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A SEPARATION THEOREM FOR Σ_1^1 SETS

BY

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ABSTRACT. In this paper, we show that the notion of Borel class is, roughly speaking, an effective notion. We prove that if a set A is both $\mathbf{\Pi}_\xi^0$ and Δ_1^1, it possesses a $\mathbf{\Pi}_\xi^0$-code which is also Δ_1^1. As a by-product of the induction used to prove this result, we also obtain a separation result for Σ_1^1 sets: If two Σ_1^1 sets can be separated by a $\mathbf{\Pi}_\xi^0$ set, they can also be separated by a set which is both Δ_1^1 and Π_ξ^0.

Applications of these results include a study of the effective theory of Borel classes, containing separation and reduction principles, and an effective analog of the Lebesgue-Hausdorff theorem on analytically representable functions. We also give applications to the study of Borel sets and functions with sections of fixed Borel class in product spaces, including a result on the conservation of the Borel class under integration.

In this paper, we shall be mainly interested in the properties of Borel subsets of product spaces. If X and Y are Polish spaces, and if B is a Borel subset of $X \times Y$, define for each x in X the section B_x of B at x by $B_x = \{y \in Y: (x, y) \in B\}$. Our aim is to relate properties of the sections of B with global properties of B. A typical problem (and one to which we shall give a positive solution) is the following, which we call the *section problem*. Let ξ be a countable ordinal, and suppose all sections of B are of additive Borel class ξ in Y (i.e. for all x in X, $B_x \in \mathbf{\Sigma}_\xi^0$). Is B the countable union of a sequence of Borel sets, whose sections are all of Borel class less than ξ?

Let us discuss briefly the history of this problem. Working on uniformization questions, Dellacherie proved in **[De]** that each Borel set with open sections is the countable union of rectangles of the form $B \times U$, where B is a Borel subset of X and U ranges over a basis of the topology of Y, and conjectured that there was a positive answer to the section problem. The case $\xi = 2$ has been solved by Saint-Raymond **[StR]**.

In **[Bo1]**, Bourgain states the section problem in a more general context, the Polish space X being replaced by an abstract measurable space, i.e. a set X with a σ-algebra $\mathfrak{B}$ of subsets of X, and obtains the result for $\xi = 3$ in case X is a complete probability space (i.e. $\mathfrak{B}$ is the complete σ-algebra of all subsets of X which are measurable with respect to a probability on X). Furthermore, he obtains in **[Bo2]** a partial result concerning the case $\xi = 3$ in the Polish case: If B is a Borel subset of $X \times Y$, where X and Y are Polish spaces, the set $C = \{x \in X: B_x \text{ is } F_{\sigma\delta}\}$ is coanalytic ($\mathbf{\Pi}_1^1$) in X. Some of the ideas of his proof then led to the solution of the case $\xi = 3$ of the section problem; independently by Bourgain **[Bo3]** and the

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author **[Lo1]** and **[Lo2]**. The solution of the section problem for all countable ordinals is announced, for Polish spaces, in two notes of the author **[Lo3]** and **[Lo4]**. In this paper, we shall present the solution in full generality (see Theorem 3.1).

We also solve a related problem concerning Borel functions: If f is a Borel function from $X \times Y$ into $[0, 1]$, call f partially of class ξ if all partial functions f_x: $Y \to [0, 1]$, for all x in X, are of Borel class ξ. With this definition we have

THEOREM. *If ξ is a nonlimit countable ordinal, then each Borel function f: $X \times Y \to [0, 1]$ which is partially of class ξ is the pointwise limit of a sequence $(f_n)_{n \in \omega}$ of Borel functions which are partially of class less than ξ.*

This theorem is an extension of the classical Lebesgue-Hausdorff theorem on analytically representable functions. We shall also derive from it a result on the conservation of the Borel class under integration (Theorem 3.8).

The proofs of Dellacherie for case $\xi = 1$, Saint-Raymond for case $\xi = 2$ and Bourgain for case $\xi = 3$ have in common to be of classical type, that is to use only tools and methods from classical descriptive set theory. Apart from that, they are unfortunately very different from one another–and of course more and more difficult–and it does not seem possible to extract from them a general method for solving the section problem.

The method we present here is of a very different spirit. We shall deduce a solution to the section problem from a result in effective descriptive set theory. The fact that effective descriptive set theory is not only a refinement of classical descriptive set theory, but also a powerful method able to solve problems of classical type is a feeling common to many set-theorists. We think that this paper provides a new concrete example that this feeling is right.

Let X be a recursively presented Polish space, with its canonical basis of open sets. (One may think of X as being ω^ω with the usual notion of recursivity. For the background material, see §0.) One generally uses the notion of "Borel code" to encode the family of Borel subsets of X. (For a precise definition, see **[Ke]** or **[Mo]**.) Roughly speaking, a real α codes the Borel set B_α if it encodes some particular way of obtaining B_α, using countable unions and complementation, from the sets of the canonical basis. Then one can associate with each Borel code α a countable ordinal $\xi(\alpha)$, the particular Borel class, additive or multiplicative, of B_α witnessed by α. This leads to the notion of ξ-code (see **[Ke]**).

Now suppose B is a Δ_1^1 subset of X. An easy consequence of the Suslin-Kleene Theorem (see **[Mo]**) insures that B admits a recursive code, i.e. there is a recursive real α such that $B = B_\alpha$. Hence we can associate with B an ordinal $\xi_{\text{rec}}(B)$, its recursive Borel class, defined by $\xi_{\text{rec}}(B) = \inf\{\xi(\alpha)\colon \alpha \text{ is recursive and } B = B_\alpha\}$. We obtain the usual recursive hierarchy $(\Sigma_\xi^0, \Pi_\xi^0)_{\xi < \omega_1}$ among Δ_1^1 sets. The obvious inequality $\xi_{\text{rec}}(B) \geqslant \xi(B)$, where $\xi(B)$ denotes the Borel class of B, is in general not an equality: one can construct open Δ_1^1 sets of arbitrary recursive Borel class below ω_1.

Suppose now α is a Borel code which is Δ_1^1. Then clearly the Borel set B_α is also

Δ_1^1. This naturally leads to a new hierarchy among Δ_1^1 sets, which we call the Δ_1^1-recursive hierarchy, and denote by $(\Sigma_\xi^*, \Pi_\xi^*)_{\xi<\omega_1}$: We define, for each Δ_1^1 subset B of X, the Δ_1^1-recursive Borel class of B, $\xi_{\Delta_1^1}(B)$ by $\xi_{\Delta_1^1}(B) = \inf\{\xi(\alpha)\colon \alpha$ is Δ_1^1 and $B = B_\alpha\}$, and $\Sigma_\xi^* = \{B \in \Delta_1^1\colon B$ admits a Δ_1^1 additive ξ-code$\} = \bigcup_{\alpha\in\Delta_1^1} \Sigma_\xi^0(\alpha)$, $\Pi_\xi^* = \{B \in \Delta_1^1\colon B$ admits a Δ_1^1 multiplicative ξ-code$\} = \bigcup_{\alpha\in\Delta_1^1} \Pi_\xi^0(\alpha)$.

The main and somewhat surprising result of this paper is that for all $B \in \Delta_1^1$, $\xi_{\Delta_1^1}(B) = \xi(B)$, i.e. that for each Δ_1^1 set B, its Borel class is witnessed by a Δ_1^1 code. This is a consequence of the following result (Theorem A of §1).

