ON THE GROUP OF VOLUME-PRESERVING
DIEREOMORPHISMS OF $\mathbb{R}^n$

BY

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Abstract. The group of all diffeomorphisms of $\mathbb{R}^n$ which preserve a given volume form is shown to be perfect when $n \geq 3$. Some useful factorizations of such diffeomorphisms are also obtained.

In this note we prove

Theorem. Let $\Omega$ be any volume form (that is, nonvanishing $C^\infty$ $n$-form) on $\mathbb{R}^n$. Then the group $\text{Diff}_\Omega \mathbb{R}^n$ of all $C^\infty$ diffeomorphisms of $\mathbb{R}^n$ which preserve the form $\Omega$ is perfect, provided that $n \geq 3$.

Remark. It follows easily from Moser [7] (see [3]) that there are only two distinct cases of this theorem, namely $\text{vol}_\Omega \mathbb{R}^n < \infty$ and $\text{vol}_\Omega \mathbb{R}^n = \infty$.

When $n = 1$, the group $\text{Diff}_\Omega \mathbb{R}$ is either trivial or isomorphic to $\mathbb{R}$, and the theorem is trivially true in the first case and trivially false in the second. On the other hand, the case $n = 2$ is potentially interesting. It has been shown by Thurston [8] and Banyaga [1] that this is the only dimension in which the identity component of the group of compactly supported volume-preserving diffeomorphisms of $\mathbb{R}^n$ is not perfect. Also, the volume-preserving and symplectic cases coincide in dimension 2. The present arguments do not work for the group of symplectic diffeomorphisms in any dimension. However the contact case is more tractable (see Banyaga and Pulido [2]).

A proof of the above theorem in the case when $\text{vol}_\Omega \mathbb{R}^n = \infty$ is given in [6]. The present proof is easier and more direct. It also yields a factorization lemma (Lemma 1) which turns out to be crucial in working out the normal subgroups of $\text{Diff}_\Omega \mathbb{R}^n$. This, together with the generalisation to manifolds other than $\mathbb{R}^n$, will be discussed elsewhere.

The present methods owe much to Ling [4] who used them to calculate the normal subgroups of the group of all diffeomorphisms of $\mathbb{R}^n$. Extra techniques are needed here in order to deal with the difficulties which are caused by the fact that the radial maps $x \mapsto \lambda(\|x\|)x$ do not preserve volume. In particular, in order to prove the factorization lemma when $n = 3$ we use a surprising but very elementary observation about knots in $\mathbb{R}^3$ (Lemma 8).

This paper is organised as follows. In the first section, we state the main lemmas and then use them to prove the Theorem. The factorization lemmas are proved in §2 and the lemmas about cells are proved in §3. For the convenience of the reader,
the needed results about extending volume-preserving diffeomorphisms are stated in an appendix.

1. We will say that an element \( f \in \text{Diff}_{\mathbb{R}^n} \) admits a Ling factorization with \( p \) factors if it may be written as a product \( h_1 \cdots h_p \) of diffeomorphisms \( h_j \in \text{Diff}_{\mathbb{R}^n} \), each of which has support in a locally finite union \( \bigcap_{i \geq 0} C_i \) of disjoint cells. (By definition, a cell is a smoothly embedded closed \( n \)-disc.) For short, we will often call such a union \( \bigcap_{i \geq 0} C_i \) a disjoint union.

The main factorization lemma is the following.

**Lemma 1.** If \( n \geq 3 \), every element of \( \text{Diff}_{\mathbb{R}^n} \) has a Ling factorization.

**Remark.** This lemma is also true in the symplectic case when \( n > 4 \). However it is not clear that the disjoint unions \( \bigcap_{i \geq 0} C_i \) which support the various factors are contained in disjoint unions of symplectic cells (i.e. cells which are symplectically embedded discs). Therefore Lemma 4 may fail in the symplectic case.

When \( \text{vol}_{\mathbb{R}^n} \) \( < \infty \), a second factorization lemma is useful.

**Lemma 2.** If \( n \geq 3 \), any diffeomorphism \( f \in \text{Diff}_{\mathbb{R}^n} \) with support in the interior of a cell \( C \) is the product \( h_1 h_2 h_3 \) of three elements \( h_j \in \text{Diff}_{\mathbb{R}^n} \) which are supported by the interiors of cells \( E_j \), where \( E_j \cap \text{Int} C \) and \( \text{vol}_{\mathbb{R}^n} E_j < \frac{3}{2} \text{vol}_{\mathbb{R}^n} C \).

