

**AFFINE CONNECTIONS AND DEFINING FUNCTIONS
 OF REAL HYPERSURFACES IN \mathbb{C}^n**

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ABSTRACT. The affine connection and curvature introduced by Tanaka on a strongly pseudoconvex real hypersurface are computed explicitly in terms of its defining function. If Fefferman's defining function is used, then the Ricci form is shown to be a function multiple of the Levi form. The factor is computable by Fefferman's algorithm and its positivity implies the vanishing of certain cohomology groups (of the $\bar{\partial}_b$ complex) in the compact case.

Introduction. Let M be a strongly pseudoconvex real hypersurface in \mathbb{C}^n . Let θ be a nowhere vanishing real differential form on M which annihilates the complex tangent space at each point. It is well known that there exists a unique vector field ξ on M such that $\theta(\xi) = 1$ and $\xi \lrcorner d\theta = 0$. In [5], Tanaka has constructed a unique affine connection ∇ satisfying certain natural conditions with respect to ξ . Using this canonical affine connection associated to ξ (or θ), he obtained the following Bochner-Kodaira type formula for the boundary Laplacian \square_b (see [5, p. 47] and compare [4, p. 119]). Let $T'M$ be the bundle of $(1, 0)$ vectors tangent to M . Then

$$\square_b \phi = -\frac{n-q-1}{n-1} \sum_{\alpha=1}^{n-1} \nabla_{x_\alpha} \nabla_{\bar{x}_\alpha} \phi - \frac{q}{n-1} \sum_{\alpha=1}^{n-1} \nabla_{\bar{x}_\alpha} \nabla_{x_\alpha} \phi + \frac{n-q-1}{n-1} R_*(\phi), \quad (*)$$

where ϕ is a smooth section of $\Lambda^q(\overline{T'M})^*$, $\{X_\alpha\}_{\alpha=1, \dots, n-1}$ is a local orthonormal basis of $T'M$ with respect to the Levi metric defined by θ and R_* is the Ricci operator associated to ∇ . In case M is compact, (*) implies, by standard arguments, that if the selfadjoint operator R_* is positive definite everywhere, then there is no nonzero harmonic form ϕ on M and the cohomology groups $H^{0,q}(M)$ of the $\bar{\partial}_b$ complex vanish, for $q \neq 0, n-1$. $H^{0,q}(M)$ here coincides with $H^{0,q}(\mathfrak{B})$ in Kohn-Rossi [3, p. 466]. In passing, the above theory works for an abstractly given strongly pseudoconvex CR manifold, not necessarily imbedded as a real hypersurface in \mathbb{C}^n .

There is a choice of ξ (or θ) in the construction of ∇ . One naturally asks if there exists a most natural choice. For any defining function f on M , one may take $\theta = -i\partial f$. Now Fefferman [1] has constructed special defining functions which

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satisfy a Monge-Ampere equation to order $\leq n + 1$ at M . The Monge-Ampere equation arises from considerations of the Bergman kernel, and is intimately related to the geometry of M . In this paper, we will show that if f satisfies the equation to second order at M , then for the affine connection associated to $\theta = -i\partial f$, the Ricci form is a function multiple of the Levi form. Thus, we obtain a real-valued function λ on M , computable using Fefferman's algorithm, such that the Ricci operator is positive definite at a point x in M if and only if $\lambda(x) > 0$. If λ is everywhere positive, then the cohomology groups $H^{0,q}(M)$ ($q \neq 0, n - 1$) vanish. Relevant to an open problem posed in [1] concerning intrinsic development, it would be interesting to know if there is some natural choice of θ for constructing ∇ on a CR manifold.

In §1, we state explicitly all the results from [1] and [5] which we need. In §2, we consider any defining function f and derive formulas for the affine connection and curvature associated to $\theta = -i\partial f$. It is interesting to compare our formulas to the local formulas of Riemannian and Kähler geometry as in [2]. We apply the formulas to consider the ellipsoids and to prove the main theorem in the last section. The formulas apply also to real hypersurfaces in complex manifolds; we shall return to their applications to isolated singularities (cf. [5]) later.

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1. Notations and preliminaries. Throughout this paper, we use the notation of tensor calculus. Small Greek (resp. Latin) indices always run from 1 to $n - 1$ (resp. 1 to n). Summation over repeated indices is understood. z^1, \dots, z^n are the coordinates of \mathbb{C}^n , and $f_j, f_{\bar{k}}, f_{j\bar{k}}, \dots$ stand for the partial derivatives $\partial f / \partial z^j, \partial f / \partial z^{\bar{k}}, \partial^2 f / \partial z^j \partial z^{\bar{k}}, \dots$. For a C^∞ manifold M , TM, T^*M , etc. have the usual meaning. For a vector bundle V over M , $CV = V \otimes \mathbb{C}$, $\Gamma(V)$ denotes the space of C^∞ sections in V , and V_x denotes the fiber over $x \in M$.

