THE INTERIOR OPERATOR LOGIC AND PRODUCT TOPOLOGIES

BY

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ABSTRACT. In this paper we present a model theory of the interior operator on product topologies with continuous functions. The main results are a completeness theorem, an axiomatization of topological groups, and a proof of an interpolation and definability theorem.

0. Introduction and basic results. This paper develops a model theory for $L(I_n)^{\omega,\omega}$, the interior operator logic, which is analogous to the author's development for $L(Q_n)^{\omega,\omega}$, the “open set” quantifier logic, in [16]. The significant difference is that $L(I_n)^{\omega,\omega}$ satisfies an interpolation and definability theorem in marked contrast to $L(Q_n)^{\omega,\omega}$. Another contrasting result is an omitting types theorem for $L(I_n)^{\omega,\omega}$.

These results were obtained by the author shortly after the results contained in [16] on $L(Q_n)^{\omega,\omega}$ were proved. More recently, J. A. Makowsky and M. Ziegler have obtained similar results for $L(I)$ using proof theory.

In §1 we prove a completeness theorem for $L(I_n)^{\omega,\omega}$. This proof uses the ideas developed for $L(Q_n)^{\omega,\omega}$. We also show that if an $L(I_n)^{\omega,\omega}$ theory has a Hausdorff topological model then it has a 0-dimensional normal topological model. Another result presented in this section is an axiomatization for the $L(I)$ theory of topological groups.

We conclude this paper by presenting a proof of a Robinson joint-consistency theorem for $L(I_n)^{\omega,\omega}$. This result implies both interpolation and definability. Thus $L(I_n)^{\omega,\omega}$ has a more “smooth” model theory than $L(Q_n)^{\omega,\omega}$. Another result is the omitting types theorem.

We will assume that the reader is familiar with the basic results of model theory.

Definition 1. If we take a first order model $\mathfrak{A}$ and $q_i \subseteq \mathcal{P}(A)$ then we call $(\mathfrak{A}, q_1, q_2, q_3, \ldots)$ a weak model for $L(I_n)^{\omega,\omega}$. If each $q_i$ is a topology on $A$ then $(\mathfrak{A}, q_1, q_2, q_3, \ldots)$ is called topological. A topological model $(\mathfrak{A}, q_1, q_2, \ldots)$ is called complete if each $q_k$ is the $k$th topological product of $q_1$ on $A$. We will abbreviate $(\mathfrak{A}, q_1, q_2, q_3, \ldots)$ by $(\mathfrak{A}, q)$.

The formulas of $L(I_n)^{\omega,\omega}$ are defined analogously to first order logic with the additional clause:

Received by the editors October 26, 1976 and, in revised form, March 11, 1977 and August 3, 1978.


1 This research was partially supported by NSF grants #MPS-74-08550 and #MCS77-04131.

2 The author would like to thank the referee for his helpful corrections and remarks in the original version of this paper.

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0002-9947/80/0000-0103/04.50

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If \( \varphi \) is a formula and \( x_1, \ldots, x_n \) are distinct variables, then \( I^n x_1, \ldots, x_n \varphi \) is also a formula (\( x_1, \ldots, x_n \) occur free in \( I^n x_1, \ldots, x_n \varphi \)). Again we abbreviate \( I^n x_1, \ldots, x_n \varphi \) by \( Ix \varphi \).

Since the language \( L(I^n) \) is not closed under substitution of free variables, we introduce the following notation to take the place of substitution.

**Definition.** Given \( n \)-tuples of variables \( x \) and substitutable terms \( k \), \( \varphi[k] \) denotes the formula \( \exists x(\bigwedge_{i=1}^n x_i = k_i \land \varphi) \).

The notion of an \( n \)-tuple \( a_1, \ldots, a_n \in A \) satisfying a formula \( \varphi(v_1, \ldots, v_n) \) of \( L(I^n)_{n \in \omega} \) in a weak model \((\mathfrak{A}, q)\) is defined in the usual manner by induction on the complexity of \( \varphi \) and is denoted by \((\mathfrak{A}, q) \models \varphi[a]\). The only difficult case is the \( Ix \varphi \) case:

\[
(\mathfrak{A}, q) \models Ix \varphi[a] \quad \text{iff} \quad a \in \emptyset \subset \{ \varphi(x) \}^{(\mathfrak{A}, a)} \quad \text{for some } \emptyset \in q_n
\]

(\( \{ \varphi(x) \}^{(\mathfrak{A}, a)} \) denotes the set \( \{ a \in A^n : (\mathfrak{A}, q) \models \varphi[a] \} \)).

It can be shown by induction that for \( a \in A^n \), a sentence \( \varphi[a] \) holds in \((\mathfrak{A}, a, q)\) iff \((\mathfrak{A}, q) \models \varphi[a]\) as previously defined.

**Remark.** Notice that by our definition of \( \models \), \( Ix \varphi \) is the interior of the set defined by \( \varphi \) in a topological model.

The axioms for \( L(I^n)_{n \in \omega} \) are the standard first order ones where we replace \( \varphi(c) \) by \( \varphi[c] \), e.g.

\[
\bigwedge_{i=1}^n x_i = y_i \rightarrow (Ix \varphi[x] \leftrightarrow Iy \varphi[y])
\]

and

\[
Ax : \forall x(\varphi \leftrightarrow \psi) \rightarrow (Ix \varphi \leftrightarrow Ix \psi).
\]

The rules of inference are:

(i) Modus ponens, i.e. from \( \varphi, \varphi \rightarrow \psi \) infer \( \psi \),

(ii) Generalization, i.e. from \( \varphi \) infer \( \forall x \varphi \).

We can now prove the following completeness theorem for topological models.

**Theorem 2.** An \( L(I^n)_{n \in \omega} \) theory \( \Sigma \) has a topological model \((\mathfrak{A}, q)\), if and only if \( \Sigma \) is consistent with

(A0) axioms for \( L(I^n)_{n \in \omega} \),

(A1) \( Ix (y = y) \leftrightarrow y = y \),

(A2) \( Ix \varphi \rightarrow \varphi \),

(A3) \( Ix \varphi \rightarrow Ix Ix \varphi \),

(A4) \( (Ix \varphi \land Ix \psi) \leftrightarrow Ix (\varphi \land \psi) \).

**Proof (only if).** Straightforward because the axioms hold in every topology.

(If) Let \( \Sigma^* \) be a maximal consistent extension of \( \Sigma \) with witnesses using \( \varphi[c] \) instead of \( \varphi(c) \). As in the proof of the completeness theorem for first order logic take \( \mathfrak{A} \) to be the model generated by \( \Sigma^* \).

Define \( q_k = \{ [a/Ix \varphi[a] \in \Sigma^*] \mid \varphi \) is a formula of \( L(I^n)_{n \in \omega}(A) \} \).

We claim \((\mathfrak{A}, q)\) models \( \Sigma^* \). We prove this by induction on the \( I \) complexity of \( \varphi \) that, for every formula \( \varphi(x) \) and \( n \)-tuple \( a \), \( \varphi[a] \in \Sigma^* \) iff \((\mathfrak{A}, q) \models \varphi[a]\). The only difficult clause is \( Ix \varphi[a] \).
If $Ix\varphi[b] \in \Sigma^*$ then $a \in \{b \in A^n | Ix\varphi[b] \in \Sigma^* \} \in q_n$. Hence $(\mathfrak{A}, \mathfrak{q}) \models Ix\varphi[a]$ using (A2).

