ON LINEAR ALGEBRAIC SEMIGROUPS

BY

MOHAN S. PUTCHA

Abstract. Let $K$ be an algebraically closed field. By an algebraic semigroup we mean a Zariski closed subset of $K^n$ along with a polynomially defined associative operation. Let $S$ be an algebraic semigroup. We show that $S$ has ideals $I_0, \ldots, I_t$, such that $S = I_1 \supseteq \cdots \supseteq I_0$. $I_0$ is the completely simple kernel of $S$ and each Rees factor semigroup $I_k/I_{k-1}$ is either nil or completely 0-simple ($k = 1, \ldots, t$). We say that $S$ is connected if the underlying set is irreducible. We prove the following theorems (among others) for a connected algebraic semigroup $S$ with idempotent set $E(S)$: (1) If $E(S)$ is a subsemigroup, then $S$ is a semilattice of nil extensions of rectangular groups. (2) If all the subgroups of $S$ are abelian and if for all $a \in S$, there exists $e \in E(S)$ such that $ea = ae = a$, then $S$ is a semilattice of nil extensions of completely simple semigroups. (3) If all subgroups of $S$ are abelian and if $S$ is regular, then $S$ is a subdirect product of completely simple and completely 0-simple semigroups. (4) $S$ has only trivial subgroups if and only if $S$ is a nil extension of a rectangular band.

1. Preliminaries. Throughout this paper, $\mathbb{Z}^+$ will denote the set of all positive integers. If $X$ is a set, then $|X|$ denotes the cardinality of $X$. $K$ will denote a fixed algebraically closed field. If $n \in \mathbb{Z}^+$, then $K^n = K \times \cdots \times K$ is the affine $n$-space and $\mathbb{M}_n(K)$ the algebra of all $n \times n$ matrices. If $A \in \mathbb{M}_n(K)$, then $\rho(A)$ is the rank of $A$. In this paper we only consider closed sets with respect to the Zariski topology. So $X \subseteq K^n$ is closed if and only if it is the set of zeroes of a finite set of polynomials on $K^n$. Let $X \subseteq K^n$, $Y \subseteq K^n$ be closed, $\varphi: X \rightarrow Y$. If $\varphi = (\varphi_1, \ldots, \varphi_m)$ where each $\varphi_i$ is a polynomial, then $\varphi$ is a morphism. Let $p, n \in \mathbb{Z}^+$, $p < n$. Then we use, without further comment, the well-known fact that the set $T = \{A | A \in \mathbb{M}_n(K), \rho(A) < p\}$ is closed. In fact for $A \in \mathbb{M}_n(K), A \in T$ if and only if all minors of $A$ of order $\geq p$ vanish. By an algebraic semigroup we mean $(S, \circ)$ where $\circ$ is an associative operation on $S$, $S$ is a closed subset of $K^n$ for some $n \in \mathbb{Z}^+$ and the map $(x, y) \mapsto x \circ y$ is a morphism from $S \times S$ into $S$. If $S$ has an identity element then $S$ is an algebraic monoid. Polynomially defined associative operations on a field have been studied by Yoshida [21], [22], Plemmons and Yoshida [13]. Yoshida's results have been generalized to integral domains by Petrich [12]. Clark [4] has studied semigroups of matrices forming a linear variety. Algebraic monoids are briefly encountered in Demazure and Gabriel [8]. The author [15] has studied semigroups on affine spaces defined by polynomials of degree at most 2.

Let $S$ be an arbitrary semigroup. If $S$ has an identity element, then $S^1 = S$. Otherwise $S^1 = S \cup \{1\}, 1 \notin S$, with obvious multiplication. If $a \in S$, then the
centralizer of $a$ in $S$, $C_S(a) = \{x \mid x \in S, \text{ } xa = ax\}$. Then center of $S$, $C(S) = \bigcap_{a \in S} C_S(a)$. If $a, b \in S$, then $a \mid b$ (a divides b) if $b \in S^1aS^1$. $\mathbb{J}, \mathbb{R}, \mathbb{L}, \mathbb{H}$ will denote the usual Green's relations on $S$ (see [6]). If $a \in S$, then we let $J(a) = S^1aS^1$. $J_a$, $H_a$ will denote the $\mathbb{J}$-class and $\mathbb{H}$-class of $a$ in $S$, respectively. $E(S)$ will denote the set of idempotents of $S$. If $e, f \in E(S)$ then $e < f$ if $ef = fe = e$. An idempotent semigroup is called a band. A commutative band is called a semilattice. A band satisfying the identity $xyzw = xzwy$ is called a normal band. If $ab = b[ba = b]$ for all $a, b \in S$, then $S$ is a right [left] zero semigroup. A direct product of a right zero semigroup and a left zero semigroup is a rectangular band. A direct product of a rectangular band and a group is a rectangular group. A direct product of a right zero semigroup and a group is a right [left] group. Let $I$ be an ideal of $S$. If $S/I$ is a nil semigroup, then we say that $S$ is a nil extension of $I$. Let $\delta_a$ ($a \in \Gamma$) be a set of congruences on $S$. If $\cap_{a \in \Gamma} \delta_a$ is the equality congruence, then $S$ is a subdirect product of $S_a$ ($a \in \Gamma$). See [7, p. 99] for details. A congruence $\delta$ on $S$ is an $S$-congruence if $S/\delta$ is a semilattice. If $S$ is a disjoint union of subsemigroups $S_a$ ($a \in \Gamma$) and if for each $a, \beta \in \Gamma$, there exists $\gamma \in \Gamma$ such that $S_a S_\beta \cup S_\beta S_a \subseteq S_\gamma$, we will say that $S$ is a semilattice (union) of $S_a$ ($a \in \Gamma$). We also say that $S_a$ ($a \in \Gamma$) is a semilattice decomposition of $S$. There is an obvious natural correspondence between $S$-congruences and semilattice decompositions [6, p. 25]. A semigroup with no $S$-congruences other than $S \times S$ is said to be $S$-indecomposable. Let $\xi$ be the finest $S$-congruence on $S$. Throughout this paper, we let $\Omega = \Omega(S) = S/\xi$ denote the maximal semilattice image of $S$. By a theorem of Tamura [18], [19], each component of $\xi$ is $S$-indecomposable and is called the $S$-indecomposable component of $S$. An ideal $P$ of $S$ is prime if $S \setminus P$ is a subsemigroup of $S$. In such a case $\{P, S \setminus P\}$ is a semilattice decomposition of $S$. If $S$ is a commutative algebraic semigroup, then it follows from Corollary 1.4 below, and Tamura and Kimura [20] that $E(S) = \Omega(S)$. Let $S, T$ be algebraic semigroups, $\varphi: S \to T$ a (semigroup) homomorphism. Then $\varphi$ is a $*-$homomorphism if $\varphi$ is also a morphism (of varieties). If $\varphi$ is a bijection and if both $\varphi$ and $\varphi^{-1}$ are $*-$homomorphisms, then we say that $\varphi$ is a $*-$isomorphism and that $S, T$ are $*-$isomorphic. $S$ is connected if the underlying closed set is irreducible (i.e. is not a union of two proper closed subsets). The proof of the following result can be found in Demazure and Gabriel [8, II, §2, Theorem 3.3].

