

NOTES ON SQUARE-INTEGRABLE COHOMOLOGY SPACES ON CERTAIN FOLIATED MANIFOLDS

BY

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ABSTRACT. We discuss some square-integrable cohomology spaces on a foliated manifold with one-dimensional foliation whose leaves are compact and with a complete bundle-like metric. Applications to a contact manifold are given.

1. Introduction. On a compact foliated manifold with a bundle-like metric, B. L. Reinhart [9] proved that the cohomology of base-like differential forms is isomorphic to the harmonic space of a certain semidefinite Laplacian. In his paper [4], H. Kitahara discussed the square-integrable basic cohomology spaces on a foliated manifold with a complete bundle-like metric.

In this note, we discuss some square-integrable cohomology spaces on a foliated manifold with one-dimensional foliation whose leaves are compact and with a complete bundle-like metric. Moreover, we give applications to a contact manifold.

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2. Preliminaries. We assume that all objects and maps are of class C^∞ . Let M be a connected, orientable, $(n + 1)$ -dimensional Riemannian foliated manifold with one-dimensional foliation \mathcal{F} whose leaves are compact. We assume that the Riemannian metric $(,)$ on M is a bundle-like metric with respect to \mathcal{F} (cf. [8]).

Let $\{U; (x, y^1, \dots, y^n)\}$ denote a flat coordinate neighborhood system with respect to \mathcal{F} , that is, the integral manifolds of \mathcal{F} are given locally by $y^1 = c^1, \dots, y^n = c^n$, for some constants c^1, \dots, c^n (cf. [8]). We may choose, in each flat coordinate neighborhood system $\{U; (x, y^1, \dots, y^n)\}$, 1-form η and vectors v_1, \dots, v_n such that $\{\eta, dy^1, \dots, dy^n\}$ and $\{\partial/\partial x, v_1, \dots, v_n\}$ are dual bases for the cotangent and tangent spaces respectively at each point in U . Hence

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$$\begin{aligned}\eta &= dx + \sum A_i dy^i, \\ v_i &= \partial/\partial y^i - A_i \partial/\partial x, \quad i = 1, 2, \dots, n\end{aligned}\quad (1)$$

(cf. [8], [12]). Throughout this note, all local expressions for forms and vectors are taken with respect to these bases.

We may choose, in (1), $A_i = A_i(x, y)$ such that $(\partial/\partial x, v_i) = 0$, $i = 1, 2, \dots, n$, where $y = (y^1, \dots, y^n)$. Then the metric has the local expression

$$ds^2 = g_{\Delta\Delta}(x, y)\eta \cdot \eta + \sum g_{ij}(y) dy^i \cdot dy^j, \quad (2)$$

where $g_{\Delta\Delta}(x, y) = (\partial/\partial x, \partial/\partial x)$ and $g_{ij}(y) = (v_i, v_j)$, since the metric $(,)$ is a bundle-like metric with respect to \mathcal{F} (cf. [8], [12]).

DEFINITION (cf. [8], [12]). A form ϕ on M is called of type $(1, s)$ (resp. $(0, s)$) if it is expressed locally as

$$\phi = \frac{1}{1!s!} \sum \phi_{\Delta i_1 \dots i_s}(x, y) \eta \wedge dy^{i_1} \wedge \dots \wedge dy^{i_s}$$

(resp. $\phi = (1/0!s!) \sum \phi_{i_1 \dots i_s}(x, y) dy^{i_1} \wedge \dots \wedge dy^{i_s}$).

Let $\wedge^{1,s}(M)$ (resp. $\wedge^{0,s}(M)$) be the space of all forms on M of type $(1, s)$ (resp. $(0, s)$). The space $\wedge^s(M)$ of all s -forms on M is the direct sum of $\wedge^{1,s-1}(M)$ and $\wedge^{0,s}(M)$.

The operator d of exterior differentiation is decomposed as $d = d' + d'' + d'''$ where the components are of type $(1, 0)$, $(0, 1)$ and $(-1, 2)$ respectively (cf. [9], [12]). Since the foliation \mathcal{F} is of one dimension, we notice the following: (i) If $\phi \in \wedge^{1,s}(M)$, then $d\phi = d''\phi + d'''\phi$, where $d''\phi \in \wedge^{1,s+1}(M)$ and $d'''\phi \in \wedge^{0,s+2}(M)$. (ii) If $\phi \in \wedge^{0,s}(M)$, then $d\phi = d'\phi + d''\phi$, where $d'\phi \in \wedge^{1,s}(M)$ and $d''\phi \in \wedge^{0,s+1}(M)$.

Among examples we show the following one for reference below:

EXAMPLE. Let R^3 be a Euclidean space with cartesian coordinates (x, y^1, y^2) . We put $\eta = dx + (-y^2)dy^1$, then $\{\eta, dy^1, dy^2\}$ is a base for the cotangent space at each point in R^3 . Let ξ be a dual of η . Then R^3 is considered a foliated manifold whose leaves are orbits of vector field ξ . We define a metric ds^2 on R^3 by

$$ds^2 = dx \cdot dx + 2(-y^2)dx \cdot dy^1 + (1 + (y^2)^2)dy^1 \cdot dy^1 + dy^2 \cdot dy^2$$

(cf. [10]). Then we have $ds^2 = \eta \cdot \eta + dy^1 \cdot dy^1 + dy^2 \cdot dy^2$. Thus the metric ds^2 is a complete bundle-like metric with respect to the foliation.

