QUASI-LINEAR EVOLUTION EQUATIONS IN BANACH SPACES

BY

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ABSTRACT. This paper is concerned with the quasi-linear evolution equation $u'(t) + A(t, u(t))u(t) = 0$ in $[0, T]$, $u(0) = x_0$ in a Banach space setting. The spirit of this inquiry follows that of T. Kato and his fundamental results concerning linear evolution equations. We assume that we have a family of semigroup generators that satisfies continuity and stability conditions. A family of approximate solutions to the quasi-linear problem is constructed that converges to a "limit solution." The limit solution must be the strong solution if one exists. It is enough that a related linear problem has a solution in order that the limit solution be the unique solution of the quasi-linear problem. We show that the limit solution depends on the initial value in a strong way. An application and the existence aspect are also addressed.

This paper is concerned with the quasi-linear evolution equation

$$u'(t) + A(t, u(t))u(t) = 0 \quad \text{in} \quad [0, T], \\ u(0) = x_0$$

in a Banach space setting.

The spirit of this inquiry follows that of T. Kato. Kato wrote a fundamental paper on linear evolution equations in 1953 [9]; that is, investigation of

$$u'(t) + A(t)u(t) = 0 \quad \text{on} \quad [0, T], \\ u(0) = x_0.$$ 


After discussing the setting and method of attack, our theorem is stated and proved. We then give an application of the theorem using the Sobolevskii-Tanabe theory of linear evolution equations of parabolic type. A proposition relevant to our theorem is also given.

Let $X$ and $Y$ be Banach spaces, with $Y$ densely and continuously embedded in $X$. Let $x_0 \in Y$, $T > 0$, $r > r_1 > 0$, $r_2 > 0$, $W = B_X(x_0; r)$, $Z = B_X(x_0; r_1) \cap B_Y(x_0; r_2)$, and for each $t \in [0, T]$ and $w \in W$, let $-A(t, w)$ be the infinitesimal generator of a strongly continuous semigroup of bounded linear operators in $X$, with $Y \subset D(A(t, w))$.

We consider the quasi-linear evolution equation

$$v'(t) + A(t, v(t))v(t) = 0.$$ 

(QL)
Given a function $u$ from $[0, T']$ into $W$, where $0 < T' < T$, we can also consider the linearized evolution equation

$$v'(t) + A(t, u(t))v(t) = 0.$$  \hspace{1cm} (L; u)

By a solution (or strong solution) of (QL) or (L; $u$) on $[0, T']$, we mean a function $v$ on $[0, T']$ to $W$ which is absolutely continuous ($X$) and differentiable ($X'$) a.e., such that $v(t) \in Y$ a.e., ess sup{$||v(t)||_Y$} $< \infty$, and $v$ satisfies the appropriate equation, (QL) or (L; $u$), a.e. on $[0, T']$.

Our method is to produce, for each $x_1 \in Z$, a “limit solution” $u$ with initial value $x_1$ on an interval $[0, T']$, where $T' \in (0, T]$ is independent of $x_1$. For a partition $\Delta = \{t_0, t_1, \ldots, t_N\}$ of $[0, T']$, we use an iterative procedure to produce a Lipschitz continuous ($X$) function $u_\Delta$ which satisfies

$$u'_\Delta(t) + A(t_i, u_{\Delta}(t_i))u_{\Delta}(t) = 0 \quad \text{for} \quad t \in (t_i, t_{i+1})$$

and $i \in \{0, 1, \ldots, N - 1\}$, with $u_\Delta(0) = x_1$. This $u_\Delta$ is shown to be the time-ordered juxtaposition of the semigroups generated by the $-A(t_i, u_{\Delta}(t_i))$. These approximate solutions converge uniformly, as $|\Delta|$ goes to 0, to give the limit solution $u$. We show, in particular, that if $v = w$ is a solution of (QL) or (L; $u$) on $[0, T']$ with initial value $x_1$, then $w = u$. Thus, subject to an initial value, a solution of (QL) is unique if it exists, and whenever the linearized equation (L; $u$) has a solution, then so does the quasi-linear equation (QL). There are known conditions which are sufficient in order that (L; $u$) has a solution.


The term “limit solution” seems as appropriate as any to describe the function obtained by the iterative procedure; e.g., see Kobayashi [16] or Crandall and Evans [3].

**Theorem.** Assume that

(i) $\{A(t, w)\}$ is stable in $X$ with constants of stability $M, \beta$; i.e.,

$$\|(A(t_k, w_k) + \lambda)^{-1}(A(t_{k-1}, w_{k-1}) + \lambda)^{-1} \ldots (A(t_1, w_1) + \lambda)^{-1}\|_X < M(\lambda - \beta)^{-k},$$

$\lambda > \beta$, for any finite family $\{(t_j, w_j)\}$, $0 < t_1 < \ldots < t_k < T$, $k = 1, 2, \ldots$.

