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NONSTANDARD EXTENSIONS OF TRANSFORMATIONS BETWEEN BANACH SPACES

BY

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ABSTRACT. Let X and Y be (infinite-dimensional) Banach spaces and denote their nonstandard hulls with respect to an $\aleph_1$-saturated enlargement by $\hat{X}$ and $\hat{Y}$ respectively. If $\mathfrak{B}(X, Y)$ denotes the space of bounded linear transformations then a subset S of elements of $\mathfrak{B}(X, Y)$ extends naturally to a subset $\hat{S}$ of $\mathfrak{B}(\hat{X}, \hat{Y})$. This paper studies the behaviour of various kinds of transformations under this extension and introduces, in this context, the concepts of super weakly compact, super strictly singular and socially compact operators. It shows that $(\mathfrak{B}(X, Y))\hat{} \subsetneq \mathfrak{B}(\hat{X}, \hat{Y})$ provided X and Y are infinite dimensional and contrasts this with the inclusion $\mathfrak{K}(\hat{H}) \subsetneq (\mathfrak{K}(H))\hat{}$ where $\mathfrak{K}(H)$ denotes the space of compact operators on a Hilbert space.

The nonstandard hull of a uniform space was introduced by Luxemburg [14] and the construction, in the case of normed spaces, has since been investigated in a number of papers (see, for example [4], [6], [7], [8], [9], [10], [15]). Our intention here is to examine the behaviour of single bounded linear transformations and classes of bounded linear transformations under the nonstandard hull extension. We recount the principal ideas.

Consider a normed space $(X, \|\cdot\|)$ embedded in a set theoretical structure $\mathfrak{M}$ and consider an $\aleph_1$-saturated enlargement $^*\mathfrak{M}$ of $\mathfrak{M}$. An element $p \in {}^*X$ is said to be *finite* if $^*\|p\|$ is a finite element of $^*\mathbf{R}$, the set of such elements being denoted by $\operatorname{fin}(^*X)$. The set of infinitesimal elements of $^*X$, i.e., the set of $p \in {}^*X$ such that $^*\|p\|$ is an infinitesimal element of $^*\mathbf{R}$, is termed the *monad* of zero and is denoted by $\mu(0)$. Both $\operatorname{fin}(^*X)$ and $\mu(0)$ form vector spaces over the scalar field of X and the quotient space $\operatorname{fin}(^*X)/\mu(0)$ becomes a normed space if the norm of $\hat{p}$ is defined to be $\operatorname{st}(^*\|p\|)$. Here $\hat{p}$ denotes the image of p under the canonical quotient map and st denotes the standard part operator on $^*\mathbf{R}$. This new space $(X, \|\cdot\|)$, which contains X, is called the *nonstandard hull* of $(X, \|\cdot\|)$ with respect to the nonstandard model $^*\mathfrak{M}$. The nonstandard hull is complete under the assumption that $\mathfrak{M}$ is $\aleph_1$-saturated [14], and $\hat{X}$ is equal to X if and only if X is finite dimensional. If H is a Hilbert space then so is $\hat{H}$ with respect to the extended inner product

$$\langle \hat{p}, \hat{q} \rangle = \operatorname{st}(\langle p, q \rangle) \quad \text{where } p, q \in \operatorname{fin}(^*H).$$

For Banach spaces X and Y we denote the space of (bounded linear) transformations from X to Y by $\mathfrak{B}(X, Y)$, abbreviating $\mathfrak{B}(X, Y)$ to $\mathfrak{B}(X)$ if $X = Y$. An

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element $T \in \text{fin}\,{}^*\mathfrak{B}(X, Y)$ extends to an element $\hat{T} \in \mathfrak{B}(\hat{X}, \hat{Y})$ via the equation $\hat{T}(\hat{p}) = (T(p))\hat{}$ where $p \in \text{fin}({}^*X)$. If $\mathfrak{S}$ is a set of bounded linear transformations we can form a new set $\hat{\mathfrak{S}}$ as above. This set $\hat{\mathfrak{S}}$ is embedded in $\mathfrak{B}(\hat{X}, \hat{Y})$ under the action $\hat{T}(\hat{p}) = (T(p))\hat{}$ where $T \in \text{fin}\,{}^*S$ and $p \in \text{fin}\,{}^*X$. We assume for definiteness in what follows that all spaces are complex and we denote the conjugate space and the closed ball of radius r of X by X' and X_r respectively. We draw attention to the following basic result of Henson and Moore [**8**, Theorem 8.5] which we use frequently.

If X is a Banach space then the following conditions are equivalent.

(i) *X is superreflexive.*

(ii) *$\hat{X}$ is reflexive.*

(iii) $(\hat{X})' = (X')\hat{}$.

Recall that in the above a space X is *superreflexive* [**12**] if any space finitely representable in X is reflexive. For convenience we list the definitions of the following types of linear transformations.

(i) The set of *finite rank* transformations, $\mathfrak{F}(X, Y)$, consists of those elements T for which $T(X)$ is finite dimensional.

(ii) The set of *compact* transformations, $\mathfrak{K}(X, Y)$, consists of those elements T for which $T(X_1)$ has compact closure in Y.

(iii) The set of *weakly compact* transformations $\mathfrak{W}\mathfrak{K}(X, Y)$ consists of those elements T for which $T(X_1)$ has weakly compact closure in Y.

(iv) The set of *strictly singular* transformations, $\mathfrak{S}(X, Y)$, consists of those elements T for which the restriction of T to any infinite dimensional subspace of X is not an isomorphism.

(v) The set of *Fredholm* transformations, $\Phi(X, Y)$, consists of those elements T which have closed range, and for which $\alpha(T) = \dim(\ker T)$ and $\beta(T) = \dim(Y/T(X))$ are finite. The *index* $i(T)$ of such a transformation is then $\alpha(T) - \beta(T)$. The sets of semi-Fredholm transformations are defined by

$$\Phi_+(X, Y) = \{T \in \mathfrak{B}(X, Y) | T(X) \text{ is closed and } \alpha(T) < \infty\}$$

and

$$\Phi_-(X, Y) = \{T \in \mathfrak{B}(X, Y) | T(X) \text{ is closed and } \beta(T) < \infty\}.$$

(vi) The set of Riesz operators, $\mathfrak{R}(X)$, consists of those elements $T \in \mathfrak{B}(X)$ for which, for every complex nonzero λ, $\lambda I - T$ has finite ascent and descent, $\ker(\lambda I - T)^k$ is finite dimensional for $k = 1, 2, 3, \ldots$, $(\lambda I - T)^k(X)$ is closed with finite codimension for $k = 1, 2, 3, \ldots$, and the nonzero points of the spectrum of T consists of eigenvalues of T with zero as the only possible limit point (see [**3**]).