Let $\varphi_m : R^3 \rightarrow R^3$ be a map defined by

$$\varphi_m(x, y^1, y^2) = (x + m, (-1)^m y^1, (-1)^m y^2),$$

where m is an integer. Define an equivalence relation in R^3 by $(x, y^1, y^2) \sim (\tilde{x}, \tilde{y}^1, \tilde{y}^2)$ if there exists an integer m such that $\varphi_m(x, y^1, y^2) = (\tilde{x}, \tilde{y}^1, \tilde{y}^2)$. It is trivial that η and ds^2 are invariant by φ_m for any m . Thus the induced η on $M = R^3/\sim$ define a foliation on M whose leaves are of one dimension and

compact, and the induced $\overline{ds^2}$ on M is a complete bundle-like metric with respect to the foliation. We notice that the foliation is not regular (cf. [8]).

3. Spaces $\Delta^{1,*}(M)$ and $\Delta^{0,*}(M)$. Hereafter, we are interested in forms such that

$$\phi = \frac{1}{1!s!} \sum \phi_{\Delta_{i_1 \dots i_s}}(y) \eta \wedge dy^{i_1} \wedge \dots \wedge dy^{i_s} \tag{I}$$

and

$$\phi = \frac{1}{0!s!} \sum \phi_{i_1 \dots i_s}(y) dy^{i_1} \wedge \dots \wedge dy^{i_s}. \tag{II}$$

Let $\Delta^{1,s}(M)$ (resp. $\Delta^{0,s}(M)$) be the subspace of $\wedge^{1,s}(M)$ (resp. $\wedge^{0,s}(M)$) satisfying (I) (resp. (II)) and $\Delta_0^{1,s}(M)$ (resp. $\Delta_0^{0,s}(M)$) the subspace of $\Delta^{1,s}(M)$ (resp. $\Delta^{0,s}(M)$) composed of forms with compact supports.

PROPOSITION 3.1 (CF. [9]). *For $\phi \in \wedge^{0,s}(M)$, $d'\phi = 0$ if and only if $\phi \in \Delta^{0,s}(M)$.*

A form in $\Delta^{0,s}(M)$ is called a basic or base-like form (cf. [4], [9]).

Restricted to $\Delta^{0,*}(M) = \sum \Delta^{0,s}(M)$, $d''^2 = d^2 = 0$ so we may consider the cohomology of $\Delta^{0,*}(M)$ and d'' (cf. [9], [12]). Next, if we have an assumption that

$$A_i = A_i(y), \quad i = 1, 2, \dots, n, \tag{III}$$

in (1), then we have $d''^2 = d^2 = 0$ on $\Delta^{1,*}(M) = \sum \Delta^{1,s}(M)$ (cf. [12]). The example in §2 satisfies our assumption (III). Hereafter, we consider the spaces $\Delta^{1,s}(M)$ and $\Delta^{1,*}(M)$ under the assumption (III). Thus we may consider the cohomology of $\Delta^{1,*}(M)$ and d'' .

4. Square-integrable basic cohomology spaces $\tilde{H}_2^{0,s}(M)$. This and the next sections are due to H. Kitahara [4], that is, they are the special cases of Kitahara's results. The methods are analogous to those of A. Andreotti and E. Vesentini [1] and K. Okamoto and H. Ozeki [7].

The $''$ -operation in $\Delta^{0,s}(M)$ is defined by

$$''\phi = \frac{1}{(n-s)!s!} \sum g^{i_1 j_1} \dots g^{i_s j_s} \delta_{j_1 \dots j_s, k_1 \dots k_{n-s}}^{1 \dots n} \times \sqrt{\det(g_{ij})} \phi_{i_1 \dots i_s} dy^{k_1} \wedge \dots \wedge dy^{k_{n-s}}$$

where (g^{ij}) denotes the inverse matrix of (g_{ij}) and $\delta_{j_1 \dots j_s, k_1 \dots k_{n-s}}^{1 \dots n}$ the Kronecker symbol (cf. [2], [4], [9], [12]). According to B. L. Reinhart [9], we define a pre-Hilbert metric $\langle \cdot, \cdot \rangle_1$ on $\Delta_0^{0,s}(M)$ by

$$\langle \phi, \psi \rangle_1 = \int_M dx \wedge \phi \wedge ''\psi.$$

The differential operator d'' maps $\Delta^{0,s}(M)$ into $\Delta^{0,s+1}(M)$. We define $\tilde{\delta}'' : \Delta^{0,s}(M) \rightarrow \Delta^{0,s-1}(M)$ by

$$\tilde{\delta}''\phi = (-1)^{ns+n+1} *'' d'' *'' \phi.$$

Let $\tilde{L}_2^{0,s}(M)$ be the completion of $\Delta_0^{0,s}(M)$ with respect to the inner product $\langle \cdot, \cdot \rangle_1$. We denote by \tilde{d}_0 the restriction of d'' to $\Delta_0^{0,s}(M)$ and by $\tilde{\theta}_0$ the restriction of $\tilde{\delta}''$ to $\Delta_0^{0,s}(M)$. We define $\tilde{d} = (\tilde{\theta}_0)^*$ and $\tilde{\theta} = (\tilde{d})^*$, where $(\cdot)^*$ denotes the adjoint operator of (\cdot) with respect to the inner product $\langle \cdot, \cdot \rangle_1$. Then \tilde{d} (resp. $\tilde{\theta}$) is a closed, densely defined operator of $\tilde{L}_2^{0,s}(M)$ into $\tilde{L}_2^{0,s+1}(M)$ (resp. $\tilde{L}_2^{0,s-1}(M)$). Let $D_{\tilde{d}}^{0,s}$ (resp. $D_{\tilde{\theta}}^{0,s}$) be the domain of the operator \tilde{d} (resp. $\tilde{\theta}$) in $\tilde{L}_2^{0,s}(M)$ and we put

$$Z_{\tilde{d}}^{0,s}(M) = \{ \phi \in D_{\tilde{d}}^{0,s}; \tilde{d}\phi = 0 \},$$

$$Z_{\tilde{\theta}}^{0,s}(M) = \{ \phi \in D_{\tilde{\theta}}^{0,s}; \tilde{\theta}\phi = 0 \}.$$