Assume $(\mathfrak{A}, \mathfrak{q}) \models Ix\varphi[a]$. That is, there is an $\emptyset \in q_n$ such that $a \in \emptyset \subseteq [\varphi(x)]^{(\mathfrak{A}, \mathfrak{q})}$. By the definition of $q_n$ there is an $Ix\psi$ such that
\[
a \in \emptyset = [Ix\psi]^{(\mathfrak{A}, \mathfrak{q})} \subseteq [\varphi(x)]^{(\mathfrak{A}, \mathfrak{q})}.
\]
Using (A4), (A3) and the fact that $\forall x((Ix\psi) \land \varphi \leftrightarrow (Ix\psi)) \in \Sigma^*$ we obtain that $\forall x(Ix\psi \rightarrow Ix\varphi) \in \Sigma^*$. Thus we have $Ix\varphi[a] \in \Sigma^*$.

Let $q_n^k$ be the topology generated by $q_n$. By axioms (A1), (A2), (A4) we know that $q_n^k$ is a basis for $q_n^*$. We claim
\[
(\mathfrak{A}, \mathfrak{q}) \equiv L(I_n)_{n \in \omega}(\mathfrak{A}, \mathfrak{q}^*).
\]
We will prove this by showing that, for each $\psi(x_1, \ldots, x_n)$,
\[
[Ix\psi]^{(\mathfrak{A}, \mathfrak{q}^*)} = [\psi(x)]^{(\mathfrak{A}, \mathfrak{q}^*)}.
\]
Again this is proved by induction on the complexity of $\psi$. The only difficult case is $Ix\varphi$. Since $q_n$ is a basis for $q_n^*$ and from the definition of $\models$ we easily obtain
\[
[Ix\varphi]^{(\mathfrak{A}, \mathfrak{q}^*)} \subseteq [Ix\varphi]^{(\mathfrak{A}, \mathfrak{q}^*)}.
\]

Let $a \in [Ix\varphi]^{(\mathfrak{A}, \mathfrak{q}^*)}$. Then $a \in \emptyset \subseteq [\varphi(x)]^{(\mathfrak{A}, \mathfrak{q}^*)}$ where $\emptyset \in q_n^*$. Since $q_n$ is a basis for $q_n^*$, there is an $\emptyset^* \in q_n$ such that $a \in \emptyset^* \subseteq \emptyset$.

Using the induction hypothesis we see that $a \in \emptyset^* \subseteq [\varphi(x)]^{(\mathfrak{A}, \mathfrak{q})}$ whence $a \in [Ix\varphi]^{(\mathfrak{A}, \mathfrak{q})}$.

**Definition 3 (Tarski and Vaught).** $(\mathfrak{B}, \mathfrak{r})$ is said to be an elementary extension of $(\mathfrak{A}, \mathfrak{q})$, in symbols $(\mathfrak{A}, \mathfrak{q}) < (\mathfrak{B}, \mathfrak{r})$, if and only if $A \subseteq B$ and for all formulas $\varphi(x_1, \ldots, x_n)$ of $L(I_n)_{n \in \omega}$ and all $a_1, \ldots, a_n \in A$ we have $(\mathfrak{A}, \mathfrak{q}) \models \varphi[a]$ iff $(\mathfrak{B}, \mathfrak{r}) \models \varphi[a]$. A sequence $(\mathfrak{A}_\alpha, \mathfrak{q}^\alpha), \alpha < \gamma$, of weak models is said to be an elementary chain if and only if we have $(\mathfrak{A}_\alpha, \mathfrak{q}^\alpha) < (\mathfrak{A}_\beta, \mathfrak{q}^\beta)$ for all $\alpha < \beta < \gamma$.

The union of an elementary chain $(\mathfrak{A}_\alpha, \mathfrak{q}^\alpha), \alpha < \gamma$, is the weak model $(\mathfrak{A}, \mathfrak{q}) = \bigcup_{\alpha < \gamma} (\mathfrak{A}_\alpha, \mathfrak{q}^\alpha)$ such that $(\mathfrak{A} = \bigcup_{\alpha < \gamma} \mathfrak{A}_\alpha$ and $q_n = \{S \subseteq A^n | \text{for some } \beta < \gamma, \beta \leq \alpha < \gamma \text{ implies } S \cap A^n \subseteq q^\alpha \}$.

These definitions enable us to state:

**Theorem 4.** Let $(\mathfrak{A}_\alpha, \mathfrak{q}^\alpha), \alpha < \gamma$ be an elementary chain of weak models of (A0)–(A4) and let $(\mathfrak{A}, \mathfrak{q})$ be the union. Then, for all $\alpha < \gamma$, $(\mathfrak{A}_\alpha, \mathfrak{q}^\alpha) < (\mathfrak{A}, \mathfrak{q})$.

We conclude this introduction by stating an omitting types theorem whose proof is similar to the one found in Keisler [7].

**Theorem 5.** Let $\Gamma$ be a set of sentences of $L(I_n)_{n \in \omega}$ containing (A0)–(A4) and $\Sigma_n(y_{n1}, \ldots, y_{nk}), n \in \omega$, be sets of formulas of $L(I_n)_{n \in \omega}$. If $\Gamma$ is consistent and omits each $\Sigma_n$ then $\Gamma$ has a weak model which omits each $\Sigma_n$.

**Remark.** In this section we could have developed a theory of weak models using fewer axioms than (A0)–(A4) for our definition of $\models$. For brevity and directness we restricted the scope of the results to interior operators.
1. Main Theorem and applications. We will show that the $L(\Gamma^n)_{\alpha \in \omega}$ theory of continuous functions (relations) on product spaces has the following axiomatization:

   (A0), \ldots, (A4),
   (A5) $I\!x\varphi \to I\!x\varphi$,
   (A6) $I\!x_{i_1}, \ldots, x_{i_k}(\varphi(\sigma \circ x)) \leftrightarrow I\!x(\varphi(\sigma \circ x))$ where $\sigma: n + 1 \to n + 1$ and range $\sigma = \{i_1 < \cdots < i_k\}$,
   (A7) $I\!x\varphi \land I\!x\psi \leftrightarrow I\!x(\varphi \land \psi)$,
   (A8) $I\!x\varphi(y) \land \varphi(x, y \uparrow k) \to I\!x\varphi_{k+1} \ldots, \varphi_m \exists z(\varphi(x, z) \land \psi(z, y_{k+1}, \ldots, y_m))$ where $lh\ y = m$ and $k < m$.

   That is, the inverse image of a segment of an interior point concatenated with its tail is an interior point which is stronger than the usual definition of continuous but is equivalent to the usual definition of a continuous function (relation) on product spaces. Here, if $x = \langle x_1, \ldots, x_n \rangle$ and $\sigma: n + 1 \to n + 1$, we have that $\sigma \circ x(i) = x_{\sigma(i)}$.

   Suppose we have a topological model $(\mathfrak{A}, \varphi)$ satisfying (A0)-(A8), $\alpha \in J$. Let $(\mathfrak{A}, \varphi)_{\beta \in D}$ be a collection of subsets of $A$. If we add the $(\{\varphi\}_{\beta \in D})$ to $\mathfrak{A}$, and still expect to have a model satisfying (in the expanded language with a $V_\beta$ for each $\varphi\in D$) (A0)-(A8), $\alpha \in J$, what do we need to add to $\varphi_n, n \in \omega$?