**Theorem 1.1 (see [8]).** Let $S$ be an algebraic monoid. Then $S$ is $*-$isomorphic to a closed submonoid of $\mathfrak{M}_n(K)$ for some $n \in \mathbb{Z}^+$. Let $S$ be an algebraic semigroup which is a group. Let $\varphi: S \to \mathfrak{M}_n(K)$ be given by Theorem 1.1. By Hilbert's Nullstellensatz, $1/\det \varphi(x)$ is a polynomial on $S$. So $x^{-1} = \varphi^{-1}(\text{adj } \varphi(x)/\det \varphi(x))$. Hence the map $x \to x^{-1}$ is a morphism and $S$ is an algebraic group in the usual sense [2].

The following result can also be found in [8, II, §2, Corollary 3.6].

**Corollary 1.2 (see [8]).** Let $S$ be an algebraic monoid which is not a group. Then the nonunits of $S$ form a closed prime ideal of $S$. 

458 M. S. PUTCHA
Let $S$ be an algebraic semigroup, $S \subseteq K^n$. Let $T = (S \times \{0\}) \cup \{(0, 1)\} \subseteq K^{n+1}$. Define $(x, \alpha)(y, \beta) = (xy + \beta x + \alpha y, \alpha \beta)$. Then $T$ is an algebraic monoid with identity element $(0, 1)$. $S$ is *-isomorphic to the closed subsemigroup $S \times \{0\}$ of $T$. Hence we have

**Corollary 1.3.** Let $S$ be an algebraic semigroup. Then $S$ is *-isomorphic to a closed subsemigroup $\mathfrak{M}_n(K)$ for some $n \in \mathbb{Z}^+$.

The following result was pointed out to the author by Clark [5].

**Corollary 1.4 [Clark].** Let $S$ be an algebraic semigroup. Then there exists $n \in \mathbb{Z}^+$ such that for all $a \in S$, $a^n$ lies in a subgroup of $S$.

**Proof.** By Corollary 1.3, we can assume that $S$ is a closed subsemigroup of $\mathfrak{M}_n(K)$ for some $n \in \mathbb{Z}^+$. Let $A \in S$. Then without loss of generality we can assume $A = (B; C)$, where $B \in \mathfrak{M}_p(K)$ is invertible and $C \in M_{n-p}(K)$ is nilpotent. Let $T = \{(x; \alpha) | X \in M_p(K), X \text{ is invertible}\}$. Then $A^n \in T$. Let $G = \{(x; \alpha) | X \in M_p(K), \alpha \in K, \alpha \det X = 1\}$. Then $G$ is an algebraic group, $\varphi: G \to T$ given by $\varphi((x; \alpha)) = X$ is a bijective morphism. $G_1 = \varphi^{-1}(T \cap S)$ is a closed subsemigroup of $G_1$. It is well known (see [8, II, §2, Corollary 3.5]) that a closed submonoid (and hence a closed subsemigroup) of a linear algebraic group is a subgroup. Thus $G_1$ is a subgroup of $G$. So $T \cap S = \varphi(G_1)$ is a subgroup of $S$. Since $A^n \in T \cap S$, we are done. □

**Corollary 1.5.** Let $S$ be an algebraic semigroup. Then $S$ has a kernel $M$ which is closed and completely simple.

**Proof.** By Theorem 1.1, we can assume that $S$ is a closed subsemigroup of $\mathfrak{M}_n(K)$ for some $n \in \mathbb{Z}^+$. By Corollary 1.4 and Clark [3], $S$ has a completely simple kernel $M$ given by its elements of minimal rank $r$. Then $M = \{a | a \in S, \rho(a) < r + 1\}$ is closed. □

**Lemma 1.6.** Let $\mathcal{E}$ be an infinite set of idempotents in $\mathfrak{M}_n(K)$ of rank $r$. Then there exist $E, F \in \mathcal{E}$ such that $E \neq F$ and $\rho(EF) = \rho(FE) = r$.

**Proof.** If $E \in \mathcal{E}$, then let $\mathcal{E}_E = \{A | A \in \mathfrak{M}_n(K), \rho(EA) < r\}$, $\mathcal{B}_E = \{A | A \in \mathfrak{M}_n(K), \rho(AE) < r\}$. Then $\mathcal{E}_E, \mathcal{B}_E$ are closed subsets of $\mathfrak{M}_n(K)$. We claim:

There exists an infinite subset $\mathcal{F}$ of $\mathcal{E}$ such that for all $E \in \mathcal{F}$, $|\mathcal{E}_E \cap \mathcal{F}| < \infty$. (1)

Suppose (1) is false. Then there exists $E_1 \in \mathcal{E}$ such that $\mathcal{E}_1 = \mathcal{E}_{E_1} \cap \mathcal{E}$ is infinite. Again by (1), there exists $E_2 \in \mathcal{E}_1$ such that $\mathcal{E}_2 = \mathcal{E}_{E_2} \cap \mathcal{E}_1$ is infinite. Continuing, we obtain a sequence $E_1, E_2, \ldots$, in $\mathcal{E}$ such that $\rho(E_iE_j) < r$ for $i < j$. So $E_{i+1} \in \mathcal{E}_{E_i} \cap \cdots \cap \mathcal{E}_{E_{i+1}} \subset \mathcal{E}_{E_{i+1}}$. Hence

$\mathcal{E}_{E_1} \supseteq \mathcal{E}_{E_1} \cap \mathcal{E}_{E_2} \supseteq \mathcal{E}_{E_1} \cap \mathcal{E}_{E_2} \cap \mathcal{E}_{E_3} \supseteq \cdots$.

Since $\mathcal{E}_E$'s are closed sets, we have a contradiction to the Hilbert Basis Theorem. Thus (1) is true. The dual of (1) applied to $\mathcal{F}$ shows that there exists an infinite subset $\mathcal{G}$ of $\mathcal{F}$ such that for all $E \in \mathcal{G}$, $|\mathcal{B}_E \cap \mathcal{G}| < \infty$. Let $E \in \mathcal{G}$. Then
A semigroup \( S \) with the property that a power of each element lies in a subgroup of \( S \) is said to be strongly \( \pi \)-regular. The study of strongly \( \pi \)-regular rings and semigroups was initiated by Azumaya [1], Drazin [9] and Munn [11]. Clark [3] showed that a strongly \( \pi \)-regular matrix semigroup has a kernel given by its elements of minimal rank. Let \( S \) be a strongly \( \pi \)-regular semigroup. A \( \mathcal{F} \)-class of \( S \) containing an idempotent is called regular.

**Theorem 1.7.** Let \( S \) be a strongly \( \pi \)-regular subsemigroup of \( \mathbb{M}_n(K) \). Then \( S \) has only finitely many regular \( \mathcal{F} \)-classes.