Let M be a C^∞ strongly pseudoconvex real hypersurface in \mathbb{C}^n . Let $T'M$ (resp. $T''M$) be the subbundle of CTM consisting of vectors of type $(1, 0)$ (resp. $(0, 1)$) in \mathbb{C}^n . There is a subbundle $H(M)$ of TM and a homomorphism $I: H(M) \rightarrow H(M)$ such that

- (1) $CH(M) = T'M \oplus T''M$.
- (2) $I^2 = -1$ and $T'M = \{X - iIX \mid X \in H(M)\}$.

At each point $x \in M$, $H_x(M)$ is called the complex tangent space at x . Let $\theta \in \Gamma(T^*M)$ be any nowhere vanishing differential form on M which annihilates $H(M)$. There is a unique vector field ξ on M such that

$$\theta(\xi) = 1 \quad \text{and} \quad \xi \lrcorner d\theta = 0. \tag{3}$$

Extend I to a tensor field of type $(1, 1)$ on M by setting $I(\xi) = 0$. Then,

$$I^2(X) = -X + \theta(X)\xi \quad \text{for any } X \in TM. \tag{4}$$

Let $\omega = -d\theta$. We shall consider the Levi form $\langle \cdot, \cdot \rangle$ defined by

$$\langle X, Y \rangle = \omega(IX, Y), \quad X, Y \in CT_x M, \quad x \in M.$$

Further we assume that \langle , \rangle is positive definite on $H(M)$. Let E be the 1-dimensional subbundle of CTM spanned by ξ . Thus, for each $x \in M$, the fiber $E_x = C\xi_x$. For any $X \in CTM$, we denote by $X_{T'}$ (resp. $X_{T''}, X_E$) the $T'M$ - (resp. $T''M$ -, E -) component of X with respect to the decomposition $CTM = T'M \oplus T''M \oplus E$.

Now we recall Tanaka's canonical affine connection on M associated to ξ . This is a connection

$$\nabla : \Gamma(TM) \rightarrow \Gamma(TM \otimes T^*M)$$

on M satisfying the following conditions:

- (5) $\nabla_X \Gamma(H(M)) \subset \Gamma(H(M))$ for any $X \in \Gamma(TM)$ i.e. $H(M)$ is parallel.
- (6) The tensor fields ξ, I, ω are parallel.
- (7) Let T be the torsion of ∇ . For any $X, Y \in \Gamma(H(M))$,

$$T(X, Y) = -\omega(X, Y)\xi \quad \text{and} \quad T(\xi, IY) = -IT(\xi, Y).$$

We shall use the following formulas [5, p. 31]. Let $X, Y \in \Gamma(T'M)$.

- (8) $\nabla_{\bar{X}} Y = [\bar{X}, Y]_{T''}$.
- (9) $\nabla_X Y \in \Gamma(T'M)$ and $\langle \nabla_X Y, \bar{W} \rangle = X \langle Y, \bar{W} \rangle - \langle Y, \nabla_{\bar{X}} \bar{W} \rangle$ for all $W \in \Gamma(T'M)$.
- (10) $\nabla_{\xi} Y = L_{\xi} Y - \frac{1}{2} I(L_{\xi} I)Y$, where L_{ξ} denotes the Lie derivation.

Finally, we recall Fefferman's defining functions. Let

$$J(u) = (-1)^n \det \begin{pmatrix} u & u_{\bar{1}} & \cdots & u_{\bar{n}} \\ u_1 & u_{1\bar{1}} & \cdots & u_{1\bar{n}} \\ \vdots & \vdots & & \vdots \\ u_n & u_{n\bar{1}} & \cdots & u_{n\bar{n}} \end{pmatrix}, \tag{11}$$

where u is any smooth function. In [1], Fefferman considered defining equations of M satisfying

$$(11)' J(u) = 1, \text{ to high order at } M.$$

He also found a clever algorithm for u up to order $n + 1$.

2. Formulas for connection and curvature. Let f be any defining function of M . Precisely, let f be a C^∞ real-valued function defined on some neighborhood of M such that M is defined by the equation $f = 0$, and $df \neq 0$ on M . Take $\theta = -i\partial f$. The assumption on \langle , \rangle implies that $f_{j\bar{k}} w^j \bar{w}^k > 0$ for any nonzero vector $w^j \partial / \partial z^j$ satisfying $w^j f_j = 0$. The matrix $(f_{j\bar{k}})$ need not be invertible. Let $\xi = \xi^j \partial / \partial z^j + \bar{\xi}^{\bar{j}} \partial / \partial \bar{z}^{\bar{j}}$. Condition (3) is equivalent to

$$if_{\bar{k}} \bar{\xi}^{\bar{k}} = 1 \tag{3}_1$$

and

$$x^j f_j = 0 \text{ implies } x^j f_{j\bar{k}} \bar{\xi}^{\bar{k}} = 0. \tag{3}_2$$

