CONTINUOUS FUNCTIONS ON COUNTABLE COMPACT ORDERED SETS AS SUMS OF THEIR INCREMENTS

BY

GADI MORAN

ABSTRACT. Every continuous function from a countable compact linearly ordered set $A$ into a Banach space $V$ (vanishing at the least element of $A$) admits a representation as a sum of a series of its increments (in the topology of uniform convergence). This series converges to no other sum under rearrangements of its terms. A uniqueness result to the problem of representation of a regulated real function on the unit interval as a sum of a continuous and a steplike function is derived.

0. Introduction. Let $A$ be a compact linearly ordered set, $V$ a (real or complex) Banach space. Let $C(A, V)$ denote the Banach space of continuous functions from $A$ to $V$ with the supremum norm. The right neighbour $a'$ (the left neighbour 'a) of $a \in A$ is defined as the maximal (minimal) $b \in A$ such that $(a, b) = \emptyset ((b, a) = \emptyset)$. With $f: A \to V$ associate $f^+: A \to V$ defined by

$$f^+(a) = f(a') - f(a).$$

With $a \in A, v \in V$ associate a function $J_v^a: A \to V$—the $v$-jump at $a$—defined by

$$J_v^a(b) = \begin{cases} 0, & b < a, \\ v, & a < b. \end{cases}$$

Then $J_v^a(a)$ is called the increment of $f$ at $a$. Note that if $f$ is in $C(A, V)$ so is every increment of $f$. Let $m_A (M_A)$ denote the minimal (maximal) element of $A$. The purpose of this article is to prove

**Theorem 1.** Let $A$ be a countable compact ordered set, $V$ a Banach space and let $f \in C(A, V)$ satisfy $f(m_A) = 0$. Then:

(a) There is an enumeration $(J_n)_n$ of the increments of $f$ such that $f = \Sigma_n J_n$ holds in $C(A, V)$.

(b) If $(\tilde{J}_n)_n$ is any enumeration of the increments of $f$ for which $\Sigma_n \tilde{J}_n$ converges in $C(A, V)$, then $\Sigma_n \tilde{J}_n = f$.

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In the special case where \( \nu = \rho + 1 \) is a compact countable ordinal Theorem 1(a) implies: Let \( f: \nu \to V \) be continuous. Then there is an enumeration \((\mu_n)_{\nu} \) of the ordinals smaller than \( \nu \) such that for every \( \mu < \nu, f(\mu) = f(0) + \sum_{\mu < \nu} (f(\mu + 1) - f(\mu_n)) \). Moreover, the convergence is uniform in \( \mu \) (in a sense made precise in §§2, 3). Similarly, there is an enumeration \((\bar{\mu}_n)_{\nu} \) of the ordinals smaller than \( \nu \) so that the identity \( f(\mu) = f(0) + \sum_{\mu < \bar{\mu}_n} (f(\bar{\mu}_n) - f(\bar{\mu}_n + 1)) \) holds uniformly for \( \mu < \nu \).

Conditionally convergent series, every convergent rearrangement of which have the same sum, are known to exist in any infinite dimensional Banach space [Ha], [McA]. Theorem 1 provides a host of natural examples of this phenomenon. The simplest of those is the following one: Let \( A = \omega + 1 = \{0, 1, 2, \ldots, \omega\} \) and let \( V = \mathbb{R} \), the reals. Let \((a_n)_{\omega} \) be any sequence of reals such that \( \sum_n a_n \) is conditionally convergent, and \( \sum_n a_n = a \). Define \( f \in C(A, \mathbb{R}) \) by \( f(n) = \sum_{i < n} a_i, f(\omega) = a \). It is straightforward to check that \((J_n)_{\omega} \) is an enumeration of \( f \)’s increments, \( \sum_n J_n = f \), and that \( \sum_n J_n \) is convergent in \( C(A, \mathbb{R}) \) iff \( \sum_n a_n \) is convergent to \( a \). Thus, \( \sum_n J_n \) is a conditionally convergent series in \( C(A, \mathbb{R}) \), every convergent rearrangement of which has the sum \( f \).

Although \( V \) is assumed to be Banach space in Theorem 1, the theorem is true in a wider context. We may assume \( V \) to be arbitrary complete normed abelian group.1 (A typical example of such a \( V \) which is not the additive group of a Banach space is the \( p \)-adic ring \( Q_p \), see e.g. [F].)

Let \( A \) be arbitrary ordered set. Call \( a \in A \) right isolated (left isolated) iff \( a < a' \) (‘\( a < a \)’). Call \( a \) a core point iff it is neither right nor left isolated. Let \( f: A \to V \). We write \( f(a +) = v \) iff \( \forall \varepsilon > 0 \exists b > a [a < c < b \implies \|f(c) - v\| < \varepsilon] \). Define \( f(a -) \) similarly. Note that \( f(a) = f(a +) \) (\( f(a) = f(a -) \)) whenever \( a \) is right (left) isolated. Let \( f(m_A -) = f(m_A), f(M_A +) = f(M_A) \).

We call \( f: A \to V \) a regulated function iff:

(i) \( f(m_A +), f(M_A -) \) exist, as do \( f(a +), f(a -) \) for \( m_A < a < M_A \).
(ii) \( f(a) = f(a -) \) whenever \( a \) is right isolated,
\( f(a) = f(a +) \) whenever \( a \) is left isolated,
\( f(a) = f(a -) \) whenever \( a \) is a core point.

The reader will note that a regulated function is continuous if and only if it satisfies \( f(a -) = f(a +) \) for every core point \( a \).

If \( A \) is compact, the family of all regulated functions from \( A \) to \( V \) with the supremum norm is again a Banach space \( \text{Reg}(A, V) \) containing \( C(A, V) \). For \( f \in \text{Reg}(A, V) \) define \( f*: A \to V \) by \( f^*(a) = f(a +) - f(a) \) if \( a \) is a core point, and \( f^*(a) = f^*(a) \) otherwise. The increment of \( f \) at \( a \) is redefined to be \( J^* \).

Theorem 1 is equivalent to

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1By a norm on an additive group \( V \) we mean a function \( \| \| \) from \( V \) into the nonnegative real numbers, satisfying \( \|v\| = 0 \iff v = 0, \|v\| = \|-v\| \) and \( \|v + \omega\| < \|v\| + \|\omega\| \).