**Proof.** Suppose not. Then there exists an infinite set of idempotents \( \mathcal{E} \) of \( S \) such that for all \( e, f \in \mathcal{E} \), \( e^f f \) implies \( e = f \). Let \( r = 0, \ldots, n \), let \( \mathcal{E}_r = \{ e \in \mathcal{E} \mid \rho(e) = r \} \). Then \( \mathcal{E}_r \) is infinite for some \( r \). By Lemma 1.6, there exist \( e, f \in \mathcal{E}_r \) such that \( e \neq f, \rho(ef) = \rho(fe) = r \). Let \( \mathcal{V} \) be the space of all \( n \times 1 \) vectors on \( K \). Then \( ef^r = e^r f^r \), \( fe^r = f^r e^r \). Hence \( e^r = (ef)^r = f^r \). There exists an idempotent \( g \) of \( S \), \( p \in \mathbb{Z}^+ \) such that \( g^r = (ef)^r = (f^p)^r = e^r \). Let \( v \in \mathcal{V} \), then \( ev = v g^r \) and so \( ge = ev \). Hence \( f[(ef)^r] g \) \( e \). So \( f e \). Similarly \( e f \) and \( e^f f \). This contradiction proves the theorem. \( \square \)

**Corollary 1.8.** Let \( S \) be a strongly \( \pi \)-regular subsemigroup of \( \mathbb{M}_n(K) \). Then \( \mathcal{U}(S) \) is finite.

**Proof.** Let \( \varphi : S \to \mathcal{U}(S) \) denote the natural homomorphism. By Theorem 1.7, \( \varphi(E(S)) \) is finite. Let \( a \in S \). Then \( a^n \mathcal{E} \) for some \( e \in E(S) \). So \( \varphi(a) = \varphi(a^n) = \varphi(e) \). Hence \( \mathcal{U}(S) = \varphi(S) = \varphi(E(S)) \) is finite. \( \square \)

**Lemma 1.9.** Let \( S \) be a strongly \( \pi \)-regular semigroup with only finitely many regular \( \mathcal{F} \)-classes. Then there exist finitely many ideals \( I_0, \ldots, I_t \) of \( S \) such that \( S = I_0, \cdots, I_t \). \( I_0 \) is the completely simple kernel of \( S \) and each \( I_i/I_{i-1} \) is either completely 0-simple or a nil semigroup \( (i = 1, \ldots, t) \).

**Proof.** We prove by induction on the number of regular \( \mathcal{F} \)-classes of \( S \). Let \( E = E(S) \). Let \( J_{e_1}, \ldots, J_{e_n} \) be the regular \( \mathcal{F} \)-classes of \( S \) where \( e_1, \ldots, e_n \in E \). Let \( I = J(e_1) \cap \cdots \cap J(e_n) \). Then \( I \) is an ideal of \( S \). So there exists \( f \in I \cap E \). Let \( a \in S \). Then there exists \( m \in \mathbb{Z}^+ \) such that \( a^m \mathcal{F} e \) for some \( i \). So \( f \in J(a^m) \subset J(a) \). Hence \( J(f) = I_0 \) is the kernel of \( S \). By Munn [11], \( I_0 \) is completely simple. Let \( \mathcal{K} = \{ J(e) \mid e \in E \cap (S \setminus I_0) \} \). Then \( \mathcal{K} \) is finite. If \( \mathcal{K} = \emptyset \), then \( S \setminus I_0 \) has no idempotent and \( S \setminus I_0 \) is nil. So assume \( \mathcal{K} \neq \emptyset \). Then \( \mathcal{K} \) has a minimal element \( J(g) \), \( g \in E \). Let \( I_2 = J(g) \), \( I_1 = I_2 \setminus J_g \). Then \( I_0 \subset I_1 \) and \( I_1 \) is an ideal of \( S \). Let \( a \in I_1 \). Then \( a^m \mathcal{F} h \) for some \( h \in E, m \in \mathbb{Z}^+ \). Then \( h \in I_1 \). So \( J(h) \subset J(g) \). By minimality of \( J(g), h \in I_0 \). Thus \( a^m \in I_0 \). So \( I_1/I_0 \) is nil. Since \( I_2 \setminus I_1 = J_g \) and \( g \in E, I_2/I_1 \) is 0-simple. By Munn [11], \( I_2/I_1 \) is completely 0-simple. Clearly \( S/I_2 \) has lesser number of regular \( \mathcal{F} \)-classes than \( S \). We are thus done by our induction hypothesis. \( \square \)

By Theorem 1.7 and Lemma 1.9 we have the following.
Theorem 1.10. Let $S$ be a strongly $\pi$-regular subsemigroup of $\mathfrak{M}_n(K)$. Then there exist ideals $I_0, \ldots, I_t$ of $S$ such that $S = I_t \supseteq \cdots \supseteq I_0$, $I_0$ is the completely simple kernel of $S$ and each $I_i/I_{i-1}$ is either completely 0-simple or nil ($i = 1, \ldots, t$).

By Corollary 1.3, Corollary 1.4 and Theorem 1.10, we have

Corollary 1.11. Let $S$ be an algebraic semigroup. Then $S$ has ideals $I_0, \ldots, I_t$ such that $S = I_t \supseteq \cdots \supseteq I_0$, $I_0$ is the completely simple kernel of $S$ and each $I_i/I_{i-1}$ is either completely 0-simple or nil ($i = 1, \ldots, t$).

Theorem 1.12. Let $S$ be an algebraic semigroup and $P$ a prime ideal of $S$. Then $P$ is closed.

Proof. By Corollaries 1.3 and 1.4, we can assume that $S$ is a closed, strongly $\pi$-regular subsemigroup of $\mathfrak{M}_n(K)$ for some $n \in \mathbb{Z}^+$. Hence $S_1 = S \setminus P$ is strongly $\pi$-regular. By Clark [3] the kernel $T$ of $S_1$ is the set of elements of $S_1$ of minimal rank. Let $e \in E(T)$, $\rho(e) = r$. Let $a \in S_1$. Then $(eae)^r \in T$ and so $\rho((eae)^r) = r$.

Let $a \in P$. There exists $f \in E(P)$ such that $(eae)^r \not\subseteq f$. So $ef = fe = f$. Hence $\rho(f) < \rho(e) = r$. Clearly $\rho((eae)^r) = \rho(f)$. Thus $P = \{a | a \in S, \rho((eae)^r) < r\}$ is closed. □

2. Connected algebraic semigroups. Let $S$ be an algebraic semigroup, $e \in E(S)$. Then the maximal subgroup $H_e$ of $S$ need not be closed. However $H_e$ can be identified with $G = \{(a, b) | a, b \in S, ab = ba = e, ae = ea = a, be = eb = b\}$. If $(a, b), (c, d) \in G$, define $(a, b)(c, d) = (ac, db)$. Then $G$ is an algebraic group. The correspondence between $H_e$ and $G$ is given by $a \leftrightarrow (a, a^{-1})$. More precisely define $\varphi : G \rightarrow S$ as $\varphi(a, b) = a$. Then $\varphi$ is an injective *-homomorphism and $\varphi(G) = H_e$.

It is easy to show that $G$ is unique to within *-isomorphisms. It can also be easily shown that if $S$ is connected then so is $G$. However, we will not need these facts in this paper.

Theorem 2.1. Let $S$ be a connected algebraic semigroup. Then $\Omega(S)$ has an identity element.

