ADDING AND SUBTRACTING JUMPS FROM
MARKOV PROCESSES

BY

RICHARD F. BASS

Abstract. If $X_t$ is a continuous Markov process with infinitesimal generator $A$, if $n$ is a kernel satisfying certain conditions, and if $B$ is an operator given by

$$B g(x) = \int [g(y) - g(x)] n(x, dy),$$

then $A + B$ will be the generator of a Markov process that has Lévy system $(n, dt)$. Conversely, if $X_t$ has Lévy system $(n, dt)$, $n$ satisfies certain conditions, and $B$ is defined as above, then $A - B$ will be the generator of a continuous Markov process.

1. Introduction. Suppose $X_t$ is a Markov process with infinitesimal generator $A$. If we perturb $A$ by another operator $B$, will $A + B$ be the generator of a Markov process? And if so, what will the new Markov process look like? In this article we show that if $X_t$ is continuous and $B$ is given by $B g(x) = \int [g(y) - g(x)] n(x, dy)$ for some kernel $n$, then $A + B$ will be the generator of a Markov process $Y_t$ whose jump structure is completely described by $n$. Conversely, if $A$ is the generator of a Markov process $X_t$ whose jump structure is given by $n$ and $B$ is as given above, then $A - B$ will be the generator of a Markov process $Y_t$ which has no jumps; that is, all the paths are continuous.

Previous work on this problem has been done by Cook [4], in the case where $n(x, \cdot)$ is bounded in neighborhoods of $x$. There is a probabilistic construction of the new process $Y_t$ due to Ikeda, Nagasawa, and Watanabe [6], Meyer [10], and Sawyer [12] (see §6) in the case $n(x, \cdot)$ is finite and sufficiently small. In this article, $n(x, \cdot)$ is allowed to be infinite (see Example 3.7). In probabilistic terms, $n(x, \cdot)$ being finite and sufficiently small means that for each path of $X_t$ or $Y_t$, there will only be a finite number of jumps in each finite time interval; $n(x, \cdot)$ being infinite allows there to be infinitely many jumps in finite time intervals; in contrast to the $n$ finite case, there may well be zero time between jumps.

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By \( X_t \) "has jump structure described by \( n \)”, we mean \( X_t \) has Lévy system \((n, dt)\), using the Lévy system developed by Watanabe [14] and Benveniste and Jacod [2]. Since the domains of infinitesimal generators are awkward to work with, we work instead with the resolvents \( R_\lambda \) and \( S_\lambda \) of \( X_t \) and \( Y_t \), respectively. In §2, we give the necessary preliminaries. In §§3 and 4, we show that if \( BR_\lambda \) is bounded in norm, \( S_\lambda \) will be the resolvent of a semigroup in the cases where we are adding jumps and subtracting jumps, respectively. Example 3.7 is an example that shows \( n \) may be infinite. In §5, we show that \( Y_t \) has Lévy system \((n, dt)\) or that \( Y_t \) is continuous depending whether we added or subtracted jumps. In §6 we describe the probabilistic construction of \( Y_t \), when it exists.

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2. Preliminaries. We will suppose \( E \) is a compact metric space. If \( E \) is only locally compact, we can make it compact by adding the point \( \Delta \), the one point compactification. \( \| \| \) will denote the usual sup norm, both for functions and operators. \( f_n \rightarrow f \) will mean \( \| f_n - f \| \rightarrow 0 \), unless specified otherwise. Let \( f_n \) converges weakly to \( f \) mean that \( \sup \| f_n \| < \infty \) and \( f_n(x) \rightarrow f(x) \) for all \( x \).

We will let \( \mathcal{K} \) be the space of bounded, Borel measurable functions on \( E \) with sup norm, and \( \mathcal{L} \) will refer to a closed subspace of \( \mathcal{K} \). We will assume throughout that all kernels \( m \) satisfy \( m(x, \{x\}) = 0 \) for all \( x \). Following the notation of [3], \( X_t = (\Omega, \mathcal{F}_t, X_t, \theta_t, P^x) \) will be a right continuous strong Markov process with state space \( E \).

A Lévy system \((n, dH_t)\) for a process \( X_t \) is a kernel \( n(x, dy) \) and a perfect continuous additive functional \( H_t \) such that for all \( x \in E \), for all bounded stopping times \( T \), and all positive Borel measurable functions \( f \) on \( E \times E \) that are 0 on the diagonal, the Lévy system identity holds:

\[
\mathbb{E}^x \sum_{0 < t < T} f(X_{t-}, X_t) = \mathbb{E}^x \int_0^T \int f(X_t, y)n(X_t, dy) \ dH_t,
\]

where both sides may be infinite.

Benveniste and Jacod [2] proved that every Hunt process has a Lévy system. We will assume throughout that \( H_t(\omega) = t \) for all \( t \) and all \( \omega \). Since one can always perform a time change on \( X_t \) so that this is true (cf. [9, p. 150]), there is no real loss of generality. If \( X_t \) has Lévy system \((n, dt)\), and \( n'(x, dy) = n(x, dy) \) for all \( x \) except for a set of potential 0, it is clear that \((n', dt)\) satisfies the Lévy system identity. We will sometimes refer to \( n \) as the Lévy kernel for \( X_t \).

Given any kernel \( m \), define the Lévy operator \( L_m \) associated with \( m \) by \( L_m g(x) = \int (g(y) - g(x))m(x, dy) \) for those \( g \)'s and \( x \)'s for which the integral
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is well defined. By the construction of Benveniste and Jacod, if \( m \) is a Lévy kernel for some Hunt process, \( L_m g(x) \) is well defined for any \( g \) that vanishes in a neighborhood of \( x \).

Let \( m > n \) mean that except for a set of \( x \)'s of potential 0, \( m(x, F) > n(x, F) \) for all Borel sets \( F \). \( m \) is bounded if \( m(x, E) \) is a bounded function of \( x \). Let us say that \( m_j \) increases strongly to \( m \) if each \( m_j \) is bounded, and there exist Borel sets \( F_j \) contained in \( E \times E \) increasing to \( E \times E \) such that \( m_j(x, dy) = 1_{F_j}(x, y)m(x, dy) \). We will need

**Proposition 2.1.** If \( m \) is the Lévy kernel of a Hunt process, there exist kernels \( m_j \) that increase strongly to \( m \).