   Let $\mathfrak{A}_\alpha, \alpha \in J$, be a collection of $(n_\alpha, m_\alpha)$-ary relations which satisfy (A8), $\alpha \in J$. There is trouble later on if the $\varphi\alpha$ are formulas of $L(\Gamma^n)$ since $L(\Gamma^n)$ is not closed under substitution. We will restrict the $\varphi\alpha$ to be relations defined by formulas of $L$ for convenience and clarity.

   Let $\varphi^{-1}_\alpha = \{(b, a) | (a, b) \in \varphi\alpha\}$ be the inverse relation of $\varphi\alpha, \alpha \in J$. We then define a collection of (definable) relations as follows (cf. [16]): $WT_0 = \{\varphi^{-1}\}_\alpha \in J \cup \{identity\ \text{relation\ on\ each\ } n \in \omega\}$, $WT_{n+1} = WT_n \cup \{\varphi(\sigma \circ x, y)\}$ where $\sigma$ maps $n_\varphi$ into $n_\psi$ where $\varphi \in WT_n$ \cup \{\varphi(x, y)\}$ where $\sigma$ maps $m_\psi$ into $m_\varphi$ where $\varphi \in WT_n$ \cup \{\varphi(x_1, \ldots, x_n, y_1, \ldots, y_m) \land \psi(z_1, \ldots, z_n, t_1, \ldots, t_m)\}$ where $\varphi, \psi \in WT_n$ \cup \{\varphi(x_1, \ldots, c, \ldots, x_n, y_1, \ldots, y_m)\}$ where $\varphi \in WT_n$ and $c$ is an individual constant symbol \cup \{\varphi(x_1, \ldots, c, \ldots, x_n, y_1, \ldots, y_m) \land \psi(y_1, \ldots, y_n, z_1, \ldots, z_m)\}$, $k < m_\psi, k < n_\psi$, i.e. the composition of the two relations, where $\varphi, \psi \in WT_n$.

   Let $WT = \bigcup_{n \in \omega} WT_n$.

   The intuitive meaning of $WT$ is that it is the smallest collection of definable relations containing $WT_0$ and closed under composition, projection, products and mappings of the variables. Hence, since each $\varphi\alpha$ satisfies (A8), $\alpha \in J$, and $(\mathfrak{A}, \varphi)$ models (A0)-(A8), $\alpha \in J$, we have that each $\varphi \in WT$ takes definable open sets to definable open sets.

   Define $q^*_n, n \in \omega$, as follows: $q^*_n$ is the topology generated by $\{\varphi(\Pi_{\beta \in D} B_\beta)\}$ each $B_{\beta \in D}$ for $1 < j \leq k, \varphi \in WT$ and $\varphi$ maps into $A^\omega$. $\varphi(C)$ means $\{b | (\mathfrak{A}, \varphi) \vdash \varphi(c, b)\}$ and $c \in C$.

   We now state the following important lemma whose proof is analogous to Lemma 2.2 in [16].
Lemma 6. Let \((\mathfrak{M}, q^*)\) be as above. Then \((\mathfrak{M}, q^*)\) models \((A0)-(A8), \alpha \in J\), in the expanded language containing a \(V_\beta\) for each \(\forall \beta\).

Now we will proceed to prove the main completeness theorem by presenting the following lemma which tells us that for each \(c \in \emptyset\) (an open set in the \(I^n\) interpretation) we can add a \(\prod_{i=1}^n o_i\) (an open \(n\)-box) to the \(I^n\) interpretation such that \(c \subseteq \prod_{i=1}^n o_i \subseteq \emptyset\) and still keep \((A0)-(A8), \alpha \in J\).

Lemma 7. Let \(\Sigma\) be an \(L(I^n)_{n<\omega}\) theory consistent with \((A0)-(A8), \alpha \in J\). If \(c = \langle c_1, \ldots, c_n \rangle\) is an \(n\)-tuple so that \((\forall x \theta^\alpha)\lfloor c\rfloor\) is consistent with \(\Sigma\) then \(\forall x (\bigwedge_{i=1}^n V(x_i) \rightarrow \theta(x))\) is consistent with \(\Sigma\) and \((A0)-(A8), \alpha \in J\). Here the \(V_i(x_i), 1 \leq i \leq n\), are new one-place predicate symbols.

Proof. We need only prove this for countable \(\Sigma\) since then by using Theorem 2 we have it for all \(\Sigma\) and \((A0)-(A8), \alpha \in J\). Let \((\mathfrak{M}, q)\) be a topological model of \(\Sigma\). This is possible since \(\Sigma\) is consistent with \((A0)-(A4)\) and Theorem 2.

We will assume that we have some countable enumeration of the "potential" basic open sets of \((\mathfrak{M}, q^*)\), i.e., \(\emptyset^*_\beta = \bigcap q_\beta (\prod B_k)\) where without loss of generality we can assume that \(B_i = \bigcap_j V_i\) for \(1 \leq i \leq n\) and \(B_j = \emptyset_{\beta_j} \in q_{\beta_j}\) for \(k > l > n\). Also we take an enumeration of \(a_k(x_1, \ldots, x_m)\) of the formulas of \(L(I^n)_{n<\omega}(A)\).

We want to define \(o_i \subseteq A, 1 \leq i \leq n\,\) such that

\[
\prod_{i=1}^n o_i \subseteq [I\theta^\alpha]^{(\mathfrak{M}, \alpha)}
\]

and forming \((\mathfrak{M}, q^*)\) from the \(\{o_i\}_{1 \leq i \leq n}\) we have that \((\mathfrak{M}, q) \prec (\mathfrak{M}, q^*)\). To do this we will construct the \(o\)'s by induction.

Suppose we have picked \(b_1, \ldots, b_{i(k)}\) for each \(o_i\) so that

\[
\prod_{i=1}^n \{b_1, \ldots, b_{i(k)}\} \subseteq [I\theta^\alpha]^{(\mathfrak{M}, \alpha)}
\]

and \(c = b^1\). Now to pick the \(b_{i,m}\), \(1 \leq i \leq n, m < f(k+1)\), we want to insure that

\[
\prod_{i=1}^n \{b_1, \ldots, b_i, b_{i(k+1)}\} \subseteq [I\theta^\alpha]^{(\mathfrak{M}, \alpha)}
\]

and also to somehow guarantee that if \(\sigma(x)\) is a formula of \(L(I^n)_{n<\omega}\, A \in A^m\), and \((\mathfrak{M}, q) \equiv \neg I\sigma(x)[a]\) then we do not get \((\mathfrak{M}, q^*) \equiv I\sigma(x)[a]\). This is equivalent to \([\sigma(x)]^{(\mathfrak{M}, \alpha)} = [\sigma(x)]^{(\mathfrak{M}, \alpha)}\) for every \(\sigma(x)\).