THEOREM A. *Let ξ be some recursive ordinal. If B is some Δ_1^1 and $\mathbf{\Pi}_\xi^0$ subset of X, then B belongs to the class Π_ξ^*.*

In the case $\xi = 1$, the result can be found in Moschovakis' forthcoming book [**Mo**]. Cases $\xi = 2$ and $\xi = 3$ are proved in [**Lo1**], with proofs which are very similar to the proofs of classical type for the section problem given by Saint-Raymond and Bourgain. In particular, we made use of the special properties of compact sets, and derived case 3 from an ingenious lemma of general set theory due to Bourgain. The proof we present in §1 avoids all these difficulties by the systematic use of another tool we introduced in the two notes [**Lo3**] and [**Lo4**], the possibility of changing the topology on the space X, the new topologies being more adequate to the problem. In any case, we want to say how indebted we are to the work of J. Bourgain on case $\xi = 3$, which contains some of the tools allowing to attack the general case, and which also convinced us that Dellacherie's conjecture should be true.

As usual in this type of problems, the structural result about Δ_1^1 sets is obtained via a separation result about Σ_1^1 sets. If Γ is a family of subsets of X, and A and B are two subsets of X, we say that A is Γ-separable from B if there is some set $C \in \Gamma$ such that $A \subset C$ and $C \cap B = \varnothing$. (Note that this relation is not symmetric in general.) We shall prove in §1 the following separation result.

THEOREM B. *If A, B are two Σ_1^1 subsets of X and for some recursive ordinal ξ, A is $\mathbf{\Pi}_\xi^0$-separable from B, then A is Π_ξ^*-separable from B.*

This separation result is not only a refinement of Theorem A, but as we shall see, it is the right induction hypothesis which allows us to prove Theorem A by induction on ξ.

Theorems A and B are proved in §1. Before that, we recall in §0 some results in effective descriptive set theory which are needed in the sequel. The reference papers are circulated but unpublished works of Moschovakis [**Mo**] and Kechris [**Ke**]. So we briefly state some of the results we shall need, especially a "uniformization lemma" which appeared first in [**Lo1**], and which is a slight generalization of well-known uniformization results quoted in [**Mo**].

In §2, we apply the structural result on Δ_1^1 sets to the theory of Borel hierarchies of sets and functions. We give some effective analogs of well-known results of classical set theory, as separation or reduction of Borel sets, or Lebesgue's theorem on analytically representable functions.

In §3, we discuss how the effective results of §§1 and 2 can be used to solve the section problem, and the related problems discussed at the beginning of the introduction. We present the results in the most general situation, and give applications to more concrete cases.

0. Notations and prerequisites. In the sequel, we have chosen to follow the notations and terminology of Moschovakis' book **[Mo]**, not only for the descriptive theory, but also for the classical one. For example, the Borel classes are denoted by $\mathbf{\Sigma}^0_\xi$ and $\mathbf{\Pi}^0_\xi$. This corresponds in the classical terminology to sets of Borel class ξ, except for finite ξ: $\mathbf{\Sigma}^0_{n+1}$ sets are given class n in the classical terminology (see **[Ku]**).

Accordingly, we denote by $\mathbf{\Sigma}^1_1, \mathbf{\Pi}^1_1, \ldots,$ the levels of the projective hierarchy. ω is the set of integers, ω^ω the set of functions from ω into ω, called as usual reals, and denoted by letters $\alpha, \beta, \gamma, \ldots$. The class of ordinals is denoted by On, and ordinals by letters λ, η, ξ. $\aleph_1$ is the first uncountable ordinal, and ω_1 the first nonrecursive one.

In §§1 and 2, our work is developed within the frame given by Moschovakis' notion of recursively presented Polish space. Such a space is a structure $\langle X, d, r\rangle$, where X is a Polish space, d is a distance on X which generates the topology, and for which $\langle X, d\rangle$ is a complete metric space, and $r = (r_n)_{n\in\omega}$ is a sequence of points of X which is dense in X and satisfies the following condition of recursivity. The relations (on ω^4), $d(r_m, r_n) \leqslant p/(q+1)$ and $d(r_m, r_n) < p/(q+1)$, are recursive.

We shall not enter the general study of these structures. This is done in **[Mo]**. We generally abbreviate by X the recursively presented (r.p.) space, the corresponding recursive presentation $\langle d_X, r_X\rangle$ being understood. This is a bit incorrect, as two different recursive presentations on the same space may give two different theories of recursivity. It is particularly dangerous when working on classical Polish spaces, as $\omega, 2^\omega, \omega^\omega, [0,1], [0,1]^\omega, \ldots,$ but for these spaces we make the following convention. Except otherwise stated, they are supposed to be equipped with their usual recursive presentation. The same convention is made for product spaces $X \times Y$. (A precise definition of these presentations can be found in **[Mo]**.)

To each r.p. space X is canonically associated, via a fixed recursive enumeration of ω^4, a countable family of basic open balls $(N(n, X))_{n\in\omega}$. This allows to extend to these structures the classical notions of the effective theory on ω and ω^ω, as the arithmetical and analytical hierarchies of sets. Here we shall restrict our attention to the first two levels of the analytical hierarchy, i.e. Σ^0_1, Π^0_1, Σ^1_1, Π^1_1 and Δ^1_1 sets, and the corresponding relativized classes.

We shall be interested mainly in partial functions $f\colon X \to Y$, where X and Y are two r.p. spaces. For such a function, $f(x)\downarrow$ abbreviates the statement "f is defined at x", and the diagram D^f of f is defined by $D^f(x, n) \leftrightarrow f(x)\downarrow \wedge f(x) \in N(n, Y)$ (D^f is a subset of $X \times \omega$).

If Γ is a class of subsets of r.p. spaces, we say that f is a partial Γ-recursive function from X into Y if D^f is in Γ. For our purpose, this definition is relevant in two cases, when $\Gamma = \Delta^1_1$ and when $\Gamma = \Pi^1_1$. It is not hard to prove that a partial function from X into Y is Δ^1_1-recursive if and only if it is the restriction of a total

Δ_1^1-recursive function from X into Y to some Δ_1^1 subset of X, and if and only if its graph is Δ_1^1 in $X \times Y$ [**Mo**].

If f is a partial Π_1^1-recursive function from X into Y, f is not, in general, the restriction of a total Δ_1^1-recursive function to some Π_1^1 subset of X. But it can be seen that the restriction of f to any Δ_1^1 subset of its domain is Δ_1^1. So if the domain of f is Δ_1^1, hence in particular if f is total, f is Δ_1^1-recursive (see [**Mo**]). For readers familiar with the classical theory, the difference between Δ_1^1- and Π_1^1-recursive functions can be seen as the effective analog of the difference between Borel and bianalytic functions defined on an arbitrary metrizable separable space (not necessarily Polish). This is more than an analogy (see §3).

Similarly, if x is some point of the r.p. space X, we define the diagram D^x of x by $D^x(n) \leftrightarrow x \in N(n, X)$. D^x codes the set of basic neighbourhoods of x in X. If Γ is some class of subsets of r.p. spaces, we say x is Γ-recursive, or simply $x \in \Gamma$, if D^x is in Γ. The relevant classes here are Δ_1^1 and $\Delta_1^1(x)$. We remark that for the classical spaces as ω^ω, the usual recursive presentations are coherent, in the sense that $\alpha \in \omega^\omega$ is a Δ_1^1-recursive function from ω into ω if and only if it is a Δ_1^1-recursive element of ω^ω.

If f is a Π_1^1-recursive partial function from X into Y, and x is in the domain of f, then $f(x)$ is $\Delta_1^1(x)$ in Y. This gives a necessary condition in order to obtain uniformization results. It turns out to be also a sufficient condition, as it can be seen by the following theorem, which will be used repeatedly in the sequel.

Uniformization lemma [**Lo1**]. *Let X and Y be two r.p. spaces, and let A be a Π_1^1 subset of $X \times Y$. The set $A^+ = \{x \in X\colon \exists y \in \Delta_1^1(x)\, A(x, y)\}$ is Π_1^1, and there is a partial Π_1^1-recursive function f from X into Y which uniformizes A on A^+, i.e. such that f is defined on A^+ and for each $x \in A^+$, $(x, f(x)) \in A$. Moreover, if B is a Σ_1^1 subset of A^+, there is a total Δ_1^1-recursive function which uniformizes A on B.*

This lemma will be frequently used in case $A^+ = \pi_X(A)$, and even more when A^+ is all of X. But we shall need the full strength of it in §3.

