DECOMPOSITION OF NONNEGATIVE GROUP-MONOTONE MATRICES

BY

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Abstract. A decomposition of nonnegative matrices with nonnegative group inverses has been obtained. This decomposition provides a new approach to the solution of problems relating to nonnegative matrices with nonnegative group inverses. As a consequence, a number of results are derived. Our results, among other things, answer a question of Berman, extend the theorems of Berman and Plemmons, DeMarr and Flor.

1. Introduction. Let $A$ be an $m \times n$ real matrix. Consider the equations: (1) $AXA = A$, (2) $XAX = X$, (3) $(AX)^T = AX$, (4) $(XA)^T = XA$, and (5) $AX = XA$ where $X$ is an $n \times m$ real matrix and $T$ denotes the transpose. For a rectangular matrix $A$ and for a nonempty subset $\lambda$ of $\{1, 2, 3, 4, 5\}$, $X$ is called a $\lambda$-inverse of $A$ if $X$ satisfies equations $(i)$ for each $i \in \lambda$. In particular, the $\{1, 2, 3, 4\}$-inverse of $A$ is the unique Moore-Penrose generalized inverse and is denoted by $A^\dagger$. A $\{1, 2\}$-inverse of $A$ which satisfies $(5)$ is necessarily square and is called a group inverse. The group inverse of a matrix $A$, if it exists, is unique and is denoted by $A^g$.

A matrix $A$ is called group-monotone if $A^g$ exists and is nonnegative. A matrix $A = (a_{ij})$ is called 0-symmetric if $a_{ij} = 0$ implies $a_{ji} = 0$. Thus every symmetric matrix and every positive matrix is 0-symmetric. $A$ is called range-Hermitian (also called EP) if the range of $A$ is equal to the range of $A^T$, i.e., $R(A) = R(A^T)$. $A$ is range-Hermitian if and only if $AA^\dagger = A^\dagger A$ and so $A^\dagger = A^g$. An $m \times n$ matrix $A = (a_{ij})$ is called row (or column) stochastic if $a_{ij} \geq 0$ and $\Sigma_{j=1}^n a_{ij} = 1$, $1 \leq i \leq m$ (or $\Sigma_{i=1}^m a_{ij} = 1$, $1 \leq j \leq n$). If a matrix $A$ is a direct sum of matrices $S_j$, then $S_j$ will be called summands of $A$. A nonzero matrix $A$ is called a zero divisor if $AB = 0$ or $BA = 0$ for some nonzero matrix $B$. For all other terminology the reader is referred to Ben-Israel and Greville [1].

Theorem 1 of this paper characterizes all nonnegative matrices $A$ which have nonnegative group inverses; equivalently, $A^{(1,2)} = p(A) > 0$, where $p(A)$ is a polynomial in $A$ with scalar coefficients. This theorem generalizes the known results for nonnegative matrices $A$ whose $A^\dagger$ is $A$ [2] or, more generally, $A^\dagger$ is some polynomial in $A$ [7]. The solution to the problem raised by Berman of the characterization of all nonnegative matrices which are equal to a $\{1\}$- or $\{1, 2\}$-inverse of themselves also comes as a special case of Theorem 1. As a consequence of Theorem 1, we show (Corollary 2) that if $A$ is a nonnegative matrix with $A^m = A$, $m > 2$, then $A = A_1 + A_2 + \cdots + A_k$, where $A_i > 0$; $A_i^m = A_i$; $A_iA_j = 0$, $i \neq j$;
\[ d_i = \text{rank } A_i, \quad d_i|m - 1. \] This generalizes the theorem of DeMarr [5] for nonnegative idempotent matrices. Corollary 3 (Corollary 4) of Theorem 2 shows that for a nonnegative range-Hermitian (row stochastic) matrix \( A \) with \( A^\# > 0, \quad A^\# = A^+ = HA^m = A^mH \) (\( A^\# = A^+ = A^m \)) where \( H \) is a diagonal matrix with all entries positive. Theorem 4 characterizes all nonnegative rank factorizations of nonnegative group-monotone matrices. Theorems of Berman and Plemmons [3, Theorem 2 and Theorem 3] are also consequences of the characterizations obtained in Theorem 4. Our results, among others, depend on the following theorems proved in [6] and [7].

**Theorem A** ([6, Theorem 2]). If \( E \) is a nonnegative idempotent matrix of rank \( r \) with no row or column completely zero. Then there exists a permutation matrix \( P \) such that

\[
P E P^T = \begin{pmatrix}
x_1 y_1^T & 0 & \cdots & 0 \\
0 & \ddots & \cdots & 0 \\
0 & \cdots & x_r y_r^T \\
0 & \cdots & \cdots & 0
\end{pmatrix}
\]

where \( x_i, y_i \) are positive vectors with \( y_i^T x_i = 1 \). In particular, \( E \) is 0-symmetric.