We claim that the \(b_{i,k+1}, \ldots, b_{i(k+1)}\) can be picked such that

\[
\prod_{i=1}^n \{b_1, \ldots, b_{i(k+1)}\} \subseteq [I\theta^\alpha]^{(\mathfrak{M}, \alpha)}
\]

and \(\left(\bullet\right)\) if \(\emptyset^*_\beta \subseteq q^*_\alpha\) and \(\emptyset^*_\beta\) is the \((k+1)\)th basic open set \(\emptyset^*_\beta \subseteq [o_{k+1}(x)]^{(\mathfrak{M}, \alpha)}\) then there is an \(\emptyset^*_\alpha \subseteq q_m\) such that \(\emptyset^*_\beta \subseteq \emptyset^*_\alpha \subseteq \emptyset^*_\beta\).

This is done as follows. Define \((x/b)\) to be \(\prod_{i=1}^n \{x_i, b_i\}\) and let

\[
C = \bigcap_{b \in \prod_{i=1}^n \{b_1, \ldots, b^n\}} \bigcap_{t \in (x/b)} [I\theta^\alpha[t]]^{(\mathfrak{M}, \alpha)}
\]
where \( \{ I \varphi[t]\}^{(n, \zeta)} = \{(a_1, \ldots, a_n) | (\zeta, \varphi) \vdash I \varphi[k_1, \ldots, k_n] \) where \( k_i = a_i \) if \( t_i = x_i, k_i = b_i \) otherwise. (Notice that in using axioms (A5)-(A7) that \( C \) is a definable open set.)

Taking \( \mathcal{O}_{B_k}^* = \cap_{j=1}^{n} \varphi_B(\Pi_{i=1}^{n} B_k) \) we define

\[
C^n = \{ c \in A \mid c(1 \cdot k), \ldots, c(n \cdot k) \in C \) for each \( k, 1 < k < n \}
\]
\( C^n \) is open because \( C \) is and by axiom (A5).

Let \( \sigma_i: n_B \to \beta \) be a permutation for each \( i, 1 < i < n \). Then define \( \sigma = (\sigma_1, \ldots, \sigma_n) \cdot c \) where \( c \in C^n \) by \( \sigma \cdot c(i) = c(\sigma_k^{-1}(i)) \) if \( i \equiv k \mod n \). That is, we look at the coordinates of the vector \( c \) as a collection of \( n_B \) vectors of length \( n \) and permute each of its \( n \)th coordinates by \( \sigma_i \). Define \( \sigma \cdot C^n \) to be \( \{ \sigma \cdot c | c \in C^n \} \) and again we have that \( \sigma \cdot C^n \) is open by the axioms. Hence \( C^* = \cap \sigma \cdot C^n \) is open because the finite intersection of open sets is open.

We claim that if \( b \in C^* \) then

\[
t(b; \ldots, b_{n-1}) = \cap_{i=1}^{n} \varphi_B(t_i)
\]
\( \text{where } b_{k+1} = b(i - n + 1) \).

If \( t \in \Pi_{i=1}^{n} \{ b_i, \ldots, b_{k+n} \} \) then it is straightforward to see that there is a \( \sigma \) such that

\[
\langle \sigma \cdot b(n - i + 1), \ldots, \sigma \cdot b(n - (i + k)) \rangle = t
\]
for some \( 1 < i < n_B \). Then \( \sigma \cdot b \in C^n \) since \( \text{id} \cdot C^n = C^n \). Thus \( t \in C \) which implies that \( t \in \{ I \varphi[t]\}^{(n, \zeta)} \).

Remembering that

\[
\mathcal{O}_{B_k}^* = \cap_{j=1}^{n} \varphi_B(\Pi_{i=1}^{n} \gamma_i \times \prod_{i=1}^{k} B_k),
\]

define \( C^\# = \{ c \in A^{n \cdot (n + k)} | c \mid n \cdot \beta \in \mathcal{C}^\# \text{ and } \langle c(n \cdot n + k \cdot i + 1), \ldots, c(n \cdot n + k \cdot (i + 1)) \rangle \in \Pi_{j=1}^{n} B_j \) for each \( 1 < i < n_B \).

Take \( c \in C^\# \) and define \( \varphi \cdot c \) to be the vector defined by

\[
\varphi \cdot c(m \cdot i - 1) = \varphi_B(\langle c(n \cdot i + 1), \ldots, c(n \cdot (i + 1)), c(n \cdot n_B + n \cdot i + 1),
\]
\( \ldots, c(n \cdot n_B + n \cdot (i + 1)) \rangle)(1).
\]

By axiom (A8) and the other axioms we know that \( \varphi \cdot C^\# \) is open where \( \varphi \cdot C^\# = \{ \varphi \cdot c | c \in C^\# \} \).

Define \( \cap \varphi \cdot C^\# = \{ c \in A^m | \text{there is a } b \in C^\# \text{ such that } c(i) = \varphi \cdot b(k \cdot i) \) for each \( 1 < k < n_B \). Again by the axioms \( \cap \varphi \cdot C^\# \) is an open subset of \( A^m \). Now if

\[
\cap \varphi \cdot C^\# - \{ a_{k+1}(x) \}^{(n, \zeta)} \neq \emptyset
\]
then there is a \( b_{k+1}, \ldots, b_{k+n_B} \in C^\# \) such that

\[
\cap_{j=1}^{n_B} \varphi_B(b_{k+j} \times \prod B_k) - \{ a_{k+1}(x) \}^{(n, \zeta)} \neq \emptyset.
\]

Otherwise let \( b_{k+i} = b_k \) for \( 1 < i < n_B \). Set \( f(k + 1) = k + n_B \) and we are done.
Note. The proof of the corresponding result in [16] contains an error at this point. The correct proof is obtained by a direct adaptation of this one to $L(Q^*)_{n \in \omega}$.

Now we will show that if $\delta_i = \{b_i^k | k \in \omega\}$ then we have the conclusion to the lemma. Let $c \in \prod_{i=1}^{r-1} \delta_i \subseteq [I \times \varphi]^{|(\mathfrak{m}, \mathfrak{q})|}$ by ($\ast$). To show $(\mathfrak{m}, \mathfrak{q}) < (\mathfrak{m}, \mathfrak{q}^*)$ we use induction. The difficult case is the $I_x$ clause.

Since $q_i \subseteq q_i^*$ for all $i$ we have that if

$$(\mathfrak{m}, \mathfrak{q}) \models I x_{\mathfrak{q}}(x)[a]$$

then

$$(\mathfrak{m}, \mathfrak{q}^*) \models I x_{\mathfrak{q}}(x)[a]$$

for arbitrary $a \in A^m$.

Suppose that $(\mathfrak{m}, \mathfrak{q}) \models I x_{\mathfrak{q}}(x)[a]$ and $(\mathfrak{m}, \mathfrak{q}^*) \models I x_{\mathfrak{q}}(x)[a]$. $a \in \Theta^*_m \subseteq [(\mathfrak{q})(\mathfrak{m}, \mathfrak{q}^*)]$ is a basic open set of $q^*_m$. Thus by ($\ast\ast$)

$$\Theta^*_m \subseteq \Theta_m \subseteq [(\chi(x))]^{(|\mathfrak{q}|)}$$

for some $\Theta_m \in q_m$. Whence $(\mathfrak{m}, \mathfrak{q}) \models I x_{\mathfrak{q}}(x)[a]$ which is a contradiction. Thus the lemma is shown.

Now we are able to prove the main completeness theorem.