Since \tilde{d} and $\tilde{\theta}$ are closed operators, $Z_{\tilde{d}}^{0,s}(M)$ and $Z_{\tilde{\theta}}^{0,s}(M)$ are closed in $\tilde{L}_2^{0,s}(M)$. Let $B_{\tilde{d}}^{0,s}(M)$ (resp. $B_{\tilde{\theta}}^{0,s}(M)$) be the closure of $\tilde{d}(D_{\tilde{d}}^{0,s-1})$ (resp. $\tilde{\theta}(D_{\tilde{\theta}}^{0,s+1})$).

DEFINITION (CF. [4]). $\tilde{H}_2^{0,s}(M) = Z_{\tilde{d}}^{0,s}(M) \ominus B_{\tilde{d}}^{0,s}(M)$ is called the square-integrable basic cohomology space, where \ominus denotes the orthogonal complement of $B_{\tilde{d}}^{0,s}(M)$.

LEMMA 4.1 (CF. [4]). $\tilde{H}_2^{0,s}(M) = Z_{\tilde{d}}^{0,s}(M) \cap Z_{\tilde{\theta}}^{0,s}(M)$.

Since $Z_{\tilde{d}}^{0,s}(M)$ and $Z_{\tilde{\theta}}^{0,s}(M)$ are closed in $\tilde{L}_2^{0,s}(M)$, $\tilde{H}_2^{0,s}(M)$ has canonically the structure of a Hilbert space.

The following orthogonal decomposition theorem is proved analogously to L. Hörmander [3]. In fact, we have to notice that $B_{\tilde{d}}^{0,s}(M)$ and $B_{\tilde{\theta}}^{0,s}(M)$ are mutually orthogonal and that the intersection of the orthogonal complements of $B_{\tilde{d}}^{0,s}(M)$ and $B_{\tilde{\theta}}^{0,s}(M)$ is $\tilde{H}_2^{0,s}(M)$.

THEOREM 4.2 (CF. [4]).

$$\tilde{L}_2^{0,s}(M) = \tilde{H}_2^{0,s}(M) \oplus B_{\tilde{d}}^{0,s}(M) \oplus B_{\tilde{\theta}}^{0,s}(M).$$

The following diagram is commutative:

$$\begin{array}{ccc} \Delta_0^{0,s}(M) & \xrightarrow{\quad *'' \quad} & \Delta_0^{0,n-s}(M) \\ \tilde{\theta}_0 \downarrow \uparrow \tilde{d}_0 & & \tilde{d}_0 \downarrow \uparrow \tilde{\theta}_0 \\ \Delta_0^{0,s-1}(M) & \xrightarrow{\quad (-1)^s *'' \quad} & \Delta_0^{0,n-s+1}(M) \end{array}$$

Then we have the Dolbeault-Serre type theorem.

THEOREM 4.3 (CF. [4]). *If the bundle-like metric on M is complete, $\tilde{H}_2^{0,s}(M) = \tilde{H}_2^{0,n-s}(M)$ (isomorphic as Hilbert spaces).*

COROLLARY 4.4 (CF. [4]). *If the bundle-like metric on M is complete and $\dim \tilde{H}_2^{0,s}(M)$ is finite, then $\dim \tilde{H}_2^{0,s}(M) = \dim \tilde{H}_2^{0,n-s}(M)$.*

5. $\tilde{\square}$ -harmonic forms. In this section, we assume that the bundle-like metric on M is complete.

We consider a function μ on \mathbf{R} (the reals) satisfying

- (i) $0 < \mu < 1$ on \mathbf{R} ,
- (ii) $\mu(t) = 1$ for $t < 1$,
- (iii) $\mu(t) = 0$ for $t > 2$.

It is known that a geodesic orthogonal to a leaf is orthogonal to other leaves (cf. [8]). Let o be a point in M , and we fix the point o . For each point p in M , we denote by $\rho(p)$ the distance between leaves through o and p . Then we put $w_k(p) = \mu(\rho(p)/k)$, $k = 1, 2, 3, \dots$. We remark that $d'w_k = 0$ and $w_k\phi$ has compact support for each $\phi \in \Delta^{0,s}(M)$. We have that $w_k\phi \in D_3^{0,s} \cap D_\theta^{0,s}$ for any $\phi \in \Delta^{0,s}(M)$ and

$$\tilde{\delta}(w_k\phi) = d''(w_k\phi), \quad \tilde{\theta}(w_k\phi) = \tilde{\delta}''(w_k\phi). \tag{3}$$

LEMMA 5.1 (CF. [4]). *Under the above notations, there exists a positive number A , depending only on μ , such that*

$$\|d''w_k \wedge \phi\|^2 < (nA^2/k^2)\|\phi\|^2$$

and

$$\|d''w_k \wedge *''\phi\|^2 < (nA^2/k^2)\|\phi\|^2$$

for all $\phi \in \Delta^{0,s}(M)$, where $\|\phi\|^2 = \langle \phi, \phi \rangle_1$.

In order to prove this lemma, we have to notice that the function ρ is a locally Lipschitz function and, at points where the derivatives exist, it holds $\sum g^{ij}v_i(\rho)v_j(\rho) < n$. Then we have

$$|d''w_k|^2 = \sum g^{ij}v_i(w_k)v_j(w_k) < nA^2/k^2,$$

where A is a positive number depending only on $\sup|d\mu/dt|$.

