NONEXISTENCE OF CONTINUOUS SELECTIONS
OF THE METRIC PROJECTION FOR A CLASS OF
WEAK CHEBYSHEV SPACES

BY

MANFRED SOMMER

ABSTRACT. The class \( \mathcal{B} \) of all those \( n \)-dimensional weak Chebyshev subspaces of
\( C[a, b] \) whose elements have no zero intervals is considered. It is shown that there
does not exist any continuous selection of the metric projection for \( G \) if there is a
nonzero \( g \) in \( G \) having at least \( n + 1 \) distinct zeros. Together with a recent result of
Nürnberger-Sommer, the following characterization of continuous selections for \( \mathcal{B} \)
is valid: There exists a continuous selection of the metric projection for \( G \) in \( \mathcal{B} \) if
and only if each nonzero \( g \) in \( G \) has at most \( n \) distinct zeros.

If \( G \) is a nonempty subset of a normed linear space \( E \), then for each \( f \) in \( E \) we
define \( P(f) := \{ g_0 \in G \mid \| f - g_0 \| = \inf \{ \| f - g \| \mid g \in G \} \} \). \( P \) defines a set-val-
ued mapping of \( E \) into \( 2^G \) which in the literature is called the metric projection onto
\( G \). A continuous mapping \( s \) of \( E \) onto \( G \) is called a continuous selection for the
metric projection \( P \) (or, more briefly, continuous selection) if \( s(f) \) is in \( P(f) \) for each
\( f \) in \( E \). In this paper we treat the problem of the existence of continuous selections
for \( n \)-dimensional subspaces \( G \) of \( C[a, b] \), with \( C[a, b] \), as usual, the Banach space
of real-valued continuous functions on \( [a, b] \) under the uniform norm.

A. Lazar, P. Morris and D. Wulbert [3] have characterized the 1-dimensional
subspaces of \( C(X) \) with \( X \) compact Hausdorff, which admit a continuous selection.
They have raised the problem of characterizing the corresponding \( n \)-dimensional
subspaces.

Using the kind of selection established by Lazar-Morris-Wulbert, it does not
seem possible to get a general theorem for \( n \)-dimensional subspaces of \( C[a, b] \).
With new methods, however, and in the setting of weak Chebyshev subspaces,
Nürnberger-Sommer [4], [5] and Sommer [7], [8] have given both sufficient condi-
tions for the existence of continuous selections and characterization theorems of
the existence of continuous selections for several classes of \( n \)-dimensional weak
Chebyshev subspaces of \( C[a, b] \) (see also Nürnberger [6]).

In the following we refer to a result in [4].

Nürnberger-Sommer have shown that for those weak Chebyshev spaces \( G \) whose
elements \( g \ (g \neq 0) \) have at most \( n \) distinct zeros on \( [a, b] \), there exists exactly one
continuous selection.

Here we show that for those weak Chebyshev spaces \( G \) which have no elements
vanishing on intervals but which have elements \(g \ (g \not\equiv 0)\) with at least \(n + 1\) distinct zeros, there does not exist any continuous selection. To prove this result we apply a fundamental lemma of Lazar-Morris-Wulbert.

Hence we have the following characterization of the existence of continuous selections for those \(n\)-dimensional weak Chebyshev subspaces \(G\) of \(C[a, b]\) whose elements do not vanish on intervals.

There exists a continuous selection for \(G\) if and only if each \(g\) in \(G\) has at most \(n\) distinct zeros.

In the following let \(G\) be an \(n\)-dimensional subspace of \(C[a, b]\).

1. **Definition.** \(G\) is called weak Chebyshev if each \(g\) in \(G\) has at most \(n - 1\) changes of sign, i.e. there do not exist points \(a < x_0 < x_1 < \cdots < x_n < b\) such that \(g(x_i) \cdot g(x_{i+1}) < 0, \ i = 0, \ldots, n - 1\).

R. C. Jones and L. A. Karlovitz have characterized these spaces. For this characterization we need the following definition.