**Proof.** If \( F = F_1 \times F_2 \), \( 1_{F_1 \times F_2}(x, y)m(x, dy) \) is clearly a kernel that is a measure in \( dy \) and is measurable in \( x \). By a monotone class argument \( 1_{F}(x, y)m(x, dy) \) is a kernel for all \( F \) Borel in \( E \times E \).

Let \( G_i = \{ (x, y) : d(y, x) > 1/i \} \). Let \( g_i(x) = \int G_i(x, y)m(x, dy) \). Since \( m \) is a Lévy kernel, \( g_i(x) \) is finite for all \( x \). Also \( g_i \uparrow \) as \( i \to \infty \). Given a positive integer \( N \), let \( H_i = \{ x : g_i(x) < N, g_k(x) > N \text{ for } k = i + 1, i + 2, \ldots \} \). Let \( F'_N = \bigcup_{i=1}^{\infty} ((H_i \times E) \cap G_i) \). Let \( 1_{F'_N}(x, y)m(x, dy) \) is bounded by \( N \). Now let \( F_j = \bigcup_{N=1}^{\infty} F'_N \). Then \( m_j(x, E) \) is bounded by \( j(j+1)/2 \), where \( m_j(x, dy) = 1_{F_j}(x, y)m(x, dy) \). Note that if \( m_j \) increases strongly to \( m \), \( g > 0 \) and \( g(x) = 0 \), \( L_m g(x) = m_j(x, g) \to m(x, g) \).

Recall that if \( X_t \) is a Markov process, \( P_t f(x) = E^x f(X_t) \) defines a semigroup \( P_t \) on \( \mathcal{C} \). The resolvent \( R_\lambda \) of \( P_t \) is given by \( R_\lambda f = \int_0^\infty e^{-\lambda t} P_t f dt \). If \( \lambda \neq \mu \), \( R_\lambda \) satisfies the resolvent identity \( R_\lambda - R_\mu = (\mu - \lambda) R_\lambda R_\mu \).

Conversely, the Hille-Yosida theorem says that if \( \{ R_\lambda, \lambda > \lambda_0 \} \) is a family of operators satisfying the resolvent identity, if \( ||R_\lambda|| < 1/\lambda \), and if \( R_\lambda(\mathcal{C}) \) is dense (under the sup norm) in \( \mathcal{L} \), a subspace of \( \mathcal{C} \), then \( R_\lambda \) is the resolvent of a semigroup \( P_t \) that is strongly continuous on \( \mathcal{L} \). An operator \( V \) is positive if \( h > 0 \) implies \( V h > 0 \). By the construction of the Hille-Yosida theorem, if \( R_\lambda \) is positive, so is \( P_t \).

If \( R_\lambda \) is the resolvent for a strong Markov process \( X_t \), we have Dynkin’s identity,

\[
E^x R_\lambda g(X_T) - R_\lambda g(x) = E^x \int_0^T \lambda (\lambda R_\lambda - I) g(X_t) \, dt
\]

for all \( x \in E \), all \( g \in \mathcal{C} \), and all bounded stopping times \( T \), where \( I \) is the identity operator. We also know that if \( g \in \mathcal{C} \), \( R_\lambda g(X_t) \) is right continuous a.s.

If \( V \) is a positive linear operator and \( V1 = c \), note that \( ||V|| = c \), since if \( -1 < g < 1, c - Vg = V(1-g) > 0 \) and \( c + Vg = V(1+g) > 0 \).
If $B$ is an operator such that $\|BR_\lambda\| < \delta < 1$ if $\lambda >$ some $\lambda_0$, we can define $S_\lambda$ by $S_\lambda = R_\lambda(\sum_{i=0}^\infty (BR_\lambda)^i)$, and $\|S_\lambda\| < ((1 - \delta)\lambda)^{-1}$.

$S_\lambda = R_\lambda(I + BS_\lambda) = (I + R_\lambda B)S_\lambda.$

We need a few basic properties of $S_\lambda$.

**Lemma 2.2.** If $R_\lambda = Q_\lambda(I + AR_\lambda)$ and $S_\lambda = R_\lambda(I + BS_\lambda)$, then $S_\lambda = Q_\lambda(I + (A + B)S_\lambda)$.

**Proof.** $S_\lambda = Q_\lambda(I + AR_\lambda)(I + BS_\lambda) = Q_\lambda(I + BS_\lambda) + Q_\lambda AR_\lambda(I + BS_\lambda) = Q_\lambda(I + BS_\lambda) + Q_\lambda AS_\lambda = Q_\lambda(I + (A + B)S_\lambda)$.

**Lemma 2.3.** If $R_\mu$ and $S_\lambda$ each satisfy the resolvent identity for all $\mu$ and $\lambda$, and $S_\lambda = R_\lambda(I + BS_\lambda)$ for some $\lambda$, then $S_\mu = R_\mu(I + BS_\mu)$ for all $\mu > 0$.

**Proof.** is by direct computation using the resolvent identity for $R_\mu$ and $S_\mu$.

3. Adding jumps. We assume throughout §3 that $X_t$ is a strong Markov process with resolvent $R_\lambda$, $n$ a kernel, $B$ a linear operator such that $\|BR_\lambda\| < 1$ for all $\lambda >$ some $\lambda_0$. We let $S_\lambda = R_\lambda(\sum_{i=0}^\infty (BR_\lambda)^i)$, and our aim is to find conditions on $B$ so that $S_\lambda$ satisfies the conditions of the Hille-Yosida theorem. Note that $S_\lambda = R_\lambda(I + BS_\lambda) = (I + S_\lambda B)R_\lambda$ if $\lambda > \lambda_0$.