**Theorem B** ([7, Remark 3]). Let \( A \) be a nonnegative matrix and \( p(A) = \alpha_1 A^{m_1} + \cdots + \alpha_k A^{m_k}, \alpha_i \neq 0, m_i > 0, \) such that \( p(A) > 0, \) \( Ap(A) \) is 0-symmetric, \( Ap(A)A = A, \) and \( \text{rank } A = \text{rank } p(A). \) Then there exists a permutation matrix \( P \) such that \( P A P^T \) is a direct sum of matrices of the following three types (not necessarily all)

(I) \( \beta xy^T, \) where \( x \) and \( y \) are positive vectors with \( y^T x = 1, \) and \( \beta \) is some positive number satisfying \( \sum_m \alpha_i \beta^{m+1} = 1; \)

(II) 

\[
\begin{pmatrix}
0 & \beta_{12} x_1 y_2^T & 0 & \cdots & 0 \\
0 & \cdots & 0 & \beta_{23} x_2 y_3^T & \cdots \\
0 & \cdots & \cdots & \cdots & \beta_{d-1d} x_{d-1} y_d^T \\
0 & \cdots & \beta_{d1} x_1 y_1^T & 0 & \cdots & 0
\end{pmatrix}
\]

where \( x_i \) and \( y_i \) are positive vectors of the same order with \( y_i^T x_i = 1, \) \( x_i \) and \( y_j, \) \( i \neq j, \) are not necessarily of the same order, and \( \beta_{12}, \beta_{23}, \ldots, \beta_{d1} \) are arbitrary positive numbers with \( d > 1 \) and \( d|m_i + 1 \) for some \( m_i, \) such that the product \( \beta_{12} \beta_{23} \cdots \beta_{d1} \) is a common root of the following system of at most \( d \) equations in \( t: \)

\[
\sum_{d|(m_i + 1)} \alpha_i t^{(m_i + 1)/d} = 1,
\]

\[
\sum_{d|(m_i + 1 - k)} \alpha_i t^{(m_i + 1 - k)/d} = 0, \quad k \in \{1, \ldots, d - 1\},
\]
where the summation in each of the above equations runs over all those \( m_i \) for which \( d|(m_i + 1 - k), k = 0,1, \ldots, d - 1 \), with the convention that if there is no \( m_i \) for which \( d|(m_i + 1 - k), k \in \{1, \ldots, d - 1\} \), then the corresponding equation is absent.

(III) A zero matrix.

In particular, if all \( a_i > 0 \) then \( \beta \) in type (I) and the product \( \beta_{12} \beta_{23} \ldots \beta_{d1} \) in type (II) are unique. Further, in this case the positive integer \( d \), i.e. the rank of a matrix of type (II), must divide each \( m_i + 1 \).

The concept of 0-symmetry has played a crucial role in the development of this paper.

2. Main results. Let \( A \) be any \( n \times n \) matrix. Let us group the indices \( i = 1, 2, 3, \ldots, n \) into four disjoint sets according to whether the \( i \)th row and the \( i \)th column of \( A \) are both zero, or the \( i \)th row is zero but the \( i \)th column is not, and so on. Then by simultaneously rearranging rows and columns, we can find a permutation matrix \( P \) such that

\[
PAP^T = \begin{bmatrix}
K & L & 0 & 0 \\
0 & 0 & 0 & 0 \\
M & N & 0 & 0 \\
0 & 0 & 0 & 0
\end{bmatrix},
\]

where the diagonal blocks are square matrices such that \( K \) and \( L \) have no zero rows in common, and \( K \) and \( M \) have no zero columns in common. It may be noted that in certain situations some of the blocks may be absent. For example, if \( A \) is row stochastic then

\[
PAP^T = \begin{pmatrix} K & 0 \\ M & 0 \end{pmatrix}.
\]

In view of the frequent use of the above representation of a matrix throughout this paper, we record it in the following lemma.

**Lemma 1.** Let \( A \) be a square matrix. Then there exists a permutation matrix \( P \) such that

\[
PAP^T = \begin{bmatrix}
K & L & 0 & 0 \\
0 & 0 & 0 & 0 \\
M & N & 0 & 0 \\
0 & 0 & 0 & 0
\end{bmatrix},
\]

where \( K \) and \( L \) have no zero rows in common, and \( K \) and \( M \) have no zero columns in common.
Theorem 1. Let $A$ be a nonnegative matrix and $A^{(1,2)} = p(A) > 0$, where $p(A) = \alpha_1 A^m_1 + \cdots + \alpha_k A^m_k$, $\alpha_i \neq 0$, $m_i > 0$. Then there exists a permutation matrix $P$ such that

$$PAP^T = \begin{bmatrix} J & JD & 0 & 0 \\ 0 & 0 & 0 & 0 \\ CJ & CJD & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

where $C, D$ are some nonnegative matrices of appropriate sizes and $J$ is a direct sum of matrices of the following types (not necessarily both):

(I) $\beta xy^T$, where $x$ and $y$ are positive vectors with $y^Tx = 1$ and $\beta$ is a positive root of

$$\sum_{m_i} \alpha_m t^{m_i + 1} = 1.$$  \hspace{1cm} (6)

