FLOWS ON FIBRE BUNDLES

BY

J. L. NOAKES

Abstract. Conditions are given under which a fibrewise flow on a fibre bundle must have a nonempty catastrophe space.

1. The problem. When we formulate the catastrophe theory of R. Thom globally we have a fibre bundle $E$ over a connected finite CW-complex $B$. The fibre $M$ of $E$ is a closed $C^r$ manifold, and the structure group of $E$ is a subgroup of the group $\text{Diff } M$ of $C^r$ diffeomorphisms $M \to M$ with the $C^r$ topology ($r > 1$). We say that $E$ is a $C^r$ bundle for short. Then $B$ is the space of observables and $M$ is the manifold of internal variables. Let $\text{Vect } M$ be the space of $C^{-1}$ vector fields on $M$ with the $C_1$ topology. We define an action $\cdot$ of $\text{Diff } M$ on $\text{Vect } M$ by means of the identity $f \cdot V = (df)Vf^{-1}$ where $f \in \text{Diff } M$, $V \in \text{Vect } M$. In catastrophe theory the bundle with fibre $\text{Vect } M$ associated with $E$ has a cross-section. We think of this cross-section as a family $V_b (b \in B)$ of fibrewise $C^{-1}$ vector fields on $E$.

We next define an attractor of a vector field. The definition in [5, §4.1] seems to be imprecise and we use the following definition instead.

Definition. An attractor of $V \in \text{Vect } M$ is a closed invariant subspace $A$ of $M$ such that

(i) there is an invariant neighbourhood $U$ of $A$ for which $\cap_{t \geq 0} \phi_V U = A$,
(ii) some trajectory of $V$ is dense on $A$ (here $\phi_V$ is the flow on $M$ corresponding to $V$).

Perhaps an attractor ought to satisfy additional conditions but these would not affect our main results.

We suppose that we are given a convention, namely an assignment to each $b \in B$ of an attractor $A_b$ of $V_b$. We think of a convention as a physical law, and of $A_b$ as the physical state of $B$. Then $b \in B$ is said to be regular when it has a neighbourhood $W$ for which there is a fibre-preserving homeomorphism $h: E|W \to W \times E_b$ onto the trivial bundle satisfying

(i) $h|E_b$ is the identity,
(ii) $h|A_c = A_b$ and $h\phi_V = \phi_V h$ for all $c \in W$.

The points that are not regular make up the catastrophe subspace $K$ of $B$.

The purpose of this paper is to study the following problem. Suppose that we are given $E$ and a closed connected subspace $F$ of $M$. Then we wish to decide whether $K$ can be empty with $A_0 = F$, that is to say whether there is a family $V_b (b \in B)$ of fibrewise $C^{-1}$ vector fields on $E$ and a convention such that $K$ is empty and $A_0 = F$. Here $0$ is the basepoint of $B$.

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If \( K \) can be empty with \( A_0 \) a point then the assignment \( b \mapsto A_b \) defines a cross-section of \( E \). Conversely let \( s \) be a cross-section of \( E \). Then \( K \) can be empty with \( A_0 \) a point. To prove this we argue as follows. The tangent bundles \( T E_b \) of the manifolds \( E_b \) \((b \in B)\) make up a vector bundle \( TFE \) over \( E \). Choose a Riemannian metric on \( TFE \) and let \( T \) be \( TFE|s(B) \). Then \( T \) is a Riemannian vector bundle over \( B \) and the \( T_b \) are the tangent spaces at \( s(b) \) to the manifolds \( E_b \) \((b \in B) \). Using the compactness of \( B \) we choose \( \delta > 0 \) so that the fibrewise exponential map \( e: T \to E \) maps the open disc bundle \( B_\delta \) of radius \( \delta \) homeomorphically into \( E \). Then \( N = e(B_\delta) \) is a neighbourhood of \( s(B) \) in \( E \). We identify \( T \) with its own fibrewise tangent bundle and for \( e \in E_b \) we define \( V_b(e) \) to be either 0 or 
\[
(−ds_0(t))\exp(−\sec \theta)
\]
accordingly as \( e \not\in N \) or \( e = e(t) \) for \( t \in B_\delta \). Here \( \theta = \pi \|t\|/2\delta \). Taking \( A_b = \{s(b)\} \) we see that \( K \) is empty.

In general when \( K \) is empty the \( A_b \) make up a fibre bundle \( A \) over \( B \). In some cases \( A \) is a \( C^r \) bundle and there is a fibrewise \( C^r \) embedding \( f: A \to E \). By this we mean that \( f \) is a fibre-preserving map such that

(i) \( f_b: A_b \to E_b \) is a \( C^r \) embedding.