Put

$$N_{d''}^{0,s}(M) = \{\phi \in \Delta^{0,s}(M); d''\phi = 0\},$$

$$N_{\delta''}^{0,s}(M) = \{\phi \in \Delta^{0,s}(M); \delta''\phi = 0\}.$$

Then we have

PROPOSITION 5.2 (CF. [4]). *If the bundle-like metric on M is complete, then*

$$N_{d''}^{0,s}(M) \cap \tilde{L}_2^{0,s}(M) \subset Z_3^{0,s}(M),$$

$$N_{\delta''}^{0,s}(M) \cap \tilde{L}_2^{0,s}(M) \subset Z_\theta^{0,s}(M).$$

PROOF. Let ϕ be in $N_{d''}^{0,s}(M) \cap \tilde{L}_2^{0,s}(M)$. By (3), we have

$$\tilde{\delta}(w_k \phi) = d''(w_k \phi) = d''w_k \wedge \phi + w_k d''\phi = d''w_k \wedge \phi.$$

Hence, from Lemma 5.1, we have $\|\tilde{\delta}(w_k \phi)\|^2 < (nA^2/k^2)\|\phi\|^2$. Putting $\phi_k = w_k \phi$, we have

$$\tilde{\delta}\phi_k \xrightarrow[\text{(strong)}]{} 0 \quad (k \rightarrow \infty),$$

where " $\xrightarrow[\text{(strong)}]$ " means "converges strongly to". On the other hand, $\phi_k \xrightarrow[\text{(strong)}]{} \phi$ ($k \rightarrow \infty$). Since $\tilde{\delta}$ is a closed operator, ϕ is in $D_{\tilde{\delta}}^{0,s}$ and $\tilde{\delta}\phi = 0$. This proves $\phi \in Z_{\tilde{\delta}}^{0,s}(M)$. In the same way, we may prove the second part.

DEFINITION (CF. [4]). The Laplacian $\tilde{\square}$ acting on $\Delta^{0,*}(M)$ is defined by $\tilde{\square} = d''\tilde{\delta}'' + \tilde{\delta}''d''$.

Let $B(k)$ be an open tube of radius k of the leaf through the fixed point o in M and $\Delta_{B(k)}^{0,s}(M)$ the space of all forms of type $(0, s)$ with compact support contained in $B(k)$. For $\phi, \psi \in \Delta_{B(k)}^{0,s}(M)$, we put $\langle \phi, \psi \rangle_{B(k)} = \langle \phi, \psi \rangle_1$. For any $\phi \in \tilde{L}_2^{0,s}(M) \cap \Delta^{0,s}(M)$, we have

$$\langle d''\phi, d''\alpha \rangle_{B(k)} + \langle \tilde{\delta}''\phi, \tilde{\delta}''\alpha \rangle_{B(k)} = \langle \tilde{\square}\phi, \alpha \rangle_{B(k)} \quad (4)$$

for all $\alpha \in \Delta_{B(k)}^{0,s}(M)$. Putting $\alpha = w_k^2\phi$, we have

$$d''\alpha = w_k^2 d''\phi + 2w_k d''w_k \wedge \phi,$$

$$\tilde{\delta}''\alpha = w_k^2 \tilde{\delta}''\phi + (-1)^{ns+n+1} *''(2w_k d''w_k \wedge *''\phi).$$

Substituting in (4), we have

$$\begin{aligned} & \|w_k d''\phi\|_{B(k)}^2 + \|w_k \tilde{\delta}''\phi\|_{B(k)}^2 \\ & \leq |\langle \tilde{\square}\phi, w_k^2\phi \rangle_{B(k)}| + |\langle d''\phi, 2w_k d''w_k \wedge \phi \rangle_{B(k)}| \\ & \quad + |\langle \tilde{\delta}''\phi, *''(2w_k d''w_k \wedge *''\phi) \rangle_{B(k)}|. \end{aligned} \quad (5)$$

On the other hand, the Schwarz inequality gives the following

$$|\langle d''\phi, 2w_k d''w_k \wedge \phi \rangle_{B(k)}| \leq \frac{1}{2} (\|w_k d''\phi\|_{B(k)}^2 + 4\|d''w_k \wedge \phi\|_{B(k)}^2),$$

$$|\langle \tilde{\delta}''\phi, *''(2w_k d''w_k \wedge *''\phi) \rangle_{B(k)}| \leq \frac{1}{2} (\|w_k \tilde{\delta}''\phi\|_{B(k)}^2 + 4\|d''w_k \wedge *''\phi\|_{B(k)}^2)$$

and

$$|\langle \tilde{\square}\phi, w_k^2\phi \rangle_{B(k)}| \leq \frac{1}{2} \left(\frac{1}{\sigma} \|w_k \phi\|_{B(k)}^2 + \sigma \|\tilde{\square}\phi\|_{B(k)}^2 \right)$$

for every $\sigma > 0$.