(II)

$$\begin{align*}
0 & \quad \beta_{12} x_1 y_2^T & 0 & 0 & \cdots & 0 \\
0 & 0 & \beta_{23} x_2 y_3^T & 0 & \cdots & 0 \\
\vdots & \vdots & & \ddots & & \vdots \\
\beta_{d1} x_d y_1^T & 0 & 0 & \cdots & 0 & \beta_{d-1} x_{d-1} y_d^T \\
\end{align*}$$

where $x_i$ and $y_i$ are positive vectors of the same order with $y_i^T x_i = 1$; $x_i$ and $x_j$, $i \neq j$, are not necessarily of the same order; and $\beta_{12}, \ldots, \beta_{d1}$ are arbitrary positive numbers with $d > 1$ and $d|m_i + 1$ for some $m_i$ such that the product $\beta_{12} \beta_{23} \cdots \beta_{d1}$ is a common root of the following system of at most $d$ equations in $t$

$$\sum_{d|(m_i + 1)} \alpha_m t^{(m_i + 1)/d} = 1,$$ \hspace{1cm} (7)

$$\sum_{d|(m_i + 1 - k)} \alpha_m t^{(m_i + 1 - k)/d} = 0, \quad k \in \{1, 2, \ldots, d-1\},$$ \hspace{1cm} (8)

where the summation in each of the above equations runs over all those $m_i$ for which $d|(m_i + 1 - k)$, $k = 0, 1, 2, \ldots, d-1$, with the convention that if there is no $m_i$ for which $d|(m_i + 1 - k)$, $k \in \{1, \ldots, d-1\}$, then the corresponding equation is absent.

In particular, if all $\alpha_i > 0$ then $\beta$ in type (I) and the product $\beta_{12} \beta_{23} \cdots \beta_{d1}$ in type (II) are unique. Further, in this case the positive integer $d$, i.e. the rank of a matrix of type (II), must divide each $m_i + 1$.

Conversely, if for some permutation matrix $P$

$$PAP^T = \begin{bmatrix} J & JD & 0 & 0 \\ 0 & 0 & 0 & 0 \\ CJ & CJD & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$
where $C$, $D$ are arbitrary nonnegative matrices of appropriate sizes and $J$ is a direct sum of matrices of the following types (not necessarily both):

(I') $\beta x y^T$, $\beta > 0$, $x$, $y$ are positive vectors with $y^Tx = 1$.

(II')

\[
\begin{bmatrix}
0 & 0 & \cdots & 0 \\
0 & 0 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & 0
\end{bmatrix}
\begin{bmatrix}
\beta_{12} x_1 y_2^T & 0 & \cdots & 0 \\
0 & \beta_{22} x_2 y_3^T & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & \beta_{d-1,d} x_{d-1} y_d^T
\end{bmatrix}
\]

where $\beta_{ij} > 0$, $x_i$ and $y_i$ are positive vectors with $y_i^Tx_i = 1$, then $A^{(1,2)} > 0$ and is equal to some polynomial in $A$ with scalar coefficients.

**Proof.** By Lemma 1, there exists a permutation matrix $P_1$ such that

\[
P_1 A P_1^T = \begin{bmatrix}
K & L & 0 & 0 \\
0 & 0 & 0 & 0 \\
M & N & 0 & 0 \\
0 & 0 & 0 & 0
\end{bmatrix},
\]

where $K$, $L$, $M$, and $N$ are nonnegative matrices such that $K$ and $L$ have no zero rows in common, and $K$ and $M$ have no zero columns in common. Since $A^{(1,2)} = p(A)$, we have $A^2 p(A) = A$ and $A(p(A))^2 = p(A)$. From $A^2 p(A) = A$ we obtain $K^2 p(K) = K$, $K p(K) L = L$, $M K p(K) = M$, and $M p(K) L = N$. Hence $K p(K)$ is a nonnegative idempotent matrix. Since $K p(K) K = K$ and $K p(K) L = L$ have no zero rows in common, $K p(K)$ cannot have a zero row. Similarly, no column of $K p(K)$ is zero. Thus by Theorem A, $K p(K)$ is 0-symmetric. Similarly, from $A(p(A))^2 = p(A)$, we obtain $K(p(K))^2 = p(K)$, which, together with $K^2 p(K) = K$, gives rank $K = \text{rank} p(K)$. But then by Theorem B, there exists a permutation matrix $P_2$ such that $P_2 K P_2^T$ is a direct sum of matrices of types (I) or (II) (not necessarily both). Set

\[
P = \begin{pmatrix}
P_2 & 0 \\
0 & I
\end{pmatrix} P_1,
\]

where the block matrices are of suitable orders. Then $P A P^T$ is of the desired form.

The converse is trivial.

**Corollary 1.** Let $A$ be a nonnegative matrix of rank $r$ and let $A^{(1,2)} = p(A) > 0$, $p(A) = \alpha_1 A^{m_1} + \cdots + \alpha_k A^{m_k}$, $\alpha_i \neq 0$, $m_i > 0$. Then $A = A_1 + A_2 + \cdots + A_k$, where $A_i > 0$; $A_i A_j = 0$, $i \neq j$; $A_i^{(1,2)} = p(A_i)$; rank $A_i = d_i$, $\sum_{i=1}^k d_i = r$, and $d_i$ divides some $m_j + 1$.