(ii) $Y \subset D(A(t, w))$ for each $(t, w)$, which implies that $A(t, w) \in B(Y, X)$, and the map $(t, w) \rightarrow A(t, w)$ is Lipschitz continuous with constant $C_1$; i.e.,

$$\|A(t_2, w_2) - A(t_1, w_1)\|_{Y, X} < C_1(\|t_2 - t_1\| + \|w_2 - w_1\|_X).$$

(iii) There is a family $\{S(t, w)\}$ of isomorphisms of $Y$ onto $X$ such that $S(t, w)$ $A(t, w)$ $S(t, w)^{-1} = A_1(t, w)$ is the negative of the infinitesimal generator of a strongly continuous semigroup in $X$ for each $(t, w)$, and $\{A_1(t, w)\}$ is stable in $X$, with constants of stability $M_1, \beta_1$. Furthermore, there is a constant $C_2$ such that $\|S(t, w)\|_{Y, X} < C_2, \|S(t, w)^{-1}\|_{X, Y} < C_2$, and the map $(t, w) \rightarrow S(t, w)$ is Lipschitz continuous with constant $C_3$ (see (ii) above).

Then, there exists a $T'$, with $0 < T' < T$, such that for each $x_1 \in Z$ and partition $\Delta = \{t_0, t_1, \ldots, t_n\}$ of $[0, T']$, we can find a function $u_\Delta$ which is Lipschitz continuous
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(X) on [0, T'] to W, Y-bounded, and satisfies \( u_\Delta(t) + A(t, \Delta(t))u_\Delta(t) = 0 \) for \( t \in (t_i, t_{i+1}) \) and \( i \in \{0, 1, \ldots, n - 1\} \), with \( u_\Delta(0) = x_1 \). In fact, given \( \varepsilon > 0 \), there exists \( \delta > 0 \) such that \( |\Delta| < \delta \) implies that \( \|u_\Delta(t) + A(t, \Delta(t))u_\Delta(t)\|_X < \varepsilon \) except at \( t_1, \ldots, t_n \). Further, the \( u_\Delta \) converge uniformly, as \( |\Delta| \) goes to 0, to a Lipschitz continuous (X) function \( u \) on [0, T'] to W which has initial value \( x_1 \) and is bounded, independent of \( x_1 \), in the relative completion of \( Y \) in \( X \) (the set of all points in \( X \) that are the limit in \( X \)-norm of sequences from \( Y \) that are bounded in \( Y \)-norm).

If \( x_2 \in Z \) and \( w \) is constructed as above but with initial value \( x_2 \), then \( \|u(t) - w(t)\|_X < C \|x_1 - x_2\|_X \) for \( t \in [0, T'] \), with \( C \) independent of \( x_1 \) and \( x_2 \).

Now, if \( v \) is a solution of (QL) or \( (L; u) \) on [0, T'"], where \( 0 < T" < T' \), with initial value \( x_1 \), then \( v = u \) on [0, T"], and thus solutions to (QL) or \( (L; u) \) are uniquely determined by their initial values.

**Corollary 1.** If \( Y \) is reflexive, then \( (L; u) \) has a solution on [0, T'] with initial value \( x_1 \), and thus \( u \) is a solution of (QL) on [0, T'] with initial value \( x_1 \).

**Remarks.** (1) If \( D(A(t, w)) = Y \) for each \( (t, w) \) and there is a \( \lambda > \beta \) such that \( \|\lambda I + A(t, w)\|_{X,Y} < C_2 \) and \( \|\lambda I + A(t, w)\|_{Y,X} < C_2 \) for each \( (t, w) \), then (iii) is satisfied with \( S(t, w) = \lambda I + A(t, w) \).

(2) If \( Y \) is \( A(t, M>)-admissible \) (\( \{\exp(-sA(t, w))\} \) takes \( y \) to \( y \) and forms a strongly continuous semigroup on \( Y \)) for each \( (t, w) \) and \( \{A(t, w)\} \) is stable in \( Y \), then (iii) is unnecessary.

We now begin to prove the Theorem. The proofs of the above remarks and Corollary will be given later.

Let \( T^0 = \min(T, r/\|A\|_M e^{BT}(\|x_0\|_Y + r_2)) \), where \( \|A\| = \sup\{\|A(t, w)\|_{Y,X}: t \in [0, T], w \in W\} \) which is finite by (ii). Let \( K = C_2 C_3 M_1 T^0 \) and \( T" = T^0/ (1 + \|A\|_C_2 M_1 e^{K+\beta_1 T^0}(\|x_0\|_Y + r_2)) \).