**Lemma 3.1.** $S_\lambda$ satisfies the resolvent identity.

The proof is identical to that of Lemma 2.1 of Leviatan [8], a fairly straightforward calculation.

**Lemma 3.2.** $S_\lambda(\mathcal{H})$ is dense in $\mathcal{L}$.

**Proof.** $R_\lambda(\mathcal{H})$ is dense in $\mathcal{L}$. If $g = R_\lambda f$, with $f \in \mathcal{H}$, let $h = (I - BR_\lambda)f$. Then $S_\lambda h = S_\lambda (I - BR_\lambda)f = R_\lambda f$.

We first prove our result for the case where the kernel $n$ is bounded and $B$ is the Lévy operator of $n$.

**Theorem 3.3.** Suppose $n(x, E) < N$ for all $x$ and $B$ is the Lévy operator of $n$. Then $S_\lambda$ is the resolvent of a positive contraction semigroup on $\mathcal{L}$.

**Proof.** Let $\lambda > 4N$. Since $R_\lambda 1 = 1/\lambda$ and $B 1 = 0$, it follows from Lemmas 3.1, 3.2 and §2 that we need only show that if $g > 0$ and $g \in \mathcal{L}$, then $S_\lambda g > 0$. We break the proof into two steps.

1. Let $m = \inf_{x \in E} S_\lambda \delta(x)$. Let $\epsilon > 0$. Since $S_\lambda g = R_\lambda(I + BS_\lambda)g$, $S_\lambda g(X_t)$ is right continuous. Let $T = \min(U, 1)$ where $U = \inf\{t > 0: S_\lambda g(X_t) - S_\lambda g(X_0) > 2\epsilon\}$. We have $S_\lambda g(X_t) > S_\lambda g(X_0) + 2\epsilon$ provided $U < \infty$. Suppose for the remainder of the proof that $S_\lambda g(x) < m + \epsilon$. 

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If $P^x(T = U) > \frac{2}{3}$, then
\[ \frac{\epsilon}{3} < E^x S_x g(X_T) - S_x g(x) \]
\[ = E^x \int_0^T (\lambda R_\lambda - I)(I + BS_\lambda)g(X_t) \, dt < \frac{6\|g\|}{E^x T}. \]
It follows that whether or not $P^x(T = U) > \frac{2}{3}$, we have $E^x T > \min\left(\frac{1}{2}, \frac{\epsilon}{18\|g\|}\right)$.

(2) If $S_x g(y) < m + 2\epsilon$, $-BS_x g(y) < 2\epsilon N$. We then have
\[ m - S_x g(x) < E^x S_x g(X_T) - S_x g(x) = \int_0^T \lambda S_x g(X_t) - BS_x g(X_t) \, dt \]
\[ < \int_0^T \lambda S_x g(X_t) \, dt - \int_0^T BS_x g(X_t) \, dt \]
\[ < \lambda(2\epsilon N) + 2\epsilon N E^x T. \]
Now if $m$ were less than 0, we could pick $\epsilon$ small enough so that $\lambda(2\epsilon N) + 2\epsilon N < \lambda m/2$. We would then have that if $S_x g(x) < m + \epsilon$, $S_x g(x) > m + \lambda(E^x T) m/2$, a contradiction to the definition of $m$ since $E^x T$ is bounded away from 0 by step (1).

Comment. By the Hille-Yosida theorem there is a semigroup $P_t$ such that
\[ S_x = \int e^{-\mu t} P_t \, dt, \quad \lambda > 4N, \quad \mu \in \mathbb{C}. \]
We can use this equation to define $S_x f$ for all $\lambda$. By Fubini's theorem and the fact that $P_t$ is a semigroup, we have that $S_x$ satisfies the resolvent identity for all $\lambda$ and $\mu > 0$. By Lemma 2.3, $S_x = R_\lambda(I + BS_\lambda)$ for all $\lambda$, not just $\lambda > 4N$.

We now allow $B$ to be unbounded.

**Theorem 3.4.** Suppose there exists a sequence of bounded kernels $n_j$, with Lévy operators $L_{n_j}$, and $L_{n_j} S_x g \rightarrow BS_x g$ for all $g \in \mathbb{C}$. Then $S_x$ is the resolvent of a positive contraction semigroup on $\mathbb{C}$.

**Proof.** Let $\lambda > \lambda_0$. Again, as in Theorem 3.3, we need only show $S_x g > 0$ if $g \in \mathbb{C}$ and $g > 0$. Let $S_j^\mu g = \sum_{n=0}^\infty R_\mu(L_{n_j} R_\mu)^n$ for $\mu > 2\|L_{n_j}\|$. By (3.3), $S_j^\mu$ is a resolvent for a semigroup, and $S_j^\mu = R_\mu + S_j^\mu L_{n_j} R_\mu$ for all $\mu$. Since $S_x = R_\lambda + R_\lambda BS_\lambda$,
\[ S_j^\lambda(g - L_{n_j} S_x g + BS_x g) = R_\lambda g + R_\lambda BS_\lambda g \]
\[ + S_j^\lambda L_{n_j}(R_\lambda g + R_\lambda BS_\lambda g) - S_j^\lambda L_{n_j} S_x g \]
\[ = S_x g. \]
Then $\|S_j^\lambda g - S_x g\| < \|L_{n_j} S_x g - BS_x g\|/\lambda \rightarrow 0$ as $j \rightarrow \infty$. Since $S_j^\lambda g \geq 0$, our result follows.
**Corollary 3.5.** Suppose $B$ is the infinitesimal generator of a Markov process such that $BR_{\lambda}(\mathcal{E}) \subset \mathcal{E}$. Then $S_{\lambda}$ is the resolvent of a contraction semigroup on $\mathcal{E}$.