**Proof.** By Theorem 1, there exists a permutation matrix $P$ such that

\[
P A P^T = \begin{bmatrix}
J & J D & 0 & 0 \\
0 & 0 & 0 & 0 \\
C J & C J D & 0 & 0 \\
0 & 0 & 0 & 0
\end{bmatrix},
\]
where $J$ is a direct sum of the matrices of the types (I) or (II) (not necessarily both)
and $C$, $D$ are nonnegative matrices of appropriate orders. Thus

$$J = \begin{bmatrix}
S_1 & & & \\
& \ddots & & \\
& & \ddots & \\
& & & S_k
\end{bmatrix},$$

where $S_i$'s are of the types (I) or (II) and rank $S_i = d_i$. Set

$$J_i = \begin{bmatrix}
0 & & & 0 \\
& \ddots & & \\
& & \ddots & \\
0 & & & 0
\end{bmatrix}.$$  

Then $A = \Sigma_{i=1}^{k} A_i$, where

$$A_i = PT \begin{bmatrix}
J_i & J_iD & 0 & 0 \\
0 & 0 & 0 & 0 \\
CJ_i & CJ_iD & 0 & 0 \\
0 & 0 & 0 & 0
\end{bmatrix} P.$$  

It can be easily verified that $A_i > 0$; $A_iA_j = 0$, $i \neq j$; $A_i^{(1,2)} = p(A_i)$; rank $A_i = d_i$, $\Sigma_{i=1}^{k} d_i = r$, and $d_i$ divides some $m_i + 1$.

The corollary which follows generalizes DeMarr's theorem for nonnegative idempotent matrices [5].

**Corollary 2.** Let $A$ be a nonnegative matrix of rank $r$ and let $A = A^m$, where $m > 2$ is a positive integer. Then $A = A_1 + A_2 + \cdots + A_k$, where $A_i > 0$; $A_iA_j = 0$, $i \neq j$; $A_i^{m} = A_i$; rank $A_i = d_i$, $d_i|m - 1$, $\Sigma_{i=1}^{k} d_i = r$.

**Proof.** Follows from Corollary 1.

**Remark 1.** Theorem 1 provides a complete solution, in a more general case, to the problem raised by Berman of characterization of the class of nonnegative matrices $A$ with $\{1\}$-inverse or $\{1, 2\}$-inverse equal to $A$ itself [2, Remark 5].

Henceforth by matrices of types (I) or (II), we will mean the matrices of types (I) or (II) described in Theorem 1.

**Theorem 2.** Let $A$ be a nonnegative matrix having a nonnegative group inverse $A^\#$. Then $A^\# = K_1A^m = A^mK_2$, where

$$K_1 = PT \begin{bmatrix}
K & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
CK & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{bmatrix} P, \quad K_2 = PT \begin{bmatrix}
K & KD & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{bmatrix} P.$$
$P$ is a permutation matrix, $K$ is a diagonal matrix with positive diagonal entries, $C, D$ are some nonnegative matrices of appropriate sizes, and $m$ is a positive integer. Indeed, we may also choose

$$K_1 = K_2 = P^T \begin{bmatrix} K & KD & 0 & 0 \\ 0 & 0 & 0 & 0 \\ CK & CKD & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} P.$$ 

**Proof.** By Theorem 1 there exists a permutation matrix $P$ such that

$$PAP^T = \begin{bmatrix} J & JD & 0 & 0 \\ 0 & 0 & 0 & 0 \\ CJ & CJD & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

where $C, D$ are some nonnegative matrices of appropriate sizes and $J$ is a direct sum of matrices of types (I) or (II) (not necessarily both). We note that if $S$ is a summand of type (I) then $S^{(1,2)} = \beta^{-1}xy^T = \beta^{-2}S = \cdots = \beta^{-k-1}S^k$, for any positive integer $k$. We show that if $S$ is a summand of type (II) then $S^{(1,2)} = (\beta_{12}\beta_{23}\cdots\beta_{d1})^{-k}S^{kd-1}$ for any positive integer $k$. A straightforward verification shows that

$$S^{(1,2)} = \begin{bmatrix} 0 & 0 & 0 & \frac{1}{\beta_d}x_1y_1^T \\ \frac{1}{\beta_{12}}x_2y_1^T & \ddots & \vdots & 0 \\ 0 & \frac{1}{\beta_{23}}x_3y_2^T & \ddots & \vdots \\ \vdots & \vdots & \ddots & 0 \\ 0 & 0 & \cdots & \frac{1}{\beta_{d-1d}}x_dy_{d-1}^T \\ 0 & 0 & \cdots & 0 \end{bmatrix} = (\beta_{12}\beta_{23}\cdots\beta_{d1})^{-1}S^{d-1}.$$ 