(vii) For $p \geqslant 1$, the *p-absolutely summing* transformations, $\mathfrak{P}^{(p)}(X, Y)$, consists of those transformations T for which there is a constant M with the property that, for every choice of integer n and vectors $x_1, x_2, \ldots, x_n$ in X,

$$\left(\sum_{i=1}^{n} \|Tx_i\|^p\right)^{1/p} \leqslant M \sup\left\{\left(\sum_{i=1}^{n} |f(x_i)|^p\right)^{1/p} : f \in X' \text{ with } \|f\| \leqslant 1\right\}.$$

(viii) The set of *nuclear* transformations, $\mathfrak{N}(X, Y)$, consists of those T for which there exists a sequence of elements $f_n \in X'$ and a sequence of elements $y_n \in Y$ with $\Sigma\|f_n\|\,\|y_n\| < \infty$ such that $T(x) = \Sigma f_n(x)y_n$ for $x \in X$. T belongs to the set of *quasinuclear* transformations, $\mathfrak{N}_0(X, Y)$, if there is a sequence of elements $f_n \in X'$ with $\Sigma_{n=1}^{\infty}\|f_n\| < \infty$ such that $\|Tx\| \leqslant \Sigma_{n=1}^{\infty}|f_n(x)|$ for $x \in X$.

(ix) If H and K are Hilbert spaces then T is a *Hilbert-Schmidt* operator, written $T \in \mathcal{HS}(H, K)$, if for two complete orthonormal families $\{e_i\colon i \in I\}$ in H and $\{f_j\colon j \in J\}$ in K the inequality $\Sigma_{i,j}|\langle Te_i, f_j\rangle|^2 < \infty$ holds.

(x) If H is a Hilbert space, T belongs to the family of *trace-class* operators $\mathfrak{T}(H)$, if T can be written as a product, AB, of Hilbert-Schmidt operators A and B.

It is well known (see [15]) that an element $T \in \mathfrak{B}(X, Y)$ is compact if and only if $\hat{T} \in \mathfrak{B}(\hat{X}, \hat{Y})$ is compact. It is also easy to check that the analogous result holds for finite rank transformations. When considering the behaviour of weakly compact transformations the earlier stated result of Henson and Moore suggests the following definition and theorem. We say $T \in \mathfrak{B}(X, Y)$ is *super weakly compact* if for all positive reals r there exists a positive integer n such that there do not exist finite sequences $\{x_1, x_2, \ldots, x_n\}$ in X and $\{f_1, f_2, \ldots, f_n\}$ in Y' with $\|x_i\| = \|f_i\| = 1$ $(i = 1, 2, \ldots, n)$, $\mathrm{Re}(f_j(Tx_i)) > r$ $(1 \leqslant j \leqslant i \leqslant n)$ and $f_j(Tx_i) = 0$ $(1 \leqslant i < j \leqslant n)$.

THEOREM 1. *Suppose X and Y are Banach spaces and let $T \in \mathfrak{B}(X, Y)$. Then the following four conditions are equivalent.*

(i) *T is super weakly compact.*

(ii) *$\hat{T}$ is super weakly compact.*

(iii) $\hat{T} \in \mathcal{WH}(\hat{X}, \hat{Y})$.

(iv) $\hat{T}'((\hat{Y})') \subseteq (X')\hat{}$.

PROOF. We establish the chain of implications (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) $\Rightarrow$ (iv) $\Rightarrow$ (i). (i) *implies* (ii). Suppose (ii) does not hold. Then there is an $r \in \mathbf{R}^+$ such that for all $n \in \mathbf{N}$ there exist finite sequences $\{p_1, p_2, \ldots, p_n\}$ in ${}^*X$ and $\{\phi_1, \phi_2, \ldots, \phi_n\}$ in $(\hat{Y})'$ such that $\|p_i\| = 1$ and $\|\phi_i\| < 1$ $(i = 1, 2, \ldots, n)$, $\mathrm{Re}(\phi_j(\hat{T}\hat{p}_i)) > r$ $(1 \leqslant j \leqslant i \leqslant n)$ and $\phi_j(\hat{T}\hat{p}_i) = 0$ $(1 \leqslant i < j \leqslant n)$. But it follows easily [8, Lemma 8.2] that for each $j = 1, 2, \ldots, n$ there is an $f_j \in {}^*Y'$ with $\|f_j\| < 1$ such that $f_j(Tp_i) = \phi_j(\hat{T}\hat{p}_i)$ for $i = 1, 2, \ldots, n$. We thus have $\mathrm{Re}(f_j(Tp_i)) > r$ $(1 \leqslant j \leqslant i \leqslant n)$ and $f_j(Tp_i) = 0$ $(1 \leqslant i < j \leqslant n)$, and so by invoking the transfer principle (for the given r and arbitrary fixed n) it follows that (i) fails.

(ii) *implies* (iii). If (iii) does not hold then by a theorem of James [11, Theorem 3] there is a positive real r for which there exist sequences $\{x_i\}$ and $\{f_i\}$ of elements of norm 1 in X and Y' respectively such that $\mathrm{Re}(f_j(Tx_i)) > r$ for $j \leqslant i$ and $f_j(Tx_i) = 0$ for $j > i$. Thus (ii) does not hold.

(iii) *implies* (iv). Since $\hat{T}$ is weakly compact the adjoint map $\hat{T}'\colon (\hat{Y})' \to (\hat{X})'$ is continuous from the weak* to the weak topology [5, p. 484]. But $(Y')\hat{}$ is weak* dense in $(\hat{Y})'$ [8, Lemma 8.2] and so $\hat{T}'((Y')\hat{})$ is dense in $\hat{T}'((\hat{Y})')$ in the weak topology, and thus in the norm toplology. Therefore $\hat{T}'((\hat{Y})')$ is contained in the norm closure of $\hat{T}'((Y')\hat{})$, and so is in $(X')\hat{}$ as claimed.

(iv) *implies* (i). We suppose (i) is false whilst (iv) holds. Then there is a positive real r such that "for each positive integer ω there exist finite sequences $\{p_n: 1 \leqslant n \leqslant \omega\}$ and $\{f_n: 1 \leqslant n \leqslant \omega\}$ in X and Y' respectively with $\|p_n\| = \|f_n\| = 1$ $(1 \leqslant n \leqslant \omega)$, $\mathrm{Re}(f_j(Tp_i)) > r$ $(1 \leqslant j \leqslant i \leqslant \omega)$ and $f_j(Tp_i) = 0$ $(1 \leqslant i < j \leqslant \omega)$". Since the sentence in quotation marks holds in $\mathfrak{M}$ it holds in $^*\mathfrak{M}$ and therefore we can assume that $\omega \in {}^*\mathbf{N} - \mathbf{N}$ etc. Then by the Banach-Alaoglu theorem the sequence $\{\hat{f}_j: j = 1, 2, 3, \ldots\}$ has a weak* limit point ϕ. We have $(\hat{T}'\phi)(\hat{p}_i) = \phi(\hat{T}\hat{p}_i) = 0$ $(i = 1, 2, 3, \ldots)$ and $\mathrm{Re}((\hat{T}'\phi)(\hat{p}_i)) \geqslant r$ $(i \in {}^*\mathbf{N} - \mathbf{N}$ and $i \leqslant \omega)$. But by (iv), $\hat{T}'\phi = \hat{f}$ for some $f \in {}^*X'$ and it thus follows that the set $\mathbf{N} = \{n: 1 \leqslant n \leqslant \omega$ and $\mathrm{Re}(f(p)) < r/2\}$ is internal. This gives the contradiction and completes the proof of the theorem.