Proof. Let $\Omega = \Omega(S), \varphi : S \rightarrow \Omega$ be the canonical homomorphism. By Corollary 1.8, $\Omega$ is a finite semilattice. Suppose $\Omega$ has two maximal elements $e, f$. Then $\Omega_1 = \Omega \setminus \{e\}, \Omega_2 = \Omega \setminus \{f\}$ are prime ideals of $\Omega$, $\Omega = \Omega_1 \cup \Omega_2$. So $S = P_1 \cup P_2$ where $P_i = \varphi^{-1}(\Omega_i), i = 1, 2$. But $P_1, P_2$ are prime ideals of $S$ and hence closed by Theorem 1.12. This contradiction shows that $\Omega$ has a maximum element $e$. So $e$ is the identity element of $\Omega$. □

In the above notation, we call $\varphi^{-1}(e)$ the top $\mathcal{S}$-indecomposable component of $S$. If $S$ is a monoid, then the top $\mathcal{S}$-indecomposable component of $S$ is the group of units of $S$.

Proposition 2.2. Let $S$ be a connected algebraic semigroup, $e, f \in E(S)$. Then $eS, Se, eSf$ are connected, closed subsemigroups of $S$. If $SeS$ is closed, then $SeS$ is also connected.
Proof. $eS = \{x | x \in S, ex = x\}$, $eSf = \{x | x \in S, ex = x = xf\}$. Hence $eS$, $Se$, $eSf$ are closed. Define $\varphi_1 : S \rightarrow eS$ as $\varphi_1(x) = ex$. Since $\varphi_1$ is a surjective morphism, $eS$ is connected. Define $\varphi_2 : S \rightarrow eSf$ as $\varphi_2(x) = exf$. Since $\varphi_2$ is a surjective morphism, $eSf$ is connected. Define $\varphi_3 : S \times S \rightarrow SeS$ as $\varphi_3(x, y) = xey$. If $SeS$ is closed, then $\varphi_3(S \times S) = SeS$ is also connected.

**Theorem 2.3.** Let $S$ be a connected algebraic semigroup. Then
1. all maximal subgroups of $S$ are closed if and only if $S$ is a nil extension of a completely simple semigroup.
2. all subgroups of $S$ are trivial if and only if $S$ is a nil extension of a rectangular band.

Proof. (2) follows trivially from (1). So we prove (1). First assume that all maximal subgroups of $S$ are closed. Let $e \in E(S)$. By Proposition 2.2, $eSe$ is connected. By hypothesis $H_e$ is closed. By Corollary 1.2, $eSe \setminus H_e$ is also closed. Hence $eSe = H_e$. Thus $a/e$ for all $a \in S$. Hence $e \in T = \text{kernel of } S$. Thus $E(S) \subseteq T$. By Corollary 1.11, $T$ is completely simple and $S/T$ is nil. Conversely assume $S/T$ is nil where $T$ is the completely simple kernel of $S$. Then for $e \in E(S) = E(T)$, $H_e = eSe$ is closed.

**Theorem 2.4.** Let $S$ be a connected algebraic semigroup. Then the following conditions are equivalent.
1. All subgroups of the top $S$-indecomposable component of $S$ are abelian.
2. All subgroups of $S$ are abelian.
3. $eSe$ is commutative for all $e \in E(S)$.

Proof. (1) $\Rightarrow$ (3). Let $T$ be the top $S$-indecomposable component of $S$. Then by Theorem 1.12, $P = S \setminus T$ is closed. Let $e \in E(T)$. Then $H_e$ is abelian. Let $S_1 = eSe, P_1 = S_1 \setminus H_e$. Then $P_1$ is closed, $S_1$ is closed and connected. $S_1 = P_1 \cup H_e$. Let $a \in H_e$. Then $H_e \subseteq C_{S_1}(a)$ and so $S_1 = P_1 \cup C_{S_1}(a)$. Hence $C_{S_1}(a) = S_1$. Thus $H_e \subseteq C(S_1)$ and $S_1 = P_1 \cup C(S_1)$. Hence $C(S_1) = S_1$ and $S_1$ is commutative. Let $a \in T$. By Corollary 1.4 there exists $n \in \mathbb{Z}^+$ such that $a^nSe$ for some $e \in E(T)$. So $a^nSe \subseteq eSe$ is commutative. Let $T_1 = \{a | a \in S, a^nSe \text{ is commutative}\}$. Then $T_1$ is closed, $T \subseteq T_1$. Since $S = P \cup T_1, T_1 = S$. Hence $eSe$ is commutative for all $e \in E(S)$. That (3) $\Rightarrow$ (2) $\Rightarrow$ (1) is obvious.

**Theorem 2.5.** Let $S$ be a connected algebraic semigroup such that all subgroups of $S$ are abelian. Suppose further that for each $a \in S$, there exists $e \in E(S)$ such that $ea = ae = a$. Then $S$ is a semilattice of nil extensions of completely simple semigroups and the top $S$-indecomposable component of $S$ is completely simple.

Proof. By Theorem 2.4, $eSe$ is commutative for all $e \in E(S)$. Let $a \in S$. Then there exists $e \in E(S)$ such that $ea = ae = a$. Let $x, y \in S^1$. Then $xay = x(eae)(eyxe)(eae)y = x(eae)^2(eyxe)y = xa^2yxye$. Hence $a^2|(xay)^2$. By a paper by the author [14, Theorem 2.13], $S$ is a semilattice of nil extensions of completely simple semigroups. Let $T$ be the top $S$-indecomposable component of $S$. Then $T$ is a nil extension of a completely simple semigroup. Let $T_1 = \text{kernel of } T$. Then
ON LINEAR ALGEBRAIC SEMIGROUPS

Let $E(T) \subseteq T$. Let $a \in T$. Then there exists $e \in E(S)$ such that $ea = a$. Clearly $e \in E(T)$. Hence $a \in T$ and $T = T$ is completely simple. □

A semigroup is regular if $a \in aSa$ for all $a \in S$.

THEOREM 2.6. Let $S$ be a regular, connected algebraic semigroup such that all subgroups of $S$ are abelian. Then $S$ is a finite subdirect product of semigroups, each of which is either completely simple or completely 0-simple.

