NECESSARY CONDITIONS FOR THE CONVERGENCE OF CARDINAL HERMITE SPLINES AS THEIR DEGREE TENDS TO INFINITY

BY

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ABSTRACT. Let $S_{n,s}$ denote the class of cardinal Hermite splines of degree $n$ having knots of multiplicity $s$ at the integers. In this paper we show that if $f_n \to f$ uniformly on $\mathbb{R}$, where $f_n \in S_{n,s}$, $s_n \to \infty$ as $n \to \infty$, and $f$ is bounded, then $f$ is the restriction to $\mathbb{R}$ of an entire function of exponential type $< \sigma$. In proving this result, we need to derive some extremal properties of certain splines $S_{n,s}$, in particular that $||S^e||_0$ minimises $||S||_0$ over $S \in S_{n,s}$ with $||S^e||_0 = ||S^e||_0$.

1. Introduction. For $n = 1, 2, \ldots$ and $1 < s < n$, let

$S_{n,s} = \{ f \in C^{n-s}(\mathbb{R}) : f'(\nu, \nu + 1) \in C^{n-1}((\nu, \nu + 1]) \}$

and

$f^{(n-1)}$ absolutely continuous on $(\nu, \nu + 1), \forall \nu \in \mathbb{Z}$.

We let $S_{n,s}$ denote the set of all cardinal spline functions of degree $n$ in $S_{n,s}$, i.e.,

$S_{n,s} = \{ S \in C^{n-s}(\mathbb{R}) : S'((\nu, \nu + 1) \in \pi_n, \forall \nu \in \mathbb{Z} \}$,

where $\pi_n$ denotes the set of all polynomials of degree at most $n$.

Throughout this paper, $||f||$ will denote $\text{ess sup}_{x \in \mathbb{R}} |f(x)|$.

In [6] Lipow and Schoenberg have shown that for odd $n, 1 < s < \frac{1}{2}(n + 1)$, and any function $f$ with $f^{(s)}$ of power growth on $\mathbb{R}, \nu = 0, 1, \ldots, s - 1, \nu = 0, 1, \ldots, s - 1$, there is a unique $S_{n,s} \in S_{n,s}$ of power growth such that $S_{n,s}^{(s)}$ interpolates $f^{(s)}$ at the integers. In [8] Marsden and Riemenschneider have shown that if $f$ is the Fourier-Stieltjes transform of a measure on $(-\pi, \pi)$, then $S_{n,s}^{(s)} \to f^{(s)}$ uniformly on $\mathbb{R}$ as $n \to \infty, \nu = 0, 1, \ldots, s - 1$. The case $s = 1$ had previously been proved by Schoenberg [10] who established in [11] the partial converse that if $f$ is bounded on $\mathbb{R}$ and $S_{n,1} \to f$ uniformly on $\mathbb{R}$ as $n \to \infty$, then $f$ is the restriction to $\mathbb{R}$ of an entire function of exponential type $< \pi$.

In § of this paper we generalise Schoenberg’s result by showing, in particular, that for any $s = 1, 2, \ldots$, if $f$ is bounded on $\mathbb{R}$ and $S_{n,s} \to f$ uniformly on $\mathbb{R}$ as $n \to \infty$, then $f$ is the restriction to $\mathbb{R}$ of an entire function

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of exponential type \( s \pi \). To establish this result we need some extremal properties of certain splines \( \mathcal{S}_{n,s} \in \mathcal{S}_{n,s} \) which may be regarded as generalisations of the Euler splines employed in [11]. For odd \( s \) these were defined by Cavaretta in [1]. In §2 we define \( \mathcal{S}_{n,s} \) for even \( s \) and show that for all \( s \), \( f \in \mathcal{S}_{n,s} \), \( \|f\| < 1 = \|\mathcal{S}_{n,s}\| \) and \( \|f^{(s)}\| < \|\mathcal{S}_{n,s}^{(s)}\| \) implies 
\[
|f^{(k)}(v + 1)| < |\mathcal{S}_{n,s}^{(k)}(v + 1)|, \quad \forall \ v \in \mathbb{Z} \text{ and } k = n - s, \ldots, n - 1.
\]

In [1] Cavaretta shows that for odd \( s \), \( S = \mathcal{S}_{n,s} \) minimises \( \|S\| \) over all \( S \in \mathcal{S}_{n,s} \) with
\[
\mathcal{S}_{n,s}^{(n)}(v, v + 1) = (-1)^v \|\mathcal{S}_{n,s}^{(n)}\|, \quad \forall v \in \mathbb{Z}.
\]

In §3 we show that for all \( s \), \( S = \mathcal{S}_{n,s} \) actually minimises \( \|S\| \) over all \( S \in \mathcal{S}_{n,s} \) with \( \|S^{(n)}\| = \|\mathcal{S}_{n,s}^{(n)}\| \).