We note the basic properties of super weakly compact transformations as a corollary to this theorem. Note first that if I is the identity operator on a nonsuperreflexive, reflexive space then I is weakly compact but not super weakly compact.

COROLLARY 1. (i) *If either X or Y is superreflexive then every $T \in \mathfrak{B}(X, Y)$ is super weakly compact.*

(ii) *The set of super weakly compact transformations is closed in $\mathfrak{B}(X, Y)$.*

(iii) *Linear combinations of super weakly compact transformations are super weakly compact. The product of a super weakly compact transformation and a bounded linear transformation is super weakly compact.*

(iv) *An element $T \in \mathfrak{B}(X, Y)$ is super weakly compact if and only if its adjoint $T' \in \mathfrak{B}(Y', X')$ is super weakly compact.*

PROOF. (i) This is an immediate consequence of the Henson-Moore result and the corresponding result for weakly compact transformations [**5**, p. 483].

(ii) If $T_n \to T$ in $\mathfrak{B}(X, Y)$ then $\hat{T}_n \to \hat{T}$ in $\mathfrak{B}(\hat{X}, \hat{Y})$ and it follows [**5**, p. 483] that $\hat{T}$ is weakly compact if each $\hat{T}_n$ is.

(iii) Again this follows from the corresponding result for weakly compact transformations [**5**, p. 485].

(iv) Suppose T is super weakly compact, so that $\hat{T}: \hat{X} \to \hat{Y}$ is weakly compact. Then, by a theorem of Gantmacher [**5**, p. 486], $\hat{T}': (\hat{Y})' \to (\hat{X})'$ is weakly compact. But $(Y')\hat{}$ and $(X')\hat{}$ are closed subspaces of $(\hat{Y})'$ and $(\hat{X})'$ respectively, so the restricted map $(Y')\hat{} \to (X')\hat{}$ is weakly compact. This means T' is super weakly compact. On the other hand if T' is super weakly compact, then so is T'', and this clearly implies that T is super weakly compact.

A similar phenomenon occurs when we investigate the behaviour of transformations in $\mathfrak{S}(X, Y)$. We say $T \in \mathfrak{B}(X, Y)$ is *super strictly singular* if for each $\delta > 0$ there exists a positive integer n such that whenever M is a subspace of X with dimension $\geqslant n$ there exists an $x \in M$ with $\|x\| = 1$ such that $\|Tx\| < \delta$.

THEOREM 2. *Let X and Y be Banach spaces and suppose $T \in \mathfrak{B}(X, Y)$. Then the following conditions are equivalent.*

(i) *T is super strictly singular.*

(ii) *$\hat{T}$ is super strictly singular.*

(iii) $\hat{T} \in \mathfrak{S}(\hat{X}, \hat{Y})$.

PROOF. We show (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) $\Rightarrow$ (iv). Suppose (ii) is not true. Then there is a positive δ such that for all positive integers n there is a subspace M of $\hat{X}$ with $\dim M \geqslant n$ such that $\|\hat{T}m\| > \delta\|m\|$ when $m \in M$. It follows that for each n the formal sentence

$$\text{“}\exists \text{ subspace } M \text{ of } X \text{ with } \dim M \geqslant n;\ \forall m \in M\ \|Tm\| > \delta\|m\|\text{”}$$

holds in $^*\mathfrak{M}$. But then this sentence holds in $\mathfrak{M}$ and it immediately follows that (i) is not true. So (i) implies (ii).

Since (ii) trivially implies (iii) we are finished once we have (iii) implies (i). Suppose that (i) is false. Then there is a positive δ such that the formal sentence

$$\text{“}\forall n \in \mathbf{N}\ \exists \text{ subspace } M \text{ of } X \text{ with } \dim M \geqslant n\colon \forall m \in M\ \|Tm\| > \delta\|m\|\text{”}$$

holds in $\mathfrak{M}$ and so holds in $^*\mathfrak{M}$. Thus if we choose ω to belong to $^*\mathbf{N} - \mathbf{N}$ there exists a subspace M of $^*X$ with $\dim M \geqslant \omega$ such that $\|Tm\| > \delta\|m\|$ whenever $m \in M$. But it then follows that $\hat{T}$ restricted to the infinite-dimensional subspace $\hat{M}$ of $\hat{X}$ is an isomorphism. Thus (iii) does not hold and we have proved the theorem.

The set of super strictly singular transformations of $\mathfrak{B}(X, Y)$ forms a closed subspace of $\mathfrak{S}(X, Y)$. In general it is a proper closed subspace as the following example shows. Consider a sequence of finite-dimensional spaces M_n ($n = 1, 2, 3, \ldots$) where $\dim M_n = n$ ($n = 1, 2, 3, \ldots$). Let X be the subspace of the product πM_n consisting of those $x = (x_1, x_2, x_3, \ldots)$ for which $\|x\| = \Sigma\|x_i\| < \infty$ and let Y be the subspace of those $y = (y_1, y_2, y_3, \ldots)$ for which $\|y\|^2 = \Sigma\|y_i\|^2 < \infty$. Then in X every weakly convergent sequence is convergent whilst Y is reflexive. Thus the identity map $I\colon X \to Y$ is strictly singular [**18**, Theorem 1.2], though it fails to be super strictly singular.

When $S \in \operatorname{fin} {}^*\mathfrak{B}(X, Y)$, $\hat{S}(\hat{X})$ is generally a proper subspace of $(S(X))\hat{}$ and when $\hat{S}(\hat{X})$ has finite codimension in $(S(X))\hat{}$ we let $\gamma(S)$ denote this codimension.