Substituting in (5),

$$\|w_k d''\phi\|_{B(k)}^2 + \|w_k \tilde{\delta}''\phi\|_{B(k)}^2 < \sigma \|\tilde{\square}\phi\|_{B(k)}^2 + \left(\frac{1}{\sigma} + \frac{8nA^2}{k^2} \right) \|\phi\|_{B(k)}^2.$$

Letting $k \rightarrow \infty$, we have

$$\|d''\phi\|^2 + \|\tilde{\delta}''\phi\|^2 < \sigma \|\tilde{\square}\phi\|^2 + \frac{1}{\sigma} \|\phi\|^2$$

for every $\sigma > 0$. In particular, setting $\tilde{\square}\phi = 0$ and letting $\sigma \rightarrow \infty$, we have

LEMMA 5.3 (CF. [4]). *Let the bundle-like metric on M be complete. If $\phi \in \tilde{L}_2^{0,s}(M) \cap \Delta^{0,s}(M)$ such that $\tilde{\square}\phi = 0$, then $d''\phi = 0$ and $\tilde{\delta}''\phi = 0$, i.e. $\phi \in N_d^{0,s}(M) \cap N_{\tilde{\delta}''}^{0,s}(M)$.*

From Proposition 5.2 and Lemma 5.3, we have the following theorem.

THEOREM 5.4 (CF. [4]). *Let the bundle-like metric on M be complete. If $\phi \in \tilde{L}_2^{0,s}(M) \cap \Delta^{0,s}(M)$ such that $\tilde{\square}\phi = 0$, then $\phi \in \tilde{H}_2^{0,s}(M)$.*

6. Square-integrable cohomology spaces $H_2^{0,s}(M)$ and $H_2^{1,s}(M)$. In this section, we set situations under the assumptions

$$A_i = A_i(y) \quad \text{and} \quad g_{\Delta\Delta}(x, y) = 1 \tag{IV}$$

in (1) and (2). The manifold given in the example in §2 satisfies (IV).

We notice that the volume element of M is

$$\begin{aligned} dV_M &= \sqrt{\det \begin{pmatrix} g_{\Delta\Delta} & 0 \\ 0 & g_{ij} \end{pmatrix}} \eta \wedge dy^1 \wedge \dots \wedge dy^n \\ &= \sqrt{\det(g_{ij})} \eta \wedge dy^1 \wedge \dots \wedge dy^n \quad (\text{from (IV)}) \\ &= \sqrt{\det(g_{ij})} dx \wedge dy^1 \wedge \dots \wedge dy^n \quad (\text{from (1)}). \end{aligned}$$

The $*$ -operation on $\Delta^{1,s}(M)$ or $\Delta^{0,s}(M)$ is defined as follows. For $\phi \in \Delta^{1,s}(M)$ and $\psi \in \Delta^{0,s}(M)$,

$$\begin{aligned} * \phi &= \frac{(-1)^{(1-1)s}}{(1-1)!(n-s)!1!s!} \sum g^{\Delta\Delta} g^{i_1 j_1} \dots g^{i_s j_s} \\ &\quad \times \delta_{\Delta j_1 \dots j_s k_1 \dots k_{n-s}}^{\Delta 1 \dots n} \sqrt{\det \begin{pmatrix} g_{\Delta\Delta} & 0 \\ 0 & g_{ij} \end{pmatrix}} \phi_{\Delta i_1 \dots i_s} dy^{k_1} \wedge \dots \wedge dy^{k_{n-s}} \\ &= \frac{1}{(n-s)!s!} \sum g^{i_1 j_1} \dots g^{i_s j_s} \delta_{j_1 \dots j_s k_1 \dots k_{n-s}}^{1 \dots n} \\ &\quad \times \sqrt{\det(g_{ij})} \phi_{\Delta i_1 \dots i_s} dy^{k_1} \wedge \dots \wedge dy^{k_{n-s}}, \\ * \psi &= \frac{(-1)^{(1-0)s}}{(1-0)!(n-s)!0!s!} \sum g^{i_1 j_1} \dots g^{i_s j_s} \delta_{j_1 \dots j_s k_1 \dots k_{n-s}}^{1 \dots n} \\ &\quad \times \sqrt{\det \begin{pmatrix} g_{\Delta\Delta} & 0 \\ 0 & g_{ij} \end{pmatrix}} \psi_{i_1 \dots i_s} \eta \wedge dy^{k_1} \wedge \dots \wedge dy^{k_{n-s}} \\ &= \frac{(-1)^s}{(n-s)!s!} \sum g^{i_1 j_1} \dots g^{i_s j_s} \delta_{j_1 \dots j_s k_1 \dots k_{n-s}}^{1 \dots n} \\ &\quad \times \sqrt{\det(g_{ij})} \psi_{i_1 \dots i_s} \eta \wedge dy^{k_1} \wedge \dots \wedge dy^{k_{n-s}} \end{aligned}$$

(cf. [2], [12]). The operator $*$ maps $\Delta^{1,s}(M)$ (resp. $\Delta^{0,s}(M)$) into $\Delta^{0,n-s}(M)$ (resp. $\Delta^{1,n-s}(M)$), since (IV) holds.

We define a pre-Hilbert metric on $\Delta_0^{1,s}(M)$ (or $\Delta_0^{0,s}(M)$) by $\langle \phi, \psi \rangle = \int_M \phi \wedge * \psi$. The differential operator d'' maps $\Delta^{1,s}(M)$ (resp. $\Delta^{0,s}(M)$) into $\Delta^{1,s+1}(M)$ (resp. $\Delta^{0,s+1}(M)$). We define $\delta'' : \Delta^{1,s}(M) \rightarrow \Delta^{1,s-1}(M)$ (or $\Delta^{0,s}(M) \rightarrow \Delta^{0,s-1}(M)$) by

$$\begin{aligned} \delta'' \phi &= (-1)^{(n+1)(s+1)+(n+1)+1} * d'' * \phi \\ &= (-1)^{ns+s+1} * d'' * \phi, \\ \delta'' \psi &= (-1)^{(n+1)s+(n+1)+1} * d'' * \psi \\ &= (-1)^{ns+s+n} * d'' * \psi \end{aligned}$$

for $\phi \in \Delta^{1,s}(M)$ and $\psi \in \Delta^{0,s}(M)$. Then we have $\langle d'' \phi, \psi \rangle = \langle \phi, \delta'' \psi \rangle$ for $\phi \in \Delta_0^{1,s}(M)$ (resp. $\Delta_0^{0,s}(M)$) and $\psi \in \Delta_0^{1,s+1}(M)$ (resp. $\Delta_0^{0,s+1}(M)$).