2. The Euler-Chebyshev splines. In [1] Cavaretta shows there are functions \( \mathcal{S}_{n,s} \) in \( \mathcal{S}_{n,s} \) for \( n = 1, 2, \ldots \) and odd \( s < n \), characterised by the following properties:

\[
\mathcal{S}_{n,s}^{(n)}(v, v + 1) = (-1)^v \|\mathcal{S}_{n,s}^{(n)}\|, \quad \forall v \in \mathbb{Z}, \tag{2.1}
\]

\( \mathcal{S}_{n,s}(x) \) equioscillates between -1 and 1 at points
\[
0 < \beta_1 < \cdots < \beta_s < 1, \tag{2.2}
\]

\( \mathcal{S}_{n,s} \) is even or odd about \( x = \frac{1}{2} \) as \( n \) is even or odd,
\[
\mathcal{S}_{n,s}^{(n)}(x) > 0 \text{ on } (0, 1). \tag{2.3}
\]

We now construct functions \( \mathcal{S}_{n,s} \) in \( \mathcal{S}_{n,s} \) for \( n = 1, 2, \ldots \) and even \( s < n \) which are also characterised by properties (2.1)–(2.4).

We shall need the following lemma. Its proof is almost identical to that of Proposition 1 in [1] and so will be omitted.

**Lemma 1.** Let \( \{f_1(x), \ldots, f_k(x)\} \) be a Chebyshev system in \([a, b]\) and define
\[
g_i(x) = (x - a)(x - b)f_i(x), \quad i = 1, \ldots, k.
\]

Let \( F(x) \) be a continuous function on \([a, b]\) which vanishes at \( a \) and \( b \). Then there exists a unique linear combination \( \sum_{i=1}^{k} a_i g_i(x) \) of best approximation in the uniform norm to \( F(x) \). This best approximation is uniquely characterised by a \((k + 1)\)-point equioscillation property, i.e. there exist \( k + 1 \) points \( a < x_1 < \cdots < x_k < b \) where the error function assumes the value of its norm with alternating signs.

We first consider the case of odd \( n \). For any \( p, q, 1 < q < p \), we define
\[
V_{2p+1,2q} = \left\{ f \in \mathcal{S}_{2p+1} : f(0) = 0, \quad i = 0, \ldots, p - q, \quad f^{(2j)}(\frac{1}{2}) = 0, \quad j = 0, \ldots, p \right\}.
\]

It follows from the theory of Jerome and Schumaker [3] and Lorentz [7]
that \( \dim V_{2p+1,2q} = q \) and any \( f \) in \( V_{2p+1,2q} \) has at most \( q + 1 \) zeros in \([0, \frac{1}{2}]\). Thus if \( x(x - \frac{1}{2})f_i(x) \), \( i = 1, \ldots, q \), form a basis for \( V_{2p+1,2q} \), then \( \{f_i(x), \ldots, f_q(x)\} \) form a Chebyshev system on \([0, \frac{1}{2}]\).