Also

$$S^{kd} = (\beta_{12}\beta_{23}\cdots\beta_{d1})^k \begin{bmatrix} x_1y_1^T & 0 \\ \vdots & \ddots \\ 0 & x_dy_d^T \end{bmatrix}$$

for any positive integer $k$. Thus $S^{(1,2)} = (\beta_{12}\beta_{23}\cdots\beta_{d1})^{-k}S^{kd-1}$ as asserted. Now if $S_{11}, S_{12}, \ldots, S_{1r}$ are summands of type (I) and $S_{21}, S_{22}, \ldots, S_{2s}$ are summands of type (II) of ranks $d_{21}, \ldots, d_{2s}$ respectively, then
where $\alpha_{ij} I$ are scalar matrices of appropriate sizes, $\alpha_{ij} > 0$. Thus $J^{(1,2)} = KJ^m = J^mK$, where
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\[
K = \begin{bmatrix}
  \alpha_{11}I & & & 0 \\
  & \ddots & & \\
  & & \alpha_{1r}I & \\
 0 & & & \alpha_{21}I \\
   & & & \ddots \\
   & & & & \alpha_{2s}I
\end{bmatrix}, \quad m = (d_{21} \cdots d_{2s}) - 1.
\]

Thus

\[
P A^{(1,2)} = \begin{bmatrix}
  K J^m & K J^m D & 0 & 0 \\
  0 & 0 & 0 & 0 \\
  C K J^m & C K J^m D & 0 & 0 \\
  0 & 0 & 0 & 0
\end{bmatrix}
= \begin{bmatrix}
  K & 0 & 0 & 0 \\
  0 & 0 & 0 & 0 \\
  C K & 0 & 0 & 0 \\
  0 & 0 & 0 & 0
\end{bmatrix}\begin{bmatrix}
  J^m & J^m D & 0 & 0 \\
  0 & 0 & 0 & 0 \\
  C J^m & C J^m D & 0 & 0 \\
  0 & 0 & 0 & 0
\end{bmatrix}
= \begin{bmatrix}
  J^m & J^m D & 0 & 0 \\
  0 & 0 & 0 & 0 \\
  C J^m & C J^m D & 0 & 0 \\
  0 & 0 & 0 & 0
\end{bmatrix}\begin{bmatrix}
  K & K D & 0 & 0 \\
  0 & 0 & 0 & 0 \\
  0 & 0 & 0 & 0 \\
  0 & 0 & 0 & 0
\end{bmatrix}
= \begin{bmatrix}
  K & K D & 0 & 0 \\
  0 & 0 & 0 & 0 \\
  0 & 0 & 0 & 0 \\
  0 & 0 & 0 & 0
\end{bmatrix}
\]

Hence

\[
A^{(1,2)} = P^T \begin{bmatrix}
  K & 0 & 0 & 0 \\
  0 & 0 & 0 & 0 \\
  C K & 0 & 0 & 0 \\
  0 & 0 & 0 & 0
\end{bmatrix} P A^m = A^m P^T \begin{bmatrix}
  K & K D & 0 & 0 \\
  0 & 0 & 0 & 0 \\
  0 & 0 & 0 & 0 \\
  0 & 0 & 0 & 0
\end{bmatrix} P.
\]

**Corollary 3.** Under the hypothesis of Theorem 2, if \( A \) is also range-Hermitian, that is, \( A^\sharp = A^\dagger \), then \( A^\dagger = A^\sharp = H A^m = A^m H \) where \( H \) is a diagonal matrix with positive diagonal entries and \( m \) is a positive integer.

**Proof.** Let \( A \) be range-Hermitian. Then \( A^\dagger = A^\sharp \). By Theorem 1,

\[
P A P^T = \begin{bmatrix}
  J & J D & 0 & 0 \\
  0 & 0 & 0 & 0 \\
  C J & C J D & 0 & 0 \\
  0 & 0 & 0 & 0
\end{bmatrix},
\]

where \( J \) is a direct sum of matrices of types (I) or (II). But then by using \( A^\dagger = A^\sharp \), we obtain that \( C \) and \( D \) must be zero. Thus

\[
P A P^T = \begin{bmatrix}
  J & 0 \\
  0 & 0
\end{bmatrix}
\]
and
\[ A^T = P^T \begin{pmatrix} K & 0 \\ 0 & 0 \end{pmatrix} P A^m = A^m P^T \begin{pmatrix} K & 0 \\ 0 & 0 \end{pmatrix} P \]
\[ = P^T \begin{pmatrix} K & 0 \\ 0 & 0 \end{pmatrix} \left( J^m \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \right) P = P^T \left( J^m \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \right) \begin{pmatrix} K & 0 \\ 0 & 0 \end{pmatrix} P \]
\[ = P^T \begin{pmatrix} K & 0 \\ 0 & I \end{pmatrix} \left( J^m \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \right) P = P^T \left( J^m \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \right) \begin{pmatrix} K & 0 \\ 0 & I \end{pmatrix} P \]
\[ = P^T \begin{pmatrix} K & 0 \\ 0 & I \end{pmatrix} P A^m = A^m P^T \begin{pmatrix} K & 0 \\ 0 & I \end{pmatrix} P \]
\[ = H A^m = A^m H, \]
where \( H \) is a diagonal matrix with positive diagonal entries.