1. The separation theorem for Σ_1^1 sets. In this section, X is a fixed recursively presented space, and we consider the family of all Δ_1^1 subsets of X. In order to code this family, it is possible to work along the lines described in the introduction. But this leads to coding by reals, and we prefer, for technical reasons, using a coding by integers. So we fix once and for all a coding pair (W, C) where W, the set of codes, is a Π_1^1 subset of ω, C is a Π_1^1 subset of $\omega \times X$ which is universal for Δ_1^1 subsets of X, with $\pi_\omega(C) = W$, and such that the relation $n \in W \wedge x \notin C_n$ is also Π_1^1. Such a coding pair exists [**Mo**].

We now define an operation on subsets of X, which we call the Δ_1^1-union.

Definition 1. A set $A \subset X$ is the Δ_1^1-union of a sequence $(B_n)_{n \in \omega}$, written $A = \bigcup_1^1 B_n$, if $A = \bigcup_n B_n$, and the subset B of $X \times \omega$, defined by $B(x, n) \leftrightarrow x \in B_n$, is in Δ_1^1. The notion of Δ_1^1-intersection is defined similarly.

As usual, if Γ is a family of sets, we denote by $\bigcup_1^1 \Gamma$ the family of all sets obtained by Δ_1^1-union performed on sequences of sets from Γ. Clearly, $\bigcup_1^1$ acts on Δ_1^1 sets and constructs Δ_1^1 sets.

DEFINITION 2. The Δ^1_1-recursive hierarchy $(\Sigma^*_\xi, \Pi^*_\xi)_{\xi \in \mathrm{On}}$ is defined by induction as follows:

$$\Sigma^*_0 = \Pi^*_0 = \{N(n, X) : n \in \omega\},$$

$$\Sigma^*_\xi = \overset{1}{\underset{1}{\bigcup}} \left(\bigcup_{\eta<\xi} \Pi^*_\eta \right)$$

and

$$\Pi^*_\xi = \neg \Sigma^*_\xi = \{A \subset X : X - A \in \Sigma^*_\xi\}.$$

From this definition, it is clear that $\Sigma^0_\xi \subset \Sigma^*_\xi \subset \mathbf{\Sigma}^0_\xi \cap \Delta^1_1$ and that $\Pi^0_\xi \subset \Pi^*_\xi \subset \mathbf{\Pi}^0_\xi \cap \Delta^1_1$, from which it follows that the induction stops at ω_1, and $\Delta^1_1 = \bigcup_{\xi<\omega_1} \Sigma^*_\xi = \bigcup_{\xi<\omega_1} \Pi^*_\xi$.

DEFINITION 3. For each recursive ordinal ξ, we define the set W_ξ of ξ-codes by $n \in W_\xi \leftrightarrow n \in W \wedge C_n \in \Pi^*_\xi$. (We could also have defined codes for the additive classes, but we do not need them in the sequel.)

PROPOSITION 4. *The relation $P(n, \alpha) \leftrightarrow \alpha \in WO \wedge n \in W_{|\alpha|}$ is Π^1_1 (in n and α). Hence, in particular, for each $\xi < \omega_1$ the sets W_ξ and $\bigcup_{\eta<\xi} W_\eta$ are Π^1_1.*

PROOF. The second statement is an immediate consequence of the first one. If ξ is recursive, choose some recursive $\alpha \in WO$ with $|\alpha| = \xi$. Then $n \in W_\xi \leftrightarrow P(n, \alpha)$ and $n \in \bigcup_{\eta<\xi} W_\eta \leftrightarrow \exists m\, P(n, \alpha_m)$ where α_m is the restriction of the ordering α to those integers which are smaller than m (with respect to α).

To prove the first statement, let ψ be the following Π^1_1 relation:

$$\psi(n, S) \leftrightarrow n \in W \wedge \Big[\exists m\, C_n = N(m, X)$$
$$\vee \Big(\exists \alpha \in \Delta^1_1\, \forall p\, \alpha(p) \in S \wedge X - C_n = \bigcup_p C_{\alpha(p)} \Big) \Big].$$

ψ is positive in S, hence it defines a Π^1_1 monotone operator P_ψ. By a result of Cenzer **[Ce]**, the relation $Q_\psi(n, \alpha) \leftrightarrow \alpha \in WO \wedge P_\psi^{|\alpha|}(n)$ is also Π^1_1, where P^ξ_ψ is defined inductively by $P^0_\psi(n) \leftrightarrow \psi(n, \varnothing)$ and $P^\xi_\psi(n) \leftrightarrow \psi(n, \bigcup_{\eta<\xi} P^\eta_\psi)$. So in order to prove the proposition, we just have to prove that for all ξ, $P^\xi_\psi = W_\xi$. We can restrict our attention to recursive ordinals.

The equality is true for $\xi = 0$, because of the equivalences

$$P^0_\psi(n) \leftrightarrow \psi(n, \varnothing) \leftrightarrow n \in W \wedge \exists m\, C_n = N(m, X) \leftrightarrow n \in W_0.$$

Suppose that we have proved the equality $P^\eta_\psi = W_\eta$ for all $\eta < \xi$. If $n \in P^\xi_\psi$, then $n \in W_0$ or there is some Δ^1_1 real α such that for all p, $\alpha(p) \in \bigcup_{\eta<\xi} P^\eta_\psi = \bigcup_{\eta<\xi} W_\eta$, and $X - C_n = \bigcup_p C_{\alpha(p)}$. As the relation $x \in C_{\alpha(p)}$ is Δ^1_1 (in x and p), it clearly implies that $C_n \in \Pi^*_\xi$, hence $n \in W_\xi$.

If $n \in W_\xi$, then by definition the set $X - C_n$ is the Δ^1_1-union of a sequence $(A_p)_{p \in \omega}$ of sets in $\bigcup_{\eta<\xi} \Pi^*_\eta$. Let $R(p, k) \leftrightarrow k \in \bigcup_{\eta<\xi} W_\eta \wedge A_p = C_k$. By the induction hypothesis $\bigcup_{\eta<\xi} W_\eta = \bigcup_{\eta<\xi} P^\eta_\psi$ is Π^1_1, and hence R is Π^1_1. Now $\forall p\, \exists k\, R(p, k)$, hence by the Uniformization Lemma, there is some Δ^1_1 real α such that for all p, $R(p, \alpha(p))$. Clearly this α witnesses that n belongs to P^ξ_ψ. $\square$

REMARK. Working along the same lines and using the Uniformization Lemma, it is not hard to see that the definition of the Δ_1^1-recursive hierarchy sketched in the introduction and the precise one given above lead to the same sets, i.e. that $\Sigma_\xi^* = \bigcup_{\alpha \in \Delta_1^1} \Sigma_\xi^0(\alpha)$ and $\Pi_\xi^* = \bigcup_{\alpha \in \Delta_1^1} \Pi_\xi^0(\alpha)$. The verification is left to the reader.

THEOREM A. *For all recursive ordinals ξ, $\Sigma_\xi^* = \mathbf{\Sigma}_\xi^0 \cap \Delta_1^1$ and $\Pi_\xi^* = \mathbf{\Pi}_\xi^0 \cap \Delta_1^1$.*

This theorem is an easy corollary of the following separation result.

THEOREM B. *Let ξ be a recursive ordinal, and let A and B be two Σ_1^1 subsets of X. If A is $\mathbf{\Sigma}_\xi^0$-separable from B, then A is Σ_ξ^*-separable from B.*

Theorem A easily follows from Theorem B. The inclusion $\Sigma_\xi^* \subset \mathbf{\Sigma}_\xi^0 \cap \Delta_1^1$ is obvious. Now if A is both Δ_1^1 and $\mathbf{\Sigma}_\xi^0$, A is Σ_1^1 and $\mathbf{\Sigma}_\xi^0$-separable from the Σ_1^1 set $X - A$. Theorem B then implies A is Σ_ξ^*.