Now take any odd \( n \) and even \( s, 4 < s < n \), and take any \( f \) in \( V_{n,2s-2} \) with \( f^{(n)} > 0 \). Let \( F \) denote the best approximation to \( f \) in the uniform norm in \( V_{n-2s-2} \). Then by Lemma 1, \( f - F \) equioscillates at points \( 0 < \beta_1 < \cdots < \beta_{s/2} < \frac{1}{2} \). Let \( G = (f - F)/\|f - F\| \) and define \( \delta_{n,s} \) in \( \mathcal{S}_{n,s} \) by
\[
\delta_{n,s}(x) = \begin{cases} 
G(x), & 0 < x < \frac{1}{2}, \\
(-1)^nG(1 - x), & \frac{1}{2} < x < 1,
\end{cases}
\] (2.5)
For \( s = 2 \), let \( G \) be the element of \( V_{n,2} \) with \( \|G\| = 1 \) and \( G^{(n)} > 0 \), and again define \( \delta_{n,s} \) by (2.5). Since \( G(0) = G(\frac{1}{2}) = 0 \), \( \exists \beta \in (0, \frac{1}{2}) \) with \( |G(\beta)| = 1 \), and so \( \delta_{n,2} \) equioscillates at \( \beta_1 \) and \( \beta_2 = 1 - \beta_1 \). Thus for all even \( s \), \( \delta_{n,s} \) is characterised by properties (2.1) to (2.4).

Next consider even \( n \). For any \( p, q, 0 < q < p \), define
\[
V_{2p,2q} = \{f \in \pi_2p\mid [0, \frac{1}{2}] : f^{(2+i)}(0) = 0, \quad i = 0, \ldots, p - q - 1, \\
f^{(2+i)}(\frac{1}{2}) = 0, \quad j = 0, \ldots, p - 1 \}. 
\]
Then \( \dim V_{2p,2q} = q + 1 \) and any \( f \) in \( V_{2p,2q} \) has at most \( q \) zeros in \([0, \frac{1}{2}]\). Thus any basis for \( V_{2p,2q} \) forms a Chebyshev system.

Now take even \( n \) and even \( s, 2 < s < n \), and take any \( f \) in \( V_{n,2s} \) with \( f^{(n)} > 0 \). Let \( F \) denote the best approximation to \( f \) in the uniform norm in \( V_{n-2s-2} \). Then \( f - F \) equioscillates at points \( 0 < \beta_1 < \cdots < \beta_{s/2+1} < \frac{1}{2} \). Now \( f' - F' \) is in \( V_{n-1,s} \) and so has at most \( \frac{s}{2} - 1 \) zeros in \((0, \frac{1}{2})\). Thus \( \beta_1 = 0 \) and \( \beta_{s/2+1} = \frac{1}{2} \). Let \( G = (f' - F')/\|f' - F'\| \) and define \( \delta_{n,s} \) in \( \mathcal{S}_{n,s} \) by (2.5). Then again \( \delta_{n,s} \) is characterised by properties (2.1) to (2.4).

We note that, for \( m = 1, 2, \ldots, \)
\[
\delta_{2m-1,1}(x) = (-1)^m\delta_{2m-1,1}(x), \\
\delta_{2m,1}(x) = (-1)^m\delta_{2m,1}(x - \frac{1}{2}),
\] (2.6)
where \( \delta_n \) denotes the Euler spline of degree \( n \), see [11].

We also note that, for \( n = 1, 2, \ldots, \)
\[
\delta_{n,n}(x) = T_n(2x - 1), \quad \forall x \in [0, 1],
\]
where \( T_n \) denotes the Chebyshev polynomial of degree \( n \).

It therefore seems appropriate to refer to \( \delta_{n,s} \) as Euler-Chebyshev splines, or ET-splines, following the similar terminology introduced by Cavaretta in [1]. They satisfy the following extremal property which is related to a theorem of Kolmogorov (see [2]).
**Theorem 1.** Suppose \( f \) in \( \mathcal{S}_{n-s} \) satisfies
\[
\|f\| < 1 \quad \text{and} \quad \|f^{(n)}\| < \|\mathcal{S}^{(n)}_{n-s}\|,
\] then
\[
|f^{(k)}(v^+)| < |\mathcal{S}^{(k)}_{n-s}(v^+)|, \quad \forall \, v \in \mathbb{Z}, \quad k = n - s, \ldots, n - 1.
\]

**Proof.** We use an elementary and powerful technique introduced by Cavaretta [2].

Without loss of generality we may take \( v = 0 \). Suppose \( f \) in \( \mathcal{S}_{n-s} \) satisfies (2.7) and is periodic of period an even integer \( K \). We shall assume \( |f^{(k)}(0^+)| > |\mathcal{S}^{(k)}_{n-s}(0^+)| \) for some \( k, \, n - s < k < n - 1 \), and reach a contradiction. Choose \( \lambda \) so that \( \lambda f^{(k)}(0^+) = \mathcal{S}^{(k)}_{n-s}(0^+) \) and let \( g = \mathcal{S}^{(k)}_{n-s} - \lambda f \), noting that \( g \) is also periodic of period \( K \).