2. **Definition.** If \(f\) is in \(C[a, b]\), then \(g\) in \(P(f)\) is called an *alternation element* (AE) of \(f\) if there exist \(n + 1\) distinct points \(a < x_0 < x_1 < \cdots < x_n < b\) such that \(e(-1)^i(f - g)(x_i) = \|f - g\|, \ i = 0, \ldots, n, e = \pm 1\). The points \(x_0, \ldots, x_n\) are called alternating extreme points of \(f - g\).

Jones and Karlovitz \([1]\) have proved the following theorems.

3. **Theorem.** \(G\) is weak Chebyshev if and only if for each \(f\) in \(C[a, b]\) there exists at least one AE in \(P(f)\).

4. **Theorem.** \(G\) is weak Chebyshev if and only if given \(a = x_0 < x_1 < \cdots < x_{n-1} < x_n = b\) there exists a \(g\) in \(G, g \not\equiv 0\), such that

\[
(-1)^{i+1}g(x) > 0, \quad x_{i-1} < x < x_i, \ i = 1, \ldots, n.
\]

In order to show the nonexistence of a continuous selection for a class of weak Chebyshev spaces, we need the following standard definition.

5. **Definition.** A zero \(x_0\) of \(f\) in \(C[a, b]\) is said to be a *simple zero* if \(f\) changes sign at \(x_0\) or if \(x_0 = a\) or \(x_0 = b\). A zero \(x_0\) of \(f\) in \(C[a, b]\) is said to be a *double zero* if \(f\) does not change sign at \(x_0\) and \(x_0\) is in \((a, b)\). Let \(Z(f) := \{x \in [a, b] | f(x) = 0\}\) the set of zeros of \(f\).

We denote by \(|Z(f)|\) the number of distinct zeros of \(f\) and by \(|Z^*(f)|\) the number of zeros of \(f\), counting simple zeros as one zero and double zeros as two zeros.

6. **Definition.** Let \(x_1, x_2\) be zeros of \(f\) in \([a, b]\). These zeros are said to be separated if there is a \(y_1\) in \([a, b]\) with

\[
x_1 < y_1 < x_2, \quad f(y_1) \not= 0.
\]

A zero \(x_0\) of \(f\) in \(G\) is said to be an *essential* zero with respect to \(G\), if there is a \(g\) in \(G\) with \(g(x_0) \not= 0\).

Nürnberger-Sommer \([4]\) have given the following sufficient condition for the existence of a continuous selection.
7. Theorem. Let $G$ be weak Chebyshev. Let $|Z(g)| < n$ for each $g$ in $G$, $g \not= 0$. Then there exists exactly one continuous selection.

Since $|Z(g)| < n$ for each $g$ in $G$, $g \not= 0$, no $g$ in $G$ has a zero interval. We will show that for those weak Chebyshev spaces $G$ which have no elements with zero intervals but which have elements $g$ with $|Z(g)| > n + 1$, there does not exist any continuous selection. For this we need the following lemmas.

8. Lemma (Lazar-Morris-Wulbert [3]). If $s$ is a continuous selection of $C[a, b]$ onto $G$ and $f$ is in $C[a, b]$, $||f|| = 1$ and $0$ is in $P(f)$, then there is a $g_0$ in $P(f)$ such that

1. for every $x \in \text{bd } Z(P(f)) \cap f^{-1}(1)$ and every $g$ in $P(f)$ there is a neighborhood $U$ of $x$ for which $g_0 > g$ on $U$ and
2. for every $x \in \text{bd } Z(P(f)) \cap f^{-1}(-1)$ and every $g$ in $P(f)$ there is a neighborhood $V$ of $x$ for which $g_0 < g$ on $V$.

Here let $Z(P(f)) := \{x \in [a, b] | g(x) = 0 \text{ for each } g \in P(f)\}$ and $\text{bd } Z(P(f))$ is the set of boundary points of $Z(P(f))$ under the topology of $[a, b]$.