Theorem B is proved by induction on ξ. Before we give the proof, we introduce a family of topologies on X, which will be our main tool.

DEFINITION 5. We define, for all ξ, $1 \leqslant \xi < \omega_1$, the topology T_ξ on X to be the topology generated by all sets which are both Σ_1^1 and in $\bigcup_{\eta<\xi} \mathbf{\Pi}_\eta^0$. Similarly, T_∞ is the topology on X generated by all Σ_1^1 subsets of X. The topology T_∞ is implicitly used in Harrington's paper **[Ha]**, via a notion of forcing previously considered by Gandy. (See our paper in **[GMS]**.) From the definition, T_1 is the usual topology on X, the topologies $(T_\xi)_{\xi<\omega_1}$ are increasing with ξ and coarser than T_∞. In the sequel, we always mention ξ when denoting the topological notions associated with T_ξ. We use the words ξ-open, ξ-closed, $\overline{A}^\xi$, int_ξ, and so on. We shall need the following proposition, which is implicit in Harrington's paper **[Ha]**.

PROPOSITION 6. *The space (X, T_∞) is a Baire space.*

PROOF. Let $(G_n)_{n\in\omega}$ be a sequence of ∞-dense ∞-open sets, and A a nonempty Σ_1^1 subset of X. We must prove that $A \cap \bigcap_n G_n \neq \varnothing$. To do that, we construct by recurrence a family $(F_n^m)_{n\leqslant m}$ of Π_1^0 subsets of $X \times \omega^\omega$ satisfying:

(i) For fixed n, $(F_n^m)_{m\geqslant n}$ is a decreasing family of Π_1^0 sets of diameter tending to 0.

(ii) $\pi_X(F_0^0) \subset A$ and for each n, $\pi_X(F_n^n) \subset G_n$.

(iii) For all m, $\bigcap_{n\leqslant m} \pi_X(F_n^m) \neq \varnothing$.

Suppose $(F_n^m)_{n\leqslant m\leqslant k}$ have been constructed. Let $A_k = \bigcap_{n\leqslant k} \pi_X(F_n^k)$. A_k is a nonempty Σ_1^1 subset of X, and by the ∞-density of G_{k+1}, there is a nonempty Σ_1^1 set $B_k \subset A_k \cap G_{k+1}$. Let F_{k+1}^{k+1} be a Π_1^0 subset of $X \times \omega^\omega$ such that $B_k = \pi_X(F_{k+1}^{k+1})$. Now, choose for each $n \leqslant k$ a basic open set $N(p_n, X \times \omega^\omega)$ of diameter less than $1/2^{k+1}$ such that $F_n^{k+1} = F_n^k \cap N(p_n, X \times \omega^\omega)$ satisfies $\bigcap_{n\leqslant k+1} F_n^{k+1} \neq \varnothing$. This is clearly possible. This defines the sequence $(F_n^m)_{n\leqslant m\leqslant k+1}$. The resulting sequence (F_n^m) clearly satisfies (i), (ii), and (iii).

As $X \times \omega^\omega$ is complete, by (i), for each n, $\bigcap_{n\leqslant m} F_n^m$ reduces to a singleton $\{(x_n, \alpha_n)\}$, and by (iii), $x_n = x$ does not depend on n. Now, by (ii), $x \in A$ and for each n, $x \in G_n$. So $A \cap (\bigcap_n G_n) \neq \varnothing$. $\square$

We now suppose Theorem B is known for all $\eta < \xi < \omega_1$.

LEMMA 7. *Let A be a Σ_1^1 subset of X. Then $\bar{A}^\xi$ is $\mathbf{\Pi}_\xi^0$ and Σ_1^1, and hence $\xi + 1$-open.*

PROOF. $\bar{A}^\xi$ is clearly $\mathbf{\Pi}_\xi^0$ (this is true for all sets A), as its complement is a countable union of ξ-open, hence $\mathbf{\Sigma}_\xi^0$ sets.

Now we have $x \notin \bar{A}^\xi \leftrightarrow \exists \eta < \xi\ \exists A' \in \Sigma_1^1 \cap \mathbf{\Pi}_\eta^0\ (x \in A' \wedge A' \cap A = \varnothing)$. Now if A' is Σ_1^1 and $\mathbf{\Pi}_\eta^0$ and disjoint from the Σ_1^1 set A, then by the induction hypothesis, A' is Π_η^*-separable from A. Hence we obtain

$$x \notin \bar{A}^\xi \leftrightarrow \exists n \in \bigcup_{\eta<\xi} W_\eta\ (x \in C_n \wedge C_n \cap A = \varnothing).$$

But this last relation is Π_1^1, hence $\bar{A}^\xi$ is Σ_1^1. □

For each set H, there is a largest ξ-open set G_H such that $H \cap G_H$ is ∞-meager. We set $\tilde{H}^\xi = X - G_H$. Clearly $\tilde{H}^\xi$ is ξ-closed and $H - \tilde{H}^\xi$ is ∞-meager. Actually we have

LEMMA 8. *If H is $\mathbf{\Pi}_\xi^0$, $\tilde{H}^\xi$ is equal to H modulo an ∞-meager set.*

PROOF. We have to prove that $\tilde{H}^\xi - H$ is ∞-meager. This is done by induction. If H is $\mathbf{\Pi}_1^0$, $\tilde{H}^1$ is contained in H. So suppose the lemma is known for all $\eta < \xi$. There is a sequence (H_n), with $H_n \in \mathbf{\Pi}_{\eta_n}^0$, $\eta_n < \xi$, such that $X - H = \bigcup_n H_n$. So

$$\tilde{H}^\xi - H \subset \tilde{H}^\xi \cap \bigcup_n H_n \subset \bigcup_n \left[(\tilde{H}^\xi \cap \tilde{H}_n^{\eta_n}) \cup (H_n - \tilde{H}_n^{\eta_n})\right].$$

Each set $H_n - \tilde{H}_n^{\eta_n}$ is ∞-meager, so we just have to show that each set $\tilde{H}^\xi \cap \tilde{H}_n^{\eta_n}$ is ∞-meager. Now $\tilde{H}^\xi$ is ξ-closed, and $\tilde{H}_n^{\eta_n}$ is η_n-closed, hence they are ∞-closed. So we have to check that $\tilde{H}^\xi \cap \tilde{H}_n^{\eta_n}$ is ∞-rare. Let A be a Σ_1^1 set contained in it. Then $\bar{A}^{\eta_n} \subset \tilde{H}_n^{\eta_n}$ and, by Lemma 7, $\bar{A}^{\eta_n}$ is $\eta_n + 1$-open, hence ξ-open. Moreover $\bar{A}^{\eta_n} \cap H \subset \tilde{H}_n^{\eta_n} \cap H \subset \tilde{H}_n^{\eta_n} - H_n$ is ∞-meager, and by the definition of $\tilde{H}^\xi$, $\bar{A}^{\eta_n} \cap \tilde{H}^\xi$ is empty. This clearly implies A is empty. □

LEMMA 9. *Let A and B be two Σ_1^1 subsets of X with A separable from B by a $\mathbf{\Sigma}_\xi^0$ set. Then $A \cap \bar{B}^\xi = \varnothing$.*

(This lemma is an intermediate separation result. In case $\xi = 1$, it is obvious. This explains why case $\xi = 1$ is easy to prove.)

