NOTES ON SQUARE-INTEGRABLE COHOMOLOGY SPACES ON CERTAIN FOLIATED MANIFOLDS

BY

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Abstract. We discuss some square-integrable cohomology spaces on a foliated manifold with one-dimensional foliation whose leaves are compact and with a complete bundle-like metric. Applications to a contact manifold are given.

1. Introduction. On a compact foliated manifold with a bundle-like metric, B. L. Reinhart [9] proved that the cohomology of base-like differential forms is isomorphic to the harmonic space of a certain semidefinite Laplacian. In his paper [4], H. Kitahara discussed the square-integrable basic cohomology spaces on a foliated manifold with a complete bundle-like metric.

In this note, we discuss some square-integrable cohomology spaces on a foliated manifold with one-dimensional foliation whose leaves are compact and with a complete bundle-like metric. Moreover, we give applications to a contact manifold.

The author wishes to express his gratitude to Professor H. Kitahara for several useful suggestions.

2. Preliminaries. We assume that all objects and maps are of class $C^\infty$. Let $M$ be a connected, orientable, $(n + 1)$-dimensional Riemannian foliated manifold with one-dimensional foliation $\mathcal{F}$ whose leaves are compact. We assume that the Riemannian metric $(\cdot, \cdot)$ on $M$ is a bundle-like metric with respect to $\mathcal{F}$ (cf. [8]).

Let $\{U; (x, y^1, \ldots, y^n)\}$ denote a flat coordinate neighborhood system with respect to $\mathcal{F}$, that is, the integral manifolds of $\mathcal{F}$ are given locally by $y^1 = c^1, \ldots, y^n = c^n$, for some constants $c^1, \ldots, c^n$ (cf. [8]). We may choose, in each flat coordinate neighborhood system $\{U; (x, y^1, \ldots, y^n)\}$, 1-form $\eta$ and vectors $v_1, \ldots, v_n$ such that $\{\eta, \partial y^1, \ldots, \partial y^n\}$ and $\{\partial / \partial x, v_1, \ldots, v_n\}$ are dual bases for the cotangent and tangent spaces respectively at each point in $U$. Hence
\[ \eta = dx + \sum A_i dy^i, \]
\[ \nu_i = \partial/\partial y^i - A_i \partial/\partial x, \quad i = 1, 2, \ldots, n \]
(cf. [8], [12]). Throughout this note, all local expressions for forms and vectors are taken with respect to these bases.

We may choose, in \((1)\), \(A_i = A_i(x, y)\) such that \((\partial/\partial x, \nu_i) = 0, \quad i = 1, 2, \ldots, n\), where \(y = (y^1, \ldots, y^n)\). Then the metric has the local expression

\[ ds^2 = g_{\Delta \Delta}(x, y) \cdot \eta \cdot \eta + \sum g_{ij}(y) \cdot dy^i \cdot dy^j, \]
where \(g_{\Delta \Delta}(x, y) = (\partial/\partial x, \partial/\partial x)\) and \(g_{ij}(y) = (\nu_i, \nu_j)\), since the metric \((, )\) is a bundle-like metric with respect to \(\mathcal{F}\) (cf. [8], [12]).

**Definition** (cf. [8], [12]). A form \(\phi\) on \(M\) is called of type \((1, s)\) (resp. \((0, s)\)) if it is expressed locally as

\[ \phi = \frac{1}{1!s!} \sum \phi_{\Delta 1 \ldots s} (x, y) \eta \wedge dy^1 \wedge \cdots \wedge dy^s. \]

(resp. \(\phi = (1/0!s!) \sum \phi_{\Delta 1 \ldots s} (x, y) dy^1 \wedge \cdots \wedge dy^s\)).

Let \(\wedge^{1,s}(M)\) (resp. \(\wedge^{0,s}(M)\)) be the space of all forms on \(M\) of type \((1, s)\) (resp. \((0, s)\)). The space \(\wedge^s(M)\) of all \(s\)-forms on \(M\) is the direct sum of \(\wedge^{1,s-1}(M)\) and \(\wedge^{0,s}(M)\).

The operator \(d\) of exterior differentiation is decomposed as \(d = d' + d'' + d'''\) where the components are of type \((1, 0)\), \((0, 1)\) and \((-1, 2)\) respectively (cf. [9], [12]). Since the foliation \(\mathcal{F}\) is of one dimension, we notice the following: (i) If \(\phi \in \wedge^{1,s}(M)\), then \(d\phi = d''\phi + d'''\phi\), where \(d''\phi \in \wedge^{1,s+1}(M)\) and \(d'''\phi \in \wedge^{0,s+2}(M)\). (ii) If \(\phi \in \wedge^{0,s}(M)\), then \(d\phi = d'\phi + d''\phi\), where \(d'\phi \in \wedge^{1,s}(M)\) and \(d''\phi \in \wedge^{0,s+1}(M)\).

Among examples we show the following one for reference below:

**Example.** Let \(R^3\) be a Euclidean space with cartesian coordinates \((x, y^1, y^2)\). We put \(\eta = dx + (-y^2)dy^1\), then \(\{\eta, dy^1, dy^2\}\) is a base for the cotangent space at each point in \(R^3\). Let \(\xi\) be a dual of \(\eta\). Then \(R^3\) is considered a foliated manifold whose leaves are orbits of vector field \(\xi\). We define a metric \(ds^2\) on \(R^3\) by

\[ ds^2 = dx \cdot dx + 2(-y^2)dx \cdot dy^1 + (1 + (y^2)^2)dy^1 \cdot dy^1 + dy^2 \cdot dy^2 \]
(cf. [10]). Then we have \(ds^2 = \eta \cdot \eta + dy^1 \cdot dy^1 + dy^2 \cdot dy^2\). Thus the metric \(ds^2\) is a complete bundle-like metric with respect to the foliation.