Before we prove Corollary 4 we record below a simple fact which we state without proof.

**Sublemma.** If \( \beta x y^T \) is a row (or column) stochastic matrix where \( \beta > 0 \) and \( x, y \) are positive vectors such that \( y^T x = 1 \) then \( \beta = 1 \).

**Corollary 4.** If \( A \) is an \( n \times n \) nonnegative row (or column) stochastic matrix such that \( A^{(1,2)} = p(A) > 0, p(A) \) is a polynomial in \( A \) with scalar coefficients, then \( A^{(1,2)} = A^m \) for some positive integer \( m \) and \( A^{(1,2)} \) is row (or column) stochastic.

**Proof.** For definiteness, let us assume that \( A \) is row stochastic. By appropriate application of Lemma 1 and Theorem 1, we can find a permutation matrix \( P \) such that
\[ P A P^T = \begin{pmatrix} J & 0 \\ C & J \end{pmatrix}, \]
where \( J \) is a direct sum of matrices of types (I) or (II) (not necessarily both). We note that if \( S \) is a summand of type (I) then by sublemma \( \beta = 1 \) and so \( S^{(1,2)} = x y^T = S \). Next, let \( S \) be a summand of type (II). Then \( S \) is a stochastic matrix and
\[ S = \begin{pmatrix} 0 & \beta_{12} x_1 y_2^T & 0 & 0 & \cdots & 0 \\ 0 & 0 & \beta_{23} x_3 y_3^T & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & 0 & \beta_{d-1, d} x_{d-1} y_d^T \\ \beta_{d1} x_d y_d^T & 0 & 0 & \cdots & 0 & 0 \end{pmatrix}, \]
where \( \beta_{ij} > 0 \) and \( x_i, y_i \) are positive vectors with \( y_i^T x_i = 1 \). A straightforward verification shows that \( S^{(1,2)} = (\beta_{12} \beta_{23} \cdots \beta_{d1})^{-1} S^{d-1} \) and \( S^{d-1} \) is a row stochastic matrix. Thus again by the sublemma we get \( (\beta_{12} \beta_{23} \cdots \beta_{d1}) = 1 \). Then as in the proof of Theorem 2, we get \( J^{(1,2)} = J^m \), and hence
\[ A^{(1,2)} = P^T \begin{pmatrix} J^m & 0 \\ C & J^m \end{pmatrix} P = A^m. \]
Theorem 3. Every nonnegative rank factorization of nonnegative matrices $J$ which are direct sum of matrices of types (I') or (II') (not necessarily both) is of the form $J = (FQ)(Q^TG)$, where $Q$ is a permutation matrix, $F$ and $G$ are respectively the direct sum of matrices of the form (1) or (2) and (1') or (2'):

(1) $\beta' x,$
(1') $\beta'' y^T,$
(2) 

$$
\begin{pmatrix}
0 & \gamma_1 x_1 & 0 & 0 & \cdots & 0 \\
0 & 0 & \gamma_2 x_2 & 0 & \cdots & 0 \\
0 & 0 & 0 & \cdots & 0 & \gamma_{d-1} x_{d-1} \\
\gamma_d x_d & 0 & 0 & \cdots & 0 & 0
\end{pmatrix}
$$

and

(2')

$$
\begin{pmatrix}
\gamma_1 y_1^T \\
\gamma_2 y_2^T \\
\vdots \\
\gamma_d y_d^T
\end{pmatrix}
$$

such that $\beta' > 0$, $\beta'' > 0$, $\gamma_i > 0$, $x_i > 0$, $y_i > 0$, $x$, $y$, $x_i$, $y_i$ are positive vectors with $y^T x = 1$ and $y_i^T x_i = 1$. Moreover, $J^p = p(J)$ where $p(t) = \sum_{i=1}^{k} a_i t^{m_i}$, $a_i \neq 0$, $m_i > 0$, is some polynomial in $t$, $\beta' \beta''$ is a root of equation (6), and the product $\gamma_1 y_1 \cdots \gamma_d y_1' y_2' \cdots y_d'$ is a common root of the system of at most $d$ equations (7) and (8). It is understood that in forming the product $(FQ)(Q^TG)$ if $F$ has a summand of type (1) or (2) in the ith place of its direct sum then $G$ has a corresponding summand of type (1') or (2') at the same ith place.

Also, for each nonnegative rank factorization $J = FG$ of $J$, $(GF)^{-1} = p(GF)$.

Proof. Let $S = \beta x y^T$ with $y^T x = 1$ be a summand of $J$ of type (I'). Clearly, the only possible nonnegative rank factorization of $S$ is $S = FG$, $F = \beta' x$, $G = \beta'' y^T$ with $\beta' \beta'' = \beta$. This gives $GF = \beta y^T x = \beta$. By equation (6), it then follows that $(GF)^{-1} = p(GF)$.