These lemmas are proved in §2. We also will need two results about cells which will be proved in §3.

**Lemma 3.** Let \( n \geq 2 \). Suppose that \( \bigcap_{i \geq 0} C_i \) is a disjoint union of cells and that \( w_i \geq \text{vol}_{\mathbb{R}^n} C_i \) for all \( i \geq 0 \). Suppose also that \( \sum (w_i - \text{vol}_{\mathbb{R}^n} C_i) < \text{vol}_{\mathbb{R}^n} (\bigcap_{i \geq 0} C_i) \), with strict inequality if both sides are finite. Then there is a disjoint union \( \bigcap_{i \geq 0} D_i \) of cells \( D_i \) which have \( \Omega \)-volume \( w_i \), contain \( C_i \) and satisfy the condition \( \text{vol}_{\mathbb{R}^n} (\bigcap_{i \geq 0} D_i) = \infty \) if \( \text{vol}_{\mathbb{R}^n} (\bigcap_{i \geq 0} C_i) = \infty \).

**Lemma 4.** If \( \bigcap_{i \geq 0} C_i \) and \( \bigcap_{i \geq 0} D_i \) are disjoint unions such that \( \text{vol}_{\mathbb{R}^n} C_i = \text{vol}_{\mathbb{R}^n} D_i \) for all \( i \) and \( \text{vol}_{\mathbb{R}^n} (\bigcap_{i \geq 0} C_i) = \text{vol}_{\mathbb{R}^n} (\bigcap_{i \geq 0} D_i) \), then there is \( g \in \text{Diff}_{\mathbb{R}^n} \) such that \( g(C_i) = D_i \) for all \( i \).

**Proof of Theorem.** Case (i). \( \text{vol}_{\mathbb{R}^n} \) \( = \infty \). (This argument is adapted from Ling [4].)

By Lemma 1 it suffices to show that any element \( h \) with support in some disjoint union \( \bigcap_{i \geq 0} C_i \) is in the commutator subgroup of \( \text{Diff}_{\mathbb{R}^n} \). By considering the restrictions of \( h \) to \( \bigcap_{i \in J} C_i \) and \( \bigcap_{i \notin J} C_i \) separately, for some suitable subset \( J \) of \( \mathbb{N} \), we may suppose that \( \text{vol}_{\mathbb{R}^n} (\bigcap_{i \geq 0} C_i) = \infty \). Then, by Lemma 3, we may replace the \( C_i \) by larger cells which satisfy the conditions \( \text{vol}_{\mathbb{R}^n} C_i < \text{vol}_{\mathbb{R}^n} C_{i+1} \) for all \( i \) and \( \text{vol}_{\mathbb{R}^n} (\bigcap_{i \geq 0} C_i) = \infty \). It now follows from Lemma 4 that there is \( g \in \text{Diff}_{\mathbb{R}^n} \) such that \( g(C_i) \subseteq C_{i+1} \) for all \( i \). (Take the \( D_i \) in that Lemma to be suitable subcells of \( C_{i+1} \).)

Define \( s \in \text{Diff}_{\mathbb{R}^n} \) by

\[
s(x) = h(g^{-1}h^{-1}) \cdots (g'h'^{-1})(x) \quad \text{if} \quad x \in C_i \quad \text{for some} \quad i \geq 0,
\]

\[= x \quad \text{otherwise}.
\]
Then \( \text{supp } s \subseteq \Pi_{i \geq 0} C_i \) and \( \text{supp } g s^{-1} \subseteq \Pi_{i \geq 1} C_i \), so that \( [s, g] = g s^{-1} g^{-1} \) has support in \( \Pi_{i \geq 0} C_i \). Also, inspection shows that within each \( C_i \) we have \( h g s^{-1} = s \). Therefore, \( [s, g] = h \), and \( h \) is a commutator as required.

**Case (ii).** \( V = \text{vol}_n \mathbb{R}^n < \infty \).

We first show that \( \text{Diff}_n \mathbb{R}^n \) is generated by elements \( h \) which are supported in the interior \( \Pi_{i \geq 0} (\text{Int } C_i) \) of some disjoint union of cells whose volumes \( v_i = \text{vol}_n C_i \) satisfy

(i) \( \frac{1}{2} v_i < v_{i+1} < v_i \) for all \( i \) and

(ii) \( \sum v_i < \frac{1}{2} V \).