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Theorem 2. Let $A$ be a compact countable ordered set, $V$ a Banach space, and let $f \in \text{Reg}(A, V)$ satisfy $f(m_A) = 0$. Then:

(a) There is an enumeration $(J_n)_n$ of the increments of $f$ such that $f = \Sigma_n J_n$ holds in $\text{Reg}(A, V)$.

(b) If $(\tilde{J}_n)_n$ is any enumeration of the increments of $f$ for which $\Sigma_n \tilde{J}_n$ converges in $\text{Reg}(A, V)$, then $\Sigma_n \tilde{J}_n = f$.

Indeed, $f^* = f^+$ for $f \in C(A, V)$ and so Theorem 2 clearly implies Theorem 1. Conversely, assume Theorem 1 and let $A$ be a countable compact ordered set. Replacing each core point $a \in A$ by a pair of points $\bar{a} < a$, we obtain another compact countable ordered set $\hat{A}$ with no core points. Define $g: \hat{A} \to A$ by $g(\bar{a}) = g(a) = a$ for a core point $a \in A$, $g(a) = a$ for a noncore point $a \in A$. For $f \in \text{Reg}(A, V)$ let $\hat{f} = f \circ g$. Then $\hat{f} \in C(\hat{A}, V)$ and it is easily checked that Theorem 1 for $\hat{f}$ implies Theorem 2 for $f$ (compare Proposition 3.1).

We apply Theorem 2 to obtain a theorem on $\text{Reg}(I, V)$, where $I = [0, 1]$ is the closed unit interval (see [GMW], [M]). $s \in \text{Reg}(I, V)$ is called steplike iff there is an enumeration $(J_n)_n$ of its increments (only countably many of which are nonzero) so that $s = \Sigma_n J_n$ holds in $\text{Reg}(I, V)$. The separation of discontinuities of an $f \in \text{Reg}(I, V)$ by means of a steplike function dates back essentially to Lebesgue’s Theorem on monotone real functions: every monotone real function has a unique representation $f = g + s$, where $g$ is a monotone continuous function and $s$ is a monotone steplike function. Similarly, every $f \in \text{Reg}(I, V)$ of bounded variation has a unique representation $f = g + s$, where $g \in C(I, V)$ and $s$ is steplike. In general, however, such a representation need not exist, and when it exists need not be unique [GMW], [M]. Call $f \in \text{Reg}(I, V)$ representable if $f$ has a representation $f = g + s$, where $g \in C(I, V)$ and $s \in \text{Reg}(I, V)$ is steplike, and uniquely representable if $f$ has precisely one such representation. Obviously, the representability and unique representability of $f \in \text{Reg}(I, V)$ depend only on $f^*$. Now $f \mapsto f^*$ is a continuous linear mapping of $\text{Reg}(I, V)$ onto $C_0(I, V) = \{h: I \to V \mid \{t: \|h(t)\| > \varepsilon\} \text{ is finite for every } \varepsilon > 0\}$ with the supremum norm. Call $h \in C_0(I, V)$ summable (uniquely summable) iff some—or equivalently, any—$f$ in $\text{Reg}(I, V)$ with $f^* = h$ is representable (uniquely representable).

The complete characterization of summable or uniquely summable members of $C_0(I, R)$ is still open. The methods of this paper suffice to characterize, however, those summable members of $C_0(I, R)$ whose support has a countable closure (to appear elsewhere). We shall address ourselves here only to the uniqueness problem.

In [M] it is shown that an $h \in C_0(I, R)$ exists, such that every $f \in \text{Reg}(I, R)$ with $f(0) = 0$ and $f^* = h$ is steplike. Obviously, the support of such an $h$ is necessarily dense in $I$, and so has $I$ for its closure. In a similar way, it
can be shown that given any countable set $A$ in $I$ of uncountable closure, there is an $h \in C_0(I, R)$ whose support is $A$ such that the family \{s \in \text{Reg}(I, R): s$ is steplike and $s^* = h\}$ is of the cardinality of the continuum. This is not anymore possible if $h$'s support is of countable closure. The following fact was stated in [M] for the case $V = R$:

**Theorem 3.** Let $A$ be a countable closed subset of $I$. Let $f \in \text{Reg}(I, V)$ satisfy \{$t: f^*(t) \neq 0$\} $\subseteq A$. If $f$ is representable, then $f$ is uniquely representable.

**Proof.** Let $W \subseteq \text{Reg}(I, V)$ be the closed subspace of those $f \in \text{Reg}(I, V)$ that vanish at 0 and are constant on every component of $I - A$. For $f: I \rightarrow V$ let $f_A: A \rightarrow V$ denote the restriction of $f$ to $A$. Clearly, $f_A \in \text{Reg}(A, V)$ whenever $f \in W$, and the mapping $Tf = f_A$ is a linear isometry of $W$ onto $\text{Reg}_0(A, V) = \{f \in \text{Reg}(A, V): f(m_A) = 0\}$.

We show now that $W = \{s \in \text{Reg}(I, V): s$ is steplike and \{$t: s^*(t) \neq 0$\} $\subseteq A\}$. Clearly $s \in W$ whenever $s$ is steplike and \{$t: s^*(t) \neq 0$\} $\subseteq A$. Conversely, let $f \in W$. By Theorem 2(a), there is an enumeration $\langle f_n \rangle_n$ of $f_A$'s increments so that $f_n = \Sigma_n J_n$. Apply $T^{-1}$ and obtain $f = \Sigma_n T^{-1}J_n$ in $\text{Reg}(I, V)$. But $\langle T^{-1}J_n \rangle_n$ is obviously an enumeration of $f$'s increments, and so $f$ is steplike.

Now assume that $f \in \text{Reg}(I, V)$ is representable, and that \{$t: f^*(t) \neq 0$\} $\subseteq A$. Let $f = g_1 + s_1 = g_2 + s_2$ where $g_1, g_2 \in C(I, V)$ and $s_1, s_2$ steplike. By $f^* = s_1^* = s_2^*$ we have $s_1, s_2 \in W$, and $(Ts_1)^* = (Ts_2)^* = f^*$. Hence $Ts_1, Ts_2 \in \text{Reg}(A, V)$ have the same increments and vanish at $m_A$. By Theorem 2(b), $Ts_1 = Ts_2$, whence $s_1 = s_2$ and $g_1 = g_2$. □

Theorem 1 is proved essentially by induction on the Cantor rank of the scattered space $A$. In §1 the class of countable compact order types is characterized as the smallest class of order types including $\emptyset$ and closed under one infinitary operation, the compact-limit operation (Definition 1.1). This yields a useful induction principle for countable compact order types. §2 is devoted to the proof of a rather technical summing lemma, which is the core of the inductive proof of Theorem 1, given in §3.