9. Lemma (Stockenberg [9]). Let $G$ be weak Chebyshev. Then no $g$ in $G$ has more than $n$ separated, essential zeros and if there is a $g$ in $G$ with $n$ separated, essential zeros $x_1 < \cdots < x_n$, then $g(x) = 0$ for all $x$ in $[a, x_1] \cup [x_n, b]$.

Now we are able to prove the nonexistence of a continuous selection for a class of weak Chebyshev spaces.

10. Theorem. Let $G$ be weak Chebyshev. Let no $g$ in $G$ ($g \not= 0$) have a zero interval but let a nontrivial function $g_0$ be in $G$ such that $|Z(g_0)| > n + 1$. Then there does not exist any continuous selection.

Proof. Let $a < z_0 < z_1 < \cdots < z_n < b$ be $n + 1$ distinct zeros of $g_0$.

First case: $a < z_0$, $z_n < b$. We will construct a function $g_0$ having exactly $n - 1$ zeros with changes of sign and two further zeros on $[a, b)$. Since $z_0, \ldots, z_n$ are separated zeros of $g_0$, by Lemma 9, there are two points $z_i, z_j \in \{z_0, \ldots, z_n\}, i < j$, such that

$$g(z_i) = g(z_j) = 0 \quad \text{for all } g \in G.$$ 

We choose $n - 1$ distinct points

$$z_n < t_1 < \cdots < t_{n-1} < b.$$ 

By Theorem 4 there exists a $g_0 \in G$, $g_0 \not= 0$, such that

$$(-1)^{i+1}g_0(x) > 0, \quad t_{i} < x < t_{i+1}, \quad i = 1, \ldots, n, \quad t_0 = a, \quad t_n = b.$$ 

Moreover $g_0(z_i) = g_0(z_j) = 0$.

Since $t_1, \ldots, t_{n-1}$ are zeros with changes of sign of $g_0$ and $G$ is weak Chebyshev of dimension $n$, the function $g_0$ has no further change of sign on $(a, b)$ and, therefore, no change of sign at $z_i$ and $z_j$.

Let $||g_0|| < 1$. We choose $n + 1$ distinct points $\{v_i\}_{i=0}^n$ satisfying

$$z_i < v_0 < z_j < v_1 < t_1 < v_2 < \cdots < t_{n-1} < v_n < b.$$
We choose $\varepsilon > 0$ such that
\[ \{z_i, z_j, t_1, \ldots, t_{n-1}, b\} \cap [v_l - \varepsilon, v_l + \varepsilon] = \emptyset, \quad l = 0, \ldots, n. \]

Now we construct an $f \in C[a, b]$ as follows:

(a) 
\[ f(z_i) = 1, \quad f(z_j) = -1, \]
\[ f(x) = 1 \text{ for all } x \in [v_0 - \varepsilon, v_0 + \varepsilon], \]
\[ f(x) = (-1)^{l+1} \text{ for all } x \in [v_l - \varepsilon, v_l + \varepsilon], l = 1, \ldots, n. \]

(b) 
\[ \max\{-1 + g_0(x), -1\} < f(x) < \min\{1 + g_0(x), 1\} \quad \text{for all } x \in [a, b]. \]

Then $\|f - 0\| = \|f - g_0\| = 1$. Because of $g(z_i) = 0$ for all $g \in G$, we always get
$\|f(z_i) - g(z_i)\| = 1$ and, therefore, 0 and $g_0$ are elements of $P(f)$.

Now let $g \in P(f)$. For each $l \in \{1, \ldots, n\}$ there exists a $y_l \in [v_l - \varepsilon, v_l + \varepsilon]$ such that $(-1)^l g(y_l) > 0$. Hence the function $g$ has at least $n - 1$ changes of sign on $(v_l - \varepsilon, b)$. Then $g \geq 0$ on $[a, v_l - \varepsilon]$.

Therefore the function $g$ has a zero in $z_l$ and also in $z_l$, if $z_l > a$.