Notice to begin with that, by Lemma 1, \( \text{Diff}_n \mathbb{R}^n \) is generated by elements \( h \) with support contained in some disjoint union \( \Pi_{i \geq 0} (\text{Int } C_i) \). By Lemma 2 we can represent each \( h \) as a \( 3^n \)-fold product of elements \( h' \) which are supported in \( \Pi_{i \geq 0} C'_i \), with \( C'_i \subseteq C_i \) and also \( v'_i < \left( \frac{1}{2} \right)^4 v''_i < \frac{1}{2} v''_i \). (Here \( v'_i = \text{vol}_n C'_i \) and \( v''_i = \text{vol}_n C''_i \).) Thus \( \sum v'_i < \frac{1}{2} V \). After renumbering, we may assume that \( v'_0 > v'_1 > \cdots \). Now define the numbers \( v_0, v_1, \ldots \) with \( v'_0 < v'_1 < v_i \) for all \( i \geq 1 \) inductively as follows. Let \( v_0 = v'_0 \). Put \( v_{i+1} = v'_{i+1} < v_i < v_{i+1} \) if \( v'_{i+1} > \frac{1}{2} v_i \), and put \( v_{i+1} = v'_{i+1} + \frac{1}{2} v_i < v_i \) otherwise. Clearly, the \( v_i \) satisfy condition (i). They also satisfy condition (ii). For, because \( v_i < v'_i + \frac{1}{2} v_{i-1} \), induction shows that \( v_i < \sum_{k=0}^{\infty} v'_i - k/2^k \). Summing over \( i \), we obtain

\[
\sum v_i < \sum_i \left( \sum_{k=0}^{\infty} 1/2^k \right) v'_i = \sum v'_i < 2\left( \frac{1}{2} V \right) = \frac{1}{2} V.
\]

Thus, by replacing the cells \( C'_i \) by larger cells \( C_i \) with volumes \( v_i \) as in Lemma 3, we may suppose that the conditions (i) and (ii) are satisfied.

We now show that any \( h \) which satisfies these conditions is a product of commutators in \( \text{Diff}_n \mathbb{R}^n \). First, use Lemma 3 to find a disjoint union \( \Pi_{i \geq 1} D_i \) such that \( C_i \subseteq D_i \) and \( \text{vol}_n D_i = v_{i-1} \) for all \( i \geq 1 \). (There is enough room for this by (ii).) Next, use Lemma 4 to find a diffeomorphism \( g \in \text{Diff}_n \mathbb{R}^n \) which takes \( C_i \) onto \( D_{i+1} \) for each \( i \geq 0 \). Because \( \left( \frac{1}{2} \right)^2 v_i < v_{i+1} \) by (i), it follows from Lemma 2 that any diffeomorphism with support in \( \text{Int } C_i \) may be written as the product of at most 9 diffeomorphisms which are supported by cells in \( \text{Int } C_i \) which have volume less than \( v_{i+1} \). Therefore, we may construct elements \( s, l_j \) and \( k_j \) for \( 1 < j < 9 \), such that

(a) \( \text{supp } s \subseteq \Pi_{i \geq 0} (\text{Int } C_i) \),

(b) \( s = k_1 \cdots k_9 \) where \( \text{supp } k_j \subseteq \Pi_{i \geq 0} (\text{Int } E_{ij}) \) and where the cells \( E_{ij} \subseteq \text{Int } C_i \) satisfy \( \text{vol}_n E_{ij} < v_{i+1} \),

(c) each \( l_j \) maps \( E_{ij} \) into \( \text{Int } C_{i+1} \) and satisfies \( \text{supp}(g^{-1} l_j) \subseteq \Pi_{i \geq 0} C_i \) so that \( l_j \) coincides with \( g \) outside the \( C_i \), and

(d) \( s = h(l_1 k_1 l_1^{-1}) \cdots (l_9 k_9 l_9^{-1}) \).