**Lemma A.** If \( u \) is Lipschitz continuous \((X)\) on [0, T'] to \( W \) with Lipschitz constant \( \|A\|_C_2 M_1 e^{K+\beta_1 T^0}(\|x_0\|_Y + r_2) \), then \( \{A(t, u(t)) : t \in [0, T']\} \) is \( Y \)-stable with constants \( C_2 M_1 e^K \) and \( \beta_1 \).

**Proof of Lemma A.** We use Kato's Proposition 4.4 [11] with \( S(t) = S(t, u(t)) \). Then we estimate the variation of \( S \) by

\[
V_S < C_3 \left( 1 + \|A\|_C_2 M_1 e^{K+\beta_1 T^0}(\|x_0\|_Y + r_2) \right) T" < C_3 T^0,
\]

whence \( \{A(t, u(t)) \) is \( Y \)-stable with constants \( C_2 M_1 e^{C_3 T^0} = C_2 M_1 e^K \) and \( \beta_1 \).

This completes the proof of Lemma A.

By an evolution operator \( \{W(t, s) : 0 < s < t < T'\} \) generated by \( \{\partial(t) : t \in [0, T']\} \subset (A(t, w) : t \in [0, T'], w \in W\) and a partition \( \Delta = \{t_0, \ldots, t_N\} \) of [0, T'], we mean the family of operators obtained by forming a time-ordered juxtaposition of the semigroups generated at the points of the partition; e.g., for \( t \in [t_i, t_{i+1}], s \in [t_j, t_{j+1}], s < t \),

\[
W(t, s) = \exp(-(t - t_i)\partial(t_i))\exp(-(t_j - t_{j-1})\partial(t_{j-1})) \ldots \exp(-(t_{j+1} - s)\partial(t_j)).
\]

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It follows from (i) and Kato's Proposition 3.3 [11] that $|| W(t, s) ||_X < Me^{B(t-s)}$. If \{ \theta(t) \} is $Y$-stable with constants $\tilde{M}$, $\tilde{B}$, then $W(t, s) Y \subset Y$ and $|| W(t, s) ||_Y < \tilde{M}e^{\tilde{B}(t-s)}$ as a result of (iii) and Kato's Propositions 2.4 and 3.3 [11]. Let $i = i_i$ if $t \in [i_i, i_{i+1})$, $i \neq N$, and $i_N = t_N$. If $f(t) = W(t, 0)x_1$ on $[0, T')$, then $f$ satisfies $f'(t) + \theta(i)f(t) = 0$ for $t \in \Delta$, with $f(0) = x_1$. The construction of an evolution operator from a family of semigroup generators and a partition, the notation $i$, and the other results above will be used from this point on without further discussion.

**Lemma B.** Suppose \{ \theta(t): t \in [0, T') \} is $Y$-stable with constants $\tilde{M}$ and $\tilde{B}$, and that \{ $W(t, s)$ \} is generated by \{ $\theta(t)$ \} and a partition $\Delta$ of $[0, T')$. Then, $f(t) = W(t, 0)x_1$ is Lipschitz continuous (X) with Lipschitz constant $||A|| \tilde{M}e^{\tilde{B}T}(||x_0||_Y + r_2)$.

The result is also true if \{ $W(t, s)$ \} is the evolution operator of Kato's Theorem 4.1 [11].

**Proof of Lemma B.** For the partition case, since $f'(t) = -\theta(i)f(t)$ except for $t \in \Delta$, we get for $s < t$

$$||f(t) - f(s)||_X = \left| - \int_s^t A(\tilde{\theta}(\xi))d\xi \right|_X < ||A|| ||f||_Y |t - s| < ||A|| \tilde{M}e^{\tilde{B}t} ||x_1||_Y |t - s| < ||A|| \tilde{M}e^{\tilde{B}T}(||x_0||_Y + r_2)|t - s|.$$

Now, the $f$ on $[0, T')$ obtained from Kato's evolution operator is the uniform (X) limit of the $f$ corresponding to the partitions $\Delta$ as $|\Delta| \to 0$. This establishes the result in the second case and completes the proof of Lemma B.

Together, Lemma A and Lemma B suggest an iteration scheme. We fix $x_1$ and $\Delta$, then obtain sequences \{ $u_n$ \}, \{ $A_n(t)$ \}, and \{ $U_n(t, s)$ \}, with $A_n(t) = A(t, u_n(t))$, \{ $U_{n+1}(t, s)$ \} the evolution operator generated by \{ $A_n(t)$ \} and $\Delta$, and $u_{n+1}(t) = U_{n+1}(t, 0)x_1$. Once Lemma A is satisfied, we have \{ $A_n(t): t \in [0, T')$ \} is $Y$-stable with constants $C_22M_1e^{K}$ and $\beta_1$; then, Lemma B applied to \{ $\theta(t)$ \} = \{ $\theta(t)$ \} and $\tilde{M} = C_22M_1e^{K}$ and $\tilde{B} = \beta_1$, implies that $u_{n+1}$ is Lipschitz continuous (X) on $[0, T']$ with Lipschitz constant $||A||C_22M_1e^{K+\beta_1T}(||x_0||_Y + r_2)$. Assuming $u_{n+1}[0, T'] \subset W$, the stage is set to apply Lemma A to \{ $A_{n+1}(t): t \in [0, T')$ \} and continue the process.