**Theorem 8.** Let $\Sigma$ be an $L(I^*|\mathcal{A})_{n \in \omega}$ theory. Then $\Sigma$ is consistent with $(A0)-(A8_{\mathfrak{q}_m})$, $\alpha \in J$, if and only if $\Sigma$ has a complete topological model $(\mathfrak{B}, \mathfrak{r})$ such that each $\varphi_\alpha$, $\alpha \in J$, is continuous.

**Proof (if).** Straightforward since $(A0)-(A8_{\mathfrak{q}_m})$, $\alpha \in J$, are true in every complete topological model where the $\varphi_\alpha$, $\alpha \in J$, are continuous.

**(Only if)** Assume $\Sigma$ is consistent with $(A0)-(A8_{\mathfrak{q}_m})$, $\alpha \in J$. Then by Theorem 2 we have a topological model $(\mathfrak{m}, \mathfrak{q})$ of $\Sigma$ and $(A0)-(A8_{\mathfrak{q}_m})$, $\alpha \in J$, which is generated by the definable open set. By repeated applications of Lemma 7 we obtain a topological model $(\mathfrak{B}, \mathfrak{r})$ of $\Sigma$, $(A0)-(A8_{\mathfrak{q}_m})$, $\alpha \in J$, and if $b \in [(\varphi(x))]^{(|\mathfrak{q}|)} \in r_n$ then there is a $b_1, \ldots, b_n \in r_1$ such that $b \in \prod_{i=1}^{r_n} b_i \subseteq [(\varphi(x))]^{(|\mathfrak{q}|)}$. Hence $(\mathfrak{B}, \mathfrak{r})$ is complete.

If we omit $(A8_{\mathfrak{q}_m})$, $\alpha \in J$, then we obtain the following interesting corollary.

**Corollary 9.** Let $\Sigma$ be an $L(I^*|\mathcal{A})_{n \in \omega}$ theory. Then $\Sigma$ is consistent with $(A0)-(A7)$ if and only if $\Sigma$ has a complete topological model.

**Proof.** This is a direct application of Theorem 8.

**Corollary 10 (Compactness Theorem).** Let $\Sigma$ be an $L(I^*|\mathcal{A})_{n \in \omega}$ theory. Then $\Sigma$ has a complete topological model where each $\varphi_\alpha$, $\alpha \in J$, is continuous if and only if every finite subset of $\Sigma$ has a complete topological model where $\varphi_\alpha$, $\alpha \in J$, is continuous.

**Proof.** An easy application of the main completeness theorem.

**Corollary 11.** The set of $L(I^*|\mathcal{A})_{n \in \omega}$ sentences valid in every complete topological model (with $\varphi_\alpha$, $\alpha \in J$, continuous) is recursively enumerable in the signature of $L$ and the set $\{\varphi_\alpha | \alpha \in J\}$.
PROOF. Theorem 8 shows that a sentence is provable from \((A_0)-(A_{89})\), \(\alpha \in J\), if and only if it is valid, so we are done.

We can now state a Löwenheim-Skolem theorem for complete topological models with continuous functions whose proof is analogous to the author's for \(L(Q^n)\) in [16].

**Theorem 12.** (a) Let \((\mathcal{M}, q)\) be an infinite complete topological model where each \(q_\alpha, \alpha \in J\), is continuous. Then for any \(\kappa > |L| + |A|\) there is a complete topological model \((\mathcal{N}, r)\) such that \((\mathcal{M}, q) < (\mathcal{N}, r), |B| = \kappa,\) and each \(q_\alpha\) is continuous in \((\mathcal{N}, r)\).

(b) Let \((\mathcal{M}, q)\) be a complete topological model where each \(q_\alpha, \alpha \in J\), is continuous. Then for any \(|L| < \kappa < |A|\) there is a complete topological model \((\mathcal{N}, r) < (\mathcal{M}, q)\) such that \(|B| = \kappa,\) and each \(q_\alpha, \alpha \in J\), is continuous in \((\mathcal{N}, r)\).

We finish the applications by presenting two other theorems whose proofs are adaptations to \(L(I^n)\) (as in the main completeness theorem) of the proof for \(L(Q^n)\) in [16].

**Theorem 13.** Let \(\Sigma\) be an \(L(I^n)\) theory and \(\kappa\) an infinite regular cardinal. Then \(\Sigma\) is consistent with \((A_0)-(A_{89})\), \(\alpha \in J\), and \(\neg Ixy(x \neq y) \iff x \neq y\) if and only if \(\Sigma\) has a 0-dimensional normal complete topological model of cardinality \(\kappa\) where each \(q_\alpha, \alpha \in J\), is continuous (complete in the model theoretic sense).

The proof is a straightforward adaptation of the proof in [16] using the following lemma.

**Lemma.** Let \(\Sigma\) be an \(L(I^n)\) theory consistent with \((A_0)-(A_{89})\), \(\alpha \in J\). Then if \(\Sigma\) is consistent with \(\forall x (Ixy(x \neq y) \iff x \neq y), \forall x (\neg q_\phi(x) \iff Ix \neg q_\phi(x)), \forall x (\neg q_\phi(x) \iff Ix \neg q_\phi(x)), \) i.e., \(\psi\) and \(\varphi\) define disjoint closed sets, then \(\forall x (\psi(x) \rightarrow U^{\psi\phi}(x)) \forall x (q_\phi(x) \rightarrow U^{\psi\phi}(x)), \forall x (Ix U^{\psi\phi}(x) \rightarrow U^{\psi\phi}(x)), \) and \(\forall x (Ix \neg U^{\psi\phi}(x) \rightarrow U^{\psi\phi}(x))\) are consistent with \(\Sigma\) and \((A_0)-(A_{89})\), \(\alpha \in J\). Here \(U^{\psi\phi}(x)\) is a new one-place predicate symbol. The conclusion means that \(U^{\psi\phi}\) and \(\neg U^{\psi\phi}\) define complete open sets which separate \(\psi\) and \(\varphi\).

**Proof.** We need only show the lemma for countable \(\Sigma\); then using the compactness theorem we obtain it for all \(\Sigma\). Let \((\mathcal{M}, q)\) be a countable topological model of \(\Sigma\) and \((A_0)-(A_{89})\), \(\alpha \in J\), where the \(q_i\) are generated by the definable open sets.

As in the proof of Lemma 7 we want to obtain \((\mathcal{M}, q^*)\) from \(\mathcal{M}, A \setminus \mathcal{U}\) and \((\mathcal{M}, q)\) such that \([q]^\mathcal{M} \subseteq \mathcal{U}\) and \([q]^\mathcal{M} \subseteq A \setminus \mathcal{U}\) and \((\mathcal{M}, q) < (\mathcal{M}, q^*)\).

We will define, as in Lemma 7, \(\mathcal{U}\) and \(A \setminus \mathcal{U}\) by induction. To do this, suppose we have defined \(r_1, \ldots, r_{k(k)}\) for \(\mathcal{U}\) and \(s_1, \ldots, s_{k(k)}\) for \(A \setminus \mathcal{U}\) up to stage \(k\). Now we will define \(r_{k(k)+1}, \ldots, r_{k(k)+1}\) for \(\mathcal{U}\) and \(s_{k(k)+1}, \ldots, s_{k(k)+1}\) for \(A \setminus \mathcal{U}\).

