( f^2(P) = \{ p_3, p_4 \} \) and \( f^3(P) = \{ p_7, p_8 \} \).

Clearly, (1) and (2) and the fact that \( P \) is a periodic orbit of period 8 imply
that
\[ f(\{P_5, P_6, P_7, P_8\}) = \{P_1, P_2, P_3, P_4\}, \]
\[ f^2(\{P_3, P_4\}) = \{P_1, P_2\}, \quad f^2(\{P_7, P_8\}) = \{P_5, P_6\}, \]
\[ f^4(P_1) = P_2, \quad f^4(P_2) = P_1, \quad f^4(P_3) = P_4, \quad f^4(P_4) = P_3, \]
\[ f^4(P_6) = P_1, \quad f^4(P_6) = P_5, \quad f^4(P_7) = P_8, \quad f^4(P_8) = P_7. \]

**Example 1.** \( f(P_1) = P_5, f(P_2) = P_6, f(P_3) = P_7, f(P_4) = P_8, f(P_5) = P_3, f(P_6) = P_4, f(P_7) = P_1, \) and \( f(P_8) = P_2. \)

In this example \( P \) is simple.

**Example 2.** \( f(P_1) = P_2, f(P_2) = P_5, f(P_3) = P_7, f(P_4) = P_8, f(P_5) = P_3, f(P_6) = P_4, f(P_7) = P_1, \) and \( f(P_8) = P_6. \)

In this example \( P \) is not simple because condition (1) above does not hold.

By Proposition 9, \( f \) has a periodic point of period \( s \), where \( s \) is odd and \( 3 < s < 21 \). By Theorem C of §1, the topological entropy of \( f \) is greater than \( \log\sqrt{2} \).

**Example 3.** \( f(P_1) = P_5, f(P_2) = P_7, f(P_3) = P_6, f(P_4) = P_8, f(P_5) = P_2, f(P_6) = P_4, f(P_7) = P_3, \) and \( f(P_8) = P_1. \)

In this example \( P \) is not simple because condition (2) does not hold (since \( f^2(P_1) = P_5 \)). By Theorem B, \( f \) has a periodic point of period 6, and by Corollary D, the topological entropy of \( f \) is greater than \( (\frac{3}{2})\log\sqrt{2} \).

**REFERENCES**


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