In fact, one can define these mappings inductively over the \( C_i \), starting with \( s = h \) on \( C_0 \). Given \( s \) on \( C_i \), one chooses the \( k_j \) and \( l_j \) on \( C_i \) satisfying (b), (c), then defines \( s \) on \( C_{i+1} \) by (d). Now, comparing formulas (b) and (d) we have \( k_1 \cdots k_9 = h(l_1 k_1 l_1^{-1}) \cdots (l_9 k_9 l_9^{-1}) \) or, in other words,

\[
h = k_1 \cdots k_9 (l_9 k_9 l_9^{-1})^{-1} \cdots (l_1 k_1 l_1^{-1})^{-1}.
\]
Clearly the right side of this equation is congruent to the identity modulo the commutator subgroup. In fact it is not difficult to express it as a product of 9 commutators. □

Remark. We will see that if \( n > 3 \) every \( f \in \text{Diff}_0 \mathbb{R}^n \) has a L"aszló factorization with at most 14 factors. It follows that there is a number \( M \) such that every element of \( \text{Diff}_0 \mathbb{R}^n \) is the product of at most \( M \) commutators. The present proof would give \( M = 28 \) if \( \text{vol} \mathbb{R}^n = \infty \), and \( M = 10206 \) otherwise, although presumably one could do much better.

2. Proof of the factorization lemmas. We will call a continuous map \( F \), from \([0, 1]\) to the group \( \text{Diff}_0 \mathbb{R}^n \) provided with the compact-open \( C^\infty \)-topology, an \( \Omega \)-isotopy from \( F(0) \) to \( F(1) \). Also, if \( X \) is an \( n \)-dimensional submanifold of \( \mathbb{R}^n \) which is closed as a subset of \( \mathbb{R}^n \), we will write \( \text{Diff}_0(X, \text{rel} \partial) \) for the subgroup of \( \text{Diff}_0 \mathbb{R}^n \) consisting of diffeomorphisms which are \( \Omega \)-isotopic to the identity by an isotopy \( t \mapsto f_t \), which is supported by the interior \( \text{Int} X = X - \partial X \) of \( X \). For convenience, we will assume throughout this section that \( \Omega \) is the standard volume form \( dx_1 \wedge \cdots \wedge dx_n \) on \( \mathbb{R}^n \) if \( \text{vol} \mathbb{R}^n = \infty \), and otherwise, that it is spherically symmetric, that is that \( \Omega(x) = f(||x||^2)dx_1 \wedge \cdots \wedge dx_n \) for some nonvanishing smooth function \( f \). This is permissible by the generalisation of Moser's theorem to noncompact manifolds [3].

The following result is surely well known. However I know of no proof in the literature.

**Lemma 5.** Every element of \( \text{Diff}_0 \mathbb{R}^n \) is \( \Omega \)-isotopic to the identity.

**Proof.** If \( \text{vol} \mathbb{R}^n = \infty \), an \( \Omega \)-isotopy from \( f \) to the identity may be constructed as follows. First compose \( f \) with translations to take it to an element \( g \) which fixes 0. Then make \( g \) linear by the standard isotopy given by

\[
g_t(x) = g(tx)/t, \quad \text{for } 0 < t < 1, \quad \text{and} \quad g_0(x) = \lim_{t \to 0} g(tx)/t.
\]

Finally, join \( g_0 \) to the identity by a path in the connected group \( SL(n, \mathbb{R}) \).

If \( \text{vol} \mathbb{R}^n < \infty \), we may identify \( (\mathbb{R}^n, \Omega) \) with \( (D, \Omega_0) \), where \( D \) is an open disc in \( \mathbb{R}^n \) centered at 0 and \( \Omega_0 = dx_1 \wedge \cdots \wedge dx_n \). Then, if \( f \) is an \( \Omega_0 \)-preserving diffeomorphism of \( D \), by restricting the isotopy described above to a suitable neighbourhood \( U \) of 0 we get a path \( f_t \) of \( \Omega_0 \)-embeddings of \( U \) into \( D \) such that \( f_0 \) is the inclusion and \( f_1 = f|U \). By Lemma A in the Appendix, there is an ambient \( \Omega_0 \)-isotopy \( h_t \) of \( D \) which equals \( f_t \) near 0. Thus \( f \) is \( \Omega_0 \)-isotopic to \( h_1^{-1} f \), an element which is the identity near 0. (Observe that this procedure is valid even if \( f \) does not fix 0 since there is no need for the isotopy \( f_t \) to fix 0.)