Let \(\varphi_m : R^3 \rightarrow R^3\) be a map defined by

\[ \varphi_m(x, y^1, y^2) = (x + m, (-1)^m y^1, (-1)^m y^2), \]
where \(m\) is an integer. Define an equivalence relation in \(R^3\) by \((x, y^1, y^2) \sim (\tilde{x}, \tilde{y}^1, \tilde{y}^2)\) if there exists an integer \(m\) such that \(\varphi_m(x, y^1, y^2) = (\tilde{x}, \tilde{y}^1, \tilde{y}^2)\). It is trivial that \(\eta\) and \(ds^2\) are invariant by \(\varphi_m\) for any \(m\). Thus the induced \(\eta\) on \(M = R^3/\sim\) define a foliation on \(M\) whose leaves are of one dimension and
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compact, and the induced $ds^2$ on $M$ is a complete bundle-like metric with respect to the foliation. We notice that the foliation is not regular (cf. [8]).

3. Spaces $\Delta^{1,*}(M)$ and $\Delta^{0,*}(M)$. Hereafter, we are interested in forms such that

$$\phi = \frac{1}{1! s!} \sum \phi_{i_1, \ldots, i_s} (y) \eta \wedge dy^{i_1} \wedge \cdots \wedge dy^{i_s} \quad (I)$$

and

$$\phi = \frac{1}{0! s!} \sum \phi_{i_1, \ldots, i_s} (y) dy^{i_1} \wedge \cdots \wedge dy^{i_s} \quad (II)$$

Let $\Delta^{1,2}(M)$ (resp. $\Delta^{0,2}(M)$) be the subspace of $\wedge^{1,2}(M)$ (resp. $\wedge^{0,2}(M)$) satisfying (I) (resp. (II)) and $\Delta^{1,2}_B(M)$ (resp. $\Delta^{0,2}_B(M)$) the subspace of $\Delta^{1,2}(M)$ (resp. $\Delta^{0,2}(M)$) composed of forms with compact supports.

**Proposition 3.1** (cf. [9]). For $\phi \in \wedge^{0,2}(M)$, $d^* \phi = 0$ if and only if $\phi \in \Delta^{0,2}(M)$.

A form in $\Delta^{0,2}(M)$ is called a basic or base-like form (cf. [4], [9]).

Restricted to $\Delta^{0,2}_B(M) = \Sigma \Delta^{0,2}_B(M)$, $d^{*2} = d^2 = 0$ so we may consider the cohomology of $\Delta^{0,2}_B(M)$ and $d^*$ (cf. [9], [12]). Next, if we have an assumption that

$$A_i = A_i (y), \quad i = 1, 2, \ldots, n, \quad (III)$$

in (I), then we have $d^{*2} = d^2 = 0$ on $\Delta^{1,*}(M) = \Sigma \Delta^{1,2}(M)$ (cf. [12]). The example in §2 satisfies our assumption (III). Hereafter, we consider the spaces $\Delta^{1,2}(M)$ and $\Delta^{1,*}(M)$ under the assumption (III). Thus we may consider the cohomology of $\Delta^{1,2}(M)$ and $d^*$.

4. Square-integrable basic cohomology spaces $\bar{H}^{0,*}_2(M)$. This and the next sections are due to H. Kitahara [4], that is, they are the special cases of Kitahara's results. The methods are analogous to those of A. Andreotti and E. Vesentini [1] and K. Okamoto and H. Ozeki [7].

The $^{**}$-operation in $\Delta^{0,2}(M)$ is defined by

$$^{**} \phi = \frac{1}{(n-s)! s!} \sum g^{i_1 j_1} \cdots g^{i_s j_s} \delta_{j_1, \ldots, j_s} \phi_{i_1, \ldots, i_s} \wedge \cdots \wedge dy^{k_{s-s}},$$

where $(g^{ij})$ denotes the inverse matrix of $(g_{ij})$ and $\delta_{j_1, \ldots, j_s k_1, \ldots, k_{s-s}}$ the Kronecker symbol (cf. [2], [4], [9], [12]). According to B. L. Reinhart [9], we define a pre-Hilbert metric $\langle \cdot, \cdot \rangle_1$ on $\Delta^{0,2}_B(M)$ by

$$\langle \phi, \psi \rangle_1 = \int_M dx \wedge \phi \wedge ^{**} \psi.$$
The differential operator $d^n$ maps $\Delta_0^0(M)$ into $\Delta_0^{0,1}(M)$. We define $\tilde{\delta}''$:

$$\tilde{\delta}'' \phi = (-1)^{n+1} d^n \phi.$$

Let $\tilde{L}_2^{0,\sigma}(M)$ be the completion of $\Delta_0^0(M)$ with respect to the inner product $\langle \cdot, \cdot \rangle_1$. We denote by $\tilde{\delta}_0$ the restriction of $d^n$ to $\Delta_0^{0,1}(M)$ and by $\tilde{\sigma}_0$ the restriction of $\tilde{\delta}''$ to $\Delta_0^{0,1}(M)$. We define $\tilde{\theta} = (\tilde{\sigma}_0)^*$ and $\tilde{\tilde{\theta}} = (\tilde{\theta})^*$, where $^*$ denotes the adjoint operator of $()$ with respect to the inner product $\langle \cdot, \cdot \rangle_1$. Then $\tilde{\theta}$ (resp. $\tilde{\tilde{\theta}}$) is a closed, densely defined operator of $\tilde{L}_2^{0,\sigma}(M)$ into $\tilde{L}_2^{0,1}(M)$ (resp. $\tilde{L}_2^{0,1-1}(M)$). Let $D^0_\theta$ (resp. $D^0_{\tilde{\tilde{\theta}}}$) be the domain of the operator $\tilde{\theta}$ (resp. $\tilde{\tilde{\theta}}$) in $\tilde{L}_2^{0,\sigma}(M)$ and we put

$$Z^0_\theta(M) = \{ \phi \in D^0_\theta; \tilde{\theta} \phi = 0 \},$$

$$Z^0_{\tilde{\tilde{\theta}}}(M) = \{ \phi \in D^0_{\tilde{\tilde{\theta}}}; \tilde{\tilde{\theta}} \phi = 0 \}.$$