Since no $g \in G$ has a zero interval, it follows that $z_l, z_j \in \text{bd}Z(P(f))$. Now we apply Lemma 8.

If there exists a continuous selection, then there exists a $\tilde{g} \in P(f)$ such that
(i) for $z_l$ and $g_0$ there is a neighborhood $U$ of $z_l$ for which $\tilde{g} > g_0$ on $U$, and
(ii) for $z_j$ and 0 there is a neighborhood $V$ of $z_j$ for which $\tilde{g} < 0$ on $V$.

Since $\tilde{g} \geq g_0$ on $U$, $\tilde{g} \equiv 0$.

Moreover, $\tilde{g} > 0$ on $[a, v_l - \varepsilon]$. Therefore, in every neighborhood $V$ of $z_j$ there is a point $\tilde{x}$ such that $\tilde{g}(\tilde{x}) > 0$. But this is a contradiction to Lemma 8.

Second case: $a < z_0, z_n < b$. We can treat this case analogously.

Third case: Let $|Z(g)| < n$ on $[a, b)$ and on $(a, b]$ for all $g \in G$. By hypothesis there is a $\tilde{g}_0 \in G, \tilde{g}_0 \equiv 0$, with exactly $n + 1$ distinct zeros $a = z_0 < z_1 < \cdots < z_n = b$. Let $a < t_1 < t_2 < \cdots < t_k < b$ be all zeros with changes of sign and $a < y_1 < y_2 < \cdots < y_{n-k-1} < b$ be all double zeros of $\tilde{g}_0$.

First we show that $k = n - 1$ or $k = n - 2$. No other possibilities are allowed.

We assume that $k < n - 3$. We choose $n - k - 1$ points $z_{n-1} < t_{k+1} < \cdots < t_{n-1} < b$. By Theorem 4 there exists a $g_0 \in G, g_0 \equiv 0$, such that
\[( -1)^{i+1} g_0(x) > 0, \quad t_{i-1} < x < t_i, i = 1, \ldots, n, t_0 = a, t_n = b.\]

We may assume that $g_0 \cdot \tilde{g}_0 \geq 0$ on $[a, t_{k+1}]$. Since $n - k > 3$, $\tilde{g}_0$ has at least $n - k - 1 \geq 2$ double zeros on $(a, b)$ and therefore $|Z^*(\tilde{g}_0)| > n + 1$ on $(a, b)$. If $g_0(y_i) \neq 0$ for all $i \in \{1, \ldots, n - k - 1\}$, then for sufficiently small $c > 0$ the function $\tilde{g}_0 - cg_0$ has at least $n + 1$ changes of sign. This is a contradiction of the hypothesis on $G$.

If there are $i_1, i_2 \in \{1, \ldots, n - k - 1\}$ such that $g_0(y_{i_1}) = g_0(y_{i_2}) = 0$, then $g_0$ has $n + 1$ distinct zeros $t_1, \ldots, t_{n-1}, y_{i_1}, y_{i_2}$ on $(a, b)$. This is also a contradiction of the hypothesis.
Therefore there is exactly one double zero \( y_\circ \) of \( g_0 \) such that \( g_0(y_\circ) = 0 \). Then \( g_0 \) has \( n \) distinct zeros \( t_1, \ldots, t_{n-1}, y_\circ \) on \((a, b)\). Then for sufficiently small \( c > 0 \) the function \( \tilde{g}_0 - cg_0 \) has at least \( k + 2(n - k - 2) = n + n - k - 4 > n - 1 \) changes of sign on \((a, b)\) because \( g_0 \) does not vanish on exactly \( n - k - 2 \) double zeros of \( g_0 \). Moreover \( \tilde{g}_0 - cg_0 \) has a further zero in \( y_\circ \) and also a further zero on a neighborhood of \( a \), because \( g_0 \cdot \tilde{g}_0 > 0 \) on \([a, t_{k+1}]\).