PROOF. Let $H \in \mathbf{\Pi}_\xi^0$ separate B from A. We claim first that $\tilde{H}^\xi \supset B$. $B - \tilde{H}^\xi$ is an ∞-open set, and is included in $H - \tilde{H}^\xi$, which is ∞-meager by Lemma 8. By the Baire category theorem for T_∞, $B - \tilde{H}^\xi$ is empty. Next we claim $A \cap \bar{B}^\xi = \varnothing$. This set is, by Lemma 7, a Σ_1^1 subset of X. Now $B \subset \tilde{H}^\xi$, so $\bar{B}^\xi \subset \tilde{H}^\xi$; $A \cap H = \varnothing$, so by Lemma 8, $A \cap \tilde{H}^\xi$ is ∞-meager. Hence $A \cap \bar{B}^\xi$ is ∞-meager, and again by the Baire category theorem for T_∞, is empty. □

END OF PROOF OF THEOREM B. Let A and B be two Σ_1^1 subsets of X, with A $\mathbf{\Sigma}_\xi^0$-separable from B. By the induction hypothesis and Lemma 9, $A \cap \bar{B}^\xi = \varnothing$. Consider the relation

$$R(x, n) \leftrightarrow x \notin A \vee \left(n \in \bigcup_{\eta<\xi} W_\eta \wedge x \in C_n \wedge C_n \cap B = \varnothing\right).$$

From Proposition 4, R is a Π_1^1 relation. Now for each x in A, $x \notin \bar{B}^\xi$. Then there is some $\eta < \xi$ and some $A' \in \Sigma_1^1 \cap \mathbf{\Pi}_\eta^0$ such that $x \in A'$ and $A' \cap B = \varnothing$. By the induction hypothesis, we infer that there is some $A'' \in \Pi_\eta^*$ such that $A'' \supset A'$ and $A'' \cap B = \varnothing$. So we have proved that $\forall x \in A\ \exists n\ R(x, n)$. By the Uniformization Lemma, there is some Δ_1^1-recursive total function f: $X \to \omega$ such that $\forall x \in A\ R(x, f(x))$. The set $f(A)$ is Σ_1^1 in ω, and contained in the Π_1^1 set $D = \{n \in \bigcup_{\eta<\xi} W_\eta\colon C_n \cap B = \varnothing\}$. By the ordinary separation theorem for Σ_1^1 sets, there is some Δ_1^1 set E such that $f(A) \subset E \subset D$. Let $G(x, n) \leftrightarrow n \in E \wedge x \in C_n$. G is Δ_1^1 in $X \times \omega$ and the set $H = \bigcup_n G_n = \bigcup_{n\in E} C_n$ is clearly a Σ_ξ^* set separating A from B. $\square$

2. Effective properties of the Borel hierarchies. This section is devoted to the effectivization of some well-known results on the Borel hierarchies of sets and functions, with the help of the results of §1. We shall particularly be interested in separation and reduction results for Borel sets, and in the Lebesgue-Hausdorff theorem about analytically representable functions. Classical proofs of these results, particularly convenient to our purpose, can be found in [**Ku**]. When looking at these proofs, one can see at once that they are constructive, so they can easily be transformed in proofs for effective versions. We discuss that possibility in one example, the reduction property of $\mathbf{\Sigma}_\xi^0$ sets. If A and B are two $\mathbf{\Sigma}_\xi^0$ sets, $\xi > 1$, in some Polish space X, there exist two disjoint $\mathbf{\Sigma}_\xi^0$ sets A' and B' such that $A' \subset A$, $B' \subset B$ and $A' \cup B' = A \cup B$. From the proof in [**Ku**] it is clear that Borel ξ-codes for A' and B' are given effectively from Borel ξ-codes for A and B. Now by our Theorem A, if A and B are Δ_1^1, they admit Δ_1^1 ξ-codes. It is then easy to verify that the ξ-codes for A' and B' are also Δ_1^1.

This phenomenon is quite general. Almost all results of the classical theory of Borel hierarchies may be effectivized that way, using Theorem A (or sometimes Theorem B). Putting down all proofs in the effective context would be quite long and uninteresting, so we just write down the effective versions, with reference to a classical proof, and sometimes give indications for modifications which may be necessary.

It is worth noticing that this "Δ_1^1-recursive theory" we sketch here is not only a refinement of the classical theory. As it will be clear in §3, it leads to nontrivial results, even of classical type.

A. *Reduction and separation results for Borel sets.* We fix a r.p. space X. It is well known that many of the properties of the Borel classes are false in general for the first classes of the Borel hierarchy, depending on the properties of disconnectedness of the space X. We shall say that X is of type 0 if each basic open set $N(n, X)$ is also closed in X.

Theorem 1 (ω-reduction of $\mathbf{\Sigma}_\xi^0$ sets). *Let A be a Δ_1^1 subset of $X \times \omega$, with all its sections A_n, $n \in \omega$, in $\mathbf{\Sigma}_\xi^0$, for $\xi > 1$. There is a Δ_1^1 set B included in A, with all its sections B_n in $\mathbf{\Sigma}_\xi^0$, such that the sets B_n are disjoint and $\bigcup_n A_n = \bigcup_n B_n$. Moreover if X is of type* 0, *the result is also true for $\xi = 1$.*

Proof. See [**Ku**, II, §30, VII, Theorem 1]. $\square$

COROLLARY 2 (ω-SEPARATION FOR $\mathbf{\Pi}^0_\xi$ SETS). *If A is a Δ^1_1 subset of $X \times \omega$, with each section A_n in $\mathbf{\Pi}^0_\xi$, for $\xi > 1$ (or $\xi \geqslant 1$ if X is of type 0), and $\bigcap_n A_n = \varnothing$, then there is some Δ^1_1 set B containing A, with all its sections in $\mathbf{\Delta}^0_\xi$, such that $\bigcap_n B_n = \varnothing$.*

Corollary 2 easily implies the effective separation of $\mathbf{\Pi}^0_\xi$ sets. Now using the Uniformization Lemma, we may infer a uniformized version of it.

COROLLARY 3 (UNIFORM SEPARATION OF $\mathbf{\Pi}^0_\xi$ SETS). *If A and B are two disjoint Δ^1_1 subsets of $X \times \omega$, with their sections in $\mathbf{\Pi}^0_\xi$, for $\xi > 1$ ($\xi \geqslant 1$ if X is of type 0), there is a Δ^1_1 set C with sections in $\mathbf{\Delta}^0_\xi$ which separates A from B.*

It is also possible, along the same lines, to obtain structural results for ambiguous sets (i.e. sets in $\mathbf{\Delta}^0_\xi$).

THEOREM 4. *Let A be a Δ^1_1 and $\mathbf{\Delta}^0_\xi$ subset of X, for $\xi > 2$ ($\xi \geqslant 2$ if X is of type 0). There is a Δ^1_1 subset B of $X \times \omega$ such that for each n the section B_n is $\mathbf{\Delta}^0_{\eta_n}$ for some $\eta_n < \xi$, with $A = \bigcap_n \bigcup_m B_{n+m} = \bigcup_n \bigcap_m B_{n+m}$. Moreover if $\xi = \lambda + 1$, with limit λ, one can find B such that for each n, $\eta_n < \lambda$.*

PROOF. See [**Ku**, II, §30; IX, Theorems 1 and 2]. □

This structure result is the key step in the proof of the effective version of the Lebesgue-Hausdorff theorem.

Another important result in the theory of Borel sets concerns the "resolution in alternated series". An effective version of it for sets which are both $\mathbf{\Delta}^0_2$ and Δ^1_1 is due to Burgess [**Bu**]. For $\xi \geqslant 3$, the problem can be reduced to the case studied by Burgess using the method of [**Ku**, III, §37; II and III], with the aid of the notion of generalized homeomorphisms.

THEOREM 5. *For each A in $\mathbf{\Delta}^0_{\xi+1} \cap \Delta^1_1$, $\xi \geqslant 2$, there is a Δ^1_1 and closed subset F of ω^ω, and a Δ^1_1-recursive function f from F onto X which is injective and continuous, such that for each open subset G of ω^ω, $f(G) \in \mathbf{\Sigma}^0_\xi$ and such that $f^{-1}(A)$ is both $\mathbf{\Delta}^0_2$ and Δ^1_1 in ω^ω.*

PROOF. See [**Ku**, III, §37; II, Theorem 1]. □

From Theorem 5 and Burgess' result, it is not hard to infer the following result.