Again without loss of generality we have \(\Theta^* = \cap \mathcal{P}(\mathcal{B})\) where \(B_1 = \mathcal{U}\) and \(B_2 = A \setminus \mathcal{U}\). We will define the \(r\)'s and \(s\)'s such that \(r_i \neq s_j\) for \(1 < i \neq j < f(k + 1)\) and such that if

\[
\Theta^* = \left[ (r_{k+1})^{(\mathcal{U}, q^*)}\right]
\]
then there is a \( \Theta \subseteq q_m \) such that

\[
\Theta^{\ast}_{n+1} \subseteq \Theta \subseteq [\sigma(x)]^{(\Re, \mathfrak{a})}.
\]

Define \( \Delta = \{ c \in A^{2n^2} | c(i) = c(j) \text{ for some } 1 \leq i < n_B \text{ and } n_B < j \leq 2n_B \} \). \( \Delta \) is closed in \( A^{2n^2} \) by the axiom \( \forall xy (Ixy(x \neq y) \leftrightarrow (x \neq y)) \). Hence \( A^{2n^2} - \Delta \cup [\varphi \vee \psi]^{(\Re, \mathfrak{a})} \) is an open subset of \( A^{2n^2} \). As in Lemma 7 take

\[
C^* = A^{2n^2} - \Delta \cup [\varphi \vee \psi]^{(\Re, \mathfrak{a})}
\]

and form \( C^* \), \( \varphi \cdot C^* \) and \( \cap \varphi \cdot C^* \) which are open subsets of \( A^{2n^2} \) and \( A \), respectively. (If \( C^* \) is finite the proof is trivial, so we will assume it is infinite.)

If

\[
\bigcap \varphi \cdot C^* \subseteq [\sigma_{k+1}(x)]^{(\Re, \mathfrak{a})} \neq \emptyset
\]

then pick \( r_{(k+1)} \), ..., \( r_{(k)+n_B} \) and \( s_{(k)+1} \), ..., \( s_{(k)+n_B} \) so that

\[
\bigcap_{i=1}^{n_B} \varphi^R \left( <r_{(k)+i}, s_{(k)+i} > \Pi B_k \right) - [\sigma_{k+1}(x)]^{(\Re, \mathfrak{a})} \neq \emptyset.
\]

Otherwise let \( r_{(k)+i} = r_{(k)} \) and \( s_{(k)+i} = s_{(k)} \) for each \( 1 \leq i < n_B \). Set \( f(k+1) = f(k) + n_B \) and we are done.

Let \( \mathfrak{U} = [\varphi]^{(\Re, \mathfrak{a})} \cup \{ r_i \}_{i \in \omega} \), and using the fact that \( \mathfrak{U} \cap \{ s_i \}_{i \in \omega} = \emptyset \) we can easily prove as in Lemma 3.1.5 in [16] that \( (\mathfrak{U}, q) < (\mathfrak{U}, q^*) \).

Note. Again the proof of Lemma 3.1.5 in [16] should be corrected along the lines of this proof since it is incorrect as published. The verification of \( (\mathfrak{U}, q) < (\mathfrak{U}, q^*) \), however, is correct.

**Corollary 14.** Let \( \Sigma \) be a countable \( L(I^n)_{n \in \omega} \) theory. Then \( \Sigma \) is consistent with \( (A_0)-(A_{n+\alpha}) \), \( \alpha \in I \), and \( Ixy(x \neq y) \leftrightarrow x \neq y \) if and only if \( \Sigma \) has a second countable 0-dimensional metrizable complete topological model where each \( \varphi_{\alpha} \) is continuous.

**Proof.** Use the fact that a second countable, regular and Hausdorff space is metrizable.

Let \( L(I) \) be the sublanguage of \( L(I^n)_{n \in \omega} \) in which the interior operator \( Ix \) is only applied to the single variable \( x \).

We now study the interrelation of \( L(I) \) theories and \( L(I^n)_{n \in \omega} \) theories. The reason for this is that in \( L(I^n)_{n \in \omega} \) we have a method of expressing the fact that a function is continuous in a product topology. It is thus natural to ask what conditions on functions (or relations) in an \( L(I) \) theory \( \Sigma \) are necessary to insure that they can be interpreted as continuous functions in some \( L(I^n)_{n \in \omega} \) theory extending \( \Sigma \).

The following definition and theorem formalize this.

**Definition 15.** \( \varphi_{\alpha}(x_1, \ldots, x_n, y_1, \ldots, y_m) \), \( \alpha \in J \), a collection of \( (n, m) \)-ary relations is called \( L(I)\)-continuous (in \( \Sigma \)) if and only if

\[
\bigwedge_{i=1}^{m} Iy_i \psi_i(y_i) \land \theta(t, y) \rightarrow Iz \exists y \left( \bigwedge_{i=1}^{m} \psi_i(y_i) \land \theta(t, y) \right)
\]
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is consistent with $\Sigma$, where the $\varphi_i(y_i)$ are arbitrary formulas of $L(I)$, $t \in ((\sigma \circ x)/z)$ and $\sigma : n + 1 \rightarrow n + 1$ (where $\theta$ is an arbitrary composition of the $\varphi_a$, i.e., $\theta \in W(T)$, and $((\sigma \circ x)/z)$ is the collection of $k$-tuples which are permuted by $\sigma$ and then any number of them are replaced by $z$).

**Theorem 16.** Let $T$ be an $L(I)$ theory and let $\varphi_a(x_1, \ldots, x_n, y_1, \ldots, y_m), \alpha \in J$, be $(n_a, m_a)$-ary $L(I)$-continuous relations. Then there is an $L(I^n)_{n \in \omega}$ theory $T^*$, such that $T \subseteq T^*$ and $\langle A0 \rangle - \langle A8_\varphi \rangle$, $\alpha \in J$, are in $T^*$. (This is to say that we can find a complete topological model $(\mathfrak{M}, q)$ of $T$ where each $\varphi_a$ is continuous in the product topology.)

Consider a group $(G, \cdot)$. Now take a topology $\tau$ on $G$. We call $(G, \cdot, \tau)$ a **topological group** if $-1$ and $\cdot$ are continuous maps into $G$. Other definitions of topological-algebraic structures, e.g. a topological vector space, often appear in mathematics. Using Theorem 16 we are now able to give an $L(I)$ axiomatization of their $L(I)$ theories. For more details on topological groups, etc., see [5].

We formalize these comments in the following corollary.

**Corollary 17.** Let $T$ be an $L(I)$ theory. Then $T$ has a topological group model if and only if $T$ is consistent with the basic $L(I)$ axioms, group axioms, and $Iy\psi(y)[t] \rightarrow Ix\psi(t)$ where

\[
\sigma: k + 1 \rightarrow k + 1 \text{ and } \epsilon: k + 1 \rightarrow \{1, -1\}.
\]

**Proof.** These axioms for topological groups are just the definition of $L(I)$-continuity for $x^{-1}$ and

**Corollary 18.** Let $T$ be an $L(I)$ theory. Then $T$ has a topological abelian group model if and only if $T$ is consistent with the basic $L(I)$ axioms, abelian group axioms, $Iy\psi(y)[x^{-1}] \rightarrow Ix\psi(x^{-1})$, and $Iy\psi(y)[x^{n} \cdot y] \rightarrow Ix\psi(x^n \cdot y)$.