Now notice that the subgroup of \( \Omega_0 \)-preserving diffeomorphisms of \( D \) which are the identity on some open disc \( D_{\lambda} = \{ x: ||x|| < \lambda \} \) may be identified with the group \( G_{\lambda} \) of diffeomorphisms of \( \mathbb{R}^n - \{ 0 \} \) which are the identity outside some open disc \( D_{\epsilon} = \{ x: ||x|| < \epsilon \} \) and which preserve \( \Omega_0 \). (Indeed, to make this identification, it suffices to construct an \( \Omega_0 \)-preserving diffeomorphism \( \psi: D - D_{\epsilon} \to D_{\lambda} - \{ 0 \} \). Choosing \( \lambda \) so that \( \text{vol}_{\Omega_0} D_{\lambda} = \text{vol}_{\Omega_0} (D - D_{\epsilon}) \), we may take \( \psi \) to be a
radial diffeomorphism of the form $x \mapsto \theta(||x||)x$, where $\theta$ is a suitable diffeomorphism $[e, \mu) \to (0, \lambda]$ and $\mu$ is the radius of $D$.) Any element $f$ of the group $G_\lambda$ is $\Omega_\lambda$- isotopic to the identity by the “Alexander” isotopy $f_t$, where $f_0 = \text{id}$ and $f_t$, $0 < t < 1$, is defined by $f_t(x) = tf(x/t)$. The result follows. \hfill \Box

We next show that, just as in the non-volume-preserving case, $\text{Diff}_0\mathbb{R}^n$ is generated by diffeomorphisms which are supported by disjoint unions of annuli, $\Pi_{i>0} A_i$. Here

$$A_i = \{ x : \lambda_{2i} < ||x|| < \lambda_{2i+1} \},$$

where $0 = \lambda_0 < \lambda_1 < \cdots$ and $\lambda_i \to \infty$. (Observe that $A_0$ is in fact a disc.)

**Lemma 6.** If $n > 2$, every $f \in \text{Diff}_0\mathbb{R}^n$ may be written as a product $g_1 g_2$ where each $g_j$ belongs to some group $\text{Diff}_{\Omega_\lambda}(\Pi_{i>0} A_i, \text{rel } \partial)$.

**Proof.** By Lemma 5 there is an $\Omega$-isotopy $f_t$, $0 < t < 1$, from $f_0 = \text{id}$ to $f_1 = f$. Let $\mu_1 = 1$ and choose a number $\mu_2$ so that the image $f_t(\mu_1 S)$ of the unit sphere $\mu_1 S$ under the isotopy $f_t$ lies inside the sphere $\mu_2 S$ of radius $\mu_2$ for all $t$. Then, by Lemma A, there is an $\Omega$-isotopy $g_t$ defined inside $\mu_2 S$ which equals $f_t$ near $\mu_1 S$ and the identity near $\mu_2 S$. Next, choose $\mu_3 < \mu_4$ so that $f_t(\mu_3 S)$ always lies outside $\mu_2 S$ and inside $\mu_4 S$. Then $g_t$ may be extended to the interior of $\mu_4 S$ in such a way that it equals the identity near $\mu_2 S$ and $\mu_4 S$ and equals $f_t$ near $\mu_3 S$. Continuing in this way, we construct $g_j$ to equal $f_t$ near each $\mu_{2j-1} S$ and to equal the identity near each $\mu_{2j} S$, where $\mu_1 < \mu_2 < \cdots$ and $\mu_i \to \infty$. Clearly, $g_1$ and $g_2 = g_1^{-1} f$ have the required form. \hfill \Box

**Lemma 7.** Let $A$ be the annulus $0 < \lambda_1 < ||x|| < \lambda_2$ and $r$ be the ray segment $\{y : \lambda_1 < ||y|| < \lambda_2 \}$, where $||y|| = 1$. Then, any element $f$ of $\text{Diff}_0(A, \text{rel } \partial)$ which equals the identity on $r$ is the product of at most 5 elements of $\text{Diff}_0(A, \text{rel } \partial)$ which are supported by cells.