PROOF. Let $e, f, g \in E(S)$ such that $e > f$, $e > g$, $f \not< g$. We claim that $f = g$. There exist $x, y \in S^1$ such that $xfy = g$. By Theorem 2.4, $eSe$ is commutative. So $g = efeyeyeyeye = eyeyeey = eyeyeey = eyeyeey = eyeyeey = eyeyeey = eyeyeey = eyeyeey = eyeyeey = eyeyeey = eyeyeey = eyeyeey = eyeyeey = eyeyeey = eyeyeey = eyeyeey = eyeyeey = eyeyeey = eyeyeey = eyeyeey = eyeyeey = eyeyeey = eyeyeey = eyeyeey = eyeyeey = eyeyeey = eyeyeey = eyeyeey = eyeyeey = eyeyeey = eyeyeey = eyeyeey = eyeyeey = eyeyeey = eyeyeey = eyeyeey = eyeyeey = eyeyeey = eyeyeey = eyeyeey = eyeyeey = eyeyeey = eyeyeey = eyeyeey = eyeyeey = eyeyeey = eyeyeey = eyeyeey = eyeyeey = eyeyeey = eyeyeey = eyeyeey = eyeyeey = eyeyeey = eyeyeey = eyeyeey = eyeyeey = eyeyeey = eyeyeey = eyeyeey = eyeyeey = eyeyeey = eyeyeey = eyeyeey = eyeyeey = eyeyeey = eyeyeey = eyeyeey = eyeyeey = eyeyeey = eyeyeey = eyeyeey = eyeyeey = eyeyeey = eyeyeey = eyeyeey = eyeyeey = eyeyeey = eyeyeey = eyeyeey = eyeyeey = eyeyeey = eyeyeey = eyeyeey = eyeyeey = eyeyeey = eyeyeey = eyeyeey = eyeyeey = eyeyeey = eyeyeey = eyeyeey = eyeyeey = eyeyeey = eyeyeey = eyeyeey = eyeyeey = eyeyeey = eyeyeey = eyeyeey = eyeyeey = eyeyeey = eyeyeey = eyeyeey = eyeyeey = eyeyeey = eyeyeey = eyeyeey = eyeyeey = eyeyeey = eyeyeey = eyeyeey = eyeyeey = eyeyeey = eyeyeey = eyeyeey = eyeyeey = eyeyeey = eyeyeey = eyeyeey = eyeyeey = eyeyeey = eyeyeey = eyeyeey = eyeyeey = eyeyeey = eyeyeey = eyeyeey = eyeyeey = eyeyeey = eyeyeey = eyeyeey = eyeyeey = eyeyeey = eyeyeey = eyeyeey = eyeyeey = eyeyeey = eyeyeey = eyeyeey = eyeyeey = eyeyeey = eyeyeey = eyeyeey = eyeyeey = eyeyeey = eyeyeey = eyeyeey = eyeyeey = eyeyeey = eyeyeey = eyeyeey = eyeyeey = eyeyeey = eyeyeey = eyeyeey = eyeyeey = eyeyeey = eyeyeey = eyeyeey = eyeyeey = eyeyeey = eyeyeey = eyeyeey = eyeyeey = eyeyeey = eyeyeey = eyeyeey = eyeyeey = eyeyeey = eyeyeey = eyeyeey = eyeyeey = eyeyeey = eyeyeey = eyeyeey = eyeyeey = eyeyeey = eyeyeey = eyeyeey = eyeyeey = eyeyeey = eyeyeey = eyeyeey = eyeyeey = eyeyeey = eyeyeey = eyeyeey = eyeyeey = eyeyeey = eyeyeey = eyeyeey = eyeyeey = eyeyeey = eyeyeey = eyeyeey = eyeyeey = eyeyeey = eyeyeey = eyeyeey = eyeyeey = eyeyeey = eyeyeey = eyeyeey = eyeyeey = eyeyeey = eyeyeey = eyeyeey = eyeyeey = eyeyeey = eyeyeey = eyeyeey = eyeyeey = eyeyeey = eyeyeey = eyeyeey = eyeyeey = eyeyeey = eyeyeey = eyeyeey = eyeyeey = eyeyeey = eyeyeey = eyeyeey = eyeyeey = eyeyeey = eyeyeey = eyeyeey = eyeyeey = eyeyeey = eyeyeey = eyeyeey = eyeyeey = eyeyeey = eyeyeey = eyeyeey = eyeyeey = eyeyeey = eyeyeey = eyeyeey = eyeyeey = eyeyeey = eyeyeey = eyeyeey = eyeyeey = eyeyeey = eyeyeey = eyeyeey = eyeyeey = eyeyeey = eyeyeey = eyeyeey = eyeyeey = eyeyeey = eyeyeey = eyeyeey = eyeyeey = eyeyeey = eyeyeey = eyeyeey = eyeyeey = eyeyeey = eyeyeey = eyeyeey = eyeyeey = eyeyeey = eyeyeey = eyeyeey = eyeyeey = eyeyeey = eyeyeey = eyeyeey = eyeyeey = eyeyeey = eyeyeey = eyeyeey = eyeyeey = eyeyeey = eyeyeey = eyeyeей |
Lemma 2.9 [Munn]. Let $S$ be a strongly $\pi$-regular semigroup. Let $a, b \in S$. If $a^2ab$ then $a \mathcal{R} ab$. If $a^2ba$, then $a \mathcal{L} ba$. If $a^2a^2$, then $a \mathcal{K} a^2$.

Proof. It suffices to consider the case $a^2ab$. There exist $x, y \in S^1$ such that $xaby = a$. Then $x(a(by)^t) = a$ for all $t \in \mathbb{Z}^+$. There exist $n \in \mathbb{Z}^+, e \in E(S)$ such that $(by)^n \mathcal{K} e$. So $a = ae \in a(by)^nS^1 \subseteq abS^1$. Hence $a \mathcal{R} ab$. □

Lemma 2.10. Let $S$ be a connected algebraic semigroup, $e, f \in E(S)$, $e \mathcal{F} f$. Then there exists $g \in E(S)$ such that $e \mathcal{R} g$ and $gf \mathcal{L} f$.

Proof. Let $E = E(S)$. Suppose the lemma is false. Then by Lemma 2.9, $gf \not\mathcal{F} f$ for all $g \in E$ with $g \mathcal{R} e$. In particular $ef \not\mathcal{F} f$. There exist $x, y \in S$ such that $xey = f$. By Corollary 1.2 and Proposition 2.2, $eSe \subseteq H_e$ and $fSe \setminus H_f$ are closed sets. Let

$$T_1 = \{a | a \in eS, faxf \in fSe \setminus H_f\}, \quad T_2 = \{a | a \in eS, ae \in eSe \setminus H_e\}.$$

Then $T_1, T_2$ are closed subsets of $eS$. If $e \not\in T_1$, then $fxef \in H_f$ and $ef \not\mathcal{F} f$, a contradiction. So $e \in T_1$. Clearly $fxef = f$ and so $ey \not\in T_1$. Thus $\emptyset \neq T_1 \subseteq eS$. Clearly $e \not\in T_2$. We claim that $ef \in T_2$. Otherwise $efe \in H_e$. Then $ef efe \not\mathcal{F} f$, a contradiction. So $ef \in T_2$. Hence $\emptyset \neq T_2 \subseteq eS$. Since $eS$ is connected by Proposition 2.2, $T_1 \cup T_2 \neq eS$. Hence there exists $a \in eS$ such that $a \not\in T_1 \cup T_2$. Then $ea = a, faxf \in H_f, ae \in H_e$. There exists $z \in S$ such that $zae = e$. So $za^2 = za = ea = a$. Hence $a^2 \mathcal{K} a$. By Lemma 2.9, $a^2 \mathcal{K} a$. By [6, Theorem 2.16], there exists $g \in E$ such that $a \mathcal{R} g$. Now $g \in a^2S = aeS \subseteq aeS = eS, e \in aeS \subseteq aS = gs$. So $e \mathcal{R} g$. Now $fxagf = faxf \in H_f$. Hence $gf \mathcal{F} f$, a contradiction. This proves the lemma. □

Theorem 2.11. Let $S$ be a connected algebraic semigroup such that $E(S)$ is a subsemigroup of $S$. Then $S$ is a semilattice of nil extensions of rectangular groups.