2. **Interpolation, definability and omitting types.** In this section we will prove a Robinson-type joint-consistency theorem and an omitting types theorem for $L(I^n)_{n \in \omega}$.

**Theorem 19.** Let $T_1, T_2$ be $L_1(I^n)_{n \in \omega}$, $L_2(I^n)_{n \in \omega}$, theories, respectively, consistent with $\langle A0 \rangle - \langle A8_\varphi \rangle$. Then if $T_1 \cap T_2$ is a complete $L_1 \cap L_2(I^n)_{n \in \omega}$ theory consistent with $\langle A0 \rangle - \langle A8_\varphi \rangle$ then $T_1 \cup T_2$ is $L_1 \cup L_2(I^n)_{n \in \omega}$ consistent with $\langle A0 \rangle - \langle A8_\varphi \rangle$.

**Proof.** Denote $L_1 \cap L_2$ by $L$ and $T_1 \cap T_2$ by $T$. Let $(\mathfrak{M}, \varphi)$ be a topological model of $T_1$. Assume $\sigma(x), \psi(x)$ are formulas of $L_1(I^n)_{n \in \omega}(A)$ such that $\sigma$ defines an open set and $\psi$ does not. Let $b \in \{[\psi(x)]^{\mathfrak{M} \varphi} - [Ix\psi]^{\mathfrak{M} \varphi} \text{ and } \varphi(b) \in Th_{L_1}((\mathfrak{M}, \varphi))$. Take $\chi(x)$ to be a formula of $L_2(I^n)_{n \in \omega}$ such that $\forall x(Ix\chi(x) \leftrightarrow \chi(x)) \in T_2$, i.e., $\chi$ is open, $\chi[b] \in T_2 \cup T_L((\mathfrak{M}, \varphi))$, and take $a_1, \ldots, a_n$ to be new constant symbols. Notice that $T_2' = T_2 \cup T_L((\mathfrak{M}, \varphi))$ is consistent since $T$ is complete and consistent.
Form

\[ T^\sharp_i = \text{Th}_L(\mathcal{X}, \mathfrak{q}) \cup \{ \neg \psi[a] \land \sigma[a] \}, \]
\[ T^\sharp_2 = T_2 \cup \{ \chi[a] \}. \]

Let \( \text{Cn}(T^\sharp_i) \) be the set of consequences of \( T^\sharp_i \) in \( L(I^n)_{n \in \omega}(A \cup \{ a_1, \ldots, a_n \}) \).

We claim that \( \text{Cn}(T^\sharp_1) \cup \text{Cn}(T^\sharp_2) \) is consistent. To prove this, suppose not; then for some \( \theta \) we have that \( \theta \in \text{Cn}(T^\sharp_1) \) and \( \neg \theta \in \text{Cn}(T^\sharp_2) \).

That is to say,

\[ \text{Th}(\mathcal{X}, \mathfrak{q}) \vdash (\neg \psi[a] \land \sigma[a]) \rightarrow \theta \]

and

\[ T^\sharp_2 \vdash \chi[a] \rightarrow \neg \theta. \]

Replacing \( a_1, \ldots, a_n \) by \( z_1, \ldots, z_n \) and generalizing we obtain

\[ \text{Th}_L(\mathcal{X}, \mathfrak{q}) \vdash \forall z((\neg \psi[z] \land \sigma[z]) \rightarrow \theta[z]), \]

or equivalently

\[ \text{Th}_L(\mathcal{X}, \mathfrak{q}) \vdash \forall z(\neg \theta[z] \rightarrow (\psi[z] \lor \neg \sigma[z])), \]

and

\[ T^\sharp_2' \vdash \forall z(\neg \theta[z] \rightarrow (\psi[z] \lor \neg \sigma[z])). \]

Using the monotonicity of the interior operator we show

\[ \text{Th}_L(\mathcal{X}, \mathfrak{q}) \vdash \forall z(\neg \theta[z] \rightarrow \text{Iz}(\psi \lor \neg \sigma)[z]) \]

and

\[ T^\sharp_2' \vdash \forall z(\text{Iz}(\psi \lor \neg \sigma)[z] \rightarrow \neg \theta[z]). \]

But \( \text{Iz}[z] \rightarrow \chi[z] \), so \( T^\sharp_2 \vdash \chi[z] \rightarrow \text{Iz}[\neg \theta[z]], \) and so

\[ T^\sharp_2' \vdash \chi[b] \rightarrow (\text{Iz}(\neg \theta[z])[b]), \]

Using the consistency and completeness of \( T^\sharp_2' \cap L(I^n)_{n \in \omega}(A) = \text{Th}_L(\mathcal{X}, \mathfrak{q}) \) we have that

\[ \text{Th}_L(\mathcal{X}, \mathfrak{q}) \vdash (\text{Iz}[\psi \lor \neg \sigma])[z][b] \rightarrow (\text{Iz}[\psi \lor \neg \sigma][z][b]) \]

so

\[ \text{Th}_L(\mathcal{X}, \mathfrak{q}) \vdash (\text{Iz}[\psi \lor \neg \sigma][z])[b]. \]

Thus, using (A4), we get \( \text{Th}_L(\mathcal{X}, \mathfrak{q}) \vdash (\text{Iz}(\sigma \land (\psi \lor \neg \theta))[z])[b]. \) However, \( \vdash \sigma \land (\psi \lor \neg \varphi) \leftrightarrow \psi \), hence \( \text{Th}_L(\mathcal{X}, \mathfrak{q}) \vdash (\text{Iz}[\psi[z])[b], \) a contradiction since \( [\text{Iz}[\psi[z]](\mathfrak{q}, \mathfrak{a}) = [\text{Iz}[\psi]](\mathfrak{q}, \mathfrak{a}). \)

Thus \( \text{Cn}(T^\sharp_1) \cup \text{Cn}(T^\sharp_2) \) is consistent. From this we easily conclude that \( \Delta = \text{Cn}(T^\sharp_1) \cup \text{Cn}(T^\sharp_2) \cup T_2 \) is consistent.

Let \( (\mathcal{B}, \mathfrak{r}) \) be a topological model of \( \Delta \) generated by the definable open sets. Then \( (\mathcal{X}, \mathfrak{q}) \prec_{L(I^\omega)_{n \in \omega}(\mathcal{B}, \mathfrak{r})} (\mathcal{B}, \mathfrak{r}) \) and \( (\mathcal{B}, \mathfrak{r}) \models T_2 \).