Choose a local C^∞ orthonormal basis $\{X_\alpha = x_\alpha^j \partial / \partial z^j\}_{\alpha=1, \dots, n-1}$ of $T'M$ with respect to \langle , \rangle . Thus,

$$x_\alpha^j f_j = 0 \tag{12}$$

and

$$x_\alpha^j f_{j\bar{k}} \overline{x_\beta^k} = \delta_\alpha^\beta. \tag{13}$$

PROPOSITION 1. Let

$$F = \begin{pmatrix} f & f_{\bar{1}} & \cdots & f_{\bar{n}} \\ f_1 & f_{1\bar{1}} & \cdots & f_{1\bar{n}} \\ \vdots & \vdots & & \vdots \\ f_n & f_{n\bar{1}} & \cdots & f_{n\bar{n}} \end{pmatrix}$$

and

$$A = \begin{pmatrix} -\langle \xi, \xi \rangle' & -i\xi^1 & \cdots & -i\xi^n \\ i\overline{\xi^1} & a^{\bar{1}1} & \cdots & a^{\bar{1}n} \\ \vdots & \vdots & & \vdots \\ i\overline{\xi^n} & a^{\bar{n}1} & \cdots & a^{\bar{n}n} \end{pmatrix},$$

where $a^{\bar{j}k} = \overline{x_\alpha^j} x_\alpha^k$ and $\langle \xi, \xi \rangle' = \frac{1}{2} \langle \xi, \xi \rangle$. Then $FA = I_{n+1}$.

PROOF. $f_j dz^j$ and $f_{j\bar{k}} \overline{\xi^k} dz^j$ annihilate all X_α , hence they are linearly dependent. Since $df \neq 0$, we may assume that $f_{j\bar{k}} \overline{\xi^k} = af_j$. Contracting with ξ^j gives $a = -i\langle \xi, \xi \rangle'$. Thus,

$$-f_j \langle \xi, \xi \rangle' + if_{j\bar{k}} \overline{\xi^k} = 0. \tag{14}$$

Writing $\xi^j = x_n^j$, we have by (3)₂ and (13)

$$\begin{pmatrix} x_1^1 & \cdots & x_1^n \\ \vdots & & \vdots \\ x_n^1 & \cdots & x_n^n \end{pmatrix} \begin{pmatrix} f_{1\bar{1}} & \cdots & f_{1\bar{n}} \\ \vdots & & \vdots \\ f_{n\bar{1}} & \cdots & f_{n\bar{n}} \end{pmatrix} \begin{pmatrix} \overline{x_1^1} & \cdots & \overline{x_1^n} \\ \vdots & & \vdots \\ \overline{x_n^1} & \cdots & \overline{x_n^n} \end{pmatrix} \\ = \begin{pmatrix} 1 & & & \mathbf{0} \\ & \ddots & & \\ & & 1 & \\ \mathbf{0} & & & \langle \xi, \xi \rangle' \end{pmatrix}. \tag{15}$$

(x_j^k) is invertible because $\xi^j \partial / \partial z^j \notin T'M$. Let $(y_j^k) = (x_j^k)^{-1}$, noting that $y_j^n = -if_j$ by (3)₁ and (12). Then

$$(f_{j\bar{i}})(\overline{x_r^i})(x_r^k) = (y_j^i) \begin{pmatrix} 1 & & & \mathbf{0} \\ & \ddots & & \\ & & 1 & \\ \mathbf{0} & & & \langle \xi, \xi \rangle' \end{pmatrix} (x_r^k). \tag{15}'$$

Now

$$f_{j\bar{i}} \overline{x_r^l} x_r^k = f_{j\bar{i}} (\overline{x_\alpha^l} x_\alpha^k + \overline{\xi^l} \xi^k) = f_{j\bar{i}} a^{\bar{i}k} - i \langle \xi, \xi \rangle' f_j \xi^k \quad \text{by (14),}$$

while the (j, k) -element on the right of (15)' is

$$y_j^\alpha x_\alpha^k + \langle \xi, \xi \rangle' y_j^n x_n^k = \delta_j^k + (\langle \xi, \xi \rangle' - 1) y_j^n x_n^k = \delta_j^k + i f_j \xi^k - i \langle \xi, \xi \rangle' f_j \xi^k.$$

Therefore, we get

$$-i f_j \xi^k + f_{j\bar{i}} a^{\bar{i}k} = \delta_j^k. \tag{16}$$

By (12),

$$f_j a^{\bar{i}k} = 0. \tag{17}$$

Since $f = 0$ on M , (3)₁, (14), (16), (17) prove the proposition.