**Proof.** First of all, it is clear that $BS_{\lambda} = \sum_{i=0}^{\infty} (BR_{\lambda})^i$ maps $\mathcal{E}$ into $\mathcal{E}$. If $T_{\lambda}$ is the resolvent of the Markov process generated by $B$, let $L_{\eta} = j(T_j - I)$:

$$L_{\eta}g(x) = \int [g(y) - g(x)] j^2T_j(x, dy).$$

Since $L_{\eta}S_{\lambda}g = j(T_j - I)S_{\lambda}g = jT_jBS_{\lambda}g \to BS_{\lambda}g$ as $j \to \infty$ if $g \in \mathcal{E}$, the result follows by Theorem 3.4.

Corollary 3.5 is also proved in [5] and [7]. As in the following example, Corollary 3.5 shows that in some cases perturbing by a drift term may be viewed as the limit of perturbation by jumps.

**Example 3.6.** Suppose $R_{\lambda}$ is the resolvent of Brownian motion on the real line, $B = d/dx$. Here $\mathcal{E}$ is $C_0$, the continuous functions that vanish at infinity. If $g \in C_0$, $R_{\lambda}g$ is twice differentiable, hence $BR_{\lambda}g$ is continuous, and it is easily checked that $BR_{\lambda}g \in C_0$. Also, since

$$2\|g\| > \|\lambda R_{\lambda}g - g\| = \|\lambda R_{\lambda}g\| = \|\frac{1}{2}(R_{\lambda}g)''\|,$$

where $A$ is the generator of Brownian motion, the identity $\|f''\| < 4\|f\| \|f''\|$ gives $\|BR_{\lambda}g\|^2 < 16\|R_{\lambda}g\| \|g\|$, or $\|BR_{\lambda}g\| < 4\|g\|/\lambda^{1/2}$. It follows by Corollary 3.5 that $S_{\lambda}$ is the resolvent of a semigroup on $C_0$; it is clear that the generator of this semigroup is $\frac{1}{2}d^2/dx^2 + d/dx$.

**Example 3.7.** Suppose that $R_{\lambda}$ is the resolvent of Brownian motion on the real line,

$$B_g(x) = \int [g(y) - g(x)] n(x, dy),$$

where $\int_{|y-x|<\varepsilon} |y-x|n(x, dy) \to 0$ uniformly in $x$ as $\varepsilon \to 0$ where $N = \int \min\{|y-x|, 1\} n(x, dy)$ is bounded. Note that it is possible here for $n(x, E)$ to be infinite for all $x$. Define $n_j$ by $n_j(x, dy) = 1_{D_j}(x, y)n(x, dy)$ where $D_j = \{(x, y): d(x, y) > 1/j\}$. As in Proposition 2.1, $n_j$ is a kernel bounded by $jN$. As in Example 3.6 $(R_{\lambda}g)'$ is bounded in norm if $g \in C_0$, or $|R_{\lambda}g(y) - R_{\lambda}g(x)| < c|y - x|$, where $c = 4\|g\|/\lambda^{1/2}$. Since $R_{\lambda}g_j \to R_{\lambda}g$ weakly if $g_j \to g$ weakly, a monotone class argument gives the above inequality for all $g$, not just those in $C_0$. Since $S_{\lambda}g = R_{\lambda}(I + BS_{\lambda})g$, $|S_{\lambda}g(y) - S_{\lambda}g(x)| < c'|y - x|$, where $c' = 8\|g\|$, if $\lambda$ is large enough.

$$\|BS_{\lambda}g - L_{\eta}S_{\lambda}g\| < \int_{|y-x|<1/j} |S_{\lambda}g(y) - S_{\lambda}g(x)| n(x, dy)$$

$$< 8\|g\| \int_{|y-x|<1/j} |y - x|n(x, dy) \to 0 \text{ as } j \to \infty.$$
A similar argument shows that $\|BR_x\| < 2N\|g\|/\lambda + 4N\|g\|/\lambda^{1/2}$; hence $\|BS_x\| \to 0$ as $\lambda \to \infty$, and clearly $\|BS_x\| < 1$ if $\lambda$ is large enough. It follows by Theorem 3.4 that $S_\lambda$ is the resolvent of a contraction semigroup on $C_0$. We will show in §5 that the associated Markov process has Lévy kernel $n$.

4. Subtracting jumps. We assume throughout §4 that $X_t$ is a Hunt process with resolvent $R_\lambda$ and Lévy system $(n, dt)$, that $B$ is a linear operator, and that $\|BR_\lambda\| < 1$ for all $\lambda >$ some $\lambda_0$. We let $S_\lambda = R_\lambda(\Sigma_{i=0}^\infty(-BR_\lambda)^i)$, and again our aim is to show that $S_\lambda$ satisfies the conditions of the Hille-Yosida theorem.

**Theorem 4.1.** Suppose $n$ is bounded by $N$ and $B$ is the Lévy operator of $n$. Then $S_\lambda$ is the resolvent of a positive contraction semigroup on $L^2$.

**Proof.** Let $\lambda > 4N$. As in Lemmas 3.1 and 3.2, $S_\lambda$ satisfies the resolvent identity and $S_\lambda(%)$ is dense in $L^2$. So we need only show that if $g > 0, g \in L^2$, then $S_\lambda g > 0$. $S_\lambda g(x_t) = R_\lambda(I - BS_\lambda)g(x_t)$ is right continuous. Let $m = \inf_{x \in E} S_\lambda g(x)$. Let $\epsilon > 0$. Let $T = \min(U, V, 1)$, where $U = \inf\{t > 0: S_\lambda g(x_t) - S_\lambda g(x_0) > 2\epsilon\}$ and $V = \inf\{t > 0: x_t \neq x_{t-}\}$. As in Theorem 3.3, if $U < \infty$ we have $S_\lambda g(x_U) > S_\lambda g(x_0) + 2\epsilon$. We break the remainder of the proof into two steps. Let us suppose throughout that $S_\lambda g(x) < m + \epsilon$.