Since \( \|\lambda f\| < \|\mathcal{S}^{(k)}_{n-s}\| \) and because of the equioscillation of \( \mathcal{S}^{(k)}_{n-s} \), \( g \) has at least \( Ks \) distinct zeros per period. Thus, by repeated application of Rolle’s theorem, \( g^{(n-s)} \) has at least \( Ks \) distinct zeros per period. If \( k = n - s \), then \( g^{(n-s)}(0) = 0 \) and so \( g^{(n-s+1)} \) has at least \( K(s - 1) + 1 \) zeros per period which are not at integers. If \( k > n - s \), then \( g^{(n-s+1)} \) has at least \( K(s - 1) \) zeros per period which are not at integers, and so \( g^{(k)} \) has at least \( K(n - k) \) zeros per period which are not at integers. But \( g^{(k)}(0^+) = 0 \) and so \( g^{(k+1)} \) has at least \( K(n - k - 1) + 1 \) changes of sign per period which are not at integers. Thus for all \( k \), \( g^{(n)} \) has at least one change of sign per period which is not at an integer. But this contradicts \( |\lambda f^{(n)}(x)| < \|\mathcal{S}^{(n)}_{n-s}(x)\| \) in every interval \((\nu, \nu + 1)\), \( \nu \in \mathbb{Z} \).

We may extend to nonperiodic \( f \) in precisely the same manner as in [2]. □

3. An extremal property of ET-splines. For \( n = 1,2, \ldots, \) \( 1 < s < n \), and numbers \( \alpha_1, \ldots, \alpha_n, \lambda \), we define
\[
\Pi_n(\alpha_1, \ldots, \alpha_n; \lambda)
\]
This determinant has the following properties, which follow from the work of Micchelli [9] or by using the method of Lee and Sharma [5].

For fixed $0 < \alpha_1 < \alpha_2 < \cdots < \alpha_s < 1$, $\Pi_n(\lambda) \equiv \Pi_n(\alpha_1, \ldots, \alpha_s; \lambda)$ is a polynomial in $\lambda$ with real distinct roots of sign $(-1)^{r}$. If $\alpha_i > 0$, $\Pi_n(\lambda) = a\lambda^{n-s+1} + \cdots$, where sign $a = (-1)^{(s+1)(n+s+1)}$. If $\alpha_1 = 0$, $\Pi_n(\lambda) = a\lambda^{n-s} + \cdots$, where sign $a = (-1)^{(s+1)(n+s)}$. If the nonzero $\alpha_i$, $i = 1, \ldots, s$, are symmetric about $\frac{1}{2}$, then $\Pi_n(\lambda)$ is reciprocal.

Now fix $0 < \alpha_1 < \alpha_2 < \cdots < \alpha_s < 1$ and take $r$, $1 < r < s$. For $x \in [0, 1]$ we define

$$\Pi(x, \lambda) = \Pi_n(\alpha_1, \ldots, \alpha_{r-1}, x, \alpha_{r+1}, \ldots, \alpha_s; \lambda) = p_0(x)\lambda^{n-s+1} + p_1(x)\lambda^{n-s} + \cdots + p_{n-s+1}(x).$$

Then it is easy to show that

$$\frac{\partial^j}{\partial x^j}\Pi(1, \lambda) = \lambda \frac{\partial^j}{\partial x^j}\Pi(0, \lambda), \quad j = 0, \ldots, n-s, \quad (3.1)$$

and

$$\Pi(\alpha_i, \lambda) = 0, \quad i \neq r. \quad (3.2)$$

We now define the 'B-spline'

$$B_r(x) = \begin{cases} p_\nu(x - \nu), & x \in [\nu, \nu + 1), \quad \nu = 0, \ldots, n-s+1, \\ 0, & x < 0 \text{ and } x > n-s+2. \end{cases}$$