Since $\tilde{\theta}$ and $\tilde{\tilde{\theta}}$ are closed operators, $Z^0_\theta(M)$ and $Z^0_{\tilde{\tilde{\theta}}}(M)$ are closed in $\tilde{L}_2^{0,\sigma}(M)$. Let $B^0_\theta(M)$ (resp. $B^0_{\tilde{\tilde{\theta}}}(M)$) be the closure of $\tilde{\theta}(D^0_\theta)$ (resp. $\tilde{\tilde{\theta}}(D^0_{\tilde{\tilde{\theta}}})$).

**Definition (cf. [4]):** $\tilde{H}_2^{0,\sigma}(M) = Z^0_\theta(M) \oplus B^0_\theta(M)$ is called the square-integrable basic cohomology space, where $\oplus$ denotes the orthogonal complement of $B^0_\theta(M)$.

**Lemma 4.1 (cf. [4]):** $\tilde{H}_2^{0,\sigma}(M) = Z^0_{\tilde{\tilde{\theta}}}(M)$.

Since $Z^0_\theta(M)$ and $Z^0_{\tilde{\tilde{\theta}}}(M)$ are closed in $\tilde{L}_2^{0,\sigma}(M)$, $\tilde{H}_2^{0,\sigma}(M)$ has canonically the structure of a Hilbert space.

The following orthogonal decomposition theorem is proved analogously to L. Hörmander [3]. In fact, we have to notice that $B^0_{\tilde{\theta}}(M)$ and $B^0_{\tilde{\tilde{\theta}}}(M)$ are mutually orthogonal and that the intersection of the orthogonal complements of $B^0_\theta(M)$ and $B^0_{\tilde{\tilde{\theta}}}(M)$ is $\tilde{H}_2^{0,\sigma}(M)$.

**Theorem 4.2 (cf. [4]):**

$$\tilde{L}_2^{0,\sigma}(M) = \tilde{H}_2^{0,\sigma}(M) \oplus B^0_\theta(M) \oplus B^0_{\tilde{\tilde{\theta}}}(M).$$

The following diagram is commutative:

$$\begin{array}{ccc}
\Delta_0^{0,\sigma}(M) & \xrightarrow{\tilde{\delta}''} & \Delta_0^{0,\sigma-1}(M) \\
\downarrow \tilde{\delta}_0 & & \downarrow \tilde{\delta}_0 \\
\Delta_0^{0,\sigma-1}(M) & \xrightarrow{(-1)^n} & \Delta_0^{0,\sigma-1}(M)
\end{array}$$

Then we have the Dolbeault-Serre type theorem.

**Theorem 4.3 (cf. [4]):** If the bundle-like metric on $M$ is complete, $\tilde{H}_2^{0,\sigma}(M) = \tilde{H}_2^{0,\sigma-1}(M)$ (isomorphic as Hilbert spaces).
Corollary 4.4 (cf. [4]). If the bundle-like metric on $M$ is complete and $\dim \tilde{H}^0_\omega(M)$ is finite, then $\dim \tilde{H}^0_\omega(M) = \dim \tilde{H}^0_\omega(M)$.

5. $\tilde{\omega}$-harmonic forms. In this section, we assume that the bundle-like metric on $M$ is complete.

We consider a function $\mu$ on $\mathbb{R}$ (the reals) satisfying

(i) $0 < \mu < 1$ on $\mathbb{R}$,
(ii) $\mu(t) = 1$ for $t < 1$,
(iii) $\mu(t) = 0$ for $t > 2$.

It is known that a geodesic orthogonal to a leaf is orthogonal to other leaves (cf. [8]). Let $o$ be a point in $M$, and we fix the point $o$. For each point $p$ in $M$, we denote by $\rho(p)$ the distance between leaves through $o$ and $p$. Then we put $w_k(p) = \mu(\rho(p))/k$, $k = 1, 2, 3, \ldots$. We remark that $d'w_k = 0$ and $w_k$ has compact support for each $\phi \in \Delta_\omega(M)$. We have that $w_k \phi \in D^\omega_\phi \cap D^\omega_\phi$ for any $\phi \in \Delta^\omega(M)$ and

$$\tilde{\delta}(w_k \phi) = d''(w_k \phi), \quad \tilde{\theta}(w_k \phi) = \delta''(w_k \phi). \quad (3)$$

Lemma 5.1 (cf. [4]). Under the above notations, there exists a positive number $A$, depending only on $\mu$, such that

$$\|d''w_k \wedge \phi\|^2 \leq \left(\frac{nA^2}{k^2}\right)\|\phi\|^2$$

and

$$\|d''\wedge \phi\|^2 \leq \left(\frac{nA^2}{k^2}\right)\|\phi\|^2$$

for all $\phi \in \Delta^\omega(M)$, where $\|\phi\|^2 = \langle \phi, \phi \rangle_1$.