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1. An induction principle for countable compact order types. We specify some notation first. An ordinal number is identified with the set of smaller ordinals and so $\mu < \nu$ and $\mu \in \nu$ are interchangeable. A cardinal number is an ordinal not equivalent to a smaller ordinal. $\omega$ denotes the first infinite cardinal. $|A|$ denotes the cardinality of $A$, that is, the cardinal number equivalent to the set $A$. An order on $A$ (or an ordering of $A$) is a total irreflexive and transitive relation on $A$. Let $\leq$ be an order on $A$, $B, C \subseteq A$, then $B < C$ means $b < c$ for all $b \in B$, $c \in C$. Note that $\emptyset < B < \emptyset$ holds for every $B \subseteq A$. $B < x \ (x < B)$ stands for $B < \{x\}$ $((x) < B)$. The order
Functions on Countable Compact Ordered Sets

Type of \( A \) is denoted by \( \overline{A} \). The reader is referred to [E] for the definitions of an order type and of arithmetical operations on ordered sets and order types. Intervals in an ordered set are denoted in the usual manner, e.g. \([a, b) = \{c \in A: a < c < b\}\), \((a, b) = \{c \in A: a < c < b\}\). Ordinals are considered as ordered sets, being well ordered by the membership relation. An ordered set is considered as a topological space, with the topology generated by the sets \( \{b: b < a\} \) and \( \{b: a < b\} \). The following proposition will be useful ([HK], see e.g. [Ke, p. 162]):

**Proposition 1.0.** Let \( A \) be a nonempty ordered set. Then \( A \) is compact if and only if:

(i) \( A \) has a smallest element,

(ii) every nonempty subset of \( A \) has a least upper bound in \( A \).

**Definition 1.1.** Let \( A_n \) be an ordered set and let \( \alpha_n = \overline{A}_n \) \((n \in \omega)\). Assume further that \( A_n \cap A_m = \emptyset \) for \( n \neq m, x \in \cup_{n \in \omega} A_n \) and denote by \( \prec_n \) the ordering of \( A_n \). Define the \( x \)-compact-limit-sum of \( (A_n)_{n \in \omega} \), denoted \( x\text{-CL}_n A_n \), as follows. The domain of \( x\text{-CL}_n A_n \) is \( \{x\} \cup \cup_{n \in \omega} A_n \). The ordering on \( x\text{-CL}_n A_n \) is defined by:

1. \( a < b \) iff \( a <_n b \) for \( a, b \in A_n \).
2. \( A_{2n} < A_{2n+2} < x < A_{2n+3} < A_{2n+1}, n \in \omega \).

Let \( \alpha \) be the order type of \( x\text{-CL}_n A_n \). Then \( \alpha \) is defined as the compact-limit-sum of \( (\alpha_n)_{n \in \omega} \), denoted by \( \alpha = \text{CL}_n \alpha_n \). The rest of this section is devoted to proving

**Theorem 1.2.** Let \( \mathcal{C} \) denote the class of countable compact order types. Then \( \mathcal{C} \) is the smallest class of order types including \( \emptyset \) and closed under the compact-limit-sum operation.

As a corollary we have the following

**Induction Principle.** Let \( P(A) \) be a statement on ordered sets. If \( P(\emptyset) \) is true, and \( P(x\text{-CL}_n A_n) \) is true whenever \( P(A_n) \) is true for every \( n \in \omega \), then \( P(A) \) is true for every countable compact ordered set.

Let \( \mathcal{C}_0 \) denote the smallest class of order types including \( \emptyset \) and closed under the compact-limit-sum operation. Since \( x\text{-CL}_n A_n \) is countable whenever each \( A_n \) is, and by Proposition 1.0 it is compact whenever each \( A_n \) is, we have \( \mathcal{C}_0 \subseteq \mathcal{C} \).

The reverse inclusion is proved essentially by induction or the Cantor rank \( \text{rk}(\alpha) \) of \( \alpha \). We first resume some notation and facts (see e.g. [Ke], [Ku]). For a topological space, \( X, X' \) denotes the set of nonisolated points of \( X \). Define by induction \( X^{(\omega)} \) for every ordinal \( \mu \) as follows. \( X^{(0)} = X, X^{(\mu+1)} = X^{(\mu)'}, \) and \( X^{(\omega)} = \cap_{\mu < \nu} X^{(\mu)} \) when \( \nu \) is a limit ordinal. Then \( X^{(\omega)} \subseteq X^{(\mu)} \) for \( \mu < \nu, \) and \( \text{rk}(X) \) is defined as the least ordinal \( \mu \) for which \( X^{(\mu)} = X^{(\mu+1)} \). \( X \) is called scattered iff \( X^{(\omega)} = \emptyset \) for \( \nu > \text{rk}(X) \).
Let $X$ be a scattered compact nonempty Hausdorff space. Then $\text{rk}(X) = \mu + 1$ for some ordinal $\mu$, and $X^{(\mu)}$ is finite. Define $\text{ch}(X) = (\mu, m)$ where $\text{rk}(X) = \mu + 1$ and $|X^{(\mu)}| = m > 0$. Also let $\text{ch}(\emptyset) = (0, 0)$.

Every countable compact set of reals $X$ is scattered, its induced topology coincides with its order topology, and its rank is countable. Every countable compact ordered set is order isomorphic with a compact countable set of reals, and so $\mathcal{C}$ is actually the class of order types of countable compact sets of reals.

We define $\text{ch}(\alpha)$ for a compact order type $\alpha$ by $\text{ch}(\alpha) = \text{ch}(A)$, where $A$ is an ordered set of type $\alpha$. In view of the previous remarks the following is clear.

**Proposition 1.3.** Let $\alpha \in \mathcal{C}$, $\text{ch}(\alpha) = (\mu, m)$. Then there are $\alpha_i, i < m$ such that $\alpha = \sum_{i<m} \alpha_i$ and $\text{ch}(\alpha_i) = (\mu, 1)$.