Hence \( g_0 - cg_0 \) has at least \( n + 1 \) distinct zeros on \([a, b)\). This is a contradiction of the hypothesis of this case. Hence we have shown that \( n - k = 1 \) or \( n - k = 2 \).

We distinguish these two cases:

(i) \( n - k = 1 \). Therefore \( \tilde{g}_0 \) has exactly \( n - 1 \) changes of sign on \((a, b)\). Let \( g \in G \), \( g \not\equiv 0 \). Then \( g(a) = g(b) = 0 \), because otherwise the function \( \tilde{g}_0 - cg \) has \( n \) changes of sign for sufficiently small \( c \).

Therefore \( g(a) = g(b) \) for all \( g \in G \).

Now we proceed as we did in the first case. We choose \( n \) distinct points \( \{v_l\}_{l=1}^{n} \) satisfying

\[
a < v_1 < z_1 < v_2 < z_2 < \cdots < v_{n-1} < z_{n-1} < v_n < b.
\]

Let \( \|\tilde{g}_0\| < 1 \) and \( \tilde{g}_0 > 0 \) on \([a, z_1]\). We choose \( \epsilon > 0 \) such that

\[
\{z_0, \ldots, z_n\} \cap [v_l - \epsilon, v_l + \epsilon] = \emptyset, \quad l = 1, \ldots, n.
\]

We construct an \( f \in C[a, b] \) as follows:

(a)

\[
f(a) = 1,
\]

\[
f(x) = (-1)^{l-1} \quad \text{for all } x \in [v_l - \epsilon, v_l + \epsilon], l = 1, \ldots, n,
\]

\[
f(b) = (-1)^n;
\]

(b)

\[
\max\{-1 + \tilde{g}_0(x), -1\} < f(x) < \min\{1 + \tilde{g}_0(x), 1\} \quad \text{for all } x \in [a, b].
\]

Then \( \|f - 0\| = \|f - \tilde{g}_0\| = 1 \) and \( 0, \tilde{g}_0 \in P(f) \).

It is easy to show that each \( g \in P(f) \) has exactly \( n - 1 \) changes of sign. Moreover \( g(a) = g(b) = 0 \) for all \( g \in P(f) \). Therefore \( a, b \in \text{bd}Z(P(f)) \). Applying Lemma 8 to the point \( a \) and \( \tilde{g}_0 \in P(f) \) and to the point \( b \) and \( 0 \in P(f) \) we get a contradiction of the hypothesis that there exists a continuous selection.

(ii) \( n - k = 2 \). Therefore, \( \tilde{g}_0 \) has exactly \( n - 2 \) zeros with changes of sign and exactly one double zero \( z_1 \) on \((a, b)\).

We choose \( n - 1 \) distinct points \( \{v_l\}_{l=1}^{n-1} \) satisfying

\[
a < z_1 < v_1 < z_2 < v_2 < \cdots < v_{n-2} < z_{n-1} < v_{n-1} < b.
\]

Let \( \|\tilde{g}_0\| < 1 \) and \( \tilde{g}_0 > 0 \) on \([a, z_1]\). We choose \( \epsilon > 0 \) such that

\[
\{z_0, \ldots, z_n\} \cap [v_l - \epsilon, v_l + \epsilon] = \emptyset, \quad l = 1, \ldots, n - 1
\]

and \( a + \epsilon < z_1 \).
We construct an \( f \in C[a, b] \) as follows:

(a) 
\[
\begin{align*}
    f(x) &= 1 \quad \text{for all } x \in [a, a + \epsilon], \\
    f(x) &= (-1)^i \quad \text{for all } x \in [v_l - \epsilon, v_l + \epsilon], \; l = 1, \ldots, i - 1, \\
    f(x) &= (-1)^i' \quad \text{for all } x \in [v_l - \epsilon, v_l + \epsilon], \; l = i, \ldots, n - 1, \\
    f(b) &= (-1)^n;
\end{align*}
\]

(b) 
\[
\max\{-1 + \tilde{g}_0(x), -1\} < f(x) < \min\{1 + \tilde{g}_0(x), 1\} \quad \text{for all } x \in [a, b].
\]

Then \( \|f - 0\| = \|f - \tilde{g}_0\| = 1 \) and \( 0, \tilde{g}_0 \in P(f) \), since \( f - 0 \) has \( n + 1 \) alternating extreme points.