We shall use the following notations:

$$F = \begin{pmatrix} f_{0\bar{0}} & \cdots & f_{0\bar{n}} \\ \vdots & & \vdots \\ f_{n\bar{0}} & \cdots & f_{n\bar{n}} \end{pmatrix}, \tag{18}$$

$$A = \begin{pmatrix} a^{\bar{0}0} & \cdots & a^{\bar{0}n} \\ \vdots & & \vdots \\ a^{\bar{n}0} & \cdots & a^{\bar{n}n} \end{pmatrix}. \tag{18}'$$

Proposition 1 implies that F is invertible in a neighborhood of M . Let $A = F^{-1}$ and use (18)' to extend the functions $a^{\bar{k}l}, a^{\bar{k}0} = i \xi^k$ and $a^{\bar{0}0} = a$. Then in a neighborhood of M , we have

$$af + i f_{\bar{k}} \overline{\xi^k} = 1, \tag{3}'_1$$

$$af_j + i f_{j\bar{k}} \overline{\xi^k} = 0, \tag{14}'$$

$$-i f_j \xi^l + f_{j\bar{k}} a^{\bar{k}l} = \delta_j^l, \tag{16}'$$

$$-i f_j \xi^l + f_{\bar{k}} a^{\bar{k}l} = 0. \tag{17}'$$

Note that $\xi^j \partial / \partial z^j + \overline{\xi^j} \partial / \partial \bar{z}^j$ is not the vector field corresponding to $\theta = -i\partial f$ on each level real hypersurface unless $a = 0$.

We now give local formulas corresponding to (8), (9) and (10).

PROPOSITION 2. Let $X = x^j \partial / \partial z^j$ and $Y = y^j \partial / \partial z^j$ (with $x^j f_j = y^j f_j = 0$) be local C^∞ sections in $T'M$. Then

(a) $\nabla_{\bar{X}} Y = (\bar{X}(y^k) + x^j y^l \Gamma_{j\bar{l}}^k) \partial / \partial z^k$, where $\Gamma_{j\bar{l}}^k = -i f_{j\bar{l}} \xi^k$.

(b) $\nabla_X Y = (X(y^k) + x^j y^l \Gamma_{jl}^k) \partial / \partial z^k$,

where $\Gamma_{jl}^k = \sum_{A=0}^n a^{\bar{A}k} (\partial / \partial z^j) f_{\bar{A}} = \Gamma_{j\bar{l}}^k$.

(c) $\nabla_\xi Y = \{\xi(y^k) - Y(\xi^k)\} \partial / \partial z^k$.

PROOF. We first derive

$$\nabla_\xi Y = [\xi, Y]_{T'}. \tag{10}'$$

Recall $\theta(Y) = 0$ and $IY = iY$. Using (3), note that $\theta([\xi, Y]) = 0$. Then

$$\begin{aligned} \nabla_{\xi} Y &= [\xi, Y] + \frac{1}{2}I(I[\xi, Y] - [\xi, IY]) \quad \text{by (10)} \\ &= [\xi, Y] - \frac{1}{2}[\xi, Y] - \frac{i}{2}I[\xi, Y] \quad \text{by (4)} \\ &= \frac{1}{2}([\xi, Y]_{T'} + [\xi, Y]_{T''}) - \frac{i}{2}(i[\xi, Y]_{T'} - i[\xi, Y]_{T''}) = [\xi, Y]_{T'}. \end{aligned}$$

Next note that for any $Z = a^j\partial/\partial z^j + b^{\bar{j}}\partial/\partial \bar{z}^{\bar{j}}$ in CTM,

$$Z_E = \theta(Z)\xi, \quad \theta(Z) = -ia^j f_j = ib^{\bar{j}} \bar{f}_{\bar{j}}. \tag{19}_a$$

$$Z_{T'} = (a^k + ia^j f_j \xi^k) \frac{\partial}{\partial z^k}. \tag{19}_b$$

$$Z_{T''} = (b^{\bar{k}} - ib^{\bar{j}} \bar{f}_{\bar{j}} \bar{\xi}^{\bar{k}}) \frac{\partial}{\partial \bar{z}^{\bar{k}}}. \tag{19}_c$$

(a) and (c) then follow easily from (8) and (10)', using $\theta([\bar{X}, Y]) = i\langle Y, \bar{X} \rangle$ and $\theta([\xi, Y]) = 0$.