Next let

$$
S = \begin{pmatrix}
0 & \beta_{12} x_1 y_2^T & 0 & 0 & \cdots & 0 \\
0 & 0 & \beta_{23} x_2 y_3^T & 0 & \cdots & 0 \\
0 & 0 & 0 & \cdots & 0 & \beta_d x_{d-1} y_{d-1}^T \\
\beta_{d1} x_d y_1^T & 0 & 0 & \cdots & 0 & 0
\end{pmatrix}
$$
be a summand of \( J \) of type (II') of rank \( d \). Let \( S = FG \) be a nonnegative rank factorization of \( S \). Partition \( F = (F_y) \) and \( G = (G_y) \) into matrix blocks such that for all \( i,j,k, F_yG_{jk} \) is defined and is of the same order as that of the \((i,k)\)th block entry in \( S \). Since \( F \) is of full column rank, no column of \( F \) is zero. Also no row of \( F \) can be zero. For otherwise, \( S = FG \) shall have a zero row which is not true. Similarly, no row or column of \( G \) is zero. Since no column of \( F \) is zero, for each \( j \) there exists an \( i \) (depending on \( j \)) such that \( F_y \neq 0 \). But then \( G_{jk} = 0 \) for all \( k \neq i + 1 \). Thus each row of the partitioned matrix \( G \) has at most one (and hence exactly one) nonzero entry. Clearly, then each column of \( G \) has also exactly one nonzero entry. The same reason yields that the partitioned matrix \( F \) has exactly one nonzero entry in each row and in each column. This implies there exists a permutation \( \sigma \in S_d \) such that \( F_{yj} = 0 \), for all \( j \neq \sigma(i) \), \( G_{\sigma(i)k} = 0 \), for all \( k \neq i + 1 \) and \( F_{\sigma(i)}G_{\sigma(i)j+1} = \beta_{i,j+1}x_jy_{i+1}^T \). But then the only solutions for \( F_{\sigma(i)} \) and \( G_{\sigma(i),i+1} \) are given by \( F_{\sigma(i)} = \gamma_i x_i \) and \( G_{\sigma(i),i+1} = \gamma_i y_i^T \) where \( \gamma_i \beta_i = \beta_{i,j+1} \). It is now clear that \( J = (FQ)(Q^T G) \) where \( F \) and \( G \) are, respectively, the direct sum of matrices of the form (1) or (2) and (1') or (2') and \( Q \) is some permutation matrix. For summand of \( J \) of type (I') we have already shown that \((GF)^{-1} = p(GF)\). We now show the same for summand of type (II'). It is sufficient to prove the result for \( S = FG \) where

\[
F = \begin{bmatrix}
0 & \gamma_1 x_1 & 0 & 0 & \cdots & 0 \\
0 & 0 & \gamma_2 x_2 & 0 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots & \ddots & \ddots \\
0 & 0 & 0 & \cdots & 0 & \gamma_d x_d \\
\gamma_d x_d & 0 & 0 & \cdots & 0 & 0 \\
\end{bmatrix}
\]

and

\[
G = \begin{bmatrix}
\gamma_1 y_1^T & 0 \\
\gamma_2 y_2^T \\
\vdots \\
\gamma_d y_d^T \\
\end{bmatrix}
\]

Then

\[
GF = \begin{bmatrix}
0 & \gamma_1 y_1' & 0 & 0 & \cdots & 0 \\
0 & 0 & \gamma_2 y_2' & 0 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots & \ddots & \ddots \\
0 & 0 & 0 & \cdots & 0 & \gamma_d y_d' \\
\gamma_d y_d' & 0 & 0 & \cdots & 0 & 0 \\
\end{bmatrix}
\]

and \((GF)^d = (\gamma_1 y_2' \cdots \gamma_d y_1' \gamma_1 \cdots \gamma_d)\).

From equations (7) and (8), it follows that \((\gamma_1 y_2' \cdots \gamma_d y_1' \gamma_1 \cdots \gamma_d)^{1/d} \) is a root of \( \sum_{i=1}^k a_i (GF)^{m+1} = I \). Thus \( \sum_{i=1}^k a_i (GF)^{m+1} = I \), since \((GF)^d = (\gamma_1 y_2' \cdots \gamma_d y_1' \gamma_1 \cdots \gamma_d)I \). Hence \((GF)p(GF) = I \), completing the proof.

The next theorem describes all nonnegative rank factorizations of a nonnegative matrix with a nonnegative group inverse.
Theorem 4. (a) Let \( A > 0 \) and \( P \) be a permutation matrix such that

\[
PAP^T = \begin{pmatrix} J & JD & 0 & 0 \\ 0 & 0 & 0 & 0 \\ CJ & CJD & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},
\]

\( J \) is a direct sum of matrices of types (I') or (II'), \( C, D \) are some nonnegative matrices of suitable sizes (equivalently, \( A > 0 \) and \( A^q = p(A) > 0 \)). Then we have the following:

(a) If \( J = FG \) is a nonnegative rank factorization of \( J \) then it "lifts" to a nonnegative rank factorization

\[
A = P^T \begin{pmatrix} F \\ 0 \\ CF \\ 0 \end{pmatrix} (G, GD, 0, 0)P
\]

of \( A \).