**Proof.** Case (i). $n > 4$. Taking the derivative $df_y$ of $f$ at points $ty$ of $r$, one gets a loop $t \mapsto df_y$, $\lambda_1 < t < \lambda_2$, in $SL(n, \mathbb{R})$. This loop is contractible because $f$ is $\Omega$-isotopic to the identity relative to the boundary of $A$. Since $f = \text{id}$ on $r$, these derivatives $df_y$ in fact lie in the subgroup $G_n$ of $SL(n, \mathbb{R})$ consisting of matrices with first row $(1, 0, \ldots, 0)$. Because $G_n \simeq SL(n-1, \mathbb{R})$ and

$$\pi_1 SL(n-1, \mathbb{R}) \simeq \pi_1 SL(n, \mathbb{R}) \simeq \mathbb{Z}/2\mathbb{Z} \text{ if } n > 4,$$

this loop contracts in $G_n$ as well. It follows that there is $h \in \text{Diff}_0(A, \text{rel } \partial)$ which has support in a cell containing $r$ such that $f h = \text{id}$ near $r$. To see this, first construct a (not necessarily volume-preserving) isotopy $f_t$ with $f_0 = f$, $f_1 = f$ outside a small neighbourhood of $r$ in $A$ for all $t$, and $f_t$ is in a tubular neighbourhood $T$ of $r$. Then, by Lemma A, there is an $\Omega$-isotopy $h_t$ in $\text{Diff}_0(A, \text{rel } \partial)$ extending $f^{-1} \partial T$, which is supported by a cell containing $r$ and is such that $h_1 = f^{-1}$ on $T$. Clearly, we may take $h = h_1$. Thus $fh$ has support in a cell of the form $A - (\text{nbhd of } r)$, and so $f = (fh)h^{-1}$ may be factored into 2 elements of the required type.

Case (ii). $n = 3$. In this case the loop $t \mapsto df_y$, $\lambda_1 < t < \lambda_2$, need not contract in $G_3$. Instead it is homotopic in $G_3$ to a loop in $SO(2)$ which represents an element of
the kernel of the homomorphism $\mathbb{Z} \approx \pi_1 SO(2) \to \mathbb{Z}/2\mathbb{Z} \approx \pi_1 SL(3, \mathbb{R})$. For each integer $k$, let $\gamma_k$ be the loop representing the element $2k \in \mathbb{Z} \approx \pi_1 SO(2)$ which is defined as follows. $\gamma_k(t)$, for $t \in [\lambda_1, \lambda_2]$, is the rotation through the angle $4\pi\theta(k, t)$, where $\theta(k, \cdot)$ is a smooth function of $t$ which has graph as in Figure 1 if $k > 0$. We put $\theta(0, t) = 0$ for all $t$, and $\theta(-k, t) = -\theta(k, t)$.

Consider the diffeomorphism $g_k$ of $A$ which rotates each sphere of radius $t$, $\lambda_1 < t < \lambda_2$, about the axis $r$ and through the angle $4\pi\theta(k, t)$. (Thus $g_0 = id$ and $g_{-k} = g_k^{-1}$.) Since $\Omega$ is spherically symmetric, $g_k$ preserves $\Omega$. Also, because $\gamma_k$ contracts in $SL(3, \mathbb{R})$, it is not hard to see that $g_k \in \text{Diff}_{00}(A, \text{rel } \partial)$. Given any $f \in \text{Diff}_{00}(A, \text{rel } \partial)$, we may choose $k$ so that the loop $t \mapsto d(g_k^{-1}f)_{\gamma}$ contracts in $G_3$. The argument of case (i) then applies to show that $g_k^{-1}f$ may be factored into $2$ elements which are supported by cells. Therefore, in order to factor $f$ into $5$ factors, it suffices to factor each $g_k$ into $3$ factors.

When $k = 1$ this may be done as follows. Clearly, it suffices to find two elements $h_1, h_2 \in \text{Diff}_{00}(A, \text{rel } \partial)$ which are supported by cells of the form $A - (\text{nbhd of ray})$ and are such that $h_1h_2 = g_1$ near the ray $\gamma$, since then we have $g_1 = h_1 \cdot h_2 \cdot (h_1h_2)^{-1}g_1$. Such $h_1$ and $h_2$ may be found because of the well-known fact that a ribbon in $\mathbb{R}^3$ with a total twist of $4\pi$ between its fixed ends $E$ and $F$ may be untwisted by passing it once around $F$. (See Figure 2.) Thus $g_1$, and hence also $g_{-1} = g_1^{-1}$, has the required factorization.

When $|k| > 1$, observe that $g_k$ is the product of $k$ "disjoint copies" of $g_1$ or $g_{-1}$, one supported in each annulus $A_i = \{x: \lambda_1 + ia < \|x\| < \lambda_1 = (i + 1)a\}$. Therefore, each $g_k$ also factors as the product of $3$ elements of $\text{Diff}_{00}(A, \text{rel } \partial)$ which are supported by cells. (Each of these cells will contain the union of $k$ disjoint cells, one in each $A_i$.)
The last ingredient in the proof of Lemma 1 is the following result about knots.