THEOREM 6 (RESOLUTION IN ALTERNATED SERIES). *Let ξ be a recursive ordinal, $\xi \geqslant 1$, and let A be a Δ^1_1 and $\mathbf{\Delta}^0_{\xi+1}$ subset of X. There is a recursive real $\alpha \in WO$, of length $|\alpha| = \lambda$, and Δ^1_1 sets C and C' in $X \times \omega$, such that if $C_\eta = \{x \in X\colon (x, n) \in C\}$, and $C'_\eta = \{x \in X\colon (x, n) \in C'\}$, for $\eta < \lambda$, $\eta = |\alpha_n|$, then*

(i) *C_η and C'_η are in $\mathbf{\Pi}^0_\xi$.*
(ii) *If $\eta < \eta'$, $C_\eta \supset C'_\eta \supset C_{\eta'} \supset C'_{\eta'}$.*
(iii) *$\bigcap_{\eta<\lambda} C_\eta = \varnothing$.*
(iv) *$A = \bigcup_{\eta<\lambda} (C_\eta - C'_\eta)$.*

B. *The hierarchy of Δ^1_1-recursive functions.* The hierarchy of Borel functions from some Polish space X into some Polish space Y is usually defined via the inverse images of open sets. The family B_ξ of functions of class ξ is defined by: $f \in B_\xi$ if for each open subset G of Y, $f^{-1}(G)$ is in Σ_ξ^0. (We remark that this definition does not agree with the classical one for finite ξ. Here continuous functions are given the class 1.)

There is another classification related to the operation of taking pointwise limits of sequences of functions. Starting from B_1 in case X is totally disconnected or $Y = [0, 1]$, and from B_2 in the general case, it leads to the hierarchy of analytically representable functions and the Lebesgue-Hausdorff theorem. (See [**Ku**, II, §31; VIII; IX].) If we restrict our attention to Δ_1^1-recursive functions, this classification must be adapted. We introduce the notion of Δ_1^1-limit. We say that a function f is the Δ_1^1-limit of a sequence $(f_n)_{n\in\omega}$ of functions from X into Y (where X and Y are r.p. spaces), if f is the limit of the sequence (f_n) and if the function g: $X \times \omega \to Y$, defined by $g(x, n) = f_n(x)$, is Δ_1^1-recursive.

THEOREM 1. *Suppose Y is compact. Let f be a Δ_1^1-recursive function, from X into Y, of Borel class $\xi + 1$, for $1 < \xi < \omega_1$. Then f is the Δ_1^1-limit of a sequence (f_n) of functions of Borel class ξ. Moreover if ξ is a limit ordinal, each function f_n can be chosen of class less than ξ.*

If X is of type 0, the conclusion is also true for $\xi = 1$.

For proving this theorem, the general case is first reduced to the case of finite Y; this only uses the fact that Y is compact, and the uniform separation corollary of §2A (see [**Ku**, II, §31; VIII, Theorem 3]). The case of finite Y is then proved by an easy application of Theorem 4 (§2A). (See [**Ku**, II, §31; VIII, Theorem 4].) □

Theorem 1, as the Lebesgue-Hausdorff theorem, is false in general for functions in B_2. Take for example $X = [0, 1]$, $Y = \{0, 1\}$, and for f the characteristic function of $\{0\}$. Clearly f is Δ_1^1 and of class 2, but is not the limit of a sequence of continuous functions from X into Y, as such functions are constant.

A particular case when Theorem 1 is true for case $\xi = 1$ is when X is of type 0, as quoted in the statement of Theorem 1. But there is also another important case when $Y = [0, 1]$. This is related to the effective normality of r.p. spaces.

PROPOSITION 2. *Let X be a r.p. space. Then X is Δ_1^1-normal, i.e. for each pair of disjoint Δ_1^1 and closed subsets F_1 and F_2 of X, there exist disjoint Δ_1^1 and open sets G_1 and G_2 such that $F_1 \subset G_1$ and $F_2 \subset G_2$.*

PROOF. By a result in [**Mo**], there exist two Δ_1^1 reals α and β such that $X - F_2 = \bigcup_n N(\alpha(n), X) = \bigcup_n \overline{N(\alpha(n), X)}$ and $X - F_1 = \bigcup_n N(\beta(n), X) = \bigcup_n \overline{N(\beta(n), X)}$. Set

$$G_1(x, n) \leftrightarrow x \in N(\alpha(n), X) \wedge \forall k < n\; x \notin \overline{N(\beta(k), X)},$$

and

$$G_2(x, n) \leftrightarrow x \in N(\beta(n), X) \wedge \forall k \leqslant n\; x \notin \overline{N(\alpha(k), X)}.$$

Then $G_1 = \bigcup_n G_1(n)$ and $G_2 = \bigcup_n G_2(n)$ are Δ_1^1 and open sets, $G_1 \cap G_2 = \varnothing$, and $F_1 \subset G_1$ and $F_2 \subset G_2$. □

Using this result and the methods of proof of Urisohn's lemma and Tietze theorem, we can infer the following results.

PROPOSITION 3. *Let A and B be two disjoint Δ_1^1 closed sets in X. There is a Δ_1^1-recursive and continuous function $f\colon X \to [0, 1]$ such that $f(x) = 0$ for $x \in A$ and $f(x) = 1$ for $x \in B$.*

PROPOSITION 4. *Let f be a Δ_1^1-recursive and continuous function from a closed and Δ_1^1 subset A of X into $[0, 1]$. There is a total Δ_1^1-recursive and continuous function $\bar{f}$: $X \to [0, 1]$ which extends the function f.*

THEOREM 5. *A Δ_1^1-recursive function from X into $[0, 1]$ belongs to B_2 (i.e. is of the first class, in the classical terminology) if and only if it is the Δ_1^1-limit of a sequence of continuous functions from X into $[0, 1]$.*

PROOF. See [**Ku**, II, §31; VIII, Theorem 7].

C. *The relativized classes*. Until now, in §§1 and 2, we restricted our attention to the class Δ_1^1, mainly for simplicity of notations. But clearly all our proofs work as well for the relativized classes $\Delta_1^1(x)$.

Define, for x in some r.p. space X, the $\Delta_1^1(x)$-recursive hierarchy on some r.p. space Y by closing successively the canonical basis of Y under complementation and $\Delta_1^1(x)$-union (with an obvious definition). The closure ordinal of this hierarchy is ω_1^x, the first ordinal nonrecursive in x, and the hierarchy $(\Pi_\xi^*(x), \Sigma_\xi^*(x))_{\xi<\omega_1^x}$ obtained that way satisfies the analogs of Theorems A and B.

THEOREM A′. *For all ξ, $1 \leqslant \xi < \omega_1^x$, $\Sigma_\xi^*(x) = \mathbf{\Sigma}_\xi^0 \cap \Delta_1^1(x)$ and $\Pi_\xi^*(x) = \mathbf{\Pi}_\xi^0 \cap \Delta_1^1(x)$.*

THEOREM B′. *If A and B are two $\Sigma_1^1(x)$ subsets of Y, and A is $\mathbf{\Sigma}_\xi^0$-separable from B, then A is $\Sigma_\xi^*(x)$-separable from B.*

From these two theorems, one can derive the relativized versions of all the results stated in §§2A and 2B. The precise statements are left to the reader. They will be used in §3 in order to derive noneffective results in product spaces. It will be done by using the Uniformization Lemma, and a particular uniform coding of the classes $\Pi_\xi^*(x)$. We fix from now on a pair $\langle \mathbf{W}, \mathbf{C}\rangle$ of Π_1^1 relations, $\mathbf{W} \subset X \times \omega$, $\mathbf{C} \subset X \times \omega \times Y$, such that for all x in X, $\pi_\omega(\mathbf{C}(x)) = \mathbf{W}(x)$, $\mathbf{C}(x)$ is universal for $\Delta_1^1(x)$ subsets of Y, and the relation $\mathbf{W}(x, n) \wedge \neg\mathbf{C}(x, n, y)$ is Π_1^1. If we then define the set W_ξ^x of codes for $\Pi_\xi^*(x)$ sets by the relation $n \in W_\xi^x \leftrightarrow \mathbf{W}(x, n) \wedge \mathbf{C}_{x,n} \in \Pi_\xi^*(x)$, then by the same argument as in the proof of Proposition 4 of §1, we obtain that the relation (in x, n, and α), $\alpha \in WO \wedge n \in W_{|\alpha|}^x$ is Π_1^1.