(ii) \( f_b \) varies continuously in the \( C^r \) topology with \( b \in B \).

In §§2 and 3 we prove conditions necessary for the existence of fibrewise \( C^r \) embeddings, and in §4 we apply these results to our original problem. Our main results assert that under certain conditions the catastrophe space \( K \) must be nonempty.

2. A related problem. Let \( E, A \) be \( C^r \) bundles with fibres closed \( C^r \) manifolds \( M, F \) over a connected finite CW-complex \( B \) \((r > 1)\).

**Lemma 1.** Let \( f: A \to E \) be a fibrewise \( C^r \) embedding. Then the complement \( E - f(A) \) is a \( C^r \) bundle over \( B \).

This requires only a local proof, and so we suppose that \( E = B \times M \) and that there is a \( C^r \) trivialization \( g: B \times F \to A \). Then it suffices to extend \( fg \) for each \( b \in B \) over some neighbourhood \( U \) of \( b \) to a \( C^r \) trivialization \( h: U \times M \to E|U = U \times M \). But this can be done because of the result, due to R. S. Palais, that the evaluation map on spaces of \( C^r \) embeddings is locally trivial. We refer to [2] for a short proof of Palais' theorem.

Two fibrewise \( C^r \) embeddings \( f_0, f_1: A \to E \) are said to be *isotopic* when there is a fibrewise \( C^r \) embedding \( F: A \times [0, 1] \to E \times [0, 1] \) over \( B \times [0, 1] \) such that

\[
F|A \times \{0\} = f_0, \quad F|A \times \{1\} = f_1.
\]

Applying Lemma 1 to \( F \) and using [3, 11.4] we have the following lemma.

**Lemma 2.** Let \( f_0, f_1: A \to E \) be isotopic fibrewise \( C^r \) embeddings. Then the \( C^r \) bundles \( E - f_0(A) \), \( E - f_1(A) \) are equivalent.

Let \( E, A \) be orthogonal sphere bundles of fibre dimensions \( q > p > 1 \) and let \( f: A \to E \) be a fibrewise \( C^r \) embedding. We suppose that \( f \) is orthogonal when
restricted to the fibre over the basepoint 0 so that, in particular, the \( f_0: A_b \to E_b \) are unknotted when \( q = p + 2 \).

**Problem.** (i) Is there a fibrewise orthogonal embedding of \( A \) in \( E \)?

(ii) If so then is \( f \) isotopic to a fibrewise orthogonal embedding?

An affirmative answer to (i) would mean that \( E \) was the fibre join of \( A \) with an orthogonal \( q - p - 1 \)-sphere bundle. The Whitney duality theorem would then give conditions on the Stiefel-Whitney classes of \( E \) necessary for the existence of \( f \).

**Remark.** If \( A \) has a cross-section \( s \) (for example if \( A \) is an oriented circle bundle and \( H^2(B; \mathbb{Z}) = 0 \) then the \( df(TFA_0) \) define an orthogonal subbundle of \( E \). We can identify this subbundle with \( A \) and so there is a fibrewise orthogonal embedding of \( A \) in \( E \).

When \( A \) does not have a cross-section we can still obtain a condition necessary for the existence of \( f \) by pulling everything back over the principal bundle associated with \( A \). We can then apply our remark, together with the Whitney duality theorem. However we shall do better than this.

Let \( W \) be the path component of the orthogonal embeddings in the space of \( C \) embeddings of \( S^p \) in \( S^q \) with the \( C' \) topology. Note that any two orthogonal embeddings of \( S^p \) in \( S^q \) are isotopic, since \( p < q \). We define an action \( \cdot \) of the direct product of the orthogonal groups \( O(p + 1) \times O(q + 1) \) on \( W \) by means of the identity

\[
((P, Q) \cdot g)(x) = Qg(P^{-1}x)
\]

where \( (P, Q) \in O(p + 1) \times O(q + 1) \), \( g \in W \), \( x \in S^p \).

Let \( L \) be the fibre product of the principal bundles associated with \( A, E \). Let \( D \) be the bundle associated with \( L \) and with fibre \( W \). Then \( f \) corresponds to a cross-section of \( D \). The Stiefel manifold \( V_{q+1,p+1} \) is the \( O(p + 1) \times O(q + 1) \)-invariant subspace \( V \) of \( W \) consisting of the orthogonal embeddings. Let \( C \) be the bundle associated with \( L \) and with fibre \( V \). Then the inclusion \( j \) of \( V \) in \( W \) extends to a fibre-preserving map from \( C \) to \( D \).