In order to prove this lemma, we have to notice that the function $\rho$ is a locally Lipschitz function and, at points where the derivatives exist, it holds $\Sigma g^\omega_{ij}(\rho) v_j(\rho) < n$. Then we have

$$\|d''w_k\|^2 = \Sigma g^\omega_{ij}(w_k)v_j(w_k) < nA^2/k^2,$$

where $A$ is a positive number depending only on $\sup|d\mu/dt|$.

Put

$$N^\omega_\mu(M) = \{\phi \in \Delta^\omega(M); d''\phi = 0\},$$
$$N^\omega_\delta(M) = \{\phi \in \Delta^\omega(M); \delta''\phi = 0\}.$$

Then we have

Proposition 5.2 (cf. [4]). If the bundle-like metric on $M$ is complete, then

$$N^\omega_\mu(M) \cap \tilde{\Omega}^\omega_\mu(M) \subset Z^\omega_\mu(M),$$
$$N^\omega_\delta(M) \cap \tilde{\Omega}^\omega_\delta(M) \subset Z^\omega_\delta(M).$$
Proof. Let \( \phi \) be in \( N^{0,s}_{-}(M) \cap \tilde{L}_{2}^{0,s}(M) \). By (3), we have
\[
\tilde{\delta}(w_{k}\phi) = d''(w_{k}\phi) = d''w_{k} \wedge \phi + w_{k}d''\phi = d''w_{k} \wedge \phi.
\]
Hence, from Lemma 5.1, we have \( \|\tilde{\delta}(w_{k}\phi)\|^2 \leq (nA^2/k^2)\|\phi\|^2 \). Putting \( \phi_k = w_{k}\phi \), we have
\[
\tilde{\delta}\phi_k \to 0 \quad (k \to \infty),
\]
where "\( \to_{(\text{strong})} \) means "converges strongly to". On the other hand, \( \phi_k \to_{(\text{strong})} \phi \) \((k \to \infty)\). Since \( \tilde{\delta} \) is a closed operator, \( \phi \) is in \( D^{0,s}_{\delta} \) and \( \tilde{\delta}\phi = 0 \). This proves \( \phi \in Z^{0,s}_{\delta}(M) \). In the same way, we may prove the second part.

Definition (cf. [4]). The Laplacian \( \Box \) acting on \( \Delta^{0,s}(M) \) is defined by
\[
\Box = d''\delta'' + \delta''d''.
\]

Let \( B(k) \) be an open tube of radius \( k \) of the leaf through the fixed point \( o \) in \( M \) and \( \Delta^{0,s}_{B(k)}(M) \) the space of all forms of type \((0, s)\) with compact support contained in \( B(k) \). For \( \phi, \psi \in \Delta^{0,s}_{B(k)}(M) \), we put \( \langle \phi, \psi \rangle_{B(k)} = \langle \phi, \psi \rangle_{1} \). For any \( \phi \in \tilde{L}_{2}^{0,s}(M) \cap \Delta^{0,s}(M) \), we have
\[
\langle d''\phi, d''\alpha \rangle_{B(k)} + \langle \delta''\phi, \delta''\alpha \rangle_{B(k)} = \langle \Box\phi, \alpha \rangle_{B(k)} \quad (4)
\]
for all \( \alpha \in \Delta^{0,s}_{B(k)}(M) \). Putting \( \alpha = w_{k}^{2}\phi \), we have
\[
d''\alpha = w_{k}^{2}d''\phi + 2w_{k}d''w_{k} \wedge \phi,
\]
\[\delta''\alpha = w_{k}^{2}\delta''\phi + (-1)^{n+s+1}*(2w_{k}d''w_{k} \wedge *\phi).\]
Substituting in (4), we have
\[
\|w_{k}d''\phi\|_{B(k)}^{2} + \|w_{k}\delta''\phi\|_{B(k)}^{2} < \|\langle \Box\phi, w_{k}^{2}\phi \rangle_{B(k)}\| + \|\langle d''\phi, 2w_{k}d''w_{k} \wedge \phi \rangle_{B(k)}\| + \|\langle \delta''\phi, *\star (2w_{k}d''w_{k} \wedge \star\phi) \rangle_{B(k)}\|. \quad (5)
\]
On the other hand, the Schwarz inequality gives the following
\[
\|\langle d''\phi, 2w_{k}d''w_{k} \wedge \phi \rangle_{B(k)}\| \leq \frac{1}{2}(\|w_{k}d''\phi\|_{B(k)}^{2} + 4\|d''w_{k} \wedge \phi\|_{B(k)}^{2}),
\]
\[
\|\langle \delta''\phi, *\star (2w_{k}d''w_{k} \wedge \star\phi) \rangle_{B(k)}\| \leq \frac{1}{2}(\|w_{k}\delta''\phi\|_{B(k)}^{2} + 4\|d''w_{k} \wedge \star\phi\|_{B(k)}^{2})
\]
and
\[
\|\langle \Box\phi, w_{k}^{2}\phi \rangle_{B(k)}\| \leq \frac{1}{\sigma}(\|w_{k}\phi\|_{B(k)}^{2} + \sigma\|\Box\phi\|_{B(k)}^{2})
\]
for every \( \sigma > 0 \).
Substituting in (5),
\[
\|w_{k}d''\phi\|_{B(k)}^{2} + \|w_{k}\delta''\phi\|_{B(k)}^{2} < \sigma\|\Box\phi\|_{B(k)}^{2} + \left(\frac{1}{\sigma} + \frac{8nA^{2}}{k^{2}}\right)\|\phi\|_{B(k)}^{2}.
\]
Letting \( k \to \infty \), we have
\[
\|d''\phi\|^{2} + \|\delta''\phi\|^{2} < \sigma\|\Box\phi\|^{2} + \frac{1}{\sigma}\|\phi\|^{2}
\]
for every \( \sigma > 0 \). In particular, setting \( \tilde{\phi} = 0 \) and letting \( \sigma \to \infty \), we have

**Lemma 5.3 (cf. [4])**. Let the bundle-like metric on \( M \) be complete. If \( \phi \in \tilde{L}_2^{0s}(M) \cap \Delta^{0s}(M) \) such that \( \tilde{\phi} = 0 \), then \( d''\phi = 0 \) and \( \delta''\phi = 0 \), i.e. \( \phi \in N_2^{0s}(M) \cap N_0^{0s}(M) \).