THEOREM 3. *Let X and Y be Banach spaces and suppose $T \in \mathfrak{B}(X, Y)$ and $S \in \operatorname{fin} {}^*\mathfrak{B}(X, Y)$. Then*

(i) (a) *if $T \in \Phi_+(X, Y)$ then $\hat{T} \in \Phi_+(\hat{X}, \hat{Y})$ and $\alpha(T) = \alpha(\hat{T})$,*
(b) *if $\hat{S} \in \Phi_+(\hat{X}, \hat{Y})$ then $S \in {}^*\Phi_+(X, Y)$ and $\alpha(\hat{S}) = \alpha(S) + \gamma(S)$,*
(ii) (a) *if $T \in \Phi_-(X, Y)$ then $\hat{T} \in \Phi_-(\hat{X}, \hat{Y})$ and $\beta(T) = \beta(\hat{T})$,*
(b) *if $\hat{S} \in \Phi_-(\hat{X}, \hat{Y})$ then $S \in {}^*\Phi_-(X, Y)$ and $\beta(\hat{S}) = \beta(S) + \gamma(S)$,*
(iii) (a) *if $T \in \Phi(X, Y)$ then $\hat{T} \in \Phi(\hat{X}, \hat{Y})$ and $i(T) = i(\hat{T})$,*
(b) *if $\hat{S} \in \Phi(\hat{X}, \hat{Y})$ then $S \in {}^*\Phi(X, Y)$ and $i(S) = i(\hat{S})$.*

PROOF. (i)(a) If $T(X)$ is closed in Y it follows by the open mapping theorem that the induced map $T_0\colon X/\ker T \to T(X)$ is an isomorphism. Thus there is a positive δ such that the sentence

$$\text{“}\forall x \in X\ \exists y \in \ker T\colon \|Tx\| \geqslant \delta\|x + y\|\text{”}$$

holds in $\mathfrak{M}$. Consequently it holds in $^*\mathfrak{M}$ and it follows that $\operatorname{fin} {}^*T(X) = {}^*T(\operatorname{fin} {}^*X)$. Thus $\hat{T}(\hat{X}) = (T(X))\hat{}$ and $\hat{T}(\hat{X})$ is closed since $(T(X))\hat{}$ is complete. Now consider a basis $\{x_1, x_2, \ldots, x_n\}$ for $\ker T$. We claim $\{\hat{x}_1, \hat{x}_2, \ldots, \hat{x}_n\}$ is a basis for $\ker \hat{T}$ or equivalently that it forms a spanning set for $\ker \hat{T}$. If not there is

an element $\hat{p} \in \ker \hat{T}$ such that $\|\hat{p}\| = 1$ and $d(\hat{p}, \operatorname{sp}\{\hat{x}_1, \hat{x}_2, \ldots, \hat{x}_n\}) = d > 0$. But we know there is a positive δ and a point $q \in \ker {}^*T$ such that $0 = \|\hat{T}\hat{p}\| \geqslant \delta\|\hat{p} + \hat{q}\| \geqslant \delta d$. This gives the result.

(i)(b) Choose a basis $\{p_1, \ldots, p_j\}$ for $\ker S$ in such a way that $\{\hat{p}_1, \ldots, \hat{p}_j, \hat{q}_1, \ldots, \hat{q}_k\}$ is a basis for $\ker \hat{S}$. Let ${}^*X = Q \oplus \operatorname{sp}\{q_1, \ldots, q_k\} \oplus \operatorname{sp}\{p_1, \ldots, p_k\}$ where Q is chosen so that the projections are finite. Then $S({}^*X) = S(Q) \oplus \{Sq_1, \ldots, Sq_k\}$,

$$\hat{X} = \hat{Q} \oplus \operatorname{sp}\{\hat{q}_1, \ldots, \hat{q}_k\} \oplus \operatorname{sp}\{\hat{p}_1, \ldots, \hat{p}_j\} \quad \text{and} \quad \hat{S}(\hat{X}) = \hat{S}(\hat{Q}).$$

By assumption $\hat{S}(\hat{X})$ is closed and so as $\hat{S}$ restricted to $\hat{Q}$ is $1:1$ there exists a positive δ such $\|\hat{S}\hat{q}\| > \delta\|\hat{q}\|$ for all nonzero $q \in \operatorname{fin} Q$. Consequently $\|Sq\| > \delta\|q\|$ for all nonzero $q \in \operatorname{fin} Q$, and thus for all nonzero $q \in Q$. This implies that $S(Q)$ is closed, so that $S({}^*X)$ is closed. Now $\alpha(\hat{S}) - \alpha(S) = k$ so we are left to show that $\gamma(S) = k$. We have $S({}^*X) = S(Q) \oplus \{Sq_1, \ldots, Sq_k\} = S(Q) \oplus (S(F))$ where $\dim S(F) = k$ and where $S(F)$ is chosen so that the projections are finite. Therefore $(S({}^*X))\hat{} = (S(Q))\hat{} \oplus (S(F))\hat{} = \hat{S}(\hat{Q}) \oplus (S(F))\hat{} = \hat{S}(\hat{X}) \oplus (S(F))\hat{}$. But $\dim(S(F))\hat{} = k$, and this gives the result.

(ii)(a) The proof in (i)(a) shows that $\hat{T}(\hat{X})$ is closed, and equals $(T(X))\hat{}$. If $\beta(T) = n$ there is a subspace E of Y with dimension n such that $T(X) \oplus E = Y$. By transfer, ${}^*(T(X)) \oplus {}^*E = {}^*Y$ where the projections have finite norm. This implies $(T(X))\hat{} \oplus \hat{E} = Y$ and that $\beta(\hat{T}) = n$.

(ii)(b) Suppose $\hat{S}(\hat{X}) \oplus \operatorname{sp}\{\hat{q}_1, \hat{q}_2, \ldots, \hat{q}_n\} = \hat{Y}$ where $n = \beta(\hat{S})$. Then for each $q \in {}^*Y_1$ there exists a point $p \in {}^*X$ and scalars $c_1, c_2, \ldots, c_n$ such that

$$\|q - (c_1q_1 + c_2q_2 + \cdots + c_nq_n + \operatorname{Sp})\| \leqslant 1/2.$$

Moreover, since $\hat{S}(\hat{X})$ is closed we can assume $p \in {}^*X_r$ and that each $|c_i| \leqslant M$ where r and M are positive constants independent of $q \in {}^*Y_1$. A well-known argument (see the proof of Lemma 8.5.2 in **[2]**) extends to show that given such a q there exists a point $p \in {}^*X_{2r}$ and scalars $c_1, c_2, \ldots, c_n$ such that $q = c_1q_1 + c_2q_2 + \cdots + c_nq_n + \operatorname{Sp}$. This shows that $S({}^*X)$ has codimension $\leqslant n$ in Y and moreover **[3**, p. 37] that $S({}^*X)$ is closed. If in fact $\beta(S) = k$ then the codimension of $(S(X))\hat{}$ in $\hat{Y}$ is k. This remark implies that $\gamma(S) = n - k$ yielding the equality $\beta(\hat{S}) = \beta(S) + \gamma(S)$.

(iii) These results are implied by the corresponding results of (i) and (ii).

COROLLARY 2. *If X is a Banach space then $T \in \mathfrak{R}(X)$ if and only if $\hat{T} \in \mathfrak{R}(\hat{X})$.*

PROOF. It is known **[3**, p. 36] that T is a Riesz operator if and only if $\lambda I - T \in \Phi(X)$ for each $\lambda \neq 0$. The result is therefore an immediate consequence of Theorem 3(iii).

Our next result is a known standard result. Proofs for the special case $X = Y$ can be found in **[3]**.