Taking derivatives at the basepoint of \( S^p \) defines a retraction of \( W \) onto \( V \) and so \( j_*: \pi_* V \to \pi_* W \) is the inclusion of a direct summand. By [4, Proposition 2] (see also [7]), \( j_* \) is surjective when \( k < 2q - 4p - 3 \). We use this to prove the following lemma.

**Lemma 3.** (i) If \( \dim B < 2q - 4p - 2 \) then there is a fibrewise orthogonal embedding of \( A \) in \( E \).

(ii) If \( \dim B < 2q - 4p - 3 \) then \( f \) is isotopic to a fibrewise orthogonal embedding.

For the proof let \( s \) be the cross-section of \( D \) corresponding to \( f \) and note that, since \( j_* \) is an isomorphism when \( k < 2q - 4p - 3 \), the vertical homotopy class of \( s| B^{2q-4p-3} \) comes from a cross-section \( s \) of \( C| B^{2q-4p-3} \). This proves (ii). To prove (i) let \( \theta_1 \in H^{2q-4p-2}(B; \pi_{2q-4p-3} V) \) be the obstruction to extending \( s| B^{2q-4p-4} \) to a cross-section of \( C| B^{2q-4p-2} \). Then \( j_* \theta_1 \in H^{2q-4p-2}(B; \pi_{2q-4p-3} W) \) is the obstruction to extending \( s| B^{2q-4p-4} \). \( B^{2q-4p-4} \to C \to D \) to a cross-section of \( D| B^{2q-4p-2} \). Since this obstruction is zero so also is \( \theta_1 \), and \( s| B^{2q-4p-4} \) extends to a cross-section of \( C| B^{2q-4p-2} \). But cross-sections of \( C \) correspond bijectively and
naturally to fibrewise orthogonal embeddings of $A$ in $E$. This proves Lemma 3.

3. The general case. For the applications in §4 we take $p = 1$, and the most interesting values of $q$ are probably 2, 3, 4. When $p = 1$ Lemma 3(i) holds trivially for these values of $q$, without the hypothesis that there is a fibrewise $C'$ embedding of $A$ in $E$. We therefore need a more general result.

We continue to work in the context of §2 and include $E$ in its fibre suspension $\Sigma E$ in the usual way. Following $f$ by $j$ such inclusions ($j > 1$) we obtain a fibrewise $C'$ embedding $f'$ of $A$ in $E' = \Sigma^j E$.

**Lemma 1.** The inclusion of $\Sigma^j (E - f(A))$ in $E' - f'(A)$ is a fibre homotopy equivalence.

By §2, Lemma 1 both $\Sigma^j (E - f(A))$ and $E' - f'(A)$ are fibre bundles. Therefore, by a result due to Dold [1], it suffices to prove that the inclusion is a homotopy equivalence over the basepoint. But the inclusion $S(X - Y) \to SX - Y$ is a homotopy equivalence whenever $(X, Y)$ is a polyhedral pair with $X, Y$ compact. Iterating this $j$ times with $X = M, Y = F$ we have Lemma 1.

**Lemma 2.** Let $S^p$ be embedded orthogonally in $S^q$ and let $S^{q-p-1}$ be the $q - p - 1$-sphere orthogonal to $S^p$. Then the inclusion of $S^{q-p-1}$ in $S^q - S^p$ is a homotopy equivalence.

One proof is by induction on $q$, beginning at $q = p + 1$ and arguing as in the proof of Lemma 1.

**Lemma 3.** Suppose that $j > [(\dim B + 1)/2] + 2p - q + 2$. Then $E' = \Sigma^j E$ is the fibre join of $A$ with a bundle $E''$ which is fibre homotopy equivalent to the $j$-fold fibre suspension of a homotopy $q - p - 1$-sphere bundle.

To prove the lemma we apply §2, Lemma 3(ii) to the fibrewise $C'$ embedding $f'$ of $A$ in $E'$. Thus $f'$ is isotopic to a fibrewise orthogonal embedding $f''$. Let $E''$ be the orthogonal complement to $f''(A)$ in $E'$. Then $E'$ is equivalent to the fibre join of $A$ with $E''$. By Lemma 2 the inclusion of $E''$ in $E' - f''(A)$ is a fibre homotopy equivalence. But $E' - f''(A)$ is equivalent to $E' - f'(A)$ by §2, Lemma 2, and $E' - f'(A)$ is fibre homotopy equivalent to $\Sigma^j (E - f(A))$ by Lemma 1. But $E' - f(A)$ is a homotopy $q - p - 1$-sphere bundle by Lemma 2. This completes the proof.