From Proposition 5.2 and Lemma 5.3, we have the following theorem.

**Theorem 5.4 (cf. [4])**. Let the bundle-like metric on \( M \) be complete. If \( \phi \in \tilde{L}_2^{0s}(M) \cap \Delta^{0s}(M) \) such that \( \tilde{\phi} = 0 \), then \( \phi \in \tilde{H}_2^{0s}(M) \).

6. Square-integrable cohomology spaces \( H_2^{0s}(M) \) and \( H_1^{0s}(M) \). In this section, we set situations under the assumptions

\[ A_i = A_i(y) \quad \text{and} \quad g_{\Delta\Delta}(x, y) = 1 \quad \text{(IV)} \]

in (1) and (2). The manifold given in the example in §2 satisfies (IV).

We notice that the volume element of \( M \) is

\[ dV_M = \sqrt{\det \begin{pmatrix} g_{\Delta\Delta} & 0 \\ 0 & g_{ij} \end{pmatrix}} \eta \wedge dy^1 \wedge \cdots \wedge dy^n \]

\[ = \sqrt{\det (g_{ij})} \eta \wedge dy^1 \wedge \cdots \wedge dy^n \quad \text{(from (IV))} \]

\[ = \sqrt{\det (g_{ij})} dx \wedge dy^1 \wedge \cdots \wedge dy^n \quad \text{(from (1)).} \]

The *-operation on \( \Delta^{1s}(M) \) or \( \Delta^{0s}(M) \) is defined as follows. For \( \phi \in \Delta^{1s}(M) \) and \( \psi \in \Delta^{0s}(M) \),

\[ \ast \phi = \frac{(-1)^{(1-s)}s!}{(1-1)! (n-s)! s!} \sum g^{\Delta\Delta} g^{i_1, i_2, \ldots, i_n} \times \delta_{i_1, \ldots, i_n, k_1, \ldots, k_n} \sqrt{\det \begin{pmatrix} g_{\Delta\Delta} & 0 \\ 0 & g_{ij} \end{pmatrix}} \phi_{\Delta i_1, \ldots, \psi} k_1 \wedge \cdots \wedge dy^{k_n}, \]

\[ = \frac{1}{(n-s)! s!} \sum g^{i_1, i_2, \ldots, i_n} g^{i_j, \delta_{j_1, \ldots, j_n, k_1, \ldots, k_n}} \times \sqrt{\det (g_{ij})} \phi_{\Delta i_1, \ldots, \psi} k_1 \wedge \cdots \wedge dy^{k_n}, \]

\[ \ast \psi = \frac{(-1)^{(1-0)s}}{(1-0)! (n-s)! 0! s!} \sum g^{i_1, i_2, \ldots, i_n} g^{i_j, \delta_{j_1, \ldots, j_n, k_1, \ldots, k_n}} \times \sqrt{\det \begin{pmatrix} g_{\Delta\Delta} & 0 \\ 0 & g_{ij} \end{pmatrix}} \psi_{\Delta i_1, \ldots, \psi} k_1 \wedge \cdots \wedge dy^{k_n}, \]

\[ = \frac{(-1)^{s}}{(n-s)! s!} \sum g^{i_1, i_2, \ldots, i_n} g^{i_j, \delta_{j_1, \ldots, j_n, k_1, \ldots, k_n}} \times \sqrt{\det (g_{ij})} \psi_{\Delta i_1, \ldots, \psi} k_1 \wedge \cdots \wedge dy^{k_n}, \]
The operator ∗ maps Δ^{1,−}(M) (resp. Δ^{0,−}(M)) into Δ^{0,−,(−1)}(M) (resp. Δ^{1,−,(−1)}(M)), since (IV) holds.

We define a pre-Hilbert metric on Δ^{0,−}(M) (or Δ^{0,−}(M)) by

\[ \langle \phi, \psi \rangle = \int_M \phi \wedge \psi. \]

The differential operator \( d'' \) maps \( \Delta^{1,−}(M) \) (resp. \( \Delta^{0,−}(M) \)) into \( \Delta^{1,−,(−1)}(M) \) (resp. \( \Delta^{0,−,(−1)}(M) \)). We define \( \delta'' \): \( \Delta^{1,−}(M) \to \Delta^{1,−,(−1)}(M) \) (or \( \Delta^{0,−}(M) \to \Delta^{0,−,(−1)}(M) \)) by

\[
\delta'' \phi = (-1)^{(n+1)(s+1)+(n+1)+1} \ast d'' \ast \phi
\]
\[
\delta'' \psi = (-1)^{(n+1)(s+1)+(n+1)+1} \ast d'' \ast \psi
\]

for \( \phi \in \Delta^{1,−}(M) \) and \( \psi \in \Delta^{0,−}(M) \). Then we have \( \langle d'' \phi, \psi \rangle = \langle \phi, \delta'' \psi \rangle \) for \( \phi \in \Delta^{0,−}(M) \) (resp. \( \Delta^{0,−}(M) \)) and \( \psi \in \Delta^{1,−,(−1)}(M) \) (resp. \( \Delta^{0,−,(−1)}(M) \)).