COROLLARY 3. *If $T \in \Phi_+(X, Y)$ (respectively $\Phi_-(X, Y)$, $\Phi(X, Y)$) then there exists an $\varepsilon > 0$ such that for all $S \in \mathfrak{B}(X, Y)\|S\| < \varepsilon$ implies $T + S \in \Phi_+(X, Y)$ (respectively $\Phi_-(X, Y)$, $\Phi(X, Y)$) and $\alpha(T + S) \leqslant \alpha(T)$ (respectively $\beta(T + S) \leqslant \beta(T)$, $i(T) = i(T + S)$).*

PROOF. Suppose $T \in \Phi_+(X, Y)$ and suppose $S \in {}^*\mathfrak{B}(X, Y)$ with $\mathrm{st}(\|S\|) = 0$. Then $(T + S)\hat{} = \hat{T} \in \Phi_+(\hat{X}, \hat{Y})$ and so $T + S \in {}^*\Phi_+(X, Y)$ with $\alpha(T + S) \leqslant \alpha((T + S)\hat{}) = \alpha(\hat{T}) = \alpha(T)$. Thus the set $A = \{\varepsilon > 0 | \forall S \in {}^*\mathfrak{B}(X, Y), \|S\| < \varepsilon$ implies $T + S \in {}^*\Phi_+(X, Y)$ with $\alpha(T + S) \leqslant \alpha(T)\}$ is an internal set containing the positive infinitesimals. Consequently it contains a standard positive ε and so we have the first result. The other two results follow by the same argument.

THEOREM 4. *Let X and Y be Banach spaces and suppose $T \in \mathfrak{B}(X, Y)$. Then*
(i) $T \in \mathcal{P}^{(p)}(X, Y)$ *if and only if* $\hat{T} \in \mathcal{P}^{(p)}(\hat{X}, \hat{Y})$,
(ii) (a) *if* $T \in \mathfrak{N}(X, Y)$ *then* $\hat{T} \in \mathfrak{N}(\hat{X}, \hat{Y})$,
(b) *if* $\hat{T} \in \mathfrak{N}(\hat{X}, \hat{Y})$ *then* $T \in \mathfrak{N}_0(X, Y)$.

PROOF. (i) Let $T \in \mathcal{P}^{(p)}(X, Y)$ and suppose M is a constant for which the defining inequality holds. Let us suppose further that $\{p_1, \ldots, p_n\}$ is a finite family of points from fin ${}^*X$ and that ε is a positive real number for which

$$\sum_{i=1}^{n} \|\hat{T}\hat{p}_i\|^p > M^p\left(\sum_{i=1}^{n} |\phi(\hat{p}_i)|\right) + \varepsilon, \quad \text{for } \phi \in \hat{X}' \text{ with } \|\phi\| \leqslant 1.$$

Since each $f \in {}^*(X')$ with $\|f\| \leqslant 1$ defines an element $\phi = \hat{f} \in \hat{X}'$ with $\|\phi\| \leqslant 1$ we have

$$\sum_{i=1}^{n} \|T(p_i)\|^p > M^p\left(\sum_{i=1}^{n} |f(p_i)|\right) + \varepsilon, \quad \text{for } f \in {}^*X' \text{ with } \|f\| \leqslant 1.$$

This contradicts our assumption on M by an application of the transfer principle. If $\hat{T} \in \mathcal{P}^p(\hat{X}, \hat{Y})$ then $T \in \mathcal{P}^p(X, Y)$ since, for $x_1, x_2, \ldots, x_n \in X$,

$$\sum_{i=1}^{n} \|T(p_i)\|^p = \sum_{i=1}^{n} \|\hat{T}(\hat{x}_i)\|^p \leqslant M^p \sup\left\{|\phi(\hat{x}_i)|^p = M^p \sup \sum |f(x_i)|^p.\right.$$

Here the first supremum is taken over $\phi \in \hat{X}_1'$; the second over $f \in X_1'$.

(ii)(a) Suppose $Tx = \sum_1^\infty f_n(x)y_n$ for all $x \in X$ where $\sum \|f_n\| \, \|y_n\| < \infty$. Then, for each infinite integer ω, $\mathrm{st}(\sum_\omega^\infty \|f_n\| \, \|y_n\|) = 0$ implying in turn that $(\sum_\omega^\infty f_n(p)y_n)\hat{} = 0$ whenever $p \in$ fin ${}^*X$. Thus given $p \in$ fin ${}^*X$ and ε $(> 0) \in R$ there is a standard integer k_0 such that $\|\sum_{n=k}^\infty f_n(p)y_n\| < \varepsilon$ whenever $k \geqslant k_0$ for otherwise ${}^*\mathbf{N} - \mathbf{N}$ would be internal. But ${}^*Tp = \sum_1^\infty f_n(p)y_n$ so $\|{}^*Tp - \sum_1^k f_n(p)y_n\| < \varepsilon$ whenever $k \geqslant k_0 - 1$. This forces the equality $\hat{T}(\hat{p}) = \sum_1^\infty \hat{f}_n(\hat{p})\hat{y}_n$ for all $p \in$ fin ${}^*X$.

(b) If $\hat{T} \in \mathfrak{N}(\hat{X}, \hat{Y})$ then it is clear that T, as a map from X into $\hat{Y}$, is nuclear. Therefore $T \in \mathfrak{B}(X, \hat{Y})$ is quasinuclear whence $T \in \mathfrak{B}(X, Y)$ is also quasinuclear (see [**16**, p. 57]).

We make it clear here that it is quite possible that the condition "$\hat{T} \in \mathfrak{N}(\hat{X}, \hat{Y})$" does imply "$T \in \mathfrak{N}(X, Y)$". It seems relevant to point out that it is unknown whether the assumption that the adjoint map T' is nuclear implies that T itself is nuclear [**16**, p. 53]. On a Hilbert space the nuclear and trace-class operators coincide and as we will shortly see (ii)(b) can then be improved. Further the absolutely summing transformations coincide with the Hilbert-Schmidt operators when the underlying spaces are Hilbert spaces [**16**, p. 47] so we already have the first part of the next result. However, since the proof that the two classes coincide is not trivial we include a direct proof for completion.

THEOREM 5. *Let H and K be Hilbert spaces and suppose $T \in \mathfrak{B}(H, K)$ and $S \in \mathfrak{B}(H)$. Then*

(i) $T \in \mathcal{HS}(H, K)$ *if and only if* $\hat{T} \in \mathcal{HS}(\hat{H}, \hat{K})$ *and*

(ii) $S \in \mathcal{T}(H)$ *if and only if* $\hat{S} \in \mathcal{T}(\hat{H})$.