Let $L_2^{1,s}(M)$ (resp. $L_2^{0,s}(M)$) be the completion of $\Delta_0^{1,s}(M)$ (resp. $\Delta_0^{0,s}(M)$) with respect to the inner product $\langle \cdot, \cdot \rangle$. We denote by ∂_0 the restriction of d'' to $\Delta_0^{1,s}(M)$ (or $\Delta_0^{0,s}(M)$) and by θ_0 the restriction of δ'' to $\Delta_0^{1,s}(M)$ (or $\Delta_0^{0,s}(M)$). Define $\partial = (\theta_0)^*$ and $\theta = (\partial)^*$ where $(\cdot)^*$ denotes the adjoint operator of (\cdot) with respect to the inner product $\langle \cdot, \cdot \rangle$. Then ∂ is a closed, densely defined operator of $L_2^{1,s}(M)$ (resp. $L_2^{0,s}(M)$) into $L_2^{1,s+1}(M)$ (resp. $L_2^{0,s+1}(M)$), and θ is a closed, densely defined operator of $L_2^{1,s}(M)$ (resp. $L_2^{0,s}(M)$) into $L_2^{1,s-1}(M)$ (resp. $L_2^{0,s-1}(M)$).

The following objects are defined by the same ways as in §4:

$$\begin{array}{cccc} D_{\partial}^{1,s}, & D_{\partial}^{0,s}, & D_{\theta}^{1,s}, & D_{\theta}^{0,s}, \\ Z_{\partial}^{1,s}(M), & Z_{\partial}^{0,s}(M), & Z_{\theta}^{1,s}(M), & Z_{\theta}^{0,s}(M), \\ B_{\partial}^{1,s}(M), & B_{\partial}^{0,s}(M), & B_{\theta}^{1,s}(M), & B_{\theta}^{0,s}(M). \end{array}$$

Then

DEFINITION. $H_2^{1,s}(M) = Z_{\partial}^{1,s}(M) \ominus B_{\partial}^{1,s}(M)$ and $H_2^{0,s}(M) = Z_{\partial}^{0,s}(M) \ominus B_{\partial}^{0,s}(M)$.

By the same ways as in §4, we have

LEMMA 6.1. Under the assumption (IV),

$$H_2^{1,s}(M) = Z_{\partial}^{1,s}(M) \cap Z_{\theta}^{1,s}(M)$$

and

$$H_2^{0,s}(M) = Z_{\partial}^{0,s}(M) \cap Z_{\theta}^{0,s}(M).$$

THEOREM 6.2. Under the assumption (IV),

$$L_2^{1,s}(M) = H_2^{1,s}(M) \oplus B_{\partial}^{1,s}(M) \oplus B_{\theta}^{1,s}(M)$$

and

$$L_2^{0,s}(M) = H_2^{0,s}(M) \oplus B_{\partial}^{0,s}(M) \oplus B_{\theta}^{0,s}(M).$$

THEOREM 6.3. *Under the assumption (IV), if the bundle-like metric on M is complete, then $H_2^{0,s}(M) = H_2^{1,n-s}(M)$ (isomorphic as Hilbert spaces).*

In order to prove Theorem 6.3, we have to notice that $\langle \phi, \psi \rangle = \langle * \phi, * \psi \rangle$ for $\phi, \psi \in \Delta_0^{0,s}(M)$.

COROLLARY 6.4. *Under the assumption (IV), if the bundle-like metric on M is complete and $\dim H_2^{0,s}(M)$ is finite, then $\dim H_2^{0,s}(M) = \dim H_2^{1,n-s}(M)$.*

Now, we have $\langle \phi, \psi \rangle = \langle \eta \wedge \phi, \eta \wedge \psi \rangle$ for $\phi, \psi \in \Delta_0^{0,s}(M)$. Let ξ denote the dual to η and i_ξ the interior product by ξ operator. Then we have $i_\xi \phi \in \Delta_0^{0,s}(M)$ and $\eta \wedge i_\xi \phi = \phi$ for $\phi \in \Delta^{1,s}(M)$. The following diagram is commutative.

$$\begin{array}{ccc} \Delta_0^{0,s}(M) & \xrightarrow{e(\eta)} & \Delta_0^{1,s}(M) \\ \theta_0 \downarrow \uparrow \tilde{\theta}_0 & & \theta_0 \uparrow \downarrow \tilde{\theta}_0 \\ \Delta_0^{0,s-1}(M) & \xrightarrow{(-1)e(\eta)} & \Delta_0^{1,s-1}(M) \end{array}$$

where $e(\eta)$ denotes the exterior product by η operator. Thus we have

THEOREM 6.5. *Under the assumption (IV), if the bundle-like metric on M is complete, then $H_2^{0,s}(M) = H_2^{1,s}(M)$ (isomorphic as Hilbert spaces).*

From Theorems 6.3 and 6.5, we have

THEOREM 6.6. *Under the assumption (IV), if the bundle-like metric on M is complete, then $H_2^{1,s}(M) = H_2^{1,n-s}(M)$ and $H_2^{0,s}(M) = H_2^{0,n-s}(M)$ (isomorphic as Hilbert spaces).*

Next, from (IV), it is easy to see that $\langle \phi, \psi \rangle_1 = \langle \phi, \psi \rangle$ for $\phi, \psi \in \Delta_0^{0,s}(M)$. The following diagram is commutative.

$$\begin{array}{ccc} \Delta_0^{0,s}(M) & \xrightarrow{I} & \Delta_0^{0,s}(M) \\ \tilde{\theta}_0 \downarrow \uparrow \tilde{\theta}_0 & & \theta_0 \uparrow \downarrow \theta_0 \\ \Delta_0^{0,s-1}(M) & \xrightarrow{I} & \Delta_0^{0,s-1}(M) \end{array}$$

where I denotes the identity map. Thus we have the following theorem.