Let \( L_{2^{−}}^{1,−}(M) \) (resp. \( L_{2^{−}}^{0,−}(M) \)) be the completion of \( \Delta^{0,−}(M) \) (resp. \( \Delta^{0,−}(M) \)) with respect to the inner product \( \langle \, , \, \rangle \). We denote by \( \theta_0 \) the restriction of \( d'' \) to \( \Delta^{0,−}(M) \) (or \( \Delta^{0,−}(M) \)) and by \( \theta_0 \) the restriction of \( \delta'' \) to \( \Delta^{0,−,(−1)}(M) \) (or \( \Delta^{0,−,(−1)}(M) \)). Define \( \partial = (\theta_0)^* \) and \( \vartheta = (\partial)^* \) where \( (\, )^* \) denotes the adjoint operator of \( (\, ) \) with respect to the inner product \( \langle \, , \, \rangle \). Then \( \partial \) is a closed, densely defined operator of \( L_{2^{−}}^{1,−}(M) \) (resp. \( L_{2^{−}}^{0,−}(M) \)) into \( L_{2^{−}}^{1,−,(−1)}(M) \) (resp. \( L_{2^{−}}^{0,−,(−1)}(M) \)), and \( \theta \) is a closed, densely defined operator of \( L_{2^{−}}^{1,−}(M) \) (resp. \( L_{2^{−}}^{0,−}(M) \)) into \( L_{2^{−}}^{1,−,(−1)}(M) \) (resp. \( L_{2^{−}}^{0,−,(−1)}(M) \)).

The following objects are defined by the same ways as in §4:

\[
\begin{align*}
D_{\partial,0}^{1,−}, & \quad D_{\partial,0}^{0,−}, & \quad D_{\partial,0}^{1,−}, & \quad D_{\partial,0}^{0,−}, \\
Z_{\partial,0}^{1,−}(M), & \quad Z_{\partial,0}^{0,−}(M), & \quad Z_{\partial,0}^{1,−}(M), & \quad Z_{\partial,0}^{0,−}(M), \\
B_{\partial,0}^{1,−}(M), & \quad B_{\partial,0}^{0,−}(M), & \quad B_{\partial,0}^{1,−}(M), & \quad B_{\partial,0}^{0,−}(M).
\end{align*}
\]

Then

\textbf{Definition.} \( H_{2^{−}}^{1,−}(M) = Z_{\partial,0}^{1,−}(M) \ominus B_{\partial,0}^{1,−}(M) \) and \( H_{2^{−}}^{0,−}(M) = Z_{\partial,0}^{0,−}(M) \ominus B_{\partial,0}^{0,−}(M) \).

By the same ways as in §4, we have

\textbf{Lemma 6.1. Under the assumption (IV),}

\[
H_{2^{−}}^{1,−}(M) = Z_{\partial,0}^{1,−}(M) \cap Z_{\partial,0}^{1,−}(M)
\]

and

\[
H_{2^{−}}^{0,−}(M) = Z_{\partial,0}^{0,−}(M) \cap Z_{\partial,0}^{0,−}(M).
\]

\textbf{Theorem 6.2. Under the assumption (IV),}

\[
L_{2^{−}}^{1,−}(M) = H_{2^{−}}^{1,−}(M) \oplus B_{\partial,0}^{1,−}(M) \oplus B_{\partial,0}^{1,−}(M)
\]

and

\[
L_{2^{−}}^{0,−}(M) = H_{2^{−}}^{0,−}(M) \oplus B_{\partial,0}^{0,−}(M) \oplus B_{\partial,0}^{0,−}(M).
\]
THEOREM 6.3. Under the assumption (IV), if the bundle-like metric on $M$ is complete, then $H^0_1(M) = H^1_{2,n-s}(M)$ (isomorphic as Hilbert spaces).

In order to prove Theorem 6.3, we have to notice that $\langle \phi, \psi \rangle = \langle *\phi, *\psi \rangle$ for $\phi, \psi \in \Delta^0_0(M)$.

COROLLARY 6.4. Under the assumption (IV), if the bundle-like metric on $M$ is complete and $\dim H^0_1(M)$ is finite, then $\dim H^0_1(M) = \dim H^1_{2,n-s}(M)$.

Now, we have $\langle \phi, \psi \rangle = \langle \eta \wedge \phi, \eta \wedge \psi \rangle$ for $\phi, \psi \in \Delta^0_0(M)$. Let $\xi$ denote the dual to $\eta$ and $i_\xi$ the interior product by $\xi$ operator. Then we have $i_\xi \phi \in \Delta^0_0(M)$ and $\eta \wedge i_\xi \phi = \phi$ for $\phi \in \Delta^{1-s}(M)$. The following diagram is commutative.

$\Delta^0_0(M) \xrightarrow{e(\eta)} \Delta^0_0(M) \xrightarrow{0 \downarrow \uparrow_0} \Delta^0_0(M) \xrightarrow{(-1)e(\eta)} \Delta^0_0(M)$

where $e(\eta)$ denotes the exterior product by $\eta$ operator. Thus we have

THEOREM 6.5. Under the assumption (IV), if the bundle-like metric on $M$ is complete, then $H^0_1(M) = H^1_{2,n-s}(M)$ (isomorphic as Hilbert spaces).

From Theorems 6.3 and 6.5, we have

THEOREM 6.6. Under the assumption (IV), if the bundle-like metric on $M$ is complete, then $H^1_{2,n-s}(M) = H^1_{2,n-s}(M)$ and $H^1_{2,n-s}(M) = H^1_{2,n-s}(M)$ (isomorphic as Hilbert spaces).