Proof. Let $E = E(S)$. Let $a, b \in S$ such that $e = ab, f = ba \in E$. By the author [14, Theorem 2.17], it suffices to show that $ef = f$. Now $e = ab[(ba)^2] = f$. By Lemma 2.10 there exists $g \in E$ such that $e \mathcal{R} g$, $gf \mathcal{F} f$. Since $gf \in E$, $fg = f$. Since $e \mathcal{R} g, eg = g$. So $fgf = f$. Since $fe \in E, ef = (fe)^2gf = f$. This proves the theorem. □

Theorem 2.12. Let $S$ be a connected algebraic semigroup such that all subgroups of $S$ are abelian. Then the following conditions are equivalent.

1. $E(S)$ is a band.
2. $E(S)$ is a normal band.
3. $S$ is a semilattice of nil extensions of rectangular groups.

Proof. (1) $\Rightarrow$ (3) follows from Theorem 2.11. (2) $\Rightarrow$ (1) is obvious. So we must show (3) $\Rightarrow$ (2). By Corollary 1.4, there exists $n \in \mathbb{Z}^+$ such that for all $a \in S, a^n \mathcal{K} e$ for some $e \in E(S)$. Let $E = E(S)$ and let $T$ be the top $S$-indecomposable component of $S$. If $T = S$, we are done. So assume $P = S \setminus T \neq \emptyset$. $P$ is a prime ideal of $S$ and hence closed by Theorem 1.12. $T$ is a nil extension of a rectangular...
group $T_1$. Since the subgroups of $T_1$ are abelian, $T_1$ satisfies the identity $xyzw = xzyw$. Hence $T$ satisfies the identity $x^n y^m z^n w^n = x^n y^m z^n w^n$. By [17, p. 54], $S \times S \times S \times S$ is connected. Let $M = \{(a, b, c, d) | a, b, c, d \in S, a^m b^n c^d = a^m c^n b^n d^n\}$. Then $M$ is closed and $T \times T \times T \times T \subseteq M$. Clearly

$$S \times S \times S \times S = M \cup (S \times S \times S \times P) \cup (S \times S \times P \times S) \cup (S \times P \times S \times S) \cup (P \times S \times S \times S).$$

Hence $M = S \times S \times S \times S$. Thus for all $e, f, g, h \in E(S)$, $efgh = egfh$. In particular $efef = eeff = ef$ and $E(S)$ is a normal band.

**Theorem 2.13.** Let $S$ be a connected algebraic semigroup such that $\dim S = 1$. Then $S$ is either a group, a group with zero, a null semigroup, a right zero semigroup or a left zero semigroup.

**Proof.** First assume $S$ has an identity element $1$. If $E(S) = \{1\}$, then $S$ is a group. Otherwise there exists $e \in E(S)$ such that $e \neq 1$. Then $\dim eS = \dim Se = 0$. So $eS = Se = \{e\}$. Hence $S$ has a zero $0$ and $E(S) = \{1, 0\}$. Let $G$ be the group of units of $S$. By Corollary 1.2, $M = S \setminus G$ is a closed ideal of $S$. Let $a \in M$. Consider the map $\varphi : S \to M$ given by $\varphi(x) = ax$. $\varphi$ is a morphism. Hence $T$, the closure of $\varphi(S)$ is irreducible. Since $T \subseteq M \neq S$, $\dim T = 0$. Since $0, a \in T$, $a = 0$. Thus $S = G \cup \{0\}$.

So assume $S$ does not have an identity element. Let $e \in E(S)$. Suppose $eS = S$. Then $Se \neq S$. So $Se = \{e\}$. Let $a, b \in S$. Then $ab = a(eb) = (ae)b = eb = b$. So $S$ is a right zero semigroup. Similarly $Se = S$ implies that $S$ is a left zero semigroup. So assume $eS \neq S$, $Se \neq S$ for all $e \in E(S)$. Hence $eS = Se = \{e\}$. So $S$ has a zero $0$ and $E(S) = \{0\}$. By Corollary 1.4, there exists $n \in \mathbb{Z}^+$ such that $a^n = 0$ for all $a \in S$. Let $D = \{a | a \in S, a^2 = 0\}$. Then $D$ is closed. Define $\psi : S \to D$ as $\psi(a) = a^{-1}$. Then $\psi(S) \neq \{0\}$. Let $T$ be the closure of $\psi(S)$. Then $T \subseteq D$. Since $S$ is connected, $T$ is irreducible. So $\dim T = 1$ and $S = T = D$. Let $a \in S$. Let $M = \{b | b \in S, ab = 0\}$. We claim that $M = S$. Suppose not. Clearly $M$ is closed. Define $\psi : S \to M$ as $\psi(b) = ab$. Since $M \neq S$, $\psi(S) \neq \{0\}$. If $W$ is the closure of $\psi(S)$, then $dim W = 0$, $W$ is irreducible, $W \subseteq M$. This contradiction shows that $M = S$. Hence $S^2 = \{0\}$, proving the theorem.

**Remark 2.14.** It is well known [2, p. 257] that a connected algebraic group of dimension one is *-isomorphic to either $(K, +)$ or the group $\{(a, b) | a, b \in K, ab = 1\}$ under multiplication. Let $S$ be an algebraic semigroup of dimension 1. The only case of Theorem 2.13 that needs a closer look is when $S = G^0$, $G$ is a group. Let $1$ be the identity of $S$. Then $\hat{G} = \{(a, b) | a, b \in S, ab = 1\}$ is a connected algebraic group of dimension 1. So $S$ is isomorphic to $(K, \cdot)$. The example $S = \{(x, y) | x, y \in K, x^2 = y^3\}$ under multiplication shows that in general $S$ is not *-isomorphic to $(K, \cdot)$.

**Theorem 2.15.** Let $S$ be a connected algebraic semigroup such that $\dim S = 2$. Then $E(S)$ is a band. If $S$ does not have an identity element then $E(S)$ is a normal band.
Proof. Let $M$ be the kernel of $S$. By Corollary 1.5, $M$ is closed and completely simple. Let $e \in E(M)$. Then $SeS = M$. By Proposition 2.2, $M$ is connected. First assume $M = S$. If $eS = S$ for some $e \in E(S)$, it follows (since $S$ is completely simple) that $E(S)$ is a right zero semigroup. Similarly $Se = S$ implies $E(S)$ is a left zero semigroup. So assume $eS \neq S$, $Se \neq S$ for all $e \in E(S)$. So $\dim eS = \dim Se < 1$ for all $e \in E(S)$. Let $e \in E(S)$. If $eSe \neq \{e\}$ then $\dim eSe = 1$. Since $eSe \subseteq Se \cap eS$, we obtain $eSe = Se$. But then $S$ is a group. So assume $eSe = \{e\}$ for all $e \in E(S)$. Then $S$ is a rectangular band.

Next assume $\dim M = 1$. By Theorem 2.13, $M$ is either a right zero semigroup, a left zero semigroup or a group. By symmetry assume $M$ is not a left zero semigroup. If $E(M) = E(S)$, we are done. So assume $E(M) \neq E(S)$. Suppose $S$ has an identity element $1$. Let $e \in E(S)$, $e \in M$. Then $M \supseteq eS$. So $eS = S$ and $e = 1$. Then $E(S) = E(M) \cup \{1\}$ and we are done. Next assume $S$ does not have an identity element. Let $e \in E(S) \setminus M$. As above, $eS = S$. So $Se \neq S$. Now $Me$ is closed and connected and $Me \supseteq Se$. So $\dim Me = 0$. If $Me = \{f\}$, let $\theta(e) = f \in E(M)$. So $\theta: E(S) \setminus M \rightarrow E(S)$. Let $D_1 = E(M)$, $D_2 = E(S) \setminus D_1$. Then $D_1$, $D_2$ are right zero semigroups. If $e \in D_2$, $f \in D_1$, then $ef = f$, $fe = \theta(e)$. It follows easily that $E(S)$ is a normal band.