**Proposition 1.4.** Let $A$ be a scattered ordered set and let $x$ be a nonisolated point of $A$. Then $x$ is an accumulation point of isolated points. Moreover, if $m_A < x < M_A$ is a core point, then for every $a, b \in A$ with $a < x < b$ there are isolated points $\bar{a}, \bar{b} \in A$ such that $a < \bar{a} < x < \bar{b} < b$.

**Proof.** Let $\mu(x)$ be the ordinal $\mu$ satisfying $x \in A^{(\mu)} \setminus A^{(\mu+1)}$. Then $x$ is nonisolated iff $\mu(x) > 0$. Also, if $\mu(x) > 0$ then there is an interval around $x$ containing in addition only $y$'s with $\mu(y) < \mu(x)$. The proposition follows by a straightforward induction. □

**Corollary 1.5.** Let $A$ be a countable scattered ordered set and let $x$ be a core point in $A$ satisfying $m_A < x < M_A$. Then there are sequences $(a_n)_{n < \omega}$ and $(b_n)_{n < \omega}$ of isolated points in $A$ such that $a_n < a_{n+1} < x < b_{n+1} < b_n$, and $x = \sup_n a_n = \inf_n b_n$.

Order the class $\{(\mu, m) : \mu \text{ an ordinal, } m \in \omega\}$ lexicographically: $(\mu, m) < (\tilde{\mu}, \tilde{m})$ if $\mu < \tilde{\mu}$ or $\mu = \tilde{\mu}$ and $m < \tilde{m}$. A proof by induction on $\text{ch}(A)$ or $\text{ch}(\alpha)$ will refer to this well-ordering. We prove now $\mathcal{C} \subseteq \mathcal{C}_0$. Let $\alpha \in \mathcal{C}$, let $\text{ch}(\alpha) = (\mu, m)$ and proceed by induction. Let $\mu = 0$ and proceed by induction on $m$. Since $\emptyset \in \mathcal{C}_0$ the case $m = 0$ is done. Now $\text{ch}(\alpha) = (0, m)$ iff $\alpha = \tilde{m}$. Assume $\emptyset \in \mathcal{C}_0$. Let $a_0 = \tilde{m}$, $a_n = \emptyset$ for $n > 0$. Then $[\text{CL} \alpha] = \mu + 1$, and so $\mu + 1 \in \mathcal{C}_0$.

Assume now $\mu > 0$. Let first $m = 1$ and let $A$ be an ordered set of order type $\alpha$. Let $A^{(\mu)} = \{x\}$. By Corollary 1.5 pick a monotone sequence $(a_n)_{n \in \omega}$ of isolated points converging to $x$, say $a_n < a_{n+1} < x$. Define $A_{2n}$ by $A_0 = [m_A, a_0]$, $A_{2n+2} = [a_n, a_{n+1}]$. Since $a_n$ is isolated and $a_n \neq M_A$, $a_n < a_{n+1}$ and so $A_{2n} < A_{2n+2}$ and $\cup_{n \in \omega} A_{2n} = [m_A, x)$. In a similar way define a sequence of (possibly empty) closed intervals $A_{2n+1}$ in $A$ so that $A_{2n+3} < A_{2n+1}$ and
FUNCTIONS ON COUNTABLE COMPACT ORDERED SETS

$\bigcup_{n \in \omega} A_{2n+1} = (x, M_x)$. Now, by $A^{(n)} \subseteq A^{(n+1)} = \{x\}$, $x \notin A_n$ we have $A^{(n)} = \emptyset$. Thus $A_n$ is a compact countable ordered set with $\text{ch}(A_n) = (\mu_n, m_n)$, where $\mu_n < \mu$. By the induction hypothesis, $\overline{A}_n \in \mathcal{C}_0$, and so by $A = x$-CL$A_n A_n$, $\alpha \in \mathcal{C}_0$.

Assume next that $\text{ch}(\alpha) = (\mu, m + 1)$. By Proposition 1.3 let $\alpha = \beta + \tilde{\alpha}$, where $\text{ch}(\beta) = (\mu, m)$ and $\text{ch}(\tilde{\alpha}) = (\mu, 1)$. By the previous case, $\tilde{\alpha} = \text{CL}_n \tilde{\alpha}_n$, where $\tilde{\alpha}_n \in \mathcal{C}_0$. By induction, $\beta \in \mathcal{C}_0$. Let $\alpha_0 = \beta$, $\alpha_{2n} = \tilde{\alpha}_{2n+1}$ and $\alpha_{2n+1} = \tilde{\alpha}_{2n+2}$. Then $\alpha_n \in \mathcal{C}_0$ for all $n$, and $\alpha = \text{CL}_n \alpha_n$, whence $\alpha \in \mathcal{C}_0$.

This completes the proof of Theorem 1.2.

2. The Summing Lemma. We say that a set $A$ is countable if $|A| < \omega$. An ordering $<$ of $A$ is called an $|A|$-ordering if $A$ ordered by $<$ is isomorphic to $|A|$. Let $<$ be an $|A|$-ordering of a countable set $A$, and let $a_0, a_1, \ldots$ be the enumeration of $A$ in the order $<$. Define $m(A)_n^<$ for $m, n \in \omega$ as follows.

$m(A)_n^< = \{a_i: m < i < n\}$ if $m, n < |A|$. $m(A)_n^< = \emptyset$ if $|A| < m$. $m(A)_n^< = m(A)_{n+1}^<$ if $|A| < n$. For $B \subseteq A$, $m, n \in \omega$ let $m(B)_n^< = B \cap m(A)_n^<$. We abbreviate by omitting $<$ when no confusion is possible, and write $[B]_n^<$ for $m(B)_n^<$ for $n(B)^<_A$.