3. Borel hierarchies in product spaces. Let (X, M) be a measurable space, that is a set X equipped with a σ-algebra M of subsets of X. We denote by $A(M)$ the result of the Suslin operation performed on elements of M, and by bi $A(M)$ the family of sets B in $A(M)$ such that $X - B$ also belongs to $A(M)$. bi $A(M)$ is a σ-algebra containing M, and bi $A(\text{bi } A(M)) = \text{bi } A(M)$. When X is metrizable separable, and M is the family of Borel subsets of X, sets in $A(M)$, resp. bi $A(M)$, are just called analytic, resp. bianalytic.

Let Y be a Polish space, with open basis U. We denote by $M \otimes U$ the σ-algebra on $X \times Y$ generated by $M \times U = \{B \times G\colon B \in M,\ G \in U\}$, and we let $N = \mathrm{bi}\, A(M \otimes U)$. It is easy to check that $M \otimes U \subset \mathrm{bi}\, A(M) \otimes U \subset N = \mathrm{bi}\, A(\mathrm{bi}\, A(M) \otimes U)$. In general $\mathrm{bi}\, A(M) \otimes U \neq N$, for it is easy to prove that if B is in $\mathrm{bi}\, A(M) \otimes U$, the sections B_x, $x \in X$, are of bounded Borel class below $\aleph_1$, a property which in general is not satisfied by all sets in N.

We define two hierarchies among N. The first one is related to global properties of sets in N, and the second one to properties of sections.

We define first $N_0 = \mathrm{bi}\, A(M) \times U$, $N_\xi^\Sigma = (\bigcup_{\eta<\xi} N_\eta^\Pi)_\sigma$ and $N_\xi^\Pi = \neg N_\xi^\Sigma$, where σ denotes the closure under countable unions. Next we define $S_0 = N_0$ and $S_\xi^\Sigma = \{B \in N\colon\ \forall x \in X\ \ B_x \in \Sigma_\xi^0\}$ and $S_\xi^\Pi = \neg S_\xi^\Sigma$. Then $S = \bigcup_{\xi<\aleph_1} S_\xi^\Sigma = \bigcup_{\xi<\aleph_1} S_\xi^\Pi$ is the family of elements of N with sections of bounded Borel class below $\aleph_1$.

The abstract version of the section problem stated in the introduction is answered positively by the following theorem.

THEOREM 1. *For all $\xi < \aleph_1$, $S_\xi^\Sigma = N_\xi^\Sigma$ and $S_\xi^\Pi = N_\xi^\Pi$. Hence $S =$* $\mathrm{bi}\, A(M) \otimes U$.

Similarly, we can prove a separation result.

THEOREM 2. *Let ξ be a countable ordinal, A and B two elements of $A(M \otimes U)$. If for each x in X the section A_x is Σ_ξ^0-separable from the section B_x, then A is separable from B by a set in N_ξ^Σ.*

Clearly for all ξ, N_ξ^Σ is contained in S_ξ^Σ. Hence Theorem 1 is an easy corollary of Theorem 2. The proof of this theorem is made by successive reductions of the problem to simpler cases. The first step consists in replacing the abstract space X by a metrizable separable space $\bar{X}$.

LEMMA 3. *Let A and B be two sets in $A(M \otimes U)$. There is a measurable mapping ψ from (X, M) into $(2^\omega, \Delta_1^1)$, such that, denoting by $\bar{\psi}$ the application $\bar{\psi}(x,y) = (\psi(x),y)$, and $\bar{X} = \psi(X)$, then $\bar{\psi}(A)$ and $\bar{\psi}(B)$ are analytic in $\bar{X} \times Y$.*

PROOF. This is a well-known result due to Marczewski. If A and B are in $A(M \otimes U)$, they are in $A(M' \otimes U)$ for some countably generated sub-σ-algebra M' of M. Let B_n, $n \in \omega$, generate M', and define ψ by $\psi(x) = \{n\colon x \in B_n\}$. Then clearly ψ satisfies the requirements of the lemma. $\square$

The function ψ of the lemma is clearly $\mathrm{bi}\, A(M) - \mathrm{bi}\, A(\Delta_1^1(\bar{X}))$ measurable, hence by inverse images $\bar{\psi}$ maps all classes N_ξ^Σ, N_ξ^Π, S_ξ^Σ, S_ξ^Π, defined from $(\bar{X}, \Delta_1^1)$ into the corresponding classes defined from (X, M). Thus, the section problem is reduced to the case when X is a subset of 2^ω, equipped with the σ-algebra of its Borel subsets, and A and B are two analytic subsets of $X \times Y$, that is the traces on $X \times Y$ of two Σ_1^1 subsets A' and B' of $2^\omega \times Y$.

We may suppose without loss of generality that Y is an r.p. space (by considering it if necessary as a G_δ subset of $[0, 1]^\omega$), that ξ is a recursive ordinal, and that A' and B' are Σ_1^1 in $2^\omega \times Y$ (the relativized result being proved similarly).

The next step consists in replacing X by a Π_1^1 subset of 2^ω.

LEMMA 4. *Let ξ be a recursive ordinal, and A and B two Σ_1^1 subsets of $2^\omega \times Y$. Then $\bar{X} = \{x \in 2^\omega : A_x$ is $\boldsymbol{\Sigma}_\xi^0$-separable from $B_x\}$ is Π_1^1 in 2^ω.*

PROOF. By the relativized version of Theorem B (see §2C),

$$x \in \bar{X} \leftrightarrow \exists n \in W_\xi^x (A_x \subset \mathbf{C}_{x,n} \wedge \mathbf{C}_{x,n} \cap B_x = \varnothing).$$

By the properties of the coding $\langle \mathbf{W}, \mathbf{C} \rangle$, $\bar{X}$ is Π_1^1. □

PROOF OF THEOREM 2. Suppose Theorem 2 is proved for all $\eta < \xi$. By Lemmas 3 and 4, we may assume that X is a Π_1^1 subset of 2^ω and A and B are two Σ_1^1 subsets of $2^\omega \times Y$, such that for all x in X, A_x is $\boldsymbol{\Sigma}_\xi^0$-separable from B_x.

Let β be a recursive real such that for each n, $\beta_n \in WO$ and the sequence $(\beta_n)_{n \in \omega}$ is nondecreasing with $\sup_n (\beta_n + 1) = \xi$. We define R by

$$R(x, \alpha) \leftrightarrow x \in X \wedge \forall n\, \alpha(n) \in W_{|\beta_n|}^x \wedge A_x \subset \bigcup_n \mathbf{C}_{x, \alpha(n)}$$
$$\wedge \bigcup_n \mathbf{C}_{x,\alpha(n)} \cap B_x = \varnothing.$$

R is a Π_1^1 relation, and by Theorem B′, $\forall x \in X\ \exists \alpha \in \Delta_1^1(x) R(x, \alpha)$. By the Uniformization Lemma, there is a partial Π_1^1-recursive function f: $2^\omega \to \omega^\omega$ such that f is defined on X and $\forall x \in X\ R(x, f(x))$.