(1) First we show $E^xT$ is bounded away from 0. We have that if $P^x(T = U) > \frac{\epsilon}{2}$,

$$\frac{\epsilon}{3} < 2\epsilon P^x(T = U) = 2\epsilon P^x(T < U) \leq E^x(S_\lambda g(X_T) - S_\lambda g(x); T = U) + E^x(S_\lambda g(X_T) - S_\lambda g(x); T < U) = E^x \int_0^T (\lambda R_\lambda - 1)(I - BS_\lambda) g(x_t) dt < 6\|g\|E^xT.$$

If $P^x(T = V) > \frac{1}{2}$, let $t_0 = 1/(6N)$. Then

$$P^x(V < t_0) < E^x \sum_{0 < t < t_0} 1_{(X_t \neq X_{t-})} E^x \int_0^t n(x_t, E) dt < Nt_0 = \frac{1}{6}.$$ 

In this case, $P^x(T > t_0) > \frac{1}{3}$, or $E^xT > 1/(36N)$.

Finally, if $P^x(T = U) < \frac{7}{12}$ and $P^x(T = V) < \frac{1}{3}$, we must have $P^x(T = 1) > \frac{1}{12}$. In any case, we have $E^xT > \epsilon = \min(\frac{1}{12}, 1/(36N)\epsilon/(18\|g\|)).$

(2) We now show $m > 0$. Note

$$S_\lambda g(X_T) - S_\lambda g(x) - (S_\lambda g(X_T) - S_\lambda g(x)) > m - S_\lambda g(x).$$

If $t < T$, $S_\lambda g(x_t) = S_\lambda g(x_{t-})$, and so

$$S_\lambda g(X_T) - S_\lambda g(x) = \sum_{0 < t < T} (S_\lambda g(x_t) - S_\lambda g(x_{t-})).$$
Then by the Lévy system identity and Dynkin's identity,

\[ m - S_\lambda g(x) < E^x S_\lambda g(X_T) - S_\lambda g(x) - E^x \sum_{0 < i < T} (S_\lambda g(X_i) - S_\lambda g(X_{i-})) \]

\[ = E^x \int_0^T (\lambda R_\lambda - I)(I - BS_\lambda)g(X_t) \, dt - E^x \int_0^T BS_\lambda g(X_t) \, dt \]

\[ < E^x \int_0^T \lambda S_\lambda g(X_t) \, dt < \lambda (m + 2\varepsilon)E^xT. \]

But if \( m \) were less than 0, we could let \( \varepsilon = |m|/4 \), and by selecting \( x \) so that \( S_\lambda g(x) \) were close enough to \( m \), we would have a contradiction.

**Theorem 4.2.** Suppose there exists a sequence of bounded kernels \( n_j \) with Lévy operators \( L_{n_j} \) such that (i) \( n_j \) strongly increases to \( n \) and (ii) \( L_{n_j} S_\lambda g \to BS_\lambda g \) for all \( g \in \mathcal{L} \). Then \( S_\lambda \) is the resolvent of a positive contraction semigroup on \( \mathcal{L} \).

**Proof.** Suppose \( g > 0, g \in \mathcal{L} \). Let \( S_j^\mu g = \sum_{t=0}^\infty R_\mu (-L_{n_j} R_\mu)^j \) for \( \mu > 2\|L_{n_j}\| \). The proof goes exactly as for Theorem 3.4 provided we show \( S_j^\mu g > 0 \). But the proof of Theorem 4.1 goes through exactly as before if we redefine \( V \) to be \( \inf\{t > 0: (X_i, X_i) \in F_j\} \) and we consider \( \Sigma(S_\lambda g(X_i) - S_\lambda g(X_{i-}))1_{(X_i, X_i) \in F_j} \) instead of \( \Sigma(S_\lambda g(X_i) - S_\lambda g(X_{i-})) \), where for each \( j, F_j \) is a set such that \( n_j(x, dy) = 1_{F_j}(x) n(x, dy) \).

**Example 4.3.** Consider Example 3.7. From the fact that if \( S_\lambda = R_\lambda + S_\lambda BR_\lambda, R_\lambda = S_\lambda(I - BR_\lambda) \), it should be clear that if the process constructed in Example 3.7 does have Lévy system \((n, dt)\), a consequence of \( \S 5 \), then the semigroup constructed as a result of Theorem 4.2 will just be that of Brownian motion.

### 5. Lévy system

At this point we make three assumptions. Suppose \( X_t \) is a Hunt process. We know by \( \S \S 3 \) and 4 that \( S_\lambda \) is the resolvent of a semigroup \( Q_\lambda \) on \( \mathcal{L} \).

I. **We assume \( Q_\lambda \) can be extended to a semigroup \( \bar{Q}_\lambda \) on \( \mathcal{K} \).**

Let \( Y_t \) be a Markov process that has \( Q_\lambda \) as its semigroup.

II. **We assume there is a version of \( Y_t \) which is a Hunt process.**

By [2], \( Y_t \) has a Lévy system \((m, dH_t)\).

III. **We assume that we can suppose that \( H_t \) is absolutely continuous with respect to Lebesgue measure for all \( \omega \).**

As a consequence of III, we may suppose that \( Y_t \) has a Lévy system of the form \((m, dt)\) (see proof of Proposition 5.1 below).

I, II, and III can be shown to hold under fairly general conditions, in particular when \( B \) is bounded, but the proofs are long. See [1]. In some cases, however, it is easy to verify I, II, and III. For example, it is well known that if \( \mathcal{L} \) is the collection of bounded continuous functions, I and II must hold. A
simple condition that guarantees III, much stronger than is necessary, is the following.

**Proposition 5.1.** Suppose that given any pair of disjoint compact sets $F_1$ and $F_2$, there exists an $f \in \mathcal{K}$ (depending on $F_1$ and $F_2$) such that $R_x f > 0$ and $R_x f$ is 0 on $F_1$ and bounded away from 0 on $F_2$. Then III holds.

**Proof.** If $F_1$ and $F_2$ are two disjoint compact sets, let $f \in \mathcal{K}$ such that $R_x f > 0$ and $R_x f$ is 0 on $F_1$ and bounded away from 0 on $F_2$, say by $\delta$.