Let $h$ be a function from the countable set $A$ into a Banach space $V$. For a finite $\mathcal{B} \subseteq A$ let $h(\mathcal{B}) = \sum_{a \in \mathcal{B}} h(a)$. Let $<$ be an $|A|$-ordering of $A$, and let $B \subseteq A$. Whenever $v = \lim_n h([B]_n^<)$ exists we write $v = \sum_{\mathcal{B}}^<_A h$, and we say that $< \text{ sums } h \text{ over } B$ we say that $< \text{ sums } h \text{ over } B$ whenever $< \text{ sums } h \text{ over } B$ to some $v$. If $\mathcal{B}$ is a family of subsets of $A$, we say that $< \text{ sums } h \text{ over } \mathcal{B}$ if $< \text{ sums } h \text{ over } B$ for every $B \in \mathcal{B}$. We say that $< \text{ sums } h \text{ uniformly over } \mathcal{B}$ if for every $\varepsilon > 0$ there is an $N \in \omega$ such that for every $B \in \mathcal{B}$ and $k, l \in \omega$ with $N < k, l$, we have $\|h([B]_k^<) - h([B]_l^<)\| < \varepsilon$.

THE SUMMING LEMMA. Let $A = \bigcup_{i \in \omega} A_i$, where $\{A_i: i \in \omega\}$ is a disjointed family of countable sets. For each $i \in \omega$ let $\mathcal{B}_i$ be a family of subsets of $A_i$, and let $\mathcal{B} = \{B \subseteq A: B \cap A_i \in \mathcal{B}_i \text{ for all } i \in \omega\}$. Let $V$ be a Banach space and let $h: A \to V$. For $i \in \omega$ let $c_i$ be a positive real number and let $<_i$ be an $|A_i|$-ordering of $A_i$ so that:

$(2.0) \sum_{i \in \omega} c_i < \infty$.

$(2.1) <_i \text{ sums } h \text{ uniformly over } \mathcal{B}_i$.

$(2.2) \|h([B]_k^<_i) - h([B]_l^<_i)\| < c_i$ for $B \in \mathcal{B}_i$, $k, l \in \omega$.

Then there is an $|A|$-ordering $< of A such that:

$(2.3) a < b \text{ iff } a <_i b \text{ for } a, b \in A_i$.

$(2.4) < \text{ sums } h \text{ uniformly over } \mathcal{B}$.

$(2.5) \text{ Let } B \in \mathcal{B} \text{ and } B_i = B \cap A_i. \text{ Then }$

$$\sum_B h = \sum_{i \in \omega} \left( \sum_{B_i} h \right).$$
Proof. Define by induction on $s \in \omega$, $m_s$, $n_s \in \omega$ ($i \in \omega$) as follows. Let $m_0 = n_0 = 0$ ($i \in \omega$).
Assume that $m_{s-1}$, $n_{s-1}$ are already defined for $i \in \omega$. By (2.0) we may choose $m_s$ so that $m_{s-1} < m_s$ and
\[
(2.6) \sum_{m_s < i} c_i < 1/2s.
\]
By (2.2) we may further choose $n_s$ for $i < m_s$ so that $n_{s-1} < n_s$ and
\[
\|h\left(k \left[ B \right]_{i}^{-1} \right)\| < \frac{1}{2 \cdot m_s \cdot s} \quad \text{for} \quad B \in \mathcal{B}_i, \quad n_s < k, l.
\]
Let $n_s = 0$ for $m_s < i$.
Define $A_s$ by
\[
A_s = n_s \left[ A_i \right]^x_{n_{s+1}}.
\]
By construction, $A_s = \bigcup_{s \in \omega} A_{i,s}$ and $A_{s} < i A_{i+1}$.

Let $< be any $|A|$-ordering of $A$ satisfying (2.3) and the condition
\[
D_\eta = \bigcup_{i < j} A_{j} \quad \text{is an initial segment of} \quad <.
\]
(For example, $A_{00} < A_{01} < A_{10} < A_{11} < A_{02} < A_{12} < A_{20} < A_{21} < A_{22} < \ldots$, i.e., $A_{i,s} < A_{i,r}$ if $\max(i, r) < \max(i, s)$ or $\max(i, r) = \max(i, s)$ and $i < i$ or $\max(i, r) = \max(i, r), i = i$ and $r < r$.)

We show next that (2.4) holds. Let $s \in \omega$, $0 < s$. Let $N_s = |D_\eta|$, let $B \in \mathcal{B}$, $N_s < k, l$ be given, and we demonstrate that $\|h_k (B)^\infty \| < 1/s$. Let $B = B \cap A_s$, $B'_s = k (B)_s^\infty$. By (2.9) $A_s^\infty = D_s$ and so, by $N_s < k$, $B'_s \cap D_s = \emptyset$. Hence there are $k_i, l_i \in \omega$ such that $n_s < k_i$, $l_i$ and $B_i = k_i (B)_i^\infty$. Thus by (2.7)
\[
\|h (B_i)\| < 1/(2 \cdot m_s \cdot s) \quad \text{for} \quad i < m_s.
\]
Also, by (2.2) $\|h_k (B_i)\| < c_i$. Thus we have
\[
\|h (B_i)\|^\infty < \sum_{i < m_s} \|h (B_i)\| + \sum_{m_s < i} c_i < m_s \cdot \frac{1}{2 \cdot m_s \cdot s} + \frac{1}{2s} = \frac{1}{s}.
\]
Finally, we prove (2.5). Let $B \in \mathcal{B}$, $B_i = B \cap A_i$ and let $v_i = \sum B_i^\infty h$. By (2.2) $\|v_i\| < c_i$ and so $\sum_{i \in \omega} v_i$ is an absolutely convergent series in $V$. Thus it is convergent to some $v \in V$ unconditionally, i.e. also under arbitrary re-arrangement of its terms. Let $v' = \sum B_i^\infty h$ and let $s$ be a positive integer. We establish (2.5) by showing $\|v - v'\| < 3/s$.