To get (b), observe that for any $W = w^i\partial/\partial z^j$ with $w^j f_j = 0$,

$$\begin{aligned} X\langle Y, \bar{W} \rangle - \langle Y, \overline{\nabla_X W} \rangle &= X(y^k) f_{k\bar{m}} \bar{w}^m + x^j y^l f_{j\bar{l}m} \bar{w}^m \quad \text{since } y^j f_{j\bar{k}} \bar{\Gamma}_{\bar{l}m}^k = 0 \\ &= (X(y^k) + x^j y^l f_{j\bar{l}i} a^{\bar{i}k}) f_{k\bar{m}} \bar{w}^m \quad \text{by (16)} \\ &= \{X(y^k) + x^j y^l (a^{\bar{i}k} f_{j\bar{l}i} - i f_{j\bar{l}} \xi^k)\} f_{k\bar{m}} \bar{w}^m \quad \text{by (3)}_2. \end{aligned}$$

Let $v^k = X(y^k) + x^j y^l (a^{\bar{i}k} f_{j\bar{l}i} - i f_{j\bar{l}} \xi^k)$. By (3)₁ and (17), $v^k f_k = 0$, hence $v^k \partial/\partial z^k \in \Gamma(T'M)$. By (9), $\nabla_X Y = v^k \partial/\partial z^k$. The expression for $\Gamma_{j\bar{l}}^k$ simply follows from notations in (18) and (18)', finishing the proof.

In the following, we consider $\Gamma_{j\bar{k}}^l$ and $\Gamma_{j\bar{k}}^l$ as functions on a neighborhood of M , defined by the formulas in Proposition 2.

PROPOSITION 3. *Let $X = x^j\partial/\partial z^j$, $Y = y^j\partial/\partial z^j$ and $W = w^j\partial/\partial z^j$ be C^∞ sections in $T'M$. Then*

$$R(X, \bar{Y})W = (\nabla_X \nabla_{\bar{Y}} - \nabla_{\bar{Y}} \nabla_X - \nabla_{[X, \bar{Y}]})W = x^j \bar{y}^k w^l R_{j\bar{k}l}^p \frac{\partial}{\partial z^p},$$

where

$$R_{j\bar{k}l}^p = \frac{\partial}{\partial z^j} \Gamma_{k\bar{l}}^p + \frac{\partial}{\partial z^{\bar{l}}} \Gamma_{kj}^p - \frac{\partial}{\partial z^k} \Gamma_{j\bar{l}}^p + \Gamma_{k\bar{l}}^r \Gamma_{jr}^p + \Gamma_{k\bar{l}}^r \Gamma_{rl}^p - \Gamma_{j\bar{l}}^r \Gamma_{kr}^p + i f_{j\bar{k}l} \xi^p$$

and

$$R_{j\bar{k}l}^p f_p \equiv 0 \quad \text{mod } f_j, \bar{f}_{\bar{k}}, f_l.$$

PROOF. Decomposing $[X, \bar{Y}]$ by (19)_{a,b,c} and using Proposition 2, one obtains by straightforward computation

$$\begin{aligned} R(X, \bar{Y})W &= \left\{ x^j \bar{y}^k w^l \left(\frac{\partial}{\partial z^j} \Gamma_{k\bar{l}}^p - \frac{\partial}{\partial z^k} \Gamma_{j\bar{l}}^p + \Gamma_{k\bar{l}}^r \Gamma_{jr}^p - \Gamma_{j\bar{l}}^r \Gamma_{kr}^p \right) \right. \\ &\quad \left. - i \langle X, \bar{Y} \rangle (\xi^r w^l \Gamma_{rl}^p + \bar{\xi}^{\bar{r}} w^l \Gamma_{\bar{r}l}^p + W(\xi^p)) \right\} \frac{\partial}{\partial z^p}. \end{aligned}$$

Now

$$-i\langle X, \bar{Y} \rangle \xi^r w^l \Gamma_{rl}^p = x^j \bar{y}^k w^l \Gamma_{kj}^r \Gamma_{rl}^p, \quad \bar{\xi}^r \Gamma_{rl} w^l = 0,$$

and

$$-i\langle X, \bar{Y} \rangle W(\xi^p) = x^j \bar{y}^k (-if_{j\bar{k}}) W(\xi^p) = x^j \bar{y}^k w^l \left(\frac{\partial}{\partial z^l} \Gamma_{kj}^p + if_{j\bar{k}l} \xi^l \right).$$

Hence

$$R(X, \bar{Y})W = x^j \bar{y}^k w^l R_{j\bar{k}l}^p \frac{\partial}{\partial z^p}$$

with $R_{j\bar{k}l}^p$ as given.

Next note that $R(X, \bar{Y})W \in \Gamma(T'M)$ for any $X, Y, W \in \Gamma(T'M)$. Hence $x^j \bar{y}^k w^l R_{j\bar{k}l}^p f_p = 0$ whenever $x^j f_j = y^j f_j = w^j f_j = 0$. The following lemma then implies that $R_{j\bar{k}l}^p f_p \equiv 0 \pmod{f_j, f_{\bar{k}}, f_l}$, finishing the proof of the proposition.