THEOREM 6.7. *Under the assumption (IV), if the bundle-like metric on M is complete, then $\tilde{H}_2^{0,s}(M) = H_2^{0,s}(M)$ (isomorphic as Hilbert spaces).*

7. □-harmonic forms. In this section, we assume that the assumption (IV) holds and that the bundle-like metric on M is complete. We put

$$\begin{aligned} N_d^{1,s}(M) &= \{ \phi \in \Delta^{1,s}(M); d'' \phi = 0 \}, \\ N_\delta^{1,s}(M) &= \{ \phi \in \Delta^{1,s}(M); \delta'' \phi = 0 \}, \\ N_d^{0,s}(M) &= \{ \phi \in \Delta^{0,s}(M); d'' \phi = 0 \}, \\ N_\delta^{0,s}(M) &= \{ \phi \in \Delta^{0,s}(M); \delta'' \phi = 0 \}. \end{aligned}$$

Then, by the same ways as in §5, we have

PROPOSITION 7.1. *Let the assumption (IV) hold and the bundle-like metric on M be complete. Then*

$$\begin{aligned} N_{d''}^{1,s}(M) \cap L_2^{1,s}(M) &\subset Z_\theta^{1,s}(M), & N_{\delta''}^{1,s}(M) \cap L_2^{1,s}(M) &\subset Z_\theta^{1,s}(M), \\ N_{d''}^{0,s}(M) \cap L_2^{0,s}(M) &\subset Z_\theta^{0,s}(M), & N_{\delta''}^{0,s}(M) \cap L_2^{0,s}(M) &\subset Z_\theta^{0,s}(M). \end{aligned}$$

DEFINITION. The Laplacian acting on $\Delta^{1,*}(M)$ (or $\Delta^{0,*}(M)$) is defined by $\square = d''\delta'' + \delta''d''$.

By the same ways as in §5, we have

LEMMA 7.2. *Let the assumption (IV) hold and the bundle-like metric on M be complete. If $\phi \in L_2^{1,s}(M) \cap \Delta^{1,s}(M)$ (resp. $L_2^{0,s}(M) \cap \Delta^{0,s}(M)$) such that $\square\phi = 0$, then $d''\phi = 0$ and $\delta''\phi = 0$, i.e. $\phi \in N_{d''}^{0,s}(M) \cap N_{\delta''}^{1,s}(M)$ (resp. $N_{d''}^{0,s}(M) \cap N_{\delta''}^{0,s}(M)$).*

From Proposition 7.1 and Lemma 7.2, we have

THEOREM 7.3. *Let the assumption (IV) hold and the bundle-like metric on M be complete. If $\phi \in L_2^{1,s}(M) \cap \Delta^{1,s}(M)$ (resp. $L_2^{0,s}(M) \cap \Delta^{0,s}(M)$) such that $\square\phi = 0$, then $\phi \in H_2^{1,s}(M)$ (resp. $H_2^{0,s}(M)$).*

From Theorems 5.4 and 6.7, we have

THEOREM 7.4. *Let the assumption (IV) hold and the bundle-like metric on M be complete. If $\phi \in \tilde{L}_2^{0,s}(M) \cap \Delta^{0,s}(M)$ such that $\tilde{\square}\phi = 0$, then $\phi \in H_2^{0,s}(M)$.*

From Theorems 6.7 and 7.3, we have

THEOREM 7.5. *Let the assumption (IV) hold and the bundle-like metric on M be complete. If $\phi \in L_2^{0,s}(M) \cap \Delta^{0,s}(M)$ such that $\square\phi = 0$, then $\phi \in \tilde{H}_2^{0,s}(M)$.*

REMARK. I. Vaisman [12], [13] already noticed that on a compact orientable Riemannian foliated manifold M , the space $\mathcal{H}^{r,s}(M)$ of foliated harmonic forms is a subspace of the de Rham cohomology space $H^{r,s}(M)$.

REMARK. For the relations between certain cohomology spaces and the existence of bundle-like metrics, see H. Kitahara and S. Yorozu [5].

8. Applications to a contact manifold. First, we cite the definition of the contact manifold. A 1-form η on a connected $(2n + 1)$ -dimensional manifold is called a contact form if $\eta \wedge (d\eta)^n \neq 0$ at each point in the manifold (cf. [10]). A connected $(2n + 1)$ -dimensional manifold with a contact form is called a contact manifold. On a contact manifold with a contact form η , there exists a global vector field ξ such that $\eta(\xi) = 1$ and $i_\xi d\eta = 0$ (cf. [10]). A connected paracompact contact manifold with a contact form η has a Riemannian metric $(,)$ such that

$$\eta(X) = (X, \xi) \tag{6}$$

for any vector field X (cf. [10]). In fact, let $(,)$ be an arbitrary Riemannian metric, and we define

$$(X, Y) = (X - \eta(X)\xi, Y - \eta(Y)\xi)' + \eta(X) \cdot \eta(Y)$$

for any vector fields X and Y . Such a metric $(,)$ satisfies (6).