Let $B_i = [B_i]_n^\infty, B_i^\infty = n_i (B_i)^\infty$. Then $v_i = h (B_i) + \sum B_i^\infty h$. By (2.7),
\[
\left\| \sum_{i \in \omega} h (B_i) \right\| < \frac{1}{2 \cdot s \cdot m_s} \quad \text{for} \quad i < m_s
\]
and so
\[
\|v_i - h (B_i)\| < \frac{1}{2 \cdot m_s \cdot s}, \quad i < m_s.
\]
Let $D = B \cap D_s$. By (2.9) we have $h(D) = \sum_{i<m_s} h(B_i)$. Hence
\[
\left\| \sum_{i<m_s} \nu_i - h(D) \right\| = \left\| \sum_{i<m_s} (\nu_i - h(B_i)) \right\| < m_s \cdot \frac{1}{2 \cdot m_s \cdot s} = \frac{1}{2s}.
\]
Also, by proof of (2.4)
\[
\|\nu' - h(D)\| < 1/s.
\]
By (2.2) and (2.6)
\[
\left\| \sum_{m_s < i} \nu_i \right\| < \sum_{m_s < i} \|\nu_i\| < \frac{1}{s}.
\]
Hence
\[
\|\nu - \nu'\| = \left\| \left( \sum_{i<m_s} \nu_i - h(D) \right) - (\nu' - h(D)) + \sum_{m_s < i} \nu_i \right\| < \frac{3}{s},
\]
and the proof of the Summing Lemma is complete. □

3. Proof of Theorem 1. Let $\Delta$ denote the class of compact ordered sets for which Theorem 1 is true, and let $\mathfrak{D} = \{ A : A \in \Delta \}$. Theorem 1 states $\mathfrak{C} \subseteq \mathfrak{D}$. We first list two obvious properties of $\mathfrak{D}$.

**PROPOSITION 3.0.** (i) $m \in \mathfrak{D}$ for every $m \in \omega$.
(ii) $\alpha, \beta \in \mathfrak{D} \Rightarrow \alpha + \beta \in \mathfrak{D}$.

Call $f \in C(A, V)$ representable (uniquely representable) iff there is an enumeration $(J_n)_n$ of $f$'s increments so that $f = \sum J_n$ holds in $C(A, V)$ (and whenever $(\hat{J}_n)_n$ is another enumeration of $f$'s increments for which $\sum \hat{J}_n$ converges, the identity $f = \sum \hat{J}_n$ holds in $C(A, V)$).

**PROPOSITION 3.1.** Let $A, B$ be ordered sets, and let $g$ be a continuous mapping of $A$ onto $B$ satisfying $g(a_1) < g(a_2)$ wherever $a_1 < a_2$. For $f \in C(B, V)$ define $f \in C(A, V)$ by $f = f \circ g$. Then:

(i) $f$ is representable (uniquely representable) iff $\hat{f}$ is representable (uniquely representable),
(ii) if $A \in \Delta$ then $B \in \Delta$.

**PROOF.** Let $\hat{C}(B, V) = \{ \hat{f} : f \in C(B, V) \}$. Clearly, $f \rightarrow \hat{f}$ is an isometric isomorphism of $C(B, V)$ onto $\hat{C}(B, V)$. For $b \in B$, $g^{-1}(b)$ is a compact interval of $A$. Let $a_b = \max g^{-1}(b)$. Let $f \in C(B, V)$, $b \in B$, $v \in V$. Then $J^*_b \in C(B, V)$ is an increment of $f$ iff $J^*_b = \hat{J}^*_b$ is an increment of $\hat{f}$. Let $(J_n)_n$ be an enumeration of $f$'s increments. Then $(\hat{J}_n)_n$ is an enumeration of $\hat{f}$'s increments, and $\sum_{n \in \omega} \hat{J}_n$ converges to $h \in C(B, V)$ iff $\sum_{n \in \omega} \hat{J}_n$ converges to $\hat{h} \in \hat{C}(B, V)$. (i) follows, and (ii) follows trivially from (i). □

---

2In fact, $\mathfrak{D}$ is the class of scattered compact order types [M1].
Let $A$ be a countable compact ordered set, and let $f, h: A \to V$. Let $\mathcal{B}_A = \{(m_A, a): a \in A\}$. We say that an $|A|$-ordering $\prec$ sums $h$ (uniformly) to $f$ iff $\prec$ sums $h$ (uniformly) over $\mathcal{B}_A$ and $f(a) = \sum_{\{m_A, a\}} h$ for every $a \in A$.

We prove Theorem 1 rephrased as follows:

**Theorem 1'.** Let $A$ be a countable compact ordered set, $V$ a Banach space, and let $f: A \to V$ be a continuous function satisfying $f(m_A) = 0$. Then: (a') There is an $|A|$-ordering $\prec$ of $A$ that sums $f^+$ uniformly to $f$.

(b') If $\prec$ is any $|A|$-ordering of $A$ that sums $f^+$ uniformly to $f$, then $f = f^+$. 

**Proof of (a').** We prove (a') by induction on $\text{ch}(A) = (p, m)$. By Proposition 3.0(i) we may assume $p > 0$ and by Proposition 3.0(ii) and Proposition 1.3 it is enough to prove (a') for $m = 1$. So assume $m = 1$ and let $A^{(p)} = \{x\}$.

We may further assume that $m_A < x < M_A$ and $x$ is a core point of $A$.

(Otherwise, $x = M_A$, $x < x'$, $x = m_A$ or $x < x$. Let $\hat{A} = A \cup \{x_n: n \in \omega\}$ where $x < x_{n+1} < x_n$ (and $x < x'$) if $x = M_A$ (if $x < x'$) and $x_n < x_{n+1} < x$ (and $x < x'$) if $x = m_A$ (if $x < x'$). Then $\text{ch}(\hat{A}) = \text{ch}(A) = (\mu, 1)$, $A^{(p)} = \{x\}$ and $x$ is a core point of $\hat{A}$. Define $g: \hat{A} \to A$ by $g(a) = a$, $a \in A$ and $g(x_n) = x$. By Proposition 3.1(i), (a') for $f \in C(A, V)$ follows from (a') for $f = f \circ g \in C(\hat{A}, V)$.)

Choose $M_n \in A$ so that the following conditions hold:

1. $M_{2n} < M_{2n+2} < x < M_{2n+3} < M_{2n+4} < M_1 = M_A$ ($n \in \omega$).
2. $M_n < x$, $n \not= 1$.
3. $x = \sup\{M_{2n}: n \in \omega\} = \inf\{M_{2n+1}: n \in \omega\}$.
4. $M_{2n} < a, b < M_{2n+3}$ implies $\|f(b) - f(a)\| < 2^{-2n+3}$.

This is possible by continuity of $f$ and Corollary 1.5. Let $m_0 = m_A$, $m_{2n+2} = M_n$, and $m_{2n+1} = M_{2n+3}$, $n \in \omega$. Define $A_n = [m_n, M_n]$. Then $A = x$-CL $A_n$ and $A_n$ is a compact ordered set satisfying $A_n^{(p)} = \emptyset$, since $A_n^{(p)} \subseteq A_n^{(p)} = \{x\}$ and $x \in A_n \supseteq A_n^{(p)}$. Thus $\text{ch}(A_n) = (\mu_n, m_n)$ where $\mu_n < \mu$.