LEMMA (QUOTIENT LAW). *If $a_{j_1 \dots j_r \bar{k}_1 \dots \bar{k}_s}$ are n^{r+s} numbers such that $a_{j_1 \dots j_r \bar{k}_1 \dots \bar{k}_s} x_1^{j_1} \dots x_r^{j_r} y_1^{k_1} \dots y_s^{k_s} = 0$ whenever $x_i^{j_i} = 0$ ($i = 1, \dots, r$) and $y_l^{k_l} = 0$ ($l = 1, \dots, s$), then $a_{j_1 \dots j_r \bar{k}_1 \dots \bar{k}_s} \equiv 0 \pmod{f_{j_1}, \dots, f_{j_r}, f_{\bar{k}_1}, \dots, f_{\bar{k}_s}}$.*

PROOF. For simplicity we prove the case $r = s = 1$; the general case is similar. Since $df \neq 0$, we assume that $f_n \neq 0$. Consider

$$x = \left(0, \dots, \frac{1}{(\alpha)}, \dots, 0, -f_\alpha/f_n \right) \quad \text{and} \quad y = \left(0, \dots, \frac{1}{(\beta)}, \dots, 0, -f_\beta/f_n \right)$$

in $a_{j\bar{k}} x^j \bar{y}^k$. We get

$$a_{\alpha\bar{\beta}} = \frac{f_\alpha}{f_n} a_{n\bar{\beta}} + \frac{f_\beta}{f_n} a_{\alpha\bar{n}} - \frac{f_\alpha f_\beta}{f_n^2} a_{n\bar{n}}$$

and verify that

$$a_{j\bar{k}} = \frac{f_j}{f_n} a_{n\bar{k}} + \frac{f_{\bar{k}}}{f_n} a_{j\bar{n}} - \frac{f_j f_{\bar{k}}}{f_n^2} a_{n\bar{n}},$$

finishing the proof.

PROPOSITION 4. *Let $X = x^j \partial / \partial z^j$, $Y = y^j \partial / \partial z^j$, $W = w^j \partial / \partial z^j$ and $U = u^j \partial / \partial z^j$ be C^∞ sections in $T'M$. Then*

$$\langle R(X, \bar{Y})W, \bar{U} \rangle = R_{j\bar{k}l\bar{m}} x^j \bar{y}^k w^l \bar{u}^m,$$

where

$$R_{j\bar{k}l\bar{m}} = -f_{j\bar{k}l\bar{m}} + f_{j\bar{l}r} a^{\bar{r}s} f_{s\bar{k}m} - i(f_{jl} f_{\bar{k}m} \xi^r - f_{\bar{k}m} f_{l\bar{s}} \xi^s) + \langle \xi, \xi \rangle (f_{j\bar{k}} f_{l\bar{m}} + f_{l\bar{k}} f_{j\bar{m}} - f_{jl} f_{\bar{k}m}).$$

PROOF. It suffices to compute $R_{j\bar{k}l}^p f_{p\bar{m}} \bmod f_{\bar{m}}$. There are seven terms:

$$\left(\frac{\partial}{\partial z^j} \Gamma_{k\bar{l}}^p\right) f_{p\bar{m}} = -\left(f_{j\bar{k}l} \xi^p f_{p\bar{m}} + f_{\bar{k}l} \frac{\partial \xi^p}{\partial z^j} f_{p\bar{m}}\right) \equiv \langle \xi, \xi \rangle' f_{j\bar{m}} f_{l\bar{k}} + i f_{\bar{k}l} f_{\bar{m}j} \xi^p \text{ by (14) and (14)'.} \tag{i}$$

$$\left(\frac{\partial}{\partial z^l} \Gamma_{k\bar{j}}^p\right) f_{p\bar{m}} = \langle \xi, \xi \rangle' f_{l\bar{m}} f_{j\bar{k}} + i f_{\bar{k}j} f_{\bar{m}l} \xi^p. \tag{i}'$$

$$\begin{aligned} -\left(\frac{\partial}{\partial z^k} \Gamma_{j\bar{l}}^p\right) f_{p\bar{m}} &= -f_{j\bar{l}i} \frac{\partial a^{\bar{i}p}}{\partial z^k} f_{p\bar{m}} - f_{j\bar{k}i} a^{\bar{i}p} f_{p\bar{m}} + i f_{j\bar{l}} \frac{\partial \xi^p}{\partial z^k} f_{p\bar{m}} + i f_{j\bar{l}k} \xi^p f_{p\bar{m}} \\ &\equiv f_{j\bar{l}i} a^{\bar{i}p} f_{p\bar{m}\bar{k}} + i f_{j\bar{l}i} f_{\bar{k}m} \xi^{\bar{i}} - f_{j\bar{k}l\bar{m}} \\ &\quad - \langle \xi, \xi \rangle' f_{j\bar{l}} f_{\bar{m}\bar{k}} - i f_{j\bar{l}} f_{\bar{m}\bar{k}p} \xi^p \text{ by (14), (14)', (16), (16)'.} \end{aligned} \tag{ii}$$