When $q = p + 1$ we also have the following result.

**Lemma 4.** Let $q = p + 1$. Then $E$ is the fibre join of a 0-sphere bundle with an orthogonal $p$-sphere bundle $A^*$ which is fibre homotopy equivalent to $A$.

To prove this note, by §2, Lemma 1, the complement $E - f(A)$ is a $C'$ bundle over $B$ whose fibre is the disjoint union of two copies of an open $q$-disc. Since the $q$-disc is contractible we may continuously assign to each $b \in B$ a subset $\{y_1, y_2\}$ of $E_b - f(A_b)$ so that $y_1, y_2$ lie in different path components. Let $A_b^*$ be the $p$-sphere orthogonal to a geodesic joining $y_1, y_2$. Let $E_b^*$ be the 0-sphere orthogonal to $A_b^*$ in $E_b$. Then the $A_b^*$, $E_b^*$ make up an orthogonal $p$-sphere bundle $A^*$ and a 0-sphere.
bundle $E^*$. But $E$ is the fibre join of $A^*$ with $E^*$. To complete the proof we note that both $A^*$ and $A$ are fibre homotopy equivalent to the bundle $E = \bigcup_{b \in B} \{ y_1, y_2 \}$.

4. Applications. We resume the discussion of §1. For $q \geq 1$ let $E$ be an orthogonal $q$-sphere bundle over a connected finite CW-complex $B$. Let $A_0 = F$ be a circle embedded orthogonally in $M = S^q$. If $K$ is empty then the attractors $A_b$ make up a circle bundle $A'$ over $B$. From the definition of an attractor each $V_b|A_b$ has at most one zero. Therefore, and since all points are regular, $A'$ is oriented. If $q = 1$ this means that $E$ is orientable. Evidently the converse also holds. If $q = 1$ and $E$ is orientable then $K$ can be empty. From now on we suppose that $q \geq 2$.

If $K$ is empty and $V_0|A_0$ has a zero then, by regularity, so has each other $V_b|A_b$ and the zeros define a cross-section of $A'$. Therefore $A'$ is trivial, and so $E$ has two orthogonal cross-sections. It follows that the trivial circle bundle $A'$ embeds fibrewise orthogonally in $E$.

If $K$ is empty and $V_0|A_0$ is never zero then, by regularity, so is each other $V_b|A_b$. Therefore the flow $\phi_{V_b}$ defines a $C^r$ embedding of $S^1$ in $E_b$ whose image is $A_b$ and whose derivative maps the clockwise unit tangent field on $S^1$ to $V_b|A_b$. Such embeddings are unique up to rotations of $S^1$. Therefore $A'$ is the image by a fibrewise $C^r$ embedding of an oriented orthogonal circle bundle $A$. Summarizing, we have the first part of the following lemma.

**Lemma 1.** (i) If $K$ can be empty then there is a fibrewise $C^r$ embedding of an oriented orthogonal bundle $A$ in $E$.
(ii) The converse also holds. We prove this in an appendix.

Comparing Lemma 1(i) with §3, Lemma 4 we have the following result.

**Theorem 1.** Let $q = 2$. If $K$ can be empty then $E$ is the fibre join of an oriented orthogonal circle bundle with a 0-sphere bundle. (Of course the converse is contained in Lemma 1(ii).)

The next result is a consequence of Lemma 1(i) and §2, Lemma 3(i).

**Theorem 2.** Suppose that $\dim B < 2q - 6$. If $K$ can be empty then $E$ is the fibre join of an oriented orthogonal circle bundle with an orthogonal $q - 2$-sphere bundle. (Of course the converse is contained in Lemma 1(ii).)
Corollary. Let $E$ be orientable.
(i) If $q > 6$ and if $K$ can be empty then for some $a \in H^2(B; \mathbb{Z})$ the sum
\[ W_q + aW_{q-2} + a^2W_{q-4} + \cdots \]
is zero.
(ii) If $\dim B < q$ and if for some $a$ the sum in (i) is zero then $K$ can be empty.

Here $W_i$ is the $i$th Stiefel-Whitney class of $F$ and we work in $H^*B$ with coefficients $\mathbb{Z}$ or $\mathbb{Z}/2\mathbb{Z}$ accordingly as $q$ is odd or even. To prove (i) we apply the Whitney duality theorem with Theorem 2. To prove (ii) we apply [6, Proposition 2.1] together with Lemma 1(ii).

Theorem 3. Let $q = 3, 4, 5$ and let $j = [(\dim B + 1)/2] - q + 4$. If $K$ can be empty then $\Sigma E$ is the fibre join of an oriented orthogonal circle bundle with a bundle which is fibre homotopy equivalent to the $j$-fold fibre suspension of a homotopy $q - 2$-sphere bundle. (Of course a converse is contained in Lemma 1(iii).)