We now work with a fixed partition $\Delta$ of $[0, T')$ and fixed $x_1 \in Z$.

Let $A_0(t) = A(t, x_1)$ for $t \in [0, T')$ and let \{ $U_i(t, s)$ \} be the evolution operator generated by \{ $A_0(t)$ \} and $\Delta$. Define $u_i(t) = U_i(t, 0)x_1$. Then, $u'_i(t) + A_0(t)u_i(t) = 0$ except at $t_1, t_2, \ldots, t_N$. Also,

$$||u_i(t) - x_1||_X = ||U_i(t, t)x_1 - U_i(t, 0)x_1||_X = \left| \int_0^t U_i(t, s)A_0(s)x_1 ds \right|_X \leq Me^{\beta T}(||x_0||_Y + r_2)t \leq r$$

by the choice of $T^0$ and $T'$. So, $u_i(t) \in W$ for each $t \in [0, T')$. This argument also works for all the following $u_n$, $n = 2, 3, \ldots$.
To start the procedure, we apply Lemma A to \( u \equiv x_1 \) and then Lemma B with 
\[ \{ \delta(t) \} = \{ d_0(t) \}, \quad \tilde{M} = C_2^2 M_1 e^K \] and \( \tilde{\beta} = \beta_1, \) proving that \( u_1 \) is Lipschitz continuous \((X)\) on \([0, T']\) with the Lipschitz constant \( \|A\|_\infty C_2^2 M_1 e^{K + \beta_1 T} (\|x_0\|_Y + r_2). \)

For the next iteration, let \( A_1(t) = A(t, u_1(t)) \) for \( t \in [0, T'] \) and let \( \{ U_2(t, s)\} \) be the evolution operator generated by \( \{ A_1(t)\} \) and \( \Delta. \) Define \( u_2(t) = U_2(t, 0)x_1. \) Then, \( u'_2(t) + A_1(t)u_2(t) = 0 \) except at \( t_1, t_2, \ldots, t_N. \) As with \( u_1, u_2(t) \in W \) for each \( t \in [0, T'] \).

As we commented before, we can continue in like manner. For convenience of notation, let \( M_2 = C_2^2 M_1 e^K. \) Then, for \( n > 1, \) we have

\[
\|u_{n+1}(t) - u_n(t)\| \leq \|U_{n+1}(t, 0)x_1 - U_n(t, 0)x_1\| X
\leq Me^{BT} C_1 M_2 e^{\beta_1 T} (\|x_0\|_Y + r_2) \cdot \int_0^t \|u_n(s) - u_{n-1}(s)\|_X ds
\leq (MM_2 e^{\beta_1 T} C_1 (\|x_0\|_Y + r_2))^n \cdot \int_0^t \|u_1(\tilde{s}) - x_0\|_X ds
\leq \left( \frac{MM_2 e^{\beta_1 T} C_1 (\|x_0\|_Y + r_2)}{n!} \right)^n \|x_0\|_X
\]

It follows that there exists a continuous function \( u_\Delta \) on \([0, T']\) to \( W \) such that \( u_n \to u_\Delta \) uniformly on \([0, T']\) as \( n \to \infty. \) The rate of convergence is independent of \( \Delta \) and \( x_1. \)

Now, let \( A_\Delta(t) = A(t, u_\Delta(t)) \) for \( t \in [0, T'] \) and let \( \{ U_\Delta(t, s)\} \) be the evolution operator generated by \( \{ A_\Delta(t)\} \) and \( \Delta. \) Define \( u(t) = U_\Delta(t, 0)x_1, \) then \( \dot{u}(t) + A_\Delta(t)\dot{u}(t) = 0 \) except at \( t_1, t_2, \ldots, t_N, \) and

\[
\|\dot{u}(t) - u_n(t)\|_X = \|U_\Delta(t, 0)x_1 - U_n(t, 0)x_1\|_X
\leq Me^{BT} C_1 M_2 e^{\beta_1 T} (\|x_0\|_Y + r_2) \cdot \int_0^t \|\dot{u}(s) - u_{n-1}(s)\|_X ds
\leq \left( \frac{MM_2 e^{\beta_1 T} C_1 (\|x_0\|_Y + r_2)}{n!} \right)^n \|x_0\|_X
\]

which tends to 0 as \( n \to \infty. \) Thus \( \dot{u}(t) = u_\Delta(t), u'_\Delta(t) + A_\Delta(t)u_\Delta(t) = 0 \) except at \( t_1, \ldots, t_N, u_\Delta(0) = x_1, u_\Delta \) is Lipschitz continuous \((X)\) with Lipschitz constant \( \|A\|_\infty C_2^2 M_1 e^{K + \beta_1 T} (\|x_0\|_Y + r_2), \) and \( \|u_\Delta(t)\|_Y = \|U_\Delta(t, 0)x_1\|_Y \leq C_2^2 M_1 e^{K + \beta_1 T} (\|x_0\|_Y + r_2), \) independent of \( t \) and \( x_1. \)