By (14) and (16),

$$\Gamma_{k\bar{l}}^r \Gamma_{j\bar{r}}^p f_{p\bar{m}} \equiv -i f_{\bar{k}l} f_{\bar{m}j} \xi^p, \tag{iii}$$

$$\Gamma_{k\bar{j}}^r \Gamma_{r\bar{l}}^p f_{p\bar{m}} \equiv -i f_{\bar{k}j} f_{\bar{m}l} \xi^p, \tag{iii}'$$

$$-\Gamma_{j\bar{l}}^r \Gamma_{k\bar{r}}^p f_{p\bar{m}} \equiv 0, \tag{iv}$$

$$i f_{j\bar{k}l} \xi^p f_{p\bar{m}} \equiv 0. \tag{v}$$

The proposition follows immediately.

Observe that we have

$$R_{j\bar{k}l\bar{m}} = R_{j\bar{m}l\bar{k}} = R_{l\bar{k}j\bar{m}} = R_{l\bar{m}j\bar{k}}, \tag{20}$$

$$\overline{R_{j\bar{k}l\bar{m}}} = R_{k\bar{j}l\bar{m}}, \tag{21}$$

corresponding to the properties [5, p. 34]

$$\begin{aligned} \langle R(X, \bar{Y})W, \bar{U} \rangle &= \langle R(X, \bar{U})W, \bar{Y} \rangle \\ &= \langle R(W, \bar{Y})X, \bar{U} \rangle = \langle R(W, \bar{U})X, \bar{Y} \rangle, \end{aligned} \tag{20}'$$

$$\langle R(X, \bar{Y})W, \bar{U} \rangle = \langle W, \overline{R(Y, \bar{X})U} \rangle. \tag{21}'$$

REMARKS. (a) Formulas for $R(\xi, \bar{Y})W$ etc. are complicated and will not be used. We shall however consider the following Ricci operator

$$R_*(Y) = \sum_{\alpha=1}^{n-1} R(X_\alpha, \bar{X}_\alpha)Y, \quad Y \in \Gamma(T'M).$$

(b) For the real hyperquadric defined by $z^\alpha \bar{z}^\alpha + i(z^n - \bar{z}^n)/2 = 0$, $\xi = 2(\partial/\partial z^n + \partial/\partial \bar{z}^n)$ and $\langle \xi, \xi \rangle' = 0$, hence $R_{j\bar{k}l\bar{m}} = 0$. This is the flat case.

3. Applications of local formulas.

(A) *Ellipsoids.* Consider the ellipsoid E in \mathbb{C}^n defined by the equation

$$\begin{aligned} f &= a_{\alpha\beta} z^\alpha \bar{z}^\beta + \overline{a_{\alpha\beta}} \bar{z}^\alpha z^\beta + b_{\alpha\bar{\beta}} z^\alpha \bar{z}^\beta \\ &\quad + a z^n \bar{z}^n + \bar{a} \bar{z}^n z^n + b z^n \bar{z}^n - 1 = 0, \end{aligned} \tag{22}$$

where $a_{\alpha\beta}$ is symmetric, $b_{\alpha\bar{\beta}} (= \overline{b_{\beta\bar{\alpha}}})$ is positive definite and b is positive. Clearly

$$R_{j\bar{k}i\bar{m}} = \langle \xi, \xi \rangle' (f_{j\bar{k}}f_{i\bar{m}} + f_{i\bar{k}}f_{j\bar{m}} - f_{j\bar{i}}f_{k\bar{m}}).$$

It is easy to solve for $\langle \xi, \xi \rangle'$ from the equation

$$\begin{pmatrix} 0 & f_{\bar{\beta}} & f_{\bar{n}} \\ f_{\alpha} & b_{\alpha\bar{\beta}} & 0 \\ f_n & 0 & b \end{pmatrix} \begin{pmatrix} -\langle \xi, \xi \rangle' \\ i \xi^{\bar{\beta}} \\ i \xi^{\bar{n}} \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}.$$

Setting $(c^{\bar{\alpha}\beta}) = (b_{\alpha\bar{\beta}})^{-1}$, one obtains

$$\langle \xi, \xi \rangle' (c^{\bar{\alpha}\beta} f_{\bar{\beta}} f_{\alpha} + b^{-1} |f_n|^2) = 1.$$

Hence $\langle \xi, \xi \rangle' > 0$. A simple computation then shows that for any local section Y and any local orthonormal basis $\{X_{\alpha}\}$ of $T'M$,

$$\langle R_{\star}(Y), \bar{Y} \rangle = \langle \xi, \xi \rangle' \sum_{\alpha} (\|X_{\alpha}\|_B^2 \|Y\|_B^2 + |\langle X_{\alpha}, \bar{Y} \rangle_B|^2 - |\langle X_{\alpha}, Y \rangle_A|^2), \quad (23)$$

where $\langle Z, \bar{W} \rangle_B = b_{\beta\bar{\gamma}} z^{\beta} \bar{w}^{\gamma} + b z^n \bar{w}^{\bar{n}}$, $\|Z\|_B^2 = \langle Z, \bar{Z} \rangle_B$ and $\langle Z, W \rangle_A = a_{\beta\gamma} z^{\beta} w^{\gamma} + a z^n w^n$.