Let $C_n = \{(x, y): x \in X \wedge y \in \mathbf{C}_{x, f(x)(n)}\}$, and $C = \bigcup_n C_n$. By the definition of R, C separates $A \cap X \times Y$ from $B \cap X \times Y$. Now, as f is Π_1^1-recursive, each C_n is Π_1^1 and the relation $x \in X \wedge y \notin \mathbf{C}_{x, f(x)(n)}$ is also Π_1^1, hence

$$C_n \in \text{bi}\, A(\Delta_1^1(X \times Y)).$$

Finally each $C_{n,x} = \mathbf{C}_{x, f(x)(n)}$ is in $\boldsymbol{\Pi}_{|\beta_n|}^0$, so $C_n \in S_{|\beta_n|}^\Pi$. But by the induction hypothesis $S_{|\beta_n|}^\Pi = N_{|\beta_n|}^\Pi$, so C is in N_ξ^Σ, and $A \cap X \times Y$ is N_ξ^Σ-separable from $B \cap X \times Y$. □

REMARK. The preceding proof shows that Theorems 1 and 2 may be improved into effective results in case X is a Π_1^1 subset of an r.p. space E. In this case, say that a subset A of X is bi Σ_1^1 in X if A and $X - A$ are Π_1^1 in E. This is clearly the effective analog of the notion of bianalytic set. If f is a partial Π_1^1-recursive function from E into Y, and $X = \text{dom}(f)$, then Graph(f) is bi Σ_1^1 in $X \times Y$. Conversely, if f is defined on some Π_1^1 subset X of E, and Graph(f) is bi Σ_1^1 in $X \times Y$, then f is a partial Π_1^1-recursive function from E into Y, for it implies that for each x in X, $f(x)$ is $\Delta_1^1(x)$ and then

$$D^f(x, n) \leftrightarrow x \in X \wedge \exists y \in \Delta_1^1(x)((x, y) \in \text{Graph}(f) \wedge y \in N(n, Y))$$

is Π_1^1. So the notion of partial Π_1^1-recursive function is the effective analog of the notion of function which is bianalytic on its domain.

The effective version of Theorems 1 and 2 is the following.

THEOREM 5. *Let ξ be a recursive ordinal, X a Π_1^1 subset of some r.p. space E and Y an r.p. space.*

(i) *Let A and B be two Σ_1^1 subsets of $E \times Y$, and suppose that for each x in X, A_x is $\mathbf{\Sigma}_\xi^0$-separable from B_x. Then there is a* bi Σ_1^1 *subset C of $\omega \times X \times Y$ such that for each n, the sections $C_{n,x}$, for x in X, are in $\mathbf{\Pi}_{\eta_n}^0$, for some $\eta_n < \xi$, and the set $D = \bigcup_n C_n$ separates $A \cap (X \times Y)$ from $B \cap (X \times Y)$.*

(ii) *In particular, if B is a* bi Σ_1^1 *subset of $X \times Y$ with all its sections in $\mathbf{\Sigma}_\xi^0$, then there is a set C as above such that $B = \bigcup_n C_n$.*

The case when X is Polish (resp. when X is an r.p. space for the effective results) is a bit simpler, as the family N of bi $A(\mathbf{\Delta}_1^1)$ subsets of $X \times Y$ reduces to the family of its Borel subsets (resp. the family of bi Σ_1^1 subsets of $X \times Y$ reduces to that of Δ_1^1 sets). Then in this case S_ξ^Σ is the family of Borel sets with sections in $\mathbf{\Sigma}_\xi^0$, and N_ξ^Σ is the family of Borel sets obtained at the ξth stage when closing the family of rectangles $B \times G$, where B is Borel in X and G is a basic open set in Y, by countable union and complementation.

We suppose for the rest of this section that X is Polish. Then it is not difficult to extend all classical results on Borel hierarchies to the hierarchy $(S_\xi^\Sigma, S_\xi^\Pi)$ we have introduced.

One way to do this is to mimic the classical proofs, and use Theorem 1 of §3, in the same manner we did in §2. Another possible way is to use directly the results of §2 together with the Uniformization Lemma, as in the proof of Theorems 1 and 2. We sketch here a third method, which seems to be of some interest in many problems of functional analysis. We first restate Theorem 1 in case of a Polish space X in a somewhat different manner.

Let $X_0 = (X, T_0)$ be a Polish space, and suppose T_1 is a finer topology on X such that $X_1 = (X, T_1)$ is also Polish. Then an easy application of the Suslin-Lusin theorem on continuous injective images of Borel sets shows that T_1 is generated by a family $(B_n)_{n\in\omega}$ of Borel sets of X_0, and X_0 and X_1 have the same Borel sets. This statement has a sort of converse. Suppose $(B_n)_{n\in\omega}$ is a sequence of Borel subsets of $X_0 = (X, T_0)$. Then there is a finer Polish topology T_1 on X such that each B_n is open in $X_1 = (X, T_1)$. Using these remarks, Theorem 1 in case X is Polish may be restated as follows.

THEOREM 6. *Let $X_0 = (X, T_0)$ and Y be two Polish spaces. Then a set A included in $X_0 \times Y$ is a Borel set with sections in $\mathbf{\Sigma}_\xi^0$ if and only if there is a finer topology T_1 on X such that $X_1 = (X, T_1)$ is Polish, and A is $\mathbf{\Sigma}_\xi^0$ in $X_1 \times Y$.*

PROOF. If A is $\mathbf{\Sigma}_\xi^0$ in $X_1 \times Y$, then all sections of A are $\mathbf{\Sigma}_\xi^0$ in Y, and A is Borel in $X_0 \times Y$ by the preceding remarks. Conversely if A is in S_ξ^Σ, then by Theorem 1, A is, for some family (B_n) of Borel subsets of X_0, in the additive ξth class obtained from the rectangles $B_n \times G$, G open in Y, by using countable union and complementation. So, if T_1 is a finer Polish topology on X such that all sets B_n are open for T_1, then A is $\mathbf{\Sigma}_\xi^0$ in $X_1 \times Y$. □

As an example of use of Theorem 6, we state the analog for product spaces of the Lebesgue-Hausdorff theorem on analytically representable functions. Let X, Y

and Z be Polish spaces, and let f be a Borel function from $X \times Y$ into Z. We say that f is partially of class ξ if for each x in X, the partial function f_x: $Y \to Z$ is of class ξ.

THEOREM 7. *Let f: $X \times Y \to Z$ be partially of class $\xi + 1$, for $\xi > 1$ (or $\xi \geqslant 1$ if Y is totally disconnected or $Z = [0, 1]$). Then f is the pointwise limit of a sequence of Borel functions which are partially of class at most ξ (less than ξ if ξ is limit).*

PROOF. Let $(A_n)_{n \in \omega}$ be a basis of open sets of Z, and $B_n = f^{-1}(A_n)$. By the hypothesis, each B_n is Borel with sections in $\Sigma^0_{\xi+1}$. So by Theorem 6, we can refine the topology T_0 of X into T_1 such that each B_n becomes $\Sigma^0_{\xi+1}$ in $X_1 \times Y$, hence f becomes of class $\xi + 1$ from $X \times Y$ into Z. (In the case $\xi = 1$ and Y is totally disconnected, we can also choose T_1 totally disconnected.) We can then apply the Lebesgue-Hausdorff theorem. f is the pointwise limit of a sequence of Borel functions from $X_1 \times Y$ into Z which are of class at most ξ (less than ξ if ξ is limit). But such a sequence clearly satisfies the conclusions of the theorem. $\square$

As an immediate corollary, we obtain the following result of conservation of the Borel class under integration. (The particular case $\xi = 2$ is due to Bourgain [**Bo1**].)

COROLLARY 8. *Let μ be a probability measure on X, and let f be a Borel function from $X \times Y$ into $[0, 1]$ which is partially of class $\xi + 1$. Then the function F: $Y \to [0, 1]$ defined by $F(y) = \int f(x, y) d\mu(x)$ is of class $\xi + 1$.*

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