Notice for the following induction that \( T^\sharp_i \cup \text{Th}_L((\mathcal{B}, \mathfrak{r})), \text{Th}_{L_2}((\mathcal{B}, \mathfrak{r})) \) are consistent and \( \text{Th}_L((\mathcal{B}, \mathfrak{r})) \) is complete.
Interchanging $T_1$ and $T_2$ in the construction and iterating $\omega$ times through all the triples of $L_1$ definable open and nonopen sets and the definable $L_2$ open sets and vice versa we produce an elementary chain $(\mathcal{U}_i, q'_i), i \in \omega$, of models of $T_1$, $(\mathcal{V}_i, r'_i), i \in \omega$, of models of $T_2$ whose topologies are generated by definable open sets such that

$$
\ldots (\mathcal{U}_i \upharpoonright L, q'_i) \prec (\mathcal{V}_i \upharpoonright L, r'_i) \prec (\mathcal{U}_{i+1} \upharpoonright L, q'_{i+1}) \prec (\mathcal{V}_{i+1} \upharpoonright L, r'_{i+1}), \ldots
$$

forms an elementary chain. We also have that if $\psi(x)$ is a formula of $L_1(I^n)_{n \in \omega}(A_k)$ such that $[\psi(x)](\mathcal{U}_i, q'_i)$ is not open in $q^k_i$, $\emptyset \in q^k_i$ and $\emptyset^* \in r^k_i$ then either

$$
\emptyset \cap \emptyset^* \subseteq [\mathcal{I}x\psi](\mathcal{U}_i, q'_i) \quad \text{or} \quad \emptyset \cap \emptyset^* \subseteq [\psi(x)](\mathcal{V}_i, r'_i) \neq \emptyset.
$$

Similarly for $L_2(I^n)_{n \in \omega}$ in place of $L_1(I^n)_{n \in \omega}$.

Let $(\mathcal{W}, q^\omega)$ be the topological model generated by $\bigcup_{i \in \omega}(\mathcal{U}_i, q'_i)$ and $(\mathcal{V}, r^\omega)$ be the topological model generated by $\bigcup_{i \in \omega}(\mathcal{V}_i, r'_i)$.

Because the $q^k_i$ and $r^k_i$ are generated by the definable open sets we see that $(\mathcal{W}, q^\omega) \models T_1$, $(\mathcal{V}, r^\omega) \models T_2$. We also know that

$$
\mathcal{W} \models L = \mathcal{V} \models L.
$$

Define $(\Omega, p) = (\mathcal{W} \cup \mathcal{V}, (r^\omega_1 \cup q^\omega_1)^*, (r^\omega_2 \cup q^\omega_2)^*, \ldots)$ where $(r^\omega_i \cup q^\omega_i)^*$ is the topology generated by the definable sets of $r^\omega_i \cup q^\omega_i$ and $\mathcal{W} \cup \mathcal{V}$ is the $L_1 \cup L_2$ model formed from $\mathcal{W}$ and $\mathcal{V}$ using $(\ast)$.

We claim that

$$
(\mathcal{W}, q^\omega) \prec (\Omega, p)
$$

and

$$
(\mathcal{V}, r^\omega) \prec (\Omega, p).
$$

This implies that $(\Omega, p) \models T_1 \cup T_2$.

We will show $(\mathcal{W}, q^\omega) \prec (\Omega, p)$ and the other equivalence follows by analogy. This assertion follows easily from the claim that

$$
[\psi(x)](\mathcal{W}, q^\omega) = [\psi(x)](\Omega, p)
$$

for each $\psi(x)$ a formula of $L_1(I^n)_{n \in \omega}(A^\omega)$ which we prove by induction on the complexity of $\psi(x)$.

The only difficult case is the $Ix$ case. Suppose $\psi(x) = Ix\varphi$. Since $q^k_i \subseteq p_i$ we obtain that $[\psi(x)](\mathcal{W}, q^\omega_i) \subseteq [\psi(x)](\Omega, p)$. Take $a \in [\psi(x)](\Omega, p)$. By definition there are $\emptyset \in q^k_i$, $\emptyset^* \in r^k_i$ such that $a \in \emptyset \cap \emptyset^* \subseteq [\varphi(x)](\Omega, p)$.

By the construction of $q^\omega_i$, $r^\omega_i$ we know that there are a $\chi$ of $L_1(I^n)_{n \in \omega}(A^\omega)$ and a $\delta$ of $L_2(I^n)_{n \in \omega}(A^\omega)$ such that

$$
a \in [\chi(x)](\mathcal{W}, q^\omega) \cap [\delta(x)](\Omega, p) \subseteq \emptyset \cap \emptyset^*.
$$

Hence, we know that by our construction

$$
a \in [\chi(x)](\mathcal{W}, q^\omega) \cap [\varphi(x)](\Omega, p) \subseteq [Ix\varphi(x)](\mathcal{W}, q^\omega)
$$

since otherwise by the construction of the elementary chain, $[\chi(x)](\mathcal{W}, q^\omega) \cap [\delta(x)](\Omega, p)$ would not be a subset of $[\delta(x)](\mathcal{W}, q^\omega) = [\varphi(x)](\Omega, p)$. Thus we are done because $(\Omega, p) \models T_1 \cup T_2$ and (A0)–(A8$_\varphi$).
Corollary 20. $L(I^n)_{n \in \omega}$ with $(A0)-(A8_\varphi)$ has an interpolation theorem.

Proof. Straightforward since the Robinson joint-consistency theorem implies interpolation for compact logics.

Corollary 21. $L(I^n)_{n \in \omega}$ with $(A0)-(A8_\varphi)$ has a Beth definability theorem.

Proof. Again the Robinson joint-consistency theorem implies the definability theorem.

Theorem 22. $L(I)$ with $(A0)-(A4)$ has a Robinson joint-consistency theorem.

Proof. The proof is the same as the $L(I^n)_{n \in \omega}$ case. In place of $I^n$ just use $I^1$.

Corollary 23. $L(I)$ with $(A0)-(A4)$ has an interpolation and a Beth definability theorem.

Proof. Use Theorem 22.

Remark. In [15] we show an interpolation theorem for a system weaker than $L(I)$. Also our methods apply to give an interpolation theorem for “ideal models” (see [0]) which was done independently by S. Shelah.

We now will state and prove an omitting types theorem for $L(I^n)_{n \in \omega}$ with $(A0)-(A8_\varphi)$.

Definition 24. A set of sentences $\Gamma$ topologically omits $\Sigma(x_1, \ldots, x_n)$ if and only if $\Gamma \cup \{(A0)-(A8_\varphi)\}$ has a model which omits $\Sigma$.

Theorem 25. Let $\Gamma$ be a countable set of sentences of $L(I^n)_{n \in \omega}$ and $\Sigma_n(y_{n1}, \ldots, y_{nk})$, $n \in \omega$, be sets of formulas of $L(I^n)_{n \in \omega}$. If $\Gamma$ is consistent with $(A0)-(A8_\varphi)$ and $\Gamma$ topologically omits each $\Sigma_n$, the $\Gamma$ has a topological model where each $\varphi$ is continuous and omits each $\Sigma_n$.

Proof. By Theorem 5 we obtain a countable weak model $(\mathfrak{A}, q)$ of $\Gamma$ and $(A0)-(A8_\varphi)$ which omits each $\Sigma_n$. If we take $q_k^\varphi$ to be the topology generated by the definable subsets of $q_k$, it is straightforward to see as in Theorem 2 that $(\mathfrak{A}, q) < (\mathfrak{A}, q^\varphi)$.

We actually proved in the proof of the main completeness theorem that for a countable topological model $(\mathfrak{A}, q^\varphi)$ there is a complete model $(\mathfrak{A}, r)$ such that $(\mathfrak{A}, r) < (\mathfrak{A}, q^\varphi)$. Hence $(\mathfrak{A}, r)$ models $\Gamma$, each $\varphi$ is continuous, and it omits each $\Sigma_n$, $n \in \omega$.

Remark. We would like to point out that we showed in [14] that $L(I^n)_{n \in \omega}$ cannot have an isomorphic ultrapowers theorem so our Theorem 19 is sharp.

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