Next, from (IV), it is easy to see that $\langle \phi, \psi \rangle = \langle \phi, \psi \rangle$ for $\phi, \psi \in \Delta^0_0(M)$. The following diagram is commutative.

$\Delta^0_0(M) \xrightarrow{I} \Delta^0_0(M) \xrightarrow{0 \downarrow \uparrow_0} \Delta^0_0(M) \xrightarrow{I} \Delta^0_0(M)$

where $I$ denotes the identity map. Thus we have the following theorem.

THEOREM 6.7. Under the assumption (IV), if the bundle-like metric on $M$ is complete, then $H^0_1(M) = H^1_{2,n-s}(M)$ (isomorphic as Hilbert spaces).

7. $\Box$-harmonic forms. In this section, we assume that the assumption (IV) holds and that the bundle-like metric on $M$ is complete. We put

$N^1_{0,2}(M) = \{ \phi \in \Delta^{1-s}(M); d^*\phi = 0 \}$,

$N^1_{0,2}(M) = \{ \phi \in \Delta^{1-s}(M); \delta^*\phi = 0 \}$,

$N^1_{0,2}(M) = \{ \phi \in \Delta^{0}(M); d^*\phi = 0 \}$,

$N^1_{0,2}(M) = \{ \phi \in \Delta^{0}(M); \delta^*\phi = 0 \}$.
Then, by the same ways as in §5, we have

**Proposition 7.1.** Let the assumption (IV) hold and the bundle-like metric on \( M \) be complete. Then

\[
N^1_\alpha(M) \cap L^2_\alpha(M) \subset Z^1_\alpha(M), \quad N^1_\beta(M) \cap L^2_\beta(M) \subset Z^1_\beta(M),
\]

\[
N^0_\alpha(M) \cap L^2_\alpha(M) \subset Z^0_\alpha(M), \quad N^0_\beta(M) \cap L^2_\beta(M) \subset Z^0_\beta(M).
\]

**Definition.** The Laplacian acting on \( \Delta^1(M) \) (or \( \Delta^0(M) \)) is defined by

\[
\square = \partial'' \delta'' + \delta'' \partial''.
\]

By the same ways as in §5, we have

**Lemma 7.2.** Let the assumption (IV) hold and the bundle-like metric on \( M \) be complete. If \( \phi \in L^2_\alpha(M) \cap \Delta^1_\alpha(M) \) (resp. \( L^2_\beta(M) \cap \Delta^0_\beta(M) \)) such that \( \square \phi = 0 \), then \( \partial'' \phi = 0 \) and \( \delta'' \phi = 0 \), i.e. \( \phi \in N^1_\alpha(M) \cap N^1_\beta(M) \) (resp. \( N^0_\alpha(M) \cap N^0_\beta(M) \)).

From Proposition 7.1 and Lemma 7.2, we have

**Theorem 7.3.** Let the assumption (IV) hold and the bundle-like metric on \( M \) be complete. If \( \phi \in L^1_\alpha(M) \cap \Delta^1_\alpha(M) \) (resp. \( L^1_\beta(M) \cap \Delta^0_\beta(M) \)) such that \( \square \phi = 0 \), then \( \phi \in H^1_\alpha(M) \) (resp. \( H^0_\beta(M) \)).

From Theorems 5.4 and 6.7, we have

**Theorem 7.4.** Let the assumption (IV) hold and the bundle-like metric on \( M \) be complete. If \( \phi \in L^0_\alpha(M) \cap \Delta^0_\alpha(M) \) such that \( \square \phi = 0 \), then \( \phi \in H^0_\alpha(M) \). From Theorems 6.7 and 7.3, we have

**Theorem 7.5.** Let the assumption (IV) hold and the bundle-like metric on \( M \) be complete. If \( \phi \in L^0_\beta(M) \cap \Delta^0_\beta(M) \) such that \( \square \phi = 0 \), then \( \phi \in H^0_\beta(M) \).

**Remark.** I. Vaisman [12], [13] already noticed that on a compact orientable Riemannian foliated manifold \( M \), the space \( Z^\alpha(M) \) of foliated harmonic forms is a subspace of the de Rham cohomology space \( H^\alpha(M) \).

**Remark.** For the relations between certain cohomology spaces and the existence of bundle-like metrics, see H. Kitahara and S. Yorozu [5].

8. Applications to a contact manifold. First, we cite the definition of the contact manifold. A 1-form \( \eta \) on a connected \((2n + 1)\)-dimensional manifold is called a contact form if \( \eta \wedge (d\eta)^n \neq 0 \) at each point in the manifold (cf. [10]). A connected \((2n + 1)\)-dimensional manifold with a contact form is called a contact manifold. On a contact manifold with a contact form \( \eta \), there exists a global vector field \( \xi \) such that \( \eta(\xi) = 1 \) and \( i_\xi d\eta = 0 \) (cf. [10]). A connected paracompact contact manifold with a contact form \( \eta \) has a Riemannian metric \( (, ,) \) such that
V(X) = (X, 0 (6) for any vector field X (cf. [10]). In fact, let ( , ) be an arbitrary Riemannian metric, and we define

\[ (X, Y) = (X - \eta(X)\xi, Y - \eta(Y)\xi) + \eta(X) \cdot \eta(Y) \]

for any vector fields X and Y. Such a metric ( , ) satisfies (6).

Now, let N be a connected (2n + 1)-dimensional contact manifold with a contact form \( \eta \) and a Riemannian metric ( , ) satisfying (6). We assume that \( \xi \) is a Killing vector field on N with respect to the metric ( , ) and that the orbits of \( \xi \) are compact. An example of such a manifold N is the manifold given in the example in §2.