PROOF. (i) If $T \in \mathcal{HS}(H, K)$ then the Hilbert-Schmidt norm $\sigma(T)$ is defined by $\sigma(T)^2 = \Sigma|\langle Te_i, f_j\rangle|^2$. Here $\sigma(T)$ is independent of the two complete orthonormal families, since using Parseval's inequality, $\sigma(T)^2 = \Sigma\|Te_i\|^2 = \Sigma\|T'f_j\|^2$. Now suppose, for such a transformation T, $\{\hat{e}_k: k \in K\}$ is a complete orthonormal family in $\hat{H}$ such that $\Sigma\|\hat{T}\hat{e}_k\|^2 = \infty$. Then there exists a finite subset K_0 such that, when summed over K_0, $\Sigma\|\hat{T}\hat{e}_k\|^2 > \sigma(T)^2$. Since the $\hat{e}_k$ form an orthonormal family the Gram-Schmidt orthonormalization process yields an orthonormal family $\{e'_k: k \in K_0\}$ in $^*H$ such that $\mathrm{st}\|e'_k - e_k\| = 0$ for each $k \in K$. But then $\Sigma\|T(e'_k)\| > \sigma(T)^2$, the summation still being over K_0. This contradicts the definition of $\sigma(T)$.

If $\hat{T} \in \mathcal{HS}(\hat{H}, \hat{K})$ consider a complete orthonormal family $\{e_i: i \in I\}$ in H. Summing over an arbitrary subset of I we have

$$\sum \|T(e_i)\|^2 = \sum \|\hat{T}(\hat{e}_i)\|^2 < \sigma(\hat{T})^2.$$

This clearly ensures that $T \in \mathcal{HS}(H, K)$.

(ii) Suppose $S \in \mathcal{T}(H)$ so that $S = AB$ where A and B are Hilbert-Schmidt operators. Then $\hat{S} = \hat{A}\hat{B}$ and, by part (i), $\hat{S} \in \mathcal{T}(\hat{H})$. Conversely suppose $\hat{S} \in \mathcal{T}(\hat{H})$ and let $S_0 = (\mathrm{abs}(S))^{1/2}$, noting that $\hat{S}_0 = (\mathrm{abs}(\hat{S}))^{1/2}$. Then $\hat{S}_0 \in \mathcal{HS}(\hat{H})$ (see [**18**, Theorem 48.2]) implying, by part (i), that $S_0 \in \mathcal{HS}(H)$. This implies $S \in \mathcal{T}(H)$.

As we have assumed [**17**, Theorem 48.2] a transformation T between Hilbert spaces H and K is a nuclear transformation if and only if its absolute value operator $|T| = (T'T)^{1/2}$ is a trace operator on H. But $|\hat{T}| = |T|\hat{}$ and so $\hat{T} \in \mathcal{N}(\hat{H}, \hat{K})$ implies $T \in \mathcal{N}(H, K)$ by Theorem 5 (ii).

In general the assumption that $\mathcal{S} \subset \mathcal{K}(X, Y)$ does not imply $\hat{\mathcal{S}} \subset \mathcal{K}(\hat{X}, \hat{Y})$. The next result clarifies under what conditions this behaviour occurs. A subset $\mathcal{S}$ of $\mathfrak{B}(X, Y)$ is *collectively compact* [**1**] if the union of the images of the unit ball X_1 has compact closure in Y, and we say $\mathcal{S}$ is *socially compact* if for all ε (> 0) there exists an integer n (depending on ε alone) such that for all $T \in \mathcal{S}$ there exist points $x_1, x_2, \ldots, x_n \in X_1$ such that

$$T(X_1) \subset \bigcup_{i=1}^{n} \{y: \|Tx_i - y\| < \varepsilon\}.$$

THEOREM 6. *Let X and Y be Banach spaces and assume $\mathcal{S} \subset \mathfrak{B}(X, Y)$. Then*

(i) *the following conditions are equivalent*:

(a) $\mathcal{S}$ *is collectively compact*;

(b) $\hat{\mathcal{S}}$ *is bounded in* $\mathfrak{B}(\hat{X}, \hat{Y})$ *and each* $T \in \hat{\mathcal{S}}$ *maps* $\hat{X}$ *into* Y;

(c) $\hat{\mathcal{S}}$ *is collectively compact*.

(ii) *The following conditions are equivalent*:

(a) $\mathcal{S}$ *is socially compact*;

(b) $\hat{\mathcal{S}}$ *is a bounded subset of* $\mathcal{K}(\hat{X}, \hat{Y})$;

(c) $\hat{\mathcal{S}}$ *is socially compact*.

PROOF. (i) We show (a) $\Rightarrow$ (b) $\Rightarrow$ (c) $\Rightarrow$ (a), noting first that the final implication is trivial. If $\mathcal{S}$ is collectively compact then $A = \bigcup \{T(X_1)\colon T \in \mathcal{S}\}$ is relatively compact and so [**14**, Theorem 3.6.1] each point in $^*A$ is near-standard. But, for each $T \in {}^*\mathcal{S}$, $T(p) \in {}^*A$ whenever $p \in {}^*X_1$. This implies $T(p)$ is near-standard whenever $p \in \operatorname{fin} {}^*X$ and hence that $\hat{T}(\hat{p}) \in Y$ for any $\hat{p} \in \hat{X}$. Further $\hat{\mathcal{S}}$ is bounded as $\mathcal{S}$ is bounded, and we thus have the first implication. Given (b) we have $\operatorname{fin} {}^*\mathcal{S} = {}^*\mathcal{S}$ and so, as $\hat{T}(\hat{X}) \subseteq Y$ whenever $T \in \operatorname{fin} {}^*\mathcal{S}$, $T(p)$ is near-standard when $T \in {}^*\mathcal{S}$ and $p \in {}^*X_1$. This means A is relatively compact. Moreover such a point $Tp \in {}^*A$ so that $\operatorname{st}(Tp) \in \bar{A}$, the closure of A. But $\hat{T}(\hat{p}) = \operatorname{st}(Tp)$ and so we have (c).

(ii) We show (a) $\Rightarrow$ (c) $\Rightarrow$ (b) $\Rightarrow$ (a), noting immediately that the implication (c) $\Rightarrow$ (b) is immediate. Let us suppose $\mathcal{S}$ is socially compact and that ε is an arbitrary positive real. If n is the corresponding fixed integer implied by the definition then the sentence

$$\text{“}\forall T \in \mathcal{S}\ \exists x_1, x_2, \ldots, x_n \in X_1\colon T(X_1) \subset \bigcup_{i=1}^{n} \{y\colon \|Tx_i - y\| < \varepsilon\}\text{”}$$

holds in $^*\mathfrak{M}$ by the transfer principle. Thus given $T \in {}^*\mathcal{S}$ there exist points $p_1, \ldots, p_n \in {}^*X$ such that $T({}^*X_1) \subset \bigcup_{i=1}^{n}\{q \in {}^*Y\colon \|Tp_i - q\| < \varepsilon\}$. But then $\hat{T}(\hat{X}_1) \subset \bigcup_{i=1}^{n}\{\hat{q} \in \hat{Y}\colon \|\hat{T}\hat{p}_i - \hat{q}\| \leqslant \varepsilon\}$ and $\hat{\mathcal{S}}$ is therefore socially compact. Finally, suppose $\mathcal{S}$ is not socially compact. Then there exists a positive ε such that the sentence