We now establish that \( \{ u_\Delta; u_\Delta(0) = x_1, \Delta \) is a partition of \([0, T']\) \} is a family of approximate solutions to (QL) on \([0, T']\) with initial value \( x_1. \) Except for \( t_1, \ldots, t_N, \) we have

\[
u'_\Delta(t) + A(t, u_\Delta(t))u_\Delta(t) = u'_\Delta(t) + A(i, u_\Delta(i))u_\Delta(t)
\leq A(t, u_\Delta(t)) - A(i, u_\Delta(i))u_\Delta(t)
= (A(t, u_\Delta(t)) - A(i, u_\Delta(i)))u_\Delta(t).\]
So,

$$\|u_\Delta'(t) + A(t, u_\Delta(t))u_\Delta(t)\|_X < C_1(||x_0\|_Y + r_2)$$

$$< C_1M_2e^{\beta T}(||x_0\|_Y + r_2) \cdot (1 + M_2e^{\beta T}||A||(||x_0\|_Y + r_2)||t - \tilde{t}||$$

$$= L|t - \tilde{t}|.$$

where $L$ is independent of $t$ in $[0, T')$ and $\Delta$. Thus $\|u_\Delta'(t) + A(t, u_\Delta(t))u_\Delta(t)\|_X < L|\Delta|$ except at $t_1, t_2, \ldots, t_N$. This verifies that we have a family of approximate solutions.

To show that the $\{u_\Delta\}$ converge as $|\Delta| \to 0$, let $\Delta_1$ and $\Delta_2$ be two partitions of $[0, T')$ with $|\Delta_1|$ and $|\Delta_2|$ small enough that both $\|f'(t) + A(t, f(t))f(t)\|_X < \epsilon$ and $\|g'(t) + A(t, g(t))g(t)\|_X < \epsilon$ for $t \in [0, T') \setminus (\Delta_1 \cup \Delta_2)$, where $f(t) = u_{\Delta_1}(t)$, $g(t) = u_{\Delta_2}(t)$, $f(0) = x_1 = g(0)$, and $\epsilon > 0$ is fixed. The preceding paragraph allows us to do this. Let $\{V(t, s)\}$ be the evolution operator obtained from Kato's Theorem 4.1 [11] for $\{A(t, f(t)) : t \in [0, T')\}$. For $s, t \in [0, T') \setminus (\Delta_1 \cup \Delta_2)$, $s < t$, we get

$$g'(s) - f'(s) = (g'(s) + A(s, g(s))g(s)) - (f'(s) + A(s, f(s))f(s))$$

$$- A(s, f(s))(g(s) - f(s)) + (A(s, f(s)) - A(s, g(s)))g(s).$$

Moving the third expression on the right to the left side of the equation and applying $V(t, s)$, we get

$$V(t, s)(g'(s) - f'(s)) + V(t, s)A(s, f(s))(g(s) - f(s))$$

$$= V(t, s)(g'(s) + A(s, g(s))g(s))$$

$$- V(t, s)(f'(s) + A(s, f(s))f(s))$$

$$+ V(t, s)(A(s, f(s)) - A(s, g(s)))g(s).$$

The left side is simply $\partial V(t, s)(g(s) - f(s))/\partial s$. Integrating both sides in $s$ from 0 to $t$, evaluating the left side at the endpoints, and recognizing that $V(t, t) = I$, we get

$$g(t) - f(t) = V(t, 0)(x_1 - x_1)$$

$$= \int_0^t V(t, s)(g'(s) + A(s, g(s))g(s)) \, ds$$

$$- \int_0^t V(t, s)(f'(s) + A(s, f(s))f(s)) \, ds$$

$$+ \int_0^t V(t, s)(A(s, f(s)) - A(s, g(s)))g(s) \, ds.$$

So,

$$\|g(t) - f(t)\|_X < T'Me^{\beta T}e + T'Me^{\beta T}e + Me^{\beta T}C_1M_2e^{\beta T}(||x_0\|_Y + r_2)$$