Applying Tanaka's results, we obtain

PROPOSITION 5. *If $|a_{\beta\gamma}|$ and $|a|$ are small such that the right-hand side of (23) is everywhere positive for all Y , then there is no nonzero harmonic scalar form on M and the cohomology groups $H^{0,q}(E)$ ($q \neq 0, n - 1$) of the $\bar{\partial}_b$ complex vanish.*

The same method was applied to the sphere by Tanaka [5, p. 63]. Both cases are however very special examples of a result of Kohn and Rossi (obtained by a different method) which states that the same cohomology groups vanish for the boundary of any bounded strongly pseudoconvex domain in \mathbb{C}^n [3, p. 467].

(B) We now consider special defining functions. Observe that for any defining function f of M , the equality

$$\frac{\partial}{\partial z^j} \log |\det F| = \sum_{B, C=0}^n a^{\bar{B}C} \frac{\partial}{\partial z^j} f_{C\bar{B}} \quad (24)$$

holds in a neighborhood of M .

THEOREM. (a) *If f satisfies $J(f) = \text{constant} + \mathcal{O}(f^s)$, then $\Gamma_{j\bar{i}}^l = \mathcal{O}(f^{s-1})$.*

(b) *If f satisfies $J(f) = \text{constant} + \mathcal{O}(f^3)$, then*

$$\langle R_{\star}(Y), \bar{Y} \rangle = -i \frac{\partial \xi^l}{\partial z^l} \langle Y, \bar{Y} \rangle \quad \text{for any } Y \in \Gamma(T'M).$$

PROOF.

$$\Gamma_{j\bar{i}}^l = a^{\bar{l}i} \frac{\partial}{\partial z^j} f_{i\bar{l}} = \frac{\partial}{\partial z^j} \log |\det F| - a^{\bar{l}0} \frac{\partial}{\partial z^j} f_{0\bar{l}}$$

by (24) and

$$-a^{\bar{l}0} \frac{\partial}{\partial z^j} f_{0\bar{l}} = a f_j + i f_{j\bar{k}} \bar{\xi}^k = 0$$

by (14)'. Hence

$$\Gamma'_{jl} = \frac{\partial}{\partial z^j} \log|\det F| = \frac{\partial}{\partial z^j} \log|J(f)|,$$

and (a) follows. One easily verifies as in the proof of Proposition 4 that

$$R'_{jk} f_p \equiv f_{j\bar{k}} f_i \langle \xi, \xi \rangle' \pmod{f_j, f_{\bar{k}}}.$$

Setting $\tilde{R}'_{j\bar{k}l} = R'_{j\bar{k}l} + i f_{j\bar{k}} f_l \langle \xi, \xi \rangle' \xi^p$, one has

$$R(X, \bar{Y})W = x^j y^{\bar{k}} w' \tilde{R}'_{j\bar{k}l} \frac{\partial}{\partial z^p},$$

where

$$\tilde{R}'_{j\bar{k}l} f_p \equiv 0 \pmod{f_j, f_{\bar{k}}}. \tag{25}$$

Now

$$\begin{aligned} \langle R_*(Y), \bar{Y} \rangle &= \sum_{\alpha} \langle R(Y, \bar{Y})X_{\alpha}, \bar{X}_{\alpha} \rangle \text{ by (20)'} \\ &= y^j \bar{y}^{\bar{k}} x_{\alpha}^i \tilde{R}'_{j\bar{k}l} f_{p\bar{m}} \bar{x}_{\alpha}^{\bar{m}} = y^j \bar{y}^{\bar{k}} \tilde{R}'_{j\bar{k}l} f_{p\bar{m}} a^{\bar{m}i} \\ &= y^j \bar{y}^{\bar{k}} \tilde{R}'_{j\bar{k}l} \text{ by (16) and (25)} \\ &= y^j \bar{y}^{\bar{k}} \left(\frac{\partial \Gamma'_{kl}}{\partial z^j} - i f_{j\bar{k}} \frac{\partial \xi^i}{\partial z^i} - f_{j\bar{k}} \langle \xi, \xi \rangle' \right) \end{aligned}$$

using (a) and Proposition 3. One finishes the proof by noting that $\partial \Gamma'_{kl} / \partial z^j \equiv \langle \xi, \xi \rangle' f_{j\bar{k}} \pmod{f_{\bar{k}}}$ by (14)'.

$-i \partial \xi^i / \partial z^i$ is the function λ referred to in the introduction.

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