By Lemma 3.2 or its counterpart from the subtracting jumps case, $R_x f = S_\lambda h$ for some $h \in \mathcal{K}$. Let $T = \inf\{t > 0: Y_t \in F_1, Y_t \in F_2\}$. Let $U = \inf\{t > 0: Y_t \in F_1\}$. By Dynkin’s identity,

$$
\delta P^x(T < t) < E^x S_\lambda h(Y_{T \wedge t}) - E^x S_\lambda h(Y_{U \wedge t})
$$

$$
= E^x \int_{U \wedge t}^{T \wedge t} (\lambda S_\lambda - I)h(Y_s) \, ds < 2\lambda \|h\| t.
$$

Let $T_0 = 0$, $T_{i+1} = T_i + T \circ \theta_{T_i}$, the $(i + 1)$st time $Y_i$ jumps from $F_1$ to $F_2$. An induction argument and the strong Markov property gives $P^x(T_i < t) < (ct)^i$, where $c = 2\lambda \|h\| / \delta$. Hence if $t < 1/(2c)$,

$$
E^x \sum_{s < t} 1_{(Y_s \in F_1, Y_s \in F_2)} < \sum_{i=0}^{\infty} i P^x(T_i < t) < 4ct.
$$

Repeating the construction of Benveniste and Jacod, one can see that one can take their additive functional $H_t$ so that $E^x H_t < c' t$ for some constant $c'$. By [11, Lemma 6.6], $H_t < c' t$ a.s.; hence $H_t = \int_0^t h(Y_s) \, du$ for some $h$, bounded by $c'$ by Proposition 5.3 below. If we let $m'(x, \theta_y) = h(x)m(x, \theta_y)$, it follows that $(m', dt)$ is a Lévy system for $Y_t$.

Since it should be clear from the context which we mean, we are using $P^x$, $E^x$ to refer to probabilities and expectations for both $X_t$ and $Y_t$.

Note that in Example 3.7, $R_\lambda (K)$ contains the collection of twice continuously differentiable functions with compact support, and hence satisfies Proposition 5.1.

We now want to show that if $X_t$ is continuous and we added jumps, $Y_t$ has Lévy system $(n, dt)$ and that if $X_t$ has Lévy system $(n, dt)$ and we subtracted jumps, $Y_t$ is continuous. We first prove some lemmas. Let us say $L_n \ll L_m$, where $L_n$ and $L_m$ are Lévy operators if: whenever $G$ is an open set, $g$ a continuous bounded function $> 0$ with support contained in $G^c$, $1_G(x)L_n g(x) < 1_G(x)L_m g(x)$ for all $x$ except possibly for a set of potential 0.

**Lemma 5.2.** A set $D$ has potential 0 with respect to $X_t$ if and only if it has potential 0 with respect to $Y_t$.

**Proof.** We have $R_\lambda 1_D = 0$. Hence $S_\lambda 1_D = R_\lambda 1_D \pm S_\lambda BR_\lambda 1_D = 0$. The “if” part follows by symmetry.
Proposition 5.3. If $Z_t$ is any Hunt process, $g > 0$, either $f > 0$ or $f$ bounded, and

$$E^x \int_0^t f(Z_u) \, du < E^x \int_0^t g(Z_u) \, du < \infty$$

for all $x$ and $t$, then $f < g$ except for a set of $Z$-potential 0.

Proof. By Proposition 2.8 of [11],

$$E^x \int_T^U f(Z_u) \, du \leq E^x \int_T^U g(Z_u) \, du$$

for all bounded stopping times $T < U$. Let $e > 0$, $K$ any compact contained in \( \{ x : f(x) > g(x) + e \} \), $F_i$ compact sets increasing to $K^c$ such that $F_i$ is disjoint from $K$. Let

$$T_{i1} = \inf \{ t : Z_t \in K \}, \quad U_{i1} = \inf \{ t > T_{i1} : Z_t \in F_i \},$$

$$T_{i2} = \inf \{ t > U_{i1} : Z_t \in K \}, \quad U_{i2} = \inf \{ t > T_{i2} : Z_t \in F_i \},$$

e tc. By quasi-left continuity, $T_j \to \infty$, $U_j \to \infty$ as $j \to \infty$. If $L$ is any real,

$$\sum_{j=1}^\infty E^x \int_{T_j \wedge L} f(Z_u) \, du = E^x \int_0^L f(Z_u) 1_{U \in [T_j, U_j]} \text{ for some } j \, du$$

$$\to E^x \int_0^L f(Z_u) 1_K(Z_u)$$

as $i \to \infty$ by dominated convergence.

Combining with a similar equation for $g$, we have

$$E^x \int_0^L f(Z_u) 1_K(Z_u) \, du < E^x \int_0^L g(Z_u) 1_K(Z_u) \, du.$$

We must have that $K$ has potential 0, and since $L$ was arbitrary, the result follows.

Lemma 5.4. Suppose $Z_t$ is a Hunt process with Lévy system $(m, dt)$, Lévy operator $L_m$, and resolvent $V_x$. Suppose $G$ is an open set, $g$ is a nonnegative, bounded continuous function with support in $G^c$. Suppose $\lambda V_x g \to g$ weakly as $\lambda \to \infty$. Suppose $\{ g_\lambda \}$ is a collection of functions that converge weakly to $g$ as $\lambda \to \infty$ and that $D$ is a bounded operator such that $Dg_\lambda \to Dg$ weakly. Suppose $T$ and $U$ are two bounded stopping times such that $T(\omega) < t < U(\omega)$ implies $Z_t(\omega) \in G$. Then

$$E^x \int_T^U \lambda^2 V_\lambda g(Z_t) \, dt \to E^x \int_T^U L_m g(Z_t) \, dt < \infty; \quad (1)$$

$$E^x \int_T^U \lambda V_\lambda Dg_\lambda(Z_t) \, dt \to E^x \int_T^U Dg(Z_t) \, dt. \quad (2)$$