We may further assume

(5) $f(M_{2n}) = f(m_{2n+2})$, $f(M_{2n+3}) = f(m_{2n+1})$, $n \in \omega$.

(Otherwise, let $\hat{A}_n = A_n \cup \{c_n\}$ where $A_n < c_n$, $c_n \in A$, and let $\hat{A} = x$-CL $\hat{A}_n$.

Define $g: \hat{A} \to A$ by $g(a) = a$ for $a \in A$, $g(c_{2k}) = m_{2k+2}$, $g(c_{2k+3}) = m_{2k+1}$, $g(c) = M_1$, and let $f = f \circ g$. Let $\hat{M}_n = M_{\hat{A}_n} = c_n$, $\hat{m}_n = m_{\hat{A}_n} = m_n$. Then $\hat{f}(\hat{M}_{2n}) = \hat{f}(\hat{m}_{2n})$, and (a') for $\hat{f}$ implies (a') for $f$ by Proposition 3.1(i).)

Define $f_n \in C(A_n, V)$ by $f_n(a) = f(a) - f(m_n)$, $n \in \omega$. Then $f_n^+ = f^+|A_n$ (by (5) this holds true also at $M_n$). By the induction hypothesis, there is an $|A_n|$-ordering $\prec_n$ of $A_n$ that sums $f_n^+$ (hence $f^+$) uniformly to $f_n$. Let $\mathcal{B}_n = \{(m_n, a): a \in A_n\} \cup \{A_n\}$. Let $K_n \in \omega$ satisfy

$$\|f^+(B) - f^+(C)\| < 2^{-n} \quad \text{for } B \in \mathcal{B}_n, K_n < k, l.$$ (6)

Let $[A_n]_{K_n} = \{a_i: 0 < i < K_n\}$ where $a_i < a_j$ for $i < j$. Modify the order $\prec_n$ into an $|A|$-ordering $\prec_n$ by reordering $[A_n]^{(p)}_{K_n}$ in the ordering inherited from
A, that is, let \( a_i <_n a_j \) for \( 0 < i < j < K_n \), and \( a <_n b \) if \( a <'_n b \) for \( a \in A_n \), \( b \in K[A_n]^{<_n} \). We shall show that \( <_n \) satisfies (2.1) and (2.2) (with \( h = f^t \) and \( c_n > 0 \) that satisfy (2.0)).

Now (2.1) is obvious, since \( <_n \) differs from \( <'_n \) only on a finite set. We prove (2.2). Let \( a_{-1} = m_n \), \( a_K = M_n \). Let \( B_j = \{ a_i : 0 < i < j \} \); and let \( a \in (a_{-1}, a_j) \). Denote \( B_a = [m_n, a] \).

\[
[B_a]^{<_n}_I = B_j \quad (0 < j < K_n).
\]

Now, by hypothesis

\[
f_n(a) = \sum_{B_a} f^t = f^t(B_j) + \sum_{K[B_a]} f^t = f^t(B_j) + \sum_{K[B_a]} f^t.
\]

Now,

\[
\sum_{K[B_a]} f^t = \lim I f_n^t(K[B_a])^{<_n}_{I_i}.
\]

and so by (6) we have

\[
\left\| f^t(K[B_a])^{<_n}_{I_i} \right\| < 2^{-n}.
\]

Hence

\[
\left\| \sum_{K[B_a]} f^t \right\| < 2^{-n}.
\]

Also, by (4),

\[
\| f_n(a) \| = | f(a) - f(m_n) | < 2^{-n} \quad (n > 1).
\]

Thus

\[
\left\| f^t(B_j) \right\| < 2 \cdot 2^{-n} \quad (n > 1).
\]

Now let \( a \in (a_{-1}, a_j) (0 < j < K_n) \), and let \( k, l \in \omega \). Then

\[
f_n^t([B_a]^{<_n}_{I_i}) = f_n^t([B_a]^{<_n}_{K_i}) + f_n^t([K[B_a]]^{<_n}_{I_i}) \quad (j < i).
\]

Hence, by (7), (8) and (9)

\[
\left\| f_n^t([B_a]^{<_n}_{I_i}) \right\| < 3 \cdot 2^{-n} \quad (n > 1).
\]

But \( f^t([k[B_a]^{<_n}_{I_i}] = f_n^t([B_a]^{<_n}_{I_i}) - f_n^t([B_a]^{<_n}_{K_i}) \) \( (k < l) \). It follows that

\[
\left\| f^t([k[B_a]^{<_n}_{I_i}] \right\| < 6 \cdot 2^{-n} \quad (n > 1).
\]

Let \( c_n = 6 \cdot 2^{-n} \) for \( n > 1 \). Then \( c_0, c_1 \) can be properly chosen so that \( \Sigma_n c_n < \infty \) and for every \( B \in \Omega_n \), \( k, l \in \omega \) we have \( \| f^t([k[B]^{<_n}_{I_i}] \| < c_n \). Thus (2.2) holds.

Let \( A' = \bigcup_{n \in \omega} A_n \) and let \( \prec \) be an \( \card{A'} \)-ordering of \( A' \) so that (2.3)–(2.5) hold.
Let \(<\) be any \(|A|\)-ordering of \(A = A' \cup \{x\}\) whose restriction to \(A'\) is \(<'\). We show that \(<\) sums \(f^i\) uniformly to \(f\).