$$\text{“}\forall n \in \mathbf{N}\ \exists T \in \mathcal{S}\ \&\ \{x_1, \ldots, x_n\} \subset X_1\colon \|Tx_i - Tx_j\| > \varepsilon\ (i \neq j)\colon i, j = 1, \ldots, n\text{”}$$

holds in $\mathfrak{M}$ and thus in $^*\mathfrak{M}$. If we take ω to be an infinite integer we then have the existence of a map $T \in {}^*\mathcal{S}$ and points $p_1, p_2, \ldots, p_\omega \in {}^*X_1$ such that $\|Tp_i - Tp_j\| > \varepsilon$ for $i \neq j$ and $i, j = 1, 2, \ldots, \omega$. If we assume $\hat{\mathcal{S}}$ is bounded then $T \in \operatorname{fin} {}^*\mathcal{S}$ and so $\|\hat{T}\hat{p}_i - \hat{T}\hat{p}_j\| \geqslant \varepsilon$ for $i \neq j$ and $i, j = 1, 2, \ldots$. Consequently $\hat{T} \notin \mathcal{K}(\hat{X}, \hat{Y})$, and we have the result (b) $\Rightarrow$ (a).

When $\mathcal{S}$ is a subset of $\mathcal{B}(X, Y)$ we denote $\{T'\colon T \in \mathcal{S}\}$ by $\mathcal{S}'$. Then (i) of the following corollary is proved in [**1**] for the case $X = Y = H$, where H is a Hilbert space. The proof given there uses the spectral theorem for compact operators.

COROLLARY 4.

(i) *If $\mathcal{S} \subset \mathcal{K}(X, Y)$ then $\mathcal{S}$ is totally bounded if and only if both $\mathcal{S}$ and $\mathcal{S}'$ are collectively compact.*

(ii) *If $\mathcal{S} \subset \mathcal{B}(X, Y)$ then $\mathcal{S}$ is socially compact if and only if $\mathcal{S}'$ is socially compact.*

PROOF. (i) If $\mathcal{S} \subset \mathcal{K}(X, Y)$ and is totally bounded then $\mathcal{S}$ and $\mathcal{S}'$ are collectively compact by [**1**, Theorem 2.5]. So let us suppose that $\mathcal{S}$ and $\mathcal{S}'$ are collectively compact. We need to show [**15**, Theorem 3.5.1] that each $T \in {}^*\mathcal{S}$ is near-standard. Now by the previous theorem such an element is finite and further $\hat{T}$ maps $\hat{X}$ into Y, whilst $(T')\hat{}$ maps $(Y')\hat{}$ into X'. Define $S\colon X \to Y$ by $Sx = \hat{T}\hat{x}$ and note that S is compact. We claim that $S = \operatorname{st} T$ or, equivalently, that $\hat{S} = \hat{T}$ on $\hat{X}$. Suppose in fact that there is a $\hat{p} \in \hat{X}$ such that $\hat{S}\hat{p} \neq \hat{T}\hat{p}$. Then, since these elements are in Y,

there is an $f \in Y'$ such that $\hat{f}(\hat{S}\hat{p}) \neq \hat{f}(\hat{T}\hat{p})$, i.e., such that $\hat{S}'\hat{f} \neq \hat{T}'\hat{f}$. But these elements are in X' so there must be an $x \in X$ such that $(\hat{S}'\hat{f})\hat{x} \neq (\hat{T}'\hat{f})\hat{x}$, i.e., such that $\hat{S}\hat{x} \neq \hat{T}\hat{x}$, and this is a contradiction.

(ii) Suppose $\mathcal{S}$ is socially compact. Then $\mathcal{S}$ is a bounded subset of $\mathcal{K}(\hat{X}, \hat{Y})$ and it follows that $\hat{\mathcal{S}}'$ is a bounded subset of $\mathcal{K}(\hat{Y}', \hat{X}')$. Since each $T \in (S')\hat{} \subset \mathcal{B}((Y')\hat{}, (X')\hat{})$ is the restriction of an element of $\hat{\mathcal{S}}'$ it follows that $(\mathcal{S}')\hat{}$ is a bounded subset of compact transformations. Thus we have the result that $\mathcal{S}'$ is socially compact. Conversely, if $\mathcal{S}'$ is socially compact then by what we have shown so too is $\mathcal{S}''$. But this implies $\mathcal{S}$ is socially compact.

The Henson-Moore result implies that $(\mathcal{B}(X, \mathbf{C}))\hat{} = \mathcal{B}(\hat{X}, \hat{\mathbf{C}}) = \mathcal{B}(\hat{X}, \mathbf{C})$ if and only if X is superreflexive. If the range space is infinite dimensional too then the analogous inequality always holds.

THEOREM 7. *Let X and Y be infinite dimensional Banach spaces. Then $(\mathcal{B}(X, Y))\hat{}$ is strictly contained in $\mathcal{B}(\hat{X}, \hat{Y})$.*

PROOF. The Henson-Moore result implies the theorem when X is not superreflexive, for then there exists a functional $f \in (\hat{X})'$ which does not belong to $(X')\hat{}$. Let q be a nonzero point in Y and define a finite rank operator $S \in \mathcal{B}(\hat{X}, \hat{Y})$ by $S(\hat{p}) = f(\hat{p})\hat{q}$ where $p \in \text{fin } {}^*X$. Suppose in fact that $S = \hat{A}$ where $A \in \text{fin}({}^*\mathcal{B}(X, Y))$. Let Z be a subspace complementary to $\overline{\text{span}\{q\}}$ in Y and let $A(p) = \lambda(p)q + r$ be the decomposition of $A(p)$ in terms of ${}^*\overline{\text{span}\{q\}}$ and ${}^*Z$. Then $\lambda \in \text{fin } {}^*X'$ and $\text{st}((\lambda(p) - f(\hat{p}))q + y) = 0$ for all $p \in \text{fin } {}^*X$. This implies $f(\hat{p}) = \text{st}(\lambda(p))$ for all $p \in \text{fin } {}^*X$, or that $f = \hat{\lambda}$ which is a contradiction.