**Lemma 8.** Let \( \gamma_1 \) and \( \gamma_2 \) be two smooth arcs from \( P \) to \( Q \) in \( \mathbb{R}^3 \) whose interiors are disjoint. Then there is a smooth arc \( \gamma_0 \) from \( P \) to \( Q \) which is unknotted with respect to both \( \gamma_1 \) and \( \gamma_2 \).

**Proof.** Arrange the knot \( \gamma_1 \cup \gamma_2 \) so that all crossings are 2-fold, and none involve \( \gamma_2 \). (See Figure 3.) Call a crossing an over-crossing if, when \( \gamma_1 \) is traversed from \( P \) to \( Q \), one goes over the crossing before going under it. Otherwise call the crossing an under-crossing. Then \( \gamma_1 \cup \gamma_2 \) may be unknotted by changing all under-crossings to over-crossings.

Let \( \gamma_0 \) be a smooth arc from \( P \) to \( Q \) which lies vertically above \( \gamma_1 \) and very close to it except near the under-crossings, where it goes over instead of under. It is easy to check that \( \gamma_0 \cup \gamma_2 \) is unknotted. To see that \( \gamma_0 \cup \gamma_1 \) is also unknotted, notice that it bounds an immersed disc which intersects itself only at the under-crossings. If we shrink the disc starting with the end \( Q \), it always happens that the first time we reach an under-crossing we are in the process of contracting the two inner strands. (See Figure 4.) This may always be done. When we return to this under-crossing there is no longer any obstruction there and the disc may be shrunk further towards \( P \). \( \square \)
PROOF OF LEMMA 1. By Lemma 6, it clearly suffices to show that if $A$ is an annulus in $\mathbb{R}^n$, then any element $f$ of $\text{Diff}_{\partial}(A, \text{rel } \partial)$ is a product of at most 7 diffeomorphisms each of which has support in some cell contained in $A$. Let $r$ be the intersection of the ray $\{\lambda y : \lambda > 0\}$ with $A$. Then, by altering $f$ near $r$ (that is, by replacing $f$ by $f_1$, where $f_1 \in \text{Diff}_{\partial}(A, \text{rel } \partial)$ is supported by a small neighbourhood of $r$), we may suppose that the intersection $r \cap f(r)$ consists of two connected arcs. It follows easily (using Lemma 8 when $n = 3$) that there is a smooth arc $r_0$, connecting the two boundary components of $A$ and disjoint from $r \cup f(r)$, which is unknotted with respect to both $r$ and $f(r)$. (See Figure 5.) This means that there is an $\Omega$-isotopy $t \mapsto g_t \in \text{Diff}_{\partial}(A, \text{rel } \partial)$ which is the identity near $r_0$ and is such that $g_t(r) = f_t$. By altering $g_1$ near $r$ we may suppose that $g_1 = f$ on $r$. Then, by Lemma 7, $g_1^{-1}f$ is the product of 5 elements of $\text{Diff}_{\partial}(A, \text{rel } \partial)$ which are supported by cells. Since $g_1$ is supported by a cell of the form $A - (\text{nbhd of } r_0)$, it follows that $f$ has the required factorization into 7 factors. (The extra factor comes from the preliminary modification of $f$.) □

REMARK. The proof shows that every element of $\text{Diff}_{\partial}(\mathbb{R}^n)$ has a Ling factorization with at most 14 factors. No attempt has been made to find the smallest number of factors which are necessary.
Proof of Lemma 2. Suppose that $f \in \text{Diff}_0 \mathbb{R}^n$ is supported by $\text{Int} \, C$, where, without loss of generality, we assume that $C$ is a disc in $\mathbb{R}^n$. Choose a closed region $W \subset C$ with volume $\frac{1}{3} \text{vol}_n C$ which is bounded by a hyperplane intersected with $C$ and let $V$ be an open $\varepsilon$-neighbourhood of $W$ in $C$ such that $\text{vol}_n (V \cup fV) < \frac{2}{3} \text{vol}_n C$. (See Figure 6. Such $V$ exists because $f = \text{id}$ near $\partial C$ so that $W$ overlaps $fW$.) Then there is a (non-volume-preserving) isotopy $g_t$, with support