From (3.1) we see that $B_r \in S_{n,s}$ and from (3.2) we have $B_r(\alpha_i + \nu) = 0$ for all $\nu \in \mathbb{Z}$ and $i \neq r$. Also

$$\sum_{r=-\infty}^{\infty} B_r(x + \nu)t^r = t^{n-s+1}\Pi(x, t^{-1}), \quad x \in [0, 1). \quad (3.3)$$

Now assume

$$\Pi_n(\alpha_1, \ldots, \alpha_s; (-1)^{r}) \neq 0. \quad (3.4)$$

Then following the method of Schoenberg [11], we may write

$$\left\{ \sum_{r=-\infty}^{\infty} B_r(\nu + \alpha_i)t^r \right\}^{-1} = \sum_{r=-\infty}^{\infty} \omega_r t^r, \quad (3.5)$$

where the series is convergent on some annulus about $|\nu| = 1$ and $|\omega_r| = O(\beta^r)$ as $\nu \to \pm \infty$ for some $0 < \beta < 1$.

We now define the 'fundamental spline'

$$L_r(x) = \sum_{r=-\infty}^{\infty} \omega_r B_r(x - \nu).$$

Then

$$L_r(k + \alpha_i) = \sum_{r=-\infty}^{\infty} \omega_r B_r(k + \alpha_i - \nu) = \delta_{k0}, \quad \forall k \in \mathbb{Z}, \quad \text{by (3.5)}.$$
It follows from the theory of [9] that if $S \in \mathcal{S}_{\alpha}$ is of power growth, then

$$S(x) = \sum_{r=1}^{s} \sum_{k=-\infty}^{\infty} S(k + \alpha_r) L_r(x - k). \quad (3.6)$$

Now take $x$ in $(0, 1)$. Then

$$\frac{\partial^n}{\partial x^n} \Pi(x, \lambda) = (-1)^{n+r+1} n! \Pi_{n-1}(\alpha_1, \ldots, \alpha_{r+1}, \ldots, \alpha_{s}; \lambda). \quad (3.7)$$

Now

$$L_r^{(n)}(k + x) = \sum_{r=-\infty}^{\infty} \omega_r B_r^{(n)}(k + x + u)$$

and so

$$\sum_{k=-\infty}^{\infty} L_r^{(n)}(k + x) t^k = \left( \sum_{r=-\infty}^{\infty} \omega_r t^r \right) \left( \sum_{r=-\infty}^{\infty} B_r^{(n)}(j + x) t^j \right).$$

So by (3.7), (3.5) and (3.3),

$$\sum_{k=-\infty}^{\infty} L_r^{(n)}(k + x) t^k = (-1)^{n+r+1} n! \Pi_{n-1}(\alpha_1, \ldots, \alpha_{r+1}, \ldots, \alpha_{s}; t^{-1}) \quad (3.8)$$

Then from (3.8) and the properties of $\Pi_n(\lambda)$, we have the following result.

$$\sum_{k=-\infty}^{\infty} L_r^{(n)}(k + x) t^k = \frac{(-1)^{r+s} K \Pi_{n-1}^{r+i+1}(1 + \mu_i t)}{\Pi_{n-1}^{r+i}(1 - \lambda_j t)} \quad \text{if } \alpha_1 > 0,$$

$$= \frac{(-1)^{r+s+1} K \Pi_{n-1}^{r+i}(1 + \mu_i t)}{\Pi_{n-1}^{r+i}(1 - \lambda_j t)} \quad \text{if } \alpha_1 = 0, \ r > 1,$$

$$= \frac{K \Pi_{n-1}^{r+i}(1 + \mu_i t)}{i \Pi_{n-1}^{r+i}(1 - \lambda_j t)} \quad \text{if } \alpha_1 = 0, \ r = 1,$$

where $K, \mu_i, \lambda_j$ are constants (depending on $r, n, \alpha_1, \ldots, \alpha_s$) with $K > 0$ and

sign $\mu_i = \text{sign } \lambda_j = (-1)^i, \forall \ i,j.$

We therefore have (see [4, p. 395]),

$$\text{sign } L_r^{(n)}(k + x) = \begin{cases} 
(-1)^{r+s+k}, & \text{if } s \text{ odd,} \\
(-1)^{r+s+1}, & \text{if } s \text{ even,}
\end{cases}$$
where

\[ q = \begin{cases} 
1, & \text{if } \alpha_1 > 0, \\
0, & \text{if } \alpha_1 = 0.
\end{cases} \quad (3.9) \]

We are now in a position to prove our result.