When X is superreflexive we can argue the following way. For $\omega \in {}^*\mathbf{N} - \mathbf{N}$ there exist *finite sequences of norm one elements, $\{p_i\colon 1 \leqslant i \leqslant \omega\}$ in ${}^*X$ and $\{f_i\colon 1 \leqslant i \leqslant \omega\}$ in ${}^*X'$ such that $f_i(p_j) = \delta_{ij}$. This follows by a result of Auerbach [**13**, p. 16]. Moreover if we invoke Dvoretzsky's theorem (see [**9**]) there exist *finite sequences of norm one elements $\{q_i\colon 1 \leqslant i \leqslant \omega\}$ in ${}^*Y$ and $\{g_i\colon 1 \leqslant i \leqslant \omega\}$ in ${}^*Y'$ such that $g_i(q_j) = \delta_{ij}$ and such that $\hat{Q}$ is isometrically isomorphic to $\hat{l}_2(\omega)$ where $Q = {}^*\text{span}\{q_1, \ldots, q_\omega\}$ and where $l_2(\omega)$ denotes the ω-dimensional Hilbert space. Now define $S\colon \hat{Q} \to (X')\hat{} = (\hat{X})'$ by $S(q_k) = f_k$ for $k \in \mathbf{N}$ and by $S = 0$ on the orthogonal complement of $\overline{\text{span}\{\hat{q}_k\colon k \in \mathbf{N}\}}$. Then $S'\colon \hat{X}'' = \hat{X} \to (\hat{Q})'$ and, identifying $(\hat{Q})'$ in the canonical manner with $\hat{Q}$, we can suppose $S'\colon \hat{X} \to \hat{Y}$. Under this last identification $\hat{g}_k$ corresponds to $\hat{q}_k$. Now suppose $S' = \hat{A}$ for some $A \in \text{fin } {}^*\mathcal{B}(X, Y)$. Then the set $M = \{k\colon 1 \leqslant k \leqslant \omega \text{ and } |g_k(Ap_k)| > 1/2\}$ is internal. But $\text{st}|g_k(Ap_k)| = \hat{g}_k(S'\hat{p}_k) = (S\hat{q}_k)(\hat{p}_k) = 1$ if $k \in \mathbf{N}$; $= 0$ otherwise. This implies that $M = \mathbf{N}$ and this is a contradiction.

Theorem 7 contrasts very much with our final result.

THEOREM 8. *Let H be an infinite-dimensional Hilbert space. Then $\mathcal{K}(\hat{H})$ is properly contained in $(\mathcal{K}(H))\hat{}$. Moreover, there exists an operator $\hat{S} \in (\mathcal{K}(H))\hat{}$ which is not essentially normal; i.e., such that $\hat{S}\hat{S}' - \hat{S}'\hat{S} \notin \mathcal{K}(\hat{H})$.*

PROOF. Suppose A is a rank 1 operator on $\hat{H}$ defined by $A(\hat{r}) = \langle \hat{r}, \hat{p} \rangle \hat{q}$, for fixed $p, q \in \text{fin } {}^*X$. Then $A = \hat{T}$ where $T \in {}^*\mathcal{B}(H)$ is the operator defined by $T(r) = \langle r, p \rangle q$. Since any finite rank operator is the finite sum of rank 1 operators we have

$\mathcal{F}(\hat{H}) \subseteq (\mathcal{K}(H))\hat{}$. But $\mathcal{K}(\hat{H})$ is the closure of $\mathcal{F}(\hat{H})$ and so $\mathcal{K}(\hat{H}) \subseteq (\mathcal{K}(H))\hat{}$ since $(\mathcal{K}(H))\hat{}$ is closed. To establish the remaining assertion observe that (for $n = 1, 2, 3, \ldots$) the operator $T_n \in \mathcal{B}(l_2)$ defined by

$$T_n\colon (x_1, x_2, x_3, \ldots) \to (0, \ldots, 0, x_1, x_2, \ldots, x_n, 0, 0, \ldots)$$

has the property that $\|(T_n'T_n - T_nT_n')(e_i - e_j)\|_\cdot = \sqrt{2}$ for $i \neq j$ and $1 \leqslant i, j \leqslant n$, where e_i is the unit basis vector $\{\delta_{ij}\colon j = 1, 2, 3, \ldots\}$. Thus the formal sentence

$$\text{“}\forall n \in \mathbf{N}\ \exists x_1, \ldots, x_n \in H_1 \& T \in (\mathcal{K}(H))_1\colon \|(TT' - T'T)(x_i - x_j)\| \geqslant \sqrt{2}\ \ (i \neq j)\text{”}$$

holds in $\mathfrak{M}$, and so, by the transfer principle, in $^*\mathfrak{M}$. Thus for $\omega \in {}^*\mathbf{N} - \mathbf{N}$ there exist points $p_1, p_2, \ldots, p_\omega \in {}^*H_1$ and $S \in {}^*(\mathcal{K}(H))_1$ such that

$$\|(SS' - S'S)(p_i - p_j)\| \geqslant \sqrt{2}\,, \qquad i \neq j \text{ and } i, j = 1, 2, \ldots, \omega.$$

But then $\|(\hat{S}\hat{S}' - \hat{S}'\hat{S})(\hat{p}_i - \hat{p}_j)\| \geqslant \sqrt{2}$, $i \neq j$ and $i, j = 1, 2, 3, \ldots$, and so $\hat{S}\hat{S}' - \hat{S}'\hat{S} \notin \mathcal{K}(\hat{H})$.

COROLLARY 5. *Let H be an infinite-dimensional Hilbert space. Then*
(i) $\mathcal{F}(\hat{H}) \subsetneqq (\mathcal{F}(H))\hat{}$,
(ii) $\mathcal{T}(\hat{H}) \subsetneqq (\mathcal{T}(H))\hat{}$,
(iii) $\mathcal{KS}(\hat{H}) \subsetneqq (\mathcal{KS}(H))\hat{}$.

PROOF. Since $\mathcal{F}(H) \subset \mathcal{T}(H) \subset \mathcal{KS}(H) \subset \mathcal{K}(H)$ and $\mathcal{F}(H)$ is dense in $\mathcal{K}(H)$ we have $(\mathcal{F}(H))\hat{} = (\mathcal{T}(H))\hat{} = (\mathcal{KS}(H))\hat{} = (\mathcal{K}(H))\hat{}$ so that the result is immediate.

Theorem 8 shows that $(\mathcal{K}(H))\hat{}$ is very much bigger than $\mathcal{K}(\hat{H})$ though, of course, $(\mathcal{K}(H))\hat{} \neq (\mathcal{B}(H))\hat{}$ since $\mathcal{K}(H)$ is a closed proper subset of $\mathcal{B}(H)$. On the other hand $(\mathcal{B}(H))\hat{}$ is strictly contained in $\mathcal{B}(\hat{H})$. Thus it seems natural to ask, for example, whether there is a classical family of operators which contains the finite rank operators and which is closed under the $\hat{}$ construction. In this context it seems pertinent to comment that the class of Riesz operators $\mathcal{R}(H)$ and the class of operators with compact powers, $\mathcal{P}K(H)$, both behave badly in this regard. On the one hand $\mathcal{R}(\hat{H})$ and $\mathcal{P}\mathcal{K}(\hat{H})$ contain external elements. For H separable the nilpotent element $T \in \mathcal{B}(\hat{H})$ defined by $T = 0$ on $H^\perp$ and $T\colon (\lambda_1, \lambda_2, \lambda_3, \lambda_4, \ldots) \to (0, \lambda_1, 0, \lambda_3, 0, \ldots)$ on H cannot be internal since $\{n\colon \|Te_n\| > 1/2\}$ equals the set of odd (finite) integers. On the other hand any projection onto a *finite dimensional subspace of $^*H$ gives rise to an operator in $(\mathcal{B}(H))\hat{}$ which is clearly not in $\mathcal{R}(\hat{H})$ or $\mathcal{P}K(\hat{H})$.

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