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Proof. (1) By the hypotheses on $g$, $T$, and $U$,
\[ E^x \int_T^U \lambda^2 V_\lambda g(Z_t) \, dt = E^x \int_T^U \lambda(\lambda V_\lambda - I)g(Z_t) \, dt \]
\[ = E^x \lambda V_\lambda g(Z_T) - E^x \lambda V_\lambda g(Z_U) \]
\[ = E^x \lambda V_\lambda g(Z_T) - E^x \lambda V_\lambda g(Z_U) + E^x \sum_{T < t < U} [g(Z_t) - g(Z_{t-})] \]
\[ = E^x \int_T^U L_m g(Z_t) \, dt. \]

(2)
\[ E^x \int_U^T \lambda^2 V_\lambda D g(Z_t) \, dt = E^x \lambda V_\lambda D g(Z_U) - E^x \lambda V_\lambda D g(Z_T) + E^x \int_T^U D g(Z_t) \, dt \]
\[ = E^x \int_T^U D g(Z_t) \, dt. \]

Lemma 5.5. Suppose $Z_t$ is a Hunt process with resolvent $V_\lambda$, Lévy system $(m, dt)$ and Lévy operator $L_m$. Let $D$ be a bounded operator such that whenever $\{f_n\}$ converges weakly to $f$, $Df_n$ converges weakly to $Df$. Suppose that $W_\lambda$ is a collection of positive linear operators such that $W_\lambda = V_\lambda(I - DW_\lambda)$, $\|W_\lambda\| < 1$, and $\lambda W_\lambda g$ converges weakly to $g$ as $\lambda \to \infty$ whenever $g$ is bounded and continuous. Then $L_m \gg D$.

Proof. Let $G$ be open, $g$ nonnegative, bounded, and continuous with support in $G^c$. Suppose $T$ and $U$ are stopping times such that if $T(\omega) < t < U(\omega)$, $Z_t(\omega) \in G$. Since $W_\lambda g > 0$,
\[ E^x \int_T^U \lambda^2 V_\lambda g(Z_t) \, dt > E^x \int_T^U \lambda V_\lambda D(\lambda W_\lambda g)(Z_t) \, dt. \]
Letting $\lambda \to \infty$, by Lemma 5.4, we get
\[ \infty > E^x \int_T^U L_m g(Z_t) \, dt > E^x \int_T^U D g(Z_t) \, dt. \]
Since $\|Dg\| < \infty$, the result now follows by an argument very similar to Lemma 5.3.

Lemma 5.6. Suppose $m$ and $n$ are two kernels, $L_m$ and $L_n$ their Lévy operators, and $L_m \gg L_n$. Then $m \gg n$.

Proof. Suppose $\{G_i\}$ is a countable open basis for the topology of $E$. For each $G_i$, select a countable dense subset $\{g_{ij}\}$ of the positive continuous functions with support in $G_i^c$. If
\[ N = \{x: 1_{G_i}(x)L_m g_{ij}(x) < 1_{G_j}(x)L_n g_{ij}(x) \text{ for some } i, j\}, \]
$N$ has potential 0. Note that if $x \in G_i$, $L_m g_{ij}(x) = m(x, g_{ij})$ and similarly for
A monotone class argument shows that if $F$ is any Borel set and $x \in N$, 
$m(x, F \cap G_i^c) > n(x, F \cap G_i^c)$, for each $i$. It follows that $m \geq n$.

**Theorem 5.7.** Suppose $X_t$, $R_\lambda$, $n$, $B$, and $S_\lambda$ are as in §4. Suppose $B$ is the Lévy operator of $n$. Suppose $n_\eta$ is a sequence of bounded kernels that increase strongly to $n$ such that if $L_n$ is the Lévy operator of $n$, $L_n R_\lambda g \to B R_\lambda g$ for all $g \in \mathcal{K}$. Then $Y_t$ is continuous a.s.

**Proof.** Suppose $Y_t$ has Lévy kernel $m > 0$ with Lévy operator $L_m$. Let $m_k$ be bounded kernels strongly increasing to $m$, $L_{m_k}$ the Lévy operators. If $V_\lambda = S_\lambda (\sum_{i=0}^\infty (-L_{m_k} S_\lambda)^i)$, $V_\lambda$ is positive by Theorem 4.1 and the proof of Theorem 4.2. Now let $W_\lambda = V_\lambda (\sum_{i=0}^\infty (B - L_n) V_\lambda)^i)$. Since $V_\lambda = S_\lambda (I - L_{m_k} V_\lambda)$, $B V_\lambda$ will have norm $< 1$ if $\lambda$ is large enough; hence for large enough $\lambda$ so will $(B - L_n) V_\lambda$. Since for all $f \in \mathcal{K}$, $W_\lambda f = R_\lambda h$ for some $h$ (Lemma 2.2), $(L_n - L_\eta) W_\lambda f \to (B - L_\eta) W_\lambda f$. Hence by Theorem 3.4, $W_\lambda$ is positive. By I and II (valid in this case by [1]), $W_\lambda$ is the resolvent of a Hunt process, hence $\lambda W_\lambda g \to g$ weakly whenever $g$ is continuous.

By Lemma 2.2, $W_\lambda = R_\lambda (I - (L_n + L_{m_k}) W_\lambda)$. By Lemma 5.5, $B \geq L_n + L_{m_k}$. Since if $g(x) = 0$, $g \geq 0$, $L_n g(x) = n_\eta (x, g) \uparrow n(x, g) = B g(x)$, letting $j \to \infty$ gives $B \geq B + L_m$. Similarly, letting $k \to \infty$, we get $0 \geq L_m$, or $m = 0$ by Lemma 5.6. Hence the expected number of jumps in finite time is 0 by the Lévy system identity, i.e., $Y_t$ is continuous a.s.

Finally we show that if $X_t$ is continuous and we added jumps, $Y_t$ has Lévy system $(m, dt)$.

**Theorem 5.8.** Suppose $X_t$, $n$, $R_\lambda$, $B$, and $S_\lambda$ are as in §3. Suppose $B$ is the Lévy operator for $n$. Suppose there exist bounded kernels $n_\eta$ strongly increasing to $n$ with Lévy operator $L_n$ such that $L_n S_\lambda \to B S_\lambda g$ for all $g \in \mathcal{K}$, then $Y_t$ has Lévy system $(m, dt)$.