**Theorem 2.** If \( S \in S_{n,s} \) satisfies \( ||S|| < 1 \), then \( ||S^{(n)}|| < ||S^{(0)}|| \).

**Proof.** Take \( \beta_1, \ldots, \beta_s \) as in (2.2). By (2.3) we know the nonzero \( \beta_p \), \( i = 1, \ldots, s \), are symmetric about \( \frac{1}{2} \) and so \( \Pi_n(\beta_1, \ldots, \beta_s; \lambda) \) is a reciprocal polynomial in \( \lambda \). If \( n \) and \( s \) are both even or both odd, then \( \beta_1 = 0 \). Otherwise \( \beta_1 > 0 \). Thus in all cases, \( \Pi_n(\beta_1, \ldots, \beta_s; \lambda) \) is a polynomial in \( \lambda \) of even degree and so

\[ \Pi_n(\beta_1, \ldots, \beta_s; (-1)^s) \neq 0. \]

Since (3.4) is satisfied, we may define the ‘fundamental spline’ \( L_r \) for \( r = 1, \ldots, s \). Then for any \( S \in S_{n,s} \) satisfying \( ||S|| < 1 \), we have from (3.6),

\[ |S^{(n)}(x)| = \left| \sum_{r=1}^{s} \sum_{k=-\infty}^{\infty} S(k + \beta_r)L_r^{(n)}(x - k) \right| < \sum_{r=1}^{s} \sum_{k=-\infty}^{\infty} |L_r^{(n)}(x - k)|, \quad \forall x \in \mathbb{R}. \quad (3.10) \]

But it follows from (3.9) and (2.2) that equality is attained in (3.10) for \( S = \delta_{n,s} \).

For \( s = 1 \) this result was proved by Schoenberg [11], and for \( s = n \) the result follows immediately from the properties of Chebyshev polynomials.

It is clear from the proof of Theorem 2 that the condition \( ||S|| < 1 \) in the statement of the theorem can be replaced by the weaker condition

\[ |S(k + \beta_i)| < 1, \quad \forall k \in \mathbb{Z}, \quad i = 1, \ldots, s. \]

4. **Limits of cardinal splines.** We need a further property of ET-splines.

**Lemma 2.** For \( s = 1,2, \ldots \), there are constants \( K_s \) such that \( ||\delta_{n,s}^{(n)}|| < K_s(s\pi)^n \) for all \( n > s \) and \( \nu = 0, \ldots, n \).

**Proof.** First suppose \( s \) is odd, \( s = 2t - 1 \). It follows from the work of [1] that for any \( n > s \),

\[ \delta_{n,s} = \delta_{n,1} + \mu_1\delta_{n-2,1} + \cdots + \mu_{t-1}\delta_{n-2t+2,1}, \quad (4.1) \]

where \( \mu_1, \ldots, \mu_{t-1} \) are chosen to minimise \( ||\delta_{n,s}|| \).

We first consider odd \( n > s \). Then it follows from (4.1) and (2.6) that we may write

\[ \delta_{n,s} = (-1)^{(n+1)/2}/\phi_n/\|\phi_n\|, \]

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where

\[ \phi_n(x) = \sum_{r=1}^{\infty} \frac{\cos((2r-1)\pi x)}{(2r-1)^{n+1}} + \lambda_{n}^{(n)} \sum_{r=1}^{\infty} \frac{\cos((2r-1)\pi x)}{(2r-1)^{n-1}} + \cdots + \lambda_{n}^{(n)} \sum_{r=1}^{\infty} \frac{\cos((2r-1)\pi x)}{(2r-1)^{n-2r+3}} \]

and \( \lambda_{n}^{(n)}, \ldots, \lambda_{n}^{(n)} \), are chosen to minimise \( \| \phi_n \| \).