$$\cdot \int_0^t \|f(s) - g(s)\|_X \, ds$$

$$= L_1 \epsilon + L_2 \int_0^t \|g(s) - f(s)\|_X \, ds.$$

This implies that

$$\|u_{\Delta_1}(t) - u_{\Delta_2}(t)\|_X = \|g(t) - f(t)\|_X = O(\epsilon)$$

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independent of $t$ in $[0, T']$. Thus, $\{u_\Delta\}$ converges uniformly to a function $u$ on $[0, T']$ to $W$ as $|\Delta| \to 0$. We note that $u$ is Lipschitz continuous ($X$) with constant $\|A\|C^2M_1e^{K_{\beta}T}(\|x_0\|_Y + r_2)$, $u(0) = x_1$, and $u$ is bounded, independent of $x_1$, by $C^2M_1e^{K_{\beta}T}(\|x_0\|_Y + r_2)$ in the relative completion of $Y$ in $X$.

We need to know that $u$ “corresponds” to $\{A(t, u(t)) : t \in [0, T']\}$. Let $\{U(t, s)\}$ be the evolution operator obtained from Kato’s Theorem 4.1 [11] for $\{A(t, u(t))\}$, and define $\bar{u}(t) = U(t, 0)x_1$. By Lemma A, $\{A(t, u(t))\}$ is $Y$-stable with constants $M_2$ and $\beta_1$. For any partition $\Delta$ of $[0, T']$ we have

$$||\bar{u}(t) - u_\Delta(t)||_X = ||U(t, 0)x_1 - U_\Delta(t, 0)x_1||_X$$

$$= \left|\int_0^t U(t, s)(A(s, u(s)) - A_\Delta(s))U_\Delta(s, 0)x_1 ds\right|_X$$

$$\leq M\epsilon^{\beta T}C_1(|\Delta| + \sup\{||u(s) - u_\Delta(s)|| : s \in [0, t]\})$$

$$\cdot M_2e^{\beta T}(\|x_0\|_Y + r_2)T' -$$

Since $u_\Delta$ converges to $u$ uniformly on $[0, T']$ as $|\Delta| \to 0$, and $u_\Delta$ is Lipschitz continuous ($X$) with a Lipschitz constant that is independent of $\Delta$, we see that $||\bar{u}(t) - u_\Delta(t)||_X$ goes to $||\bar{u}(t) - u(t)||_X$ and to 0 as $|\Delta| \to 0$. Thus $u(t) = \bar{u}(t) = U(t, 0)x_1$.

Suppose $x_2 \in Z$ and that $w_\Delta$ and $w$ are obtained in the same manner as $u_\Delta$ and $u$, except that the initial value for $w_\Delta$ and $w$ is $x_2$. Analogous to the technique employed to obtain $u$, we get

$$\frac{\partial}{\partial s} U_\Delta(t, s)(u_\Delta(s) - w_\Delta(s)) = U_\Delta(t, s)(A(s, w_\Delta(s)) - A(s, u_\Delta(s)))w_\Delta(s)$$

for $s, t \in [0, T']$, $s < t$, $s \notin \Delta$. Integrating both sides in $s$ from 0 to $t$ yields

$$u_\Delta(t) - w_\Delta(t) - U_\Delta(t, 0)(x_1 - x_2)$$

$$= \int_0^t U_\Delta(t, s)(A(s, w_\Delta(s)) - A(s, u_\Delta(s)))w_\Delta(s) ds,$$

and so

$$||u_\Delta(t) - w_\Delta(t)||_X < Me^{\beta T}||x_1 - x_2||_X + Me^{\beta T}C_1M_2e^{\beta T}(\|x_0\|_Y + r_2)$$

$$\cdot \int_0^t ||u_\Delta(\tilde{s}) - w_\Delta(\tilde{s})||_X ds.$$
value $x_1$. In fact, this also makes $u$ a solution of (QL). We note that it is not necessary that $v([0, T' \right] \subset W$.

Now suppose that $v$ is a solution to (QL) on $[0, T' \right]$ with initial value $x_1$. Then, $v'(s) + A(s, v(s))v(s) = 0$ a.e., and so

$$v'(s) + A(s, u(s))v(s) = (A(s, u(s)) - A(s, v(s)))v(s) \quad \text{a.e.}$$

Thus,

$$\frac{\partial}{\partial s} U(t, s)v(s) = U(t, s)v'(s) + U(t, s)A(s, u(s))v(s) \quad \text{a.e.}$$

$$= U(t, s)(A(s, u(s)) - A(s, v(s)))v(s) \quad \text{a.e.}$$

Integrating in $s$ from 0 to $t$, we get

$$v(t) - u(t) = U(t, t)v(t) - U(t, 0)v(0)$$

$$= \int_0^t U(t, s)(A(s, u(s)) - A(s, v(s)))v(s) \, ds.$$ 

This implies that

$$\|v(t) - u(t)\|_X \leq M e^{\beta T} \|v\|_Y C_1 \int_0^t \|u(s) - v(s)\|_X \, ds,$$

and thus $\|v(t) - u(t)\|_X = 0$ for all $t$ in $[0, T')$. This makes $u$ the unique solution of (QL) on $[0, T')$ with initial value $x_1$.