(a2) If \( A = F'G' \) is a nonnegative rank factorization of \( A \) then it "contracts" to a nonnegative rank factorization \( J = F'_{11}G'_{11} \) of \( J \) where \( F'_{11} \) \( (G'_{11}) \) consists of first \( n \) rows (columns) of \( PF' \) \( (G'P^T) \), \( n \) being the order of the matrix \( J \).

(a3) If \( \sigma \) denotes the operation of "lifting" defined in (a1), and \( \eta \) denotes the operation of "contracting" defined in (a2) then \( \sigma \eta = \text{id} = \eta \sigma. \) Thus there is a 1-1 correspondence between the class of nonnegative rank factorizations of \( J \) and the class of nonnegative rank factorizations of \( A \).

(a4) If \( A = F'G' \) and \( J = FG \) are corresponding nonnegative rank factorizations of \( A \) and \( J \), respectively, then \( GF = G'F' \) and \( (GF)^{-1} = p(GF) \), where \( A^q = p(A) \).

(b) If \( A = FG \) is a nonnegative rank factorization of \( A \) such that \( (GF)^{-1} = p(GF) \) where \( p(t) \) is some polynomial in \( t \) with scalar coefficients then \( A^q \) exists, \( A^q > 0 \), and \( A^q = p(A) \).

Proof. (a1) Straightforward verification.

(a2) Let \( n \) be the order of \( J \). Partition

\[
PAP^T = \begin{pmatrix} J & JD & 0 & 0 \\ 0 & 0 & 0 & 0 \\ CJ & CJD & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad PF' = \begin{pmatrix} F'_{11} \\ 0 \\ F'_{21} \end{pmatrix}, \quad \text{and} \quad G'P^T = (G'_{11}, G'_{12})
\]

where \( F'_{11} \) \( (G'_{11}) \) consists of first \( n \) rows (columns) of \( PF' \) \( (G'P^T) \) respectively. By comparing we get \( J = F'_{11}G'_{11} \) which is clearly nonnegative rank factorization of \( J \).

(a3) It is obvious that if we perform the operation of lifting followed by the operation of contracting then the composition is identity operation, i.e. \( \eta \sigma = \text{id} \) identity. On the other hand, it is not clear to us that \( \sigma \eta \) is also identity, in general. However, we can show \( \sigma \eta = \text{id} \) under our hypothesis. Let \( A = F'G' \) be a nonnegative rank factorization of \( A \). By performing the operation \( \eta \) we get
\( J = F'_{11}G'_{11} \) where \( F'_{11} \) and \( G'_{11} \) are respectively the first \( n \) rows and the first \( n \) columns of \( PF' \) and \( G'P^T \). Then by performing \( \sigma \), we get

\[
A = P^T \begin{pmatrix} F'_{11} & 0 & CF'_{11} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} (G'_{11} & G'_{11}D & 0 & 0)P.
\]

To prove \( \sigma \eta = \text{identity} \), we need to show

\[
PP' = \begin{pmatrix} F'_{11} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad G'P^T = (G'_{11} & G'_{11}D & 0 & 0).
\] (9)

Since by Theorem 3, \( F'_{11} (G'_{11}) \) is the direct sum of matrices of the form (1) or (2) ((1') or (2')) it is sufficient to prove (9) when \( F'_{11} \) is of the form (1) or (2) and \( G'_{11} \) is of the corresponding form (1') or (2'). Partition

\[
PP' = \begin{pmatrix} F'_{11} & F'_{21} & F'_{31} & F'_{41} \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad G'P^T = (G'_{11} & G'_{12} & G'_{13} & G'_{14})
\]

such that the size of \( F'_{j1} \), \( j = 1, 2, 3, 4 \), is the same as that of the corresponding block in

\[
\begin{pmatrix} F'_{11} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.
\]

Similarly, for the size of \( G'_{ij} \), \( j = 1, 2, 3, 4 \). Then comparing the corresponding blocks in

\[
\begin{pmatrix} F'_{11} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} (G'_{11} & G'_{11}D & 0 & 0) = \begin{pmatrix} F'_{11} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} (G'_{11} & G'_{12} & G'_{13} & G'_{14})
\]

we get the result by observing that \( F'_{11} \) and \( G'_{11} \) are not zero divisors.

(a4) Follows from (a1)–(a3) and Theorem 3.

(b) Recall that if \( A = FG \) is a rank factorization of \( A \) then \( A^g \) exists if and only if \((GF)^{-1} \) exists [4]. In this case \( A^g = F(GF)^{-2}G \). A straightforward computation then yields \( A^g = p(A) \) where \((GF)^{-1} = p(GF)\).

Remark 2. Theorem 3 along with Theorem 4(a3) characterizes all the nonnegative rank factorizations of a nonnegative matrix with nonnegative group inverse.
Remark 3. Another proof of Theorem 4(a₄) can be given on the same lines as the proof for the special case when p(A) = A given by Berman and Plemmons [3]. However, the purpose of Theorem 4 is to characterize all nonnegative rank factorizations of nonnegative matrices A with A* > 0, and (a₄) comes out as an offshoot.

References

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