We define the operators \( \delta, e(\eta), i_\xi, L \) and \( \Lambda \) on \( \wedge^*(N) \) as follows:

\[ \delta \phi = (-1)^s \ast d^{*} \phi, \quad e(\eta)\phi = \eta \wedge \phi, \]
\[ i_\xi \phi = (-1)^{s-1} \ast e(\eta) \ast \phi, \]
\[ L \phi = d\eta \wedge \phi, \quad \Lambda \phi = \ast L \ast \phi \]

(cf. [2], [6], [11]).

**Definition.** A form \( \phi \) in \( \wedge^*(N) \) is called a C-harmonic form (resp. C*-harmonic form) if \( i_\xi \phi = 0, d\phi = 0 \) and \( \delta \phi = e(\eta)\Lambda \phi \) (resp. \( e(\eta)\phi = 0, d\phi = i_\xi L \phi \) and \( \delta \phi = 0 \)).

**Remark.** The notion of C-harmonic forms was introduced by S. Tachibana [11], and Y. Ogawa [6] gave the definition of C*-harmonic forms. They discussed it on compact normal contact metric manifolds. A normal contact metric manifold is a so-called Sasakian manifold (for the definition, see [6], [10], [11]).

For each point in N, there exists a local coordinate neighborhood system \( \{ U; (x, y^1, \ldots, y^{2n}) \} \) such that

\[ \eta = dx + \sum (-y^{n+i}) dy^i \]  
\[ (i = 1, 2, \ldots, n) \]

and the orbits of \( \xi \) are given locally by

\[ y^1 = c^1, \ldots, y^n = c^n, y^{n+1} = c^{n+1}, \ldots, y^{2n} = c^{2n} \]

for the same constants \( c^1, \ldots, c^n, c^{n+1}, \ldots, c^{2n} \) (cf. [10]). \{ \eta, dy^1, \ldots, dy^n, dy^{n+1}, \ldots, dy^{2n} \} and \{ \partial/\partial x, v_1, \ldots, v_n, v_{n+1}, \ldots, v_{2n} \} are dual bases for the cotangent and tangent spaces respectively at each point in U, where

\[ v_i = \partial/\partial y^i + (y^{n+i})\partial/\partial x \quad \text{and} \quad v_{n+i} = \partial/\partial y^{n+i}. \]

We may consider N as a foliated manifold whose leaves are orbits of \( \xi \).

From (6), we have

\[ (v_i, \xi) = \eta(v_i) = 0, \]
\[ (v_{n+i}, \xi) = \eta(v_{n+i}) = 0. \]
Then, since $\xi$ is a Killing vector field on $N$, the metric $(\cdot, \cdot)$ on $N$ is a bundle-like metric with respect to the foliation, that is, the local expression of the metric $(\cdot, \cdot)$ in $U$ is

$$ds^2 = \eta \cdot \eta + \sum g_{AB} \, dy^A \cdot dy^B$$

where $A, B = 1, 2, \ldots, 2n$. Thus the contact manifold $N$ is a Riemannian foliated manifold with one-dimensional foliation $\mathcal{F}$ whose leaves are compact and the Riemannian metric $(\cdot, \cdot)$ on $N$ is a bundle-like metric with respect to $\mathcal{F}$. Moreover, the assumption (IV) in §6 is satisfied. Therefore, we may apply the discussions of above sections to the contact manifold $N$.

In order to obtain the applications to $N$, we have to prepare the decomposition of the operator $\delta$. We have the decomposition of the operator $d$:

$$d = d' + d'' + d'''$$

(cf. §2). Then, according to I. Vaisman [12], we define the operators $\delta'$, $\delta''$ and $\delta'''$ as follows:

$$\delta'\phi = (-1)^{r+s} \cdot d' \cdot \phi,$$

$$\delta''\phi = (-1)^{r+s} \cdot d'' \cdot \phi,$$

$$\delta'''\phi = (-1)^{r+s} \cdot d''' \cdot \phi,$$

where $\phi \in \wedge^r(N)$, $r = 0$ or 1. Then we have the decomposition of the operator $\delta$:

$$\delta = \delta' + \delta'' + \delta'''.$$  

We notice the following: (i) If $\phi \in \wedge^1(N)$, then $\delta\phi = \delta'\phi + \delta''\phi$, where $\delta'\phi \in \wedge^0(N)$ and $\delta''\phi \in \wedge^{1,1}(N)$. (ii) If $\phi \in \wedge^0(N)$, then $\delta\phi = \delta''\phi + \delta'''\phi$, where $\delta''\phi \in \wedge^{0,1}(N)$ and $\delta'''\phi \in \wedge^{1,2}(N)$.

We have the following lemma.

**Lemma 8.1.** For $\phi \in \Delta^1(N)$ and $\psi \in \Delta^0(N)$,

$$d'''\phi = i_\xi L\phi, \quad \delta'''\psi = e(\eta) \Lambda\psi.$$  

From Theorem 7.4 and Lemma 8.1, we have

**Theorem 8.2.** Let the metric $(\cdot, \cdot)$ on $N$ be complete. If $\phi \in \hat{L}^0(N) \cap \Delta^0(N)$ such that $\Box\phi = 0$, then $\phi$ is a $C$-harmonic form.

From Theorem 7.3 and Lemma 8.1, we have

**Theorem 8.3.** Let the metric $(\cdot, \cdot)$ on $N$ be complete. If $\phi \in \Delta^1(N)$ such that $\Box\phi = 0$, then $\phi$ is a $C^*$-harmonic form.

**Acknowledgement.** The author wishes to thank the referee for his valuable suggestions.
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