To prove Theorem 3 we compare Lemma 1(i) with §3, Lemma 3.

Corollary. Let $q = 3, 4, 5$. If $K$ can be empty then for some $a \in H^2(B; \mathbb{Z})$ the mod 2 reduction of the sum in the first part of the corollary to Theorem 2 is zero. (Of course a converse is contained in the second part of the corollary to Theorem 2.)

To prove this we apply the mod 2 Whitney duality theorem with Theorem 3.

Examples. (i) If $B$ is a sphere then $K$ can be empty if and only if $E$ has two orthogonal cross-sections. This follows from Lemma 1 together with the remark in §2.

(ii) Let $B$ be a closed oriented 4-manifold such that for all $a \in H^2(B; \mathbb{Z})$ the mod 2 reduction of $a^2$ is zero. For instance $S^2 \times S^2$ would do. Let $g : B \to S^4$ be a degree 1 map. Let $H$ be the Hopf 3-sphere bundle $S^7$ over $S^4$. Then let $E$ be the pullback $g^*\Sigma H$ of the fibre suspension $\Sigma H$. We have $W_0 = 1, W_1 = W_2 = W_3 = 0$ and $W_4 \neq 0$. Therefore by the corollary to Theorem 3 the catastrophe space $K$ must be nonempty.

There remains only an appendix for the proof of Lemma 1(ii). I wish to thank Dr. A. du Plessis and Professor M. G. Barratt for their helpful comments. I am especially grateful to Professor Barratt for his hospitality during the winter of 1976-77.

Appendix. Let $E$ be a $C^r$ bundle with fibre a closed $C^r$ manifold $M (r > 1)$ over a connected finite CW-complex $B$. Let $A'$ be an oriented orthogonal circle bundle over $B$. The purpose of this appendix is to prove the following result.

Proposition. If there is a fibrewise $C^r$ embedding of $A'$ in $E$ then $K$ can be empty with $A_0 = A'_0$.

The proof is modelled on that of the corresponding result in §1 for point attractors. As in §1 we form the vector bundles $TFE$, $TFA'$ over $E, A'$. We identify $A'$ with the image in $E$. Then the line bundle $TFA'$ is a subbundle of $TFE|A'$. We choose a Riemannian metric on $TFE|A'$. 

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If \( \text{dim } M = 1 \) then \( M \) is a disjoint union of circles and the proposition holds trivially. If \( \text{dim } M > 1 \) we define \( \mathcal{N}^\prime \) to be the orthogonal complement to \( \mathcal{T} \mathcal{A}' \) in \( \mathcal{T} \mathcal{F} \mathcal{E} \mathcal{A}' \). Then \( \mathcal{N}^\prime \) is a Riemannian vector bundle over \( \mathcal{A}' \). Using the compactness of \( B \) we choose \( \delta > 0 \) so that the fibrewise exponential map \( e: \mathcal{N}^\prime \rightarrow E \) maps the open disc bundle \( B_\delta \) of radius \( \delta \) homeomorphically into \( E \). Then \( N = e(B_\delta) \) is a neighbourhood of \( \mathcal{A}' \) in \( E \).

We identify \( \mathcal{N}^\prime \) with its own fibrewise tangent bundle and define a family \( R_b \) \((b \in B)\) of fibrewise \( C^{r-1} \) vector fields on \( N \) by means of the identity \( R_b(e) = -de_b(t) \) where \( e \in E_b \) and \( e = e(t) \) for \( t \in B_\delta \). The orientation defines a family of fibrewise \( C^{r-1} \) unit vector fields on \( \mathcal{A}' \). We extend this to a family \( C_b \) \((b \in B)\) of fibrewise \( C^{r-1} \) vector fields on \( N \). Then for \( e \in E_b \) we define \( V_b(e) \) to be either 0 or

\[
C_b(e) + R_b(e) \exp(-\sec \theta)
\]

accordingly as \( e \notin N \) or \( e = e(t) \in N \) for \( t \in B_\delta \). Here \( \theta = \pi/2\delta \). Taking \( A_b = A_b' \) we see that \( K \) is empty. This completes the proof.

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DEPARTMENT OF MATHEMATICS, UNIVERSITY OF WESTERN AUSTRALIA, NEDLANDS, WESTERN AUSTRALIA 6009, AUSTRALIA

Current address: Department of Mathematics, Massachusetts Institute of Technology, Cambridge, Massachusetts 02139