Let \( \lambda_1, \ldots, \lambda_{t-1} \) be the unique solution of the equations

\[ 1 + (2r-1)^2 \lambda_1 + \cdots + (2r-1)^{2t-2} \lambda_{t-1} = 0, \quad r = 1, \ldots, t-1. \]

Let

\[ \psi_n(x) = \sum_{r=1}^{\infty} \frac{\cos((2r-1)\pi x)}{(2r-1)^{n+1}} \{ 1 + \lambda_1(2r-1)^2 + \cdots + \lambda_{t-1}(2r-1)^{2t-2} \}. \]

Then \( \| (2t-3)^{n+1} \psi_n \| \rightarrow 0 \) as \( n \rightarrow \infty \). Since \( \| \phi_n \| < \| \psi_n \| \), \( \| (2t-3)^{n+1} \phi_n \| \rightarrow 0 \) as \( n \rightarrow \infty \) and so for \( r = 1, \ldots, t-1 \),

\[ \{ 1 + \lambda_1(2r-1)^2 + \cdots + \lambda_{t-1}(2r-1)^{2t-2} \} \rightarrow 0 \quad \text{as} \quad n \rightarrow \infty. \]

So \( \lambda_i^{(n)} \rightarrow \lambda_i \) as \( n \rightarrow \infty \), \( i = 1, \ldots, t-1 \). Thus

\[ (2t-1)^{n+1} \phi_n(x) = f_n(x) + a_n \cos((2t-1)\pi x) + O\left( \left[ \frac{2t-1}{2t+1} \right]^n \right) \]

where \( f_n(x) \) is of the form \( \sum_{r=1}^{t} b_r \cos((2r-1)\pi x) \) and

\[ a_n \rightarrow a = 1 + (2t-1)^2 \lambda_1 + \cdots + (2t-1)^{2t-2} \lambda_{t-1} \neq 0 \quad \text{as} \quad n \rightarrow \infty. \]

Now for each \( n \), there is an integer \( j, 1 < j < 2t-1 \), such that

\[ f_n\left( \frac{j}{2t-1} \right) a_n \cos j \pi > 0, \]

and so

\[ (2t-1)^{n+1} \left| f_n\left( \frac{j}{2t-1} \right) \right| > |a_n| + O\left( \left[ \frac{2t-1}{2t+1} \right]^n \right). \]

So \( \exists \delta > 0 \) such that

\[ s^{n+1} \| \phi_n \| > \delta, \quad \forall \ n \geq s. \quad (4.2) \]

Writing

\[ g_n(x) = \sum_{r=1}^{\infty} \frac{\cos((2r-1)\pi x)}{(2r-1)^{n+1}}, \]
we have

\[ \|s_n^{(v)}\| < \pi^v \left( 1 + \frac{1}{3^2} + \frac{1}{5^2} + \cdots \right) < 2\pi^v \]

for \( n = 1, 2, \ldots \) and \( v < n \). Also \( \|s_n^{(v)}\| = 2\|s_n^{(n-1)}\| < 4\pi^{n-1} \). So

\[ \|\phi_n^{(v)}\| < 4\pi^{n-1} \{ 1 + |\lambda_n^{(n)}| + \cdots + |\lambda_n^{(v)}| \}, \quad v < n, \]

and so there is a constant \( K \) such that

\[ \|\phi_n^{(v)}\| < Kn^n \quad \text{for all } n > s \text{ and } v < n. \]  \hspace{1cm} (4.3)

Thus

\[ D_n^{(s)} = \frac{K}{\delta} (s\pi)^n, \quad \forall n > s, \quad v < n, \]

by (4.2) and (4.3).

The result for even \( n \) follows similarly.

Next suppose \( s \) is even, \( s = 2t \). We first note that

\[ D_n^{(n-1)}(x)/\|D_n^{(n)}\| = x - \frac{1}{2}, \quad \forall x \in (0, 1). \]

So

\[ D_n^{(s)} = (-1)^{(s/2)} h/\|h\|, \]

where

\[ h(x) = \begin{cases} \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{2^k} \cos 2\pi k(x - \frac{1}{2}) + \sum_{k=1}^{\infty} \frac{1}{(2k)^n} & \text{if } n \text{ even,} \\ \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{2^k} \sin 2\pi k(x - \frac{1}{2}) & \text{if } n \text{ odd.} \end{cases} \]

It follows that for even \( n \),

\[ D_n^{(s)} = (-1)^{n/2} \phi_n/\|\phi_n\|, \]

where

\[ \phi_n(x) = \mu + \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{2^k} \cos 2\pi k(x - \frac{1}{2}) + \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{2^k} \cos 2\pi k(x - \frac{1}{2}) \]

\[ + \cdots + \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{2^k} \cos 2\pi k(x - \frac{1}{2}), \]

and \( \mu, \lambda_n^{(n)}, \ldots, \lambda_n^{(n-1)} \) are chosen to minimise \( \|\phi_n\| \).