Indeed, by (2.4) \(<\) sums \(f^i\) uniformly over \(\mathcal{B} = \{B \subseteq A: B \cap A_n \in \mathcal{B}_n\}\). Now for all \(a \in A\), \([m_0, a) \cap A_n \in \mathcal{B}_n\) and so \(<\) sums \(f^i\) uniformly over \(([m_0, a): a \in A\)}. We shall show that \(\sum_{(m_0, a)} f^i = f(a)\) for each \(a \in A\). By (2.3), (5) we have

\[
\sum_{A_n} f^i = \sum_{\{m_0, m_n\}} f^i = f_n(m_n),
\]

so

\[
\sum_{A_n} f^i = f(M_n) - f(m_n)
\]

and for \(a \in A_n\), by (2.3)

\[
\sum_{[m_0, a)]} f^i = f(a) - f(m_n).
\]

Now, by (2.5) (10), (11) and by \(f^i(x) = 0\) we have for every \(a \in A\):

\[
\sum_{(m_0, a)} f^i = \sum_{n \in \omega} \left( \sum_{[m_0, a)] \cap A_n} f^i \right) = \sum_{A_k \subset A_n} (f(M_k) - f(m_k)) + f(a) - f(m_n).
\]

By \(f(m_0) = 0\) we deduce for \(a \in A_{2n}\):

\[
\sum_{[m_0, a)} f^i = \sum_{k < n} (f(m_{2k+2}) - f(m_{2k})) + f(a) - f(m_{2n}) = f(a).
\]

By \(f(x) = \lim_n f(m_{2n})\) we have

\[
\sum_{[m_0, x)} f^i = \sum_{n \in \omega} (f(m_{2n+2}) - f(m_{2n})) = \lim_n f(m_{2n}) = f(x).
\]

Finally, let \(a \in A_{2n+1}\). Then

\[
\sum_{[m_0, a]} f^i = \sum_{k \in \omega} (f(m_{2k+2}) - f(m_{2k})) + \sum_{j > n} (f(m_{2j+1}) - f(m_{2j+3})) + f(a) - f(m_{2n+1})
\]

\[
f(x) + \sum_{j > n} (f(m_{2j+1}) - f(m_{2j+3})) + f(a) - f(m_{2n+1}).
\]

But \(\sum_{n < j < n} (f(m_{2j+1}) - f(m_{2j+3})) = f(m_{2n+1}) - f(m_{2n+1})\). Also, \(\lim_i f(m_{2i+1}) = f(x)\), so

\[
\sum_{[m_0, a]} f^i = f(x) + (f(m_{2n+1}) - f(x)) + (f(a) - f(m_{2n+1})) = f(a).
\]
Proof of (b'). We shall rather prove

(b') Let \(<'\) be any \(|A|\)-ordering of \(A\) that sums \(f^t\) to a continuous function \(\tilde{f}\). Then \(\tilde{f} = f\).

((b') follows, as any \(|A|\)-ordering \(<'\) that sums \(f^t\) uniformly over \([\{m_0, a\}; a \in A]\) sums it to a continuous function.)

(b") is proved using the Induction Principle (§1). It is obvious for a finite \(A\), and so we have to show that if \(A = x-CL_{n \in \omega} A_n\) and (b") is true of \(A_n, n \in \omega\), it is also true of \(A\). As before, we may assume \(A_n \neq \emptyset\) for all \(n\), and setting \(m_n = m_{A_n}, M_n = M_{A_n}\) we may assume (5).

Let \(a \in A_n\) and define \(f_n(a) = f(a) - f(m_n), \tilde{f}_n(a) = \tilde{f}(a) - \tilde{f}(m_n)\). Then, \(f_n, \tilde{f}_n \in C(A_n, V)\) and \(f_n = f^t\) for \(A_n\). By assumption, \(<'\) sums \(f^t\) over \([m_0, a]\) to \(\tilde{f}(a)\) and over \([m_0, m_n]\) to \(\tilde{f}(m_n)\), hence \(<'\) sums \(f_n^t\) over \([m_0, a] - [m_0, m_n]\) to \(\tilde{f}_n(a) = \tilde{f}(a) - \tilde{f}(m_n)\). By the induction hypothesis, \(\tilde{f}_n = f_n\). Hence \(\tilde{f}(a) = \tilde{f}(m_n) + f_n(a)\) and in particular

\[\tilde{f}(M_n) - \tilde{f}(m_n) = f(M_n) - f(m_n).\]  

(12)

It is left to show that \(\tilde{f}(m_n) = f(m_n)\) and that \(\tilde{f}(x) = f(x)\). Now

\[\tilde{f}(m_0) = \sum_{[m_0, m_0]} f^t = 0 = f(m_0).\]

Assuming \(\tilde{f}(m_{2n}) = f(m_{2n})\) we have

\[\tilde{f}(m_{2n+2}) = \sum_{[m_0, m_{2n+2}]} f^t = \sum_{[m_0, m_{2n}]} f^t + \sum_{[m_{2n}, m_{2n+2}]} f^t = f(m_{2n}) + \sum_{[m_{2n}, m_{2n+2}]} f^t.\]

By (5), (12):

\[\sum_{[m_{2n}, m_{2n+2}]} f^t = f(m_{2n+2}) - f(m_{2n+2})\]

hence \(\tilde{f}(m_{2n+2}) = f(m_{2n+2})\). Thus \(\tilde{f}(m_{2n}) = f(m_{2n}), n \in \omega\). Also \(\tilde{f}(x) = \lim_n \tilde{f}(m_{2n})\) by continuity of \(\tilde{f}\), hence \(\tilde{f}(x) = \lim_n f(m_{2n}) = f(x)\). Finally, for \(l \in \omega\) we have by (5), (12)

\[\tilde{f}(m_{2n+1}) - \tilde{f}(m_{2n+2l+1}) = \sum_{0 \leq i < l} (\tilde{f}(m_{2n+2i+1}) - \tilde{f}(m_{2n+2i+3}))\]

\[= \sum_{0 \leq i < l} (f(m_{2n+2i+1}) - f(m_{2n+2i+3}))\]

\[= f(m_{2n+1}) - f(m_{2n+2l+1}).\]

(b) is actually a consequence of (a), see [M1]. We use the Induction Principle (§1) to give an independent proof of (b").
By continuity of $\tilde{f}$, $\lim_{i} \tilde{f}(m_{2n+2i+1}) = \tilde{f}(x)$. Hence, by $\tilde{f}(x) = f(x)$ and $f(x) = \lim_{i} f(m_{2n+2i+1})$:

$$\tilde{f}(m_{2n+1}) = \tilde{f}(m_{2n+1}) - \tilde{f}(x) + f(x)$$

$$= \lim_{i} (\tilde{f}(m_{2n+1}) - \tilde{f}(m_{2n+2i+1})) + f(x)$$

$$= \lim_{i} (f(m_{2n+1}) - f(m_{2n+2i+1})) + f(x) = f(m_{2n+1}).$$

This completes the proof of Theorem 1.

**References**


Department of Mathematics, University of Haifa, Haifa 31999, Israel