Now, let N be a connected $(2n + 1)$ -dimensional contact manifold with a contact form η and a Riemannian metric $(,)$ satisfying (6). We assume that ξ is a Killing vector field on N with respect to the metric $(,)$ and that the orbits of ξ are compact. An example of such a manifold N is the manifold given in the example in §2.

We define the operators $\delta, e(\eta), i_\xi, L$ and Λ on $\wedge^s(N)$ as follows:

$$\begin{aligned} \delta\phi &= (-1)^s * d * \phi, & e(\eta)\phi &= \eta \wedge \phi, \\ i_\xi\phi &= (-1)^{s-1} * e(\eta) * \phi, \\ L\phi &= d\eta \wedge \phi, & \Lambda\phi &= * L * \phi \end{aligned}$$

(cf. [2], [6], [11]).

DEFINITION. A form ϕ in $\wedge^s(N)$ is called a C -harmonic form (resp. C^* -harmonic form) if $i_\xi\phi = 0, d\phi = 0$ and $\delta\phi = e(\eta)\Lambda\phi$ (resp. $e(\eta)\phi = 0, d\phi = i_\xi L\phi$ and $\delta\phi = 0$).

REMARK. The notion of C -harmonic forms was introduced by S. Tachibana [11], and Y. Ogawa [6] gave the definition of C^* -harmonic forms. They discussed it on compact normal contact metric manifolds. A normal contact metric manifold is a so-called Sasakian manifold (for the definition, see [6], [10], [11]).

For each point in N , there exists a local coordinate neighborhood system $\{U; (x, y^1, \dots, y^n, y^{n+1}, \dots, y^{2n})\}$ such that

$$\eta = dx + \sum (-y^{n+i}) dy^i \quad (i = 1, 2, \dots, n)$$

and the orbits of ξ are given locally by

$$y^1 = c^1, \dots, y^n = c^n, y^{n+1} = c^{n+1}, \dots, y^{2n} = c^{2n}$$

for the same constants $c^1, \dots, c^n, c^{n+1}, \dots, c^{2n}$ (cf. [10]). $\{\eta, dy^1, \dots, dy^n, dy^{n+1}, \dots, dy^{2n}\}$ and $\{\partial/\partial x, v_1, \dots, v_n, v_{n+1}, \dots, v_{2n}\}$ are dual bases for the cotangent and tangent spaces respectively at each point in U , where

$$v_i = \partial/\partial y^i + (y^{n+i})\partial/\partial x \quad \text{and} \quad v_{n+i} = \partial/\partial y^{n+i}.$$

We may consider N as a foliated manifold whose leaves are orbits of ξ . From (6), we have

$$\begin{aligned} (v_i, \xi) &= \eta(v_i) = 0, \\ (v_{n+i}, \xi) &= \eta(v_{n+i}) = 0. \end{aligned}$$

Then, since ξ is a Killing vector field on N , the metric $(,)$ on N is a bundle-like metric with respect to the foliation, that is, the local expression of the metric $(,)$ in U is

$$ds^2 = \eta \cdot \eta + \sum g_{AB} dy^A \cdot dy^B$$

where $A, B = 1, 2, \dots, 2n$. Thus the contact manifold N is a Riemannian foliated manifold with one-dimensional foliation \mathcal{F} whose leaves are compact and the Riemannian metric $(,)$ on N is a bundle-like metric with respect to \mathcal{F} . Moreover, the assumption (IV) in §6 is satisfied. Therefore, we may apply the discussions of above sections to the contact manifold N .

In order to obtain the applications to N , we have to prepare the decomposition of the operator δ . We have the decomposition of the operator d :

$$d = d' + d'' + d'''$$

(cf. §2). Then, according to I. Vaisman [12], we define the operators δ', δ'' and δ''' as follows:

$$\begin{aligned} \delta' \phi &= (-1)^{r+s} * d' * \phi, \\ \delta'' \phi &= (-1)^{r+s} * d'' * \phi, \\ \delta''' \phi &= (-1)^{r+s} * d''' * \phi, \end{aligned}$$

where $\phi \in \wedge^{r,s}(N)$, $r = 0$ or 1 . Then we have the decomposition of the operator δ :

$$\delta = \delta' + \delta'' + \delta'''.$$

We notice the following: (i) If $\phi \in \wedge^{1,s}(N)$, then $\delta\phi = \delta'\phi + \delta''\phi$, where $\delta'\phi \in \wedge^{0,s}(N)$ and $\delta''\phi \in \wedge^{1,s-1}(N)$. (ii) If $\phi \in \wedge^{0,s}(N)$, then $\delta\phi = \delta''\phi + \delta'''\phi$, where $\delta''\phi \in \wedge^{0,s-1}(N)$ and $\delta'''\phi \in \wedge^{1,s-2}(N)$.

We have easily the following lemma.

LEMMA 8.1. For $\phi \in \Delta^{1,s}(N)$ and $\psi \in \Delta^{0,s}(N)$,

$$d''' \phi = i_\xi L\phi, \quad \delta''' \psi = e(\eta)\Delta\psi.$$

From Theorem 7.4 and Lemma 8.1, we have

THEOREM 8.2. Let the metric $(,)$ on N be complete. If $\phi \in \tilde{L}_2^{0,s}(N) \cap \Delta^{0,s}(N)$ such that $\tilde{\square}\phi = 0$, then ϕ is a C-harmonic form.

From Theorem 7.3 and Lemma 8.1, we have

THEOREM 8.3. Let the metric $(,)$ on N be complete. If $\phi \in L_2^{1,s}(N) \cap \Delta^{1,s}(N)$ such that $\square\phi = 0$, then ϕ is a C*-harmonic form.

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