For odd \( n \),

\[ D_n^{(s)} = (-1)^{(n-1)/2} \phi_n/\|\phi_n\|, \]
where
\[
\phi_n(x) = \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k^n} \sin 2k\pi(x - \frac{1}{2}) + \lambda^{(n)}_1 \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k^n} \sin 2k\pi(x - \frac{1}{2}) + \ldots + \lambda^{(n)}_{2m-1} \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k^{n-2m+2}} \sin 2k\pi(x - \frac{1}{2}),
\]
and \(\lambda^{(n)}_1, \ldots, \lambda^{(n)}_{2m-1}\) are chosen to minimise \(\|\phi_n\|\).

The result now follows by the same method as for odd \(s\). □

We now apply Lemma 2 and Theorems 1 and 2 in proving the following:

**Lemma 3.** For \(s = 1, 2, \ldots\), there are constants \(L_s\) such that if \(S\) in \(\mathbb{S}_{n,s}\) satisfies \(\|S\| < 1\), then \(\|S^{(k)}\| < L_s(s\pi)^k\), for all \(n > s\) and \(k < n - s\).

**Proof.** Take \(S\) in \(\mathbb{S}_{n,s}\) with \(\|S\| < 1\). Then by Theorem 2, \(\|S^{(n)}\| < \|S^{(n)}_{n,s}\|\). So by Theorem 1,
\[
|S^{(k)}(\nu +)| < |\tilde{S}^{(k)}_{n,s}(\nu +)|, \quad \forall \nu \in \mathbb{Z}, \quad k = n - s + 1, \ldots, n - 1.
\]
So by Lemma 2,
\[
\|S^{(n)}\| < K_s(s\pi)^n \quad (4.4)
\]
and
\[
|S^{(k)}(\nu +)| < K_s(s\pi)^n, \quad \forall \nu \in \mathbb{Z}, \quad k = n - s + 1, \ldots, n - 1. \quad (4.5)
\]
It follows from (4.4) and (4.5) for \(k = n - 1\) that \(\|S^{(n-1)}\| < 2K_s(s\pi)^n\). Proceeding in this manner we deduce that
\[
\|S^{(n-s+1)}\| < sK_s(s\pi)^n. \quad (4.6)
\]
Let \(T(x) = S(Mx)\), where \(M = [\frac{1}{2} K_s(s^{n+1}\pi^n)^{-1/(n-s+1)}]\). Then
\[
|T^{(n-s+1)}(x)| = M^{n-s+1} |S^{(n-s+1)}(x)|
\leq \frac{1}{2} K_s(s^{n+1}\pi^n)^{-1/2} K_s(s\pi)^n \quad \text{(by (4.6))}
= 2\pi^{n-s} < \|S_{n-s+1}\|^n.
\]
So by a theorem of Kolmogorov (see [2]), for \(k < n - s\),
\[
\|T^{(k)}\| < \|S_{n-s+1}^{(k)}\|^k < 2\pi^k \quad \text{(see [11]).} \quad (4.7)
\]
So
\[
\|S^{(k)}\| = M^{-k} \|T^{(k)}\| < M^{-k} 2\pi^k \quad \text{(by (4.7))}
= 2\left(\frac{1}{2} K_s(s\pi)^{s/(n-s+1)}(s\pi)^{k}\right) < L_s(s\pi)^k,
\]
where \(L_s = \max(2, K_s(s\pi)^{s})\). □

By the method of Schoenberg [11], we may deduce from Lemma 3 our final result.
Theorem 3. For a given natural number \( s \), suppose \( f_n \in S_{i_n,k} \), where \( i_n \to \infty \) as \( n \to \infty \). If \( f_n \to f \) uniformly on \( \mathbb{R} \) and \( f \) is bounded, then \( f \) is the restriction to \( \mathbb{R} \) of an entire function of exponential type \( < s \).

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References


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