Let \( g \in P(f) \), \( g \not\equiv 0 \). Then it is easy to show that \( g \) has at least \( n - 2 \) changes of sign and a double zero at \( z_i \), since otherwise \( g \) has \( n \) changes of sign. This would be a contradiction of the hypothesis on \( G \).

Since \( g(a) > 0 \), \( (-1)^n g(b) > 0 \), for sufficiently small \( c > 0 \) the function \( \tilde{g}_0 - cg \) has \( n - 2 \) changes of sign, a double zero at \( z_i \) and two further zeros on neighborhoods of \( a \) and \( b \). Since by hypothesis \( |Z(\tilde{g}_0 - cg)| < n \) on \( [a, b] \) and on \( (a, b) \), the function \( \tilde{g}_0 - cg \) has two zeros at \( a \) and \( b \). Therefore \( g(a) = g(b) = 0 \) for all \( g \in P(f) \) and \( a, z_i, b \in \text{bd}Z(P(f)) \).

Applying Lemma 8 to the point \( a \) and \( \tilde{g}_0 \in P(f) \) and to the point \( z_i \) and \( 0 \in P(f) \) we get a contradiction of the hypothesis that there exists a continuous selection.

Now we give two examples showing that it is necessary to distinguish the two cases \( n - k = 1 \) and \( n - k = 2 \) in the third part of the above proof.

Example 1. \( G := \langle \sin \frac{1}{2}x, \sin x \rangle \subset C[0, 2\pi] \). Here \( |Z(g)| < 2 \) for all \( g \in G \) on \( [0, 2\pi] \) and on \( (0, 2\pi) \). The function \( \tilde{g}_0(x) = \sin x \) has exactly the distinct zeros \( 0, \pi, 2\pi \) such that \( n - k = 2 - 1 = 1 \).

Example 2. \( G := \langle x^3, |x|(1 - |x|) \rangle \subset C[-1, 1] \). Here \( |Z(g)| < 2 \) for all \( g \in G \) on \( [-1, 1] \) and on \( (-1, 1) \). There is no \( g \in G \) with three zeros \(-1 = z_0 < z_1 < z_2 = 1 \) such that \( z_1 \) is a zero with change of sign of \( g \) but the function \( \tilde{g}_0(x) = |x|(1 - |x|) \) has exactly the distinct zeros \(-1, 0, 1 \) where \( 0 \) is a double zero of \( \tilde{g}_0 \). Therefore \( n - k = 2 - 1 = 1 \).

Last we give a class of weak Chebyshev subspaces \( G \) of \( C[a, b] \) satisfying the additional condition that no \( g \) in \( G \), \( g \not\equiv 0 \), has a zero interval. Let \( g_0 \) be a nonnegative function in \( C[a, b] \) having no zero interval, but at least two distinct zeros on \( [a, b] \). Then for any \( g_0 \) having these properties, the space \( G \) spanned by the functions \( g_0(x), xg_0(x), \ldots, x^{n-2}g_0(x), x^{n-1}g_0(x) \) is a weak Chebyshev space in \( C[a, b] \), since each \( g \) in \( G \) has the representation \( g(x) = g_0(x)\Sigma_{i=0}^{n-1} a_i x^i \) and since the function \( \Sigma_{i=0}^{n-1} a_i x^i \) has at most \( n - 1 \) changes of sign on \( (a, b) \). Since \( g_0 \) has no zero interval, no \( g \) in \( G \), \( g \not\equiv 0 \), has a zero interval.
References


Institut für Angewandte Mathematik der Universität Erlangen-Nürnberg, Martensstrasse 3, 8520 Erlangen, Germany