If $Y$ is reflexive, then by Kato's Theorem 5.1 [11], we have the result that $v = u$ is a solution of (L; $u$) on $[0, T')$ with initial value $x_1$, and thus $u$ is a solution of (QL). This gives us Corollary 1.

The remarks following the statement of the Theorem and Corollary are straightforward. We also note that Remark (1) deals with a particular case of condition (iii) of the theorem. Remark (2) contains a condition which greatly simplifies the proof of the Theorem, but which would be extremely difficult to verify in the absence of conditions stronger than condition (iii); e.g., see [2].

We now turn our attention to an application of our Theorem using the Sobolevskii-Tanabe theory of linear evolution equations of parabolic type.

**Corollary 2.** Let $S$ be the sector of the complex plane $C$ consisting of 0 and $\{\lambda \in C: -\theta < \arg \lambda < \theta\}$, where $\theta \in (\pi/2, \pi)$ is fixed. We assume that conditions (i) and (ii) of the Theorem hold with $Y = D(A(t, w))$ for each $t, w$, and that (iii) the resolvent set of $-A(t, w)$ contains $S$ and

$$\|\lambda I + A(t, w)\|^{-1}_X \leq C_4/(1 + |\lambda|)$$

for each $\lambda \in S$, $t \in [0, T]$, and $w \in W$, where $C_4$ is a constant independent of $\lambda, t, \text{ and } w$.

Then, the conclusion of the Theorem holds and (L; $u$) has a continuously differentiable ($X$) solution on $[0, T')$ with initial value $x_1$, and thus $u$ is a continuously differentiable ($X$) solution of (QL) on $[0, T')$ with initial value $x_1$.

**Proof.** Under these conditions the hypotheses of the Theorem hold, where $S(t, w) = A(t, w)$ for each $t$ and $w$. This gives us the limit solution $u$. The plan of attack is to produce a solution to (L; $u$) which is continuously differentiable ($X$) on
[0, T'] and has initial value $x_0$. This is where the Sobolevskii-Tanabe theory enters. Let $A(t) = A(t, u(t))$ for each $t \in [0, T']$, and we see for $t_1, t_2, t_3 \in [0, T']$ that

$$
\|(A(t_1) - A(t_2))A(t_3)^{-1}\|_X \leq \|A(t_1) - A(t_2)\|_{Y,X}\|A(t_3)^{-1}\|_{X,Y}
$$

$$
\leq C_6\|A(t_1, u(t_1)) - A(t_2, u(t_2))\|_{Y,X}
$$

and the Lipschitz continuity of $u$, where $C_6$ is independent of the choice of $t_1, t_2, t_3$.

It follows from the Sobolevskii-Tanabe theory [14], [15], [19], [20], [21], [22] that there is an evolution operator $\{V(t, s) : 0 \leq s \leq t \leq T'\}$ such that $v(t) = V(t, 0)x_0$ defines a continuously differentiable $(X)$ function that satisfies $v'(t) + A(t)v(t) = 0$, $v(0) = x_0$. The operator also satisfies $\|A(t)V(t, 0)A(0)^{-1}\|_X \leq C_6$ on $[0, T']$, with $C_6$ independent of $t$ [19, p. 5], thus

$$
\|v(t)\|_Y = \|V(t, 0)x_0\|_Y = \|A(0)^{-1}A(t)\|_Y \|V(t, 0)A(0)^{-1}A(0)x_0\|_Y
$$

$$
\leq \|A(t)^{-1}\|_{X,Y}\|A(t)V(t, 0)A(0)^{-1}\|_X \cdot \|A(0)\|_{Y,X}\|x_0\|_Y
$$

$$
\leq C_7\|x_0\|_Y,
$$

where $C_7$ is independent of $t$. So, except for the image of $v$ lying in $W$, we have that $v$ is a solution of $(L; u)$. Since the proof of the uniqueness of a solution to $(L; u)$ does not depend on $v([0, T']) \subset W$, we still have that $v(t) = u(t)$ on $[0, T']$. Consequently, $u$ is the solution of $(QL)$ on $[0, T']$ with initial value $x_0$. In fact, $u$ is continuously differentiable $(X)$, without exception, on $[0, T']$. □

We note that in general an application of the Theorem involves finding conditions that guarantee the existence of a solution to $(L; u)$, which then implies that $u$ is the solution of $(QL)$.