Finally assume that $\dim M = 0$. Then $S$ has a zero $0$. Suppose $S$ has an identity element $1$. Let $E(M) = E(S)$, $e \neq 1, 0$. Then $\{0\} \subsetneq eS \subsetneq eS \subsetneq S$. So $eS = eSe$. Similarly $Se = eSe$ and $e \in C(S)$. Hence $E(S) \subseteq C(S)$. Next assume $S$ does not have an identity element. By symmetry we can assume that $eS \neq S$ for all $e \in E(S)$. Then $\{0\} \subsetneq eS \subsetneq eS \subsetneq S$ for all $e \in E(S)$, $e \neq 0$. So $eS = eSe$ for all $e \in E(S)$, $e \neq 0$. Let $A = \{e \in E(S), Se = s\}$. Then $A = \varnothing$ or $A$ is a left zero semigroup. Let $e \in E(S)$, $e \neq 0$, $e \notin A$. Then $\{0\} \subsetneq eSe \subsetneq eS \subsetneq S$. Hence $eSe = Se$ and $eS = Se$. So $e \in C(S)$. It follows that $E(S)$ is a normal band. □

Let $S$ be a strongly $\pi$-regular semigroup, $J$ a regular $\delta$-class of $S$. Let $J^0$ be the semigroup $J \cup \{0\}$ where $0$ is the zero of $J^0$ and for $a, b \in J$, we set $ab = 0$ if $ab \notin J$. By Munn [11], $J^0$ is completely 0-simple. By the Rees theorem [6, Theorem 3.5] we can assume that $J^0 = (\Gamma \times G \times \Lambda) \cup \{0\}$ with sandwich map $P: \Lambda \times \Gamma \rightarrow G^0$ where $G$ is a group. Multiplication in $J^0$ is given by

$$(\alpha, a, \beta)(\gamma, b, \delta) = \begin{cases} (\alpha, aP(\beta, \gamma)b, \delta) & \text{if } P(\beta, \gamma) \neq 0, \\ 0 & \text{if } P(\beta, \gamma) = 0. \end{cases}$$

(3)

Theorem 2.16. Let $S$ be a connected algebraic semigroup, $J$ a regular $\delta$-class of $S$. Let $J^0$ have the Rees representation given by (3). Then for all $\alpha, \beta \in \Gamma$, there exists $\gamma \in \Lambda$ such that $P(\gamma, \alpha) \neq 0$ and $P(\gamma, \beta) \neq 0$. For all $\gamma, \delta \in \Lambda$, there exists $\alpha \in \Gamma$ such that $P(\gamma, \alpha) \neq 0$ and $P(\delta, \alpha) \neq 0$.

Proof. The second statement being the dual of the first, we only need to prove the first. Let $\alpha, \beta \in \Gamma$. Since $J^0$ is regular, it follows [6, Lemma 3.1] that there exist $\mu, \nu \in \Lambda$ such that $P(\mu, \alpha) \neq 0$, $P(\nu, \beta) \neq 0$. Let $e = (\alpha, P(\mu, \alpha)^{-1}, \mu)$, $f = (\beta, P(\nu, \beta)^{-1}, \nu)$. Then $e, f \in E(S)$, $ef$. By Lemma 2.10, there exists $g \in E(S)$ such that $eRg$ and $gfRf$. Now $g = (\alpha, a, \gamma)$ for some $a \in G$, $\gamma \in \Gamma$. Since $g^2 = g$, $P(\gamma, \alpha) \neq 0$. Since $gf \neq 0$ in $J^0$, $P(\gamma, \beta) \neq 0$. This proves the theorem. □
Theorem 2.17. Suppose $S$ is a connected, algebraic semigroup. Assume that $S$ is a semilattice of groups and that $E(S)$ is linearly ordered. Then $|E(S)| < 2$.

Proof. By Theorem 2.8, $E(S)$ is finite and $E(S) \subseteq C(S)$. Suppose $|E(S)| > 3$. Let $E(S) = \{e_1 < e_2 < e_3 < \cdots \}$. Let $T = e_3S$. Let $T_1 = e_1S$, $T_2 = e_2S$. Then $T_1 \subseteq T_2 \subseteq T$, $e_3(T \setminus T_1) = T_2 \setminus T_1$, $e_2T_1 = T_1$. Define $\varphi: T \to T_2$, as $\varphi(x) = e_2x$. Clearly $\varphi$ is surjective and $\dim T > \dim T_2$. So [17, p. 60], $\dim \varphi^{-1}(a) > 0$ for all $a \in T_2$. In particular $\dim \varphi^{-1}(e_1) > 0$. Let $x \in \varphi^{-1}(e_1)$. Then $e_2x = e_1$. But then $x \in T_1$ and so $e_2x = x$. This contradiction proves the theorem. □

3. Examples and problems. Let $D$ be a closed subset of $K^n$. Let $\circ$ be a binary operation on $D$ such that the map $(a, b) \mapsto a \circ b$ from $D \times D$ into $D$ is a morphism. We will then say that $(D, \circ)$ is an algebraic groupoid.

Example 3.1. Let $D$ be an algebraic groupoid, $S$ a subsemigroup of $D$. Let $T$ be the closure of $S$ in $D$. Then $T$ is an algebraic semigroup. In fact let $a \in S$, $T_1 = \{b \mid b \in T, ab \in T\}$. Then $S \subseteq T_1$, and so $T_1 = T$. So a $T \subseteq T$. Let $T_2 = \{b \mid b \in T, bT \subseteq T\}$. $S \subseteq T_2$ and so $T_2 = T$. Hence $T^2 \subseteq T$. Let $a, b \in S$ and let $T_3 = \{c \mid c \in T, (ab)c = a(bc)\}$. $S \subseteq T_3$ and so $T_3 = T$. Repeating this argument twice, we see that $T$ is a semigroup.

Example 3.2. Let $X \subseteq K^n$ be closed. Let $S = \{A \mid A \in \mathfrak{N}_n(K), XA \subseteq X\}$. Then $S$ is a closed subsemigroup of $\mathfrak{N}_n(K)$.

Example 3.3. Let $X \subseteq K^n$ be a nonempty closed set. Then $X$ admits a right zero, left zero and null semigroup structures given by $ab = b$, $ab = a$, $ab = u$ where $u$ is a fixed element of $X$.

Example 3.4. Let $S$ be any finite semigroup. Then $S$ is closed subsemigroup of the finite dimensional algebra $K[S]$. Hence $S$ is an algebraic semigroup.