Suppose $Y_t$ has Lévy system $m$, Lévy operator $L_m$. We show $m \geq n$. Let

$$V_\lambda = R_\lambda (\sum_{i=0}^\infty ((B - L_n) R_\lambda)^i).$$

As in Theorem 5.7, $V_\lambda$ is positive. By Lemma 2.2, $V_\lambda = S_\lambda (I - L_n V_\lambda)$. Hence $L_m \geq L_n$. Since $L_n g(x) \uparrow B g(x)$ if $g(x) = 0$, letting $j \to \infty$ gives $L_m \geq B$. By Lemma 5.6, $m \geq n$.

Next we must show $m \leq n$. Let $m_k$ be bounded kernels strongly increasing to $m$, $L_{m_k}$ the Lévy operators. Let $j$ be fixed. Again letting $V_\lambda = R_\lambda (\sum_{i=0}^\infty ((B - L_n) R_\lambda)^i)$, we know $V_\lambda$ is positive. Let $p$ be the Lévy kernel for the process with resolvent $V_\lambda$, $L_p$ the Lévy operator. We first show $L_p = L_m - L_n$.

If $T_\lambda = S_\lambda (\sum_{i=0}^\infty (-L_{m_k} S_\lambda)^i)$, $T_\lambda$ is positive since $Y_t$ has Lévy kernel $m \geq m_k$.

$$T_\lambda = R_\lambda (I + (B - L_{m_k}) T_\lambda) = R_\lambda (I + ((B - L_n) + (L_n - L_{m_k})) T_\lambda).$$
from which it follows that \( T_\lambda = V_\lambda((I + L_\eta - L_m)T_\lambda) \). By Lemma 5.5, \( L_\eta \gg L_m - L_\eta \). Letting \( k \to \infty \), \( L_\eta \gg L_m - L_\eta \).

On the other hand, let \( p_r \) be a sequence of kernels increasing strongly to \( p \), with Lévy operators \( L_\eta \). If \( U_\lambda = V_\lambda(\Sigma_i^\infty 0(-L_\eta V_\lambda)^i) \), \( U_\lambda \) is positive, and

\[
U_\lambda = V_\lambda(I - L_\eta U_\lambda) = S_\lambda(I - (L_\eta + L_\eta)U_\lambda).
\]

Hence \( L_\eta \gg L_m - L_\eta \). Letting \( r \to \infty \), \( L_\eta \ll L_m - L_\eta \). Hence \( p = m - n \).

We know \( \eta(x, dy) = 1_{H}(x, y)n(x, dy) \) for sets \( H \), \( m_\eta(x, dy) = 1_{R}(x, y)m(x, dy) \) for sets \( R \). Let

\[
q_{jk}(x, dy) = 1_{F}(x, y)m(x, dy) - 1_{F}(x, y)n(x, dy),
\]

\[
s_{jk}(x, dy) = 1_{F}(x, y)m(x, dy) - 1_{F}(x, y)n(x, dy),
\]

the associated Lévy operators. Since \( n \ll m \ll m_\eta \), \( q_{jk} \) is bounded. \( s_{jk} < m_\eta \), and \( s_{jk} \) is also bounded. Note also that \( q_{jk} - s_{jk}(x, dy) = 1_{F}(x, y)(m - n)(x, dy) \), which is independent of \( j \).

Since \( q_{jk} < m - n \), the Lévy kernel for the process with resolvent \( V_\lambda \),

\[
W_\lambda = V_\lambda \left( \sum_{i=0}^\infty (-L_\eta V_\lambda)^i \right)
\]
is positive by §4. Then \( Z_\eta = W_\lambda(\Sigma_i^\infty 0(L_\eta W_\lambda)^i) \) is positive by Theorem 3.4. If we denote \( L_\eta = L_\eta \) simply by \( L_\eta \), \( Z_\eta = V_\lambda(I - L_\eta Z_\eta) \), hence

\[
Z_\eta = R_\lambda(I + (B - L_\eta - L_\eta)Z_\eta).
\]

Letting \( j \to \infty \), we get \( Z_\eta = R_\lambda(\Sigma_i^\infty 0(-L_\eta R_\lambda)^i) \) is positive. \( Z_\lambda = R_\lambda(I - L_\eta Z_\eta) \). Since \( X \) is continuous, the Lévy operator of \( X \) is 0, and by Lemma 5.5, \( 0 \gg L_\eta \). Letting \( k \to \infty \), by Lemma 5.6 we conclude that \( m - n < 0 \).

6. A probabilistic construction. It would be nice to have a probabilistic construction of \( Y_t \). If \( n(x, E) \) is finite and small enough, one exists. In general, as in Example 3.7, none is known.

Suppose \( X_t \) has finitely many jumps in finite time. Kill \( X_t \) at the time \( T \) of the first jump, then “restart” it with distribution \( X_{T-} \). Proceed until the first jump of the new process and kill again, etc. “Restarting” can be made precise through the work of Ikeda, Nagasawa, and Watanabe [6] or Meyer [10]. See also Stroock [13] in the case \( n \) is bounded. That this pieced together process is the same as the process constructed by the methods of this paper follows readily from [12]. If \( X_t \) has infinitely many jumps in finite time, as when \( n(x, E) \) is infinite, the time of the first jump will, in general, be identically equal to 0, and this method will fail.

To add jumps, kill the process according to the multiplicative functional \( \exp(\cdot n(X_t, E)) \). At the time of death \( T \), “restart” it with distribution \( n(X_{T-}, dy)/n(X_T, E) \). Again it is not too difficult to check that this new process is the \( Y_t \) one would have constructed through the methods of this
paper. Again, however, if \( n(x, E) \) is infinite, this method will fail since 
\[ \exp(-n(X_t, E)) \]
would be identically 0.

**References**


**Department of Mathematics, University of Washington, Seattle, Washington 98195**