It may be difficult at times to recognize that the conditions for our Theorem hold. The following Proposition gives criteria that obtain the Banach space $Y$ and verify most of condition (iii) of the Theorem. If, in particular, we are able to use $\lambda I + A(t, w)$, where $\lambda > \beta$ is fixed, for $S(t, w)$ in the Proposition, then condition (ii) of the Theorem holds as well as all of condition (iii).

**Proposition.** Let $Y$ be a dense linear subspace of $X$. Suppose for each $t \in [0, T]$ and $w \in W$ that $S(t, w)$ is an isomorphism (algebraically) from $Y$ onto $X$, $S(t, w)$ is a closed operator in $X$, $S(t, w)^{-1} \in B(X)$ with $\|S(t, w)^{-1}\|_X \leq L_1$, and the bounded linear operator $S(t, w)S(t_0, w_0)^{-1}$ satisfies

$$
\|S(t_2, w_2)S(t_0, w_0)^{-1} - S(t_1, w_1)S(t_0, w_0)^{-1}\|_X \leq L_2(|t_2 - t_1| + \|w_2 - w_1\|_X),
$$

where $L_1, L_2, t_0$ from $[0, T]$, and $w_0 \in W$ are fixed. Suppose further that $Y$ has the graph norm induced by $S(t_0, w_0)$; i.e., for $y \in Y$, $\|y\|_Y = \|y\|_X + \|S(t_0, w_0)y\|_X$.

Then,

(i) $Y$ is a Banach space under this norm, and $Y$ is continuously embedded in $X$.

(ii) $S(t, w)^{-1} \in B(X, Y)$ for each $t$ and $w$, and $\|S(t, w)^{-1}\|_{X,Y} < 1 + L_1 + L_2(T + 2r)$, where $r$ is the radius of the ball $W$.

(iii) $S(t, w) \in B(Y, X)$ for each $t$ and $w$, and $\|S(t, w)\|_{Y,X} < 1 + L_2(T + 2r) \equiv L_3$.

(iv) $\|S(t_2, w_2) - S(t_1, w_1)\|_{Y,X} \leq L_2L_3(|t_2 - t_1| + \|w_2 - w_1\|_X)$. 
Proof. Since $S(t_0, w_0)$ is a closed linear operator with domain $Y$, it is clear that $Y$ is a Banach space under the indicated norm. It is also immediate that $Y$ is continuously embedded in $X$.

Let $x \in X$, then
\[ \|S(t, w)^{-1}x\|_Y = \|S(t, w)^{-1}x\|_X + \|S(t_0, w_0)S(t, w)^{-1}x\|_X \]
\[ \leq L_1\|x\|_X + \|S(t_0, w_0)S(t, w)^{-1}x\|_X = (L_1 + 1)\|x\|_X + L_2(\|t - t_0\| + \|w - w_0\|)\|x\|_X \]
\[ \leq (L_1 + 1)\|x\|_X + L_2(T + 2r)\|x\|_X = (1 + L_1 + L_2(T + 2r))\|x\|_X. \]

So, $S(t, w)^{-1} \in B(X, Y)$ and $\|S(t, w)^{-1}\|_{X,Y} < 1 + L_1 + L_2(T + 2r)$.

Let $y \in Y$, then
\[ \|S(t, w)y\|_X = \|S(t, w)S(t_0, w_0)^{-1}S(t_0, w_0)y\|_X \]
\[ \leq (1 + L_2(\|t - t_0\| + \|w - w_0\|)) \cdot \|S(t_0, w_0)y\|_X \]
\[ \leq (1 + L_2(T + 2r))(\|y\|_Y - \|y\|_X) \]
\[ \leq (1 + L_2(T + 2r))\|y\|_Y. \]

So, $S(t, w) \in B(Y, X)$ and $\|S(t, w)\|_{Y,X} < 1 + L_2(T + 2r)$.

To show (iv), let $y \in Y$, then
\[ \|S(t_2, w_2)y - S(t_1, w_1)y\|_X = \|(S(t_2, w_2) - S(t_1, w_1))S(t_0, w_0)^{-1}S(t_0, w_0)y\|_X \]
\[ \leq \|(S(t_2, w_2) - S(t_1, w_1))S(t_0, w_0)^{-1}\|_X \cdot \|S(t_0, w_0)y\|_{Y,X} \]
\[ \leq L_2L_3(\|t_2 - t_1\| + \|w_2 - w_1\|) \cdot \|S(t_0, w_0)y\|_{Y,X} \]
\[ \leq L_2L_3(t_2 - t_1 + \|w_2 - w_1\|)\|y\|_Y. \]

We also note that condition (i) for our Theorem holds when each $A(t, w)$ satisfies $\|\exp(-sA(t, w))\|_X < e^{Bt}$.

References


3. M. G. Crandall and L. C. Evans, On the relation of the operator $\partial / \partial s + \partial / \partial t$ to evolution governed by accretive operators, Israel J. Math. 21 (1975), 261–278.


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