Example 3.5. Let $S \subseteq K^2$ be the closed set $\{(a, b) \mid a, b \in K, ab^2 = b\}$. If $(a, b)$, $(c, d) \in S$, define $(a, b)(c, d) = (abcdac + 1 – abcd, 0)$. Then $S$ is a commutative algebraic semigroup. Note that $(1, 1)S = S^2 = \{(a, 0) \mid a \in K, a \neq 0\}$ is not closed. $S^3 = \{(1, 0)\}$.

Problem 3.6. Let $S$ be an algebraic semigroup. Does there exist $n \in \mathbb{Z}^+$ such that $S^n = S^{n+1}$ is closed?

Problem 3.7. Let $S$ be an algebraic semigroup, $e \in E(S)$. Is $SeS$ necessarily closed?

Problem 3.8. Can the ideals in Corollary 1.11 be chosen to be closed?

Problem 3.9. Let $n \in \mathbb{Z}^+$. Does the number of regular $\pi$-classes of strongly $\pi$-regular subsemigroups of $\mathfrak{N}_n(K)$ have an upper bound (depending on $n$)? More generally, can $\mathfrak{S}$ in Lemma 1.6 be replaced by a sufficiently large finite set of idempotents?

Problem 3.10. Are the nil Rees factor semigroups of Theorem 1.10 and Corollary 1.11 necessarily nilpotent?

Problem 3.11. Can the Krohn-Rhodes theorem for finite semigroups be generalized to strongly $\pi$-regular subsemigroups of $\mathfrak{N}_n(K)$?

Example 3.12. Let $T_1 \subseteq K^m$, $T_2 \subseteq K^n$ be algebraic semigroups. Let $S = (T_1 \times \{0_n\} \times \{1\}) \cup ((0_m) \times T_2 \times \{0\}) \subseteq K^{m+n+1}$ where $0_m, 0_n$ are the zero vectors of $K^m, K^n$, respectively.
$K^m$ and $K^n$ respectively. Then $S$ is closed. Define multiplication in $S$ as follows.

$$(a, b, c, d, eta) = (abac, (1 - a)\beta b + a(1 - \beta)d + (1 - a)(1 - \beta)bd, a\beta).$$

Then $S$ is an algebraic semigroup. Let $\hat{T}_1 = T_1 \times \{0_m\} \times \{1\}, \hat{T}_2 = \{0_m\} \times T_2 \times \{0\}$. Then $S = \hat{T}_1 \cup \hat{T}_2, \; xy = yx = y$ for $x \in \hat{T}_1, \; y \in \hat{T}_2$. $\hat{T}_1, \hat{T}_2$ are disjoint closed subsemigroups of $S$. $\hat{T}_i$ is *-isomorphic to $T_i$ ($i = 1, 2$).

**Example 3.13.** Let $\mathcal{A}$ be a finite dimensional algebra over $K$. Then the multiplicative semigroup of $\mathcal{A}$ is a connected algebraic semigroup. $\mathcal{A}$ along with the circle operation $a \circ b = a + b - ab$ is also a connected algebraic semigroup.

**Example 3.14.** Let $S = \mathcal{M}_n(K)$. For $i = 1, \ldots, n$, let $S_i = \{a|a \in S, \; \rho(a) < i\}$. If $e \in S_i, \; e^2 = e, \; \rho(e) = i$, then $SeS = S_i$ and so by Proposition 2.2, each $S_i$ is a connected algebraic semigroup. $S_i$ is completely 0-simple and all subgroups of $S_i$ are abelian. Also $\text{dim } S_i = 2n - 1$.

Let $S, T$ be algebraic semigroups. Suppose for $a \in S, \; b \in T$ an element $a^b \in S$ is uniquely determined. Suppose the map $(a, b) \rightarrow a^b$ is a morphism and that for all $a_1, a_2 \in S, \; b_1, b_2 \in T$, $(a_1a_2)^b = a_1^b a_2^b, \; (a_1)^{b_2} = (a_1^{b_2})$. In $D = S \times T$ define $(a_1, b_1)(a_2, b_2) = (a_1a_2, b_1b_2)$. Then the semidirect product $D$ is an algebraic semigroup. If $S, T$ are connected then so is $D$. In particular if $a \in \mathcal{M}_n(K), \; b \in GL(n, K) = \{a|a \in \mathcal{M}_n(K), \; \rho(a) = n\}$, we can set $a^b = ab^{-1}$. If $G$ is any connected, closed subgroup of $GL(n, K)$, we can form the semidirect product of $S_i$ (see Example 3.14) and $G$ to again obtain a connected algebraic semigroup. By Lallement [10, Theorem 2.17], the semidirect product of $S_i$ and $G$ is a subdirect product of completely simple and completely 0-simple semigroups.

**Example 3.15.** The example $S = \{(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix})|a, b \in K\}$ shows that Theorem 2.12 is not true without the assumption that the subgroups of $S$ are abelian.

**Problem 3.16.** Let $S$ be a connected algebraic semigroup which is a semilattice of groups. Determine all possibilities for $E(S)$ and $|E(S)|$. For example, by Theorems 1.2 and 2.17, $|E(S)| \neq 3$. If $S$ is also the multiplicative semigroup of a finite-dimensional algebra, then clearly $|E(S)| = 2^n$ for some $n \in \mathbb{Z}^+$. This is not true in general as the following example shows.

**Example 3.17.** Let $T = K^4$ under multiplication and let $S = \{(a, b, c, d)|a, b, c, d \in K, \; ab = cd\}$. Then $S$ is a connected, closed subsemigroup of $T$. $S$ is also a semilattice of groups, $\text{dim } S = 3$ and $|E(S)| = 10$.

**Problem 3.18.** Determine all possibilities for $\Omega(S)$ and $|\Omega(S)|$ where $S$ is a connected algebraic semigroup.

**Example 3.19.** Let $T_1 = (K^3, \ast)$ where

$$(a_1, a_2, a_3) \ast (b_1, b_2, b_3) = (a_2b_3 + a_1 + b_1, b_2, a_3).$$

Let $T_2$ be any commutative finite-dimensional algebra with an identity element. Then $T_1$ is completely simple. $T_1, T_2, T_1 \times T_2$ are all examples of connected algebraic semigroups satisfying the hypothesis of Theorem 2.5.

**Example 3.20.** Let $S = \{(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix})|a, b \in K\}$. Then $S$ is a connected algebraic semigroup of dimension 2. $S$ is a semilattice of a nil semigroup and a right group.

**Example 3.21.** Let $P \in \mathcal{M}_n(K)$ and let $\mathcal{A} = \{A|A \in \mathcal{M}_n(K), \; A^TPA = 0\}, \; \mathcal{B} = \{A|A \in \mathcal{M}_n(K), \; A^TPA = P\}$. Then $\mathcal{A}, \mathcal{B}$ are closed subsemigroups of
\( \mathfrak{M}_n(K) \). \( \mathfrak{a} \) has a zero and \( \mathfrak{b} \) has an identity element. When is \( \mathfrak{a} \) or \( \mathfrak{b} \) connected? If \( n = 3 \) and

\[
P = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},
\]

then \( \mathfrak{a} \) is a connected algebraic semigroup of dimension 7.

**REFERENCES**


5. ______, Private communication.


15. ______, *Quadratic semigroups on affine spaces*, Linear Algebra and Appl. 26 (1979), 107–121.


Department of Mathematics, North Carolina State University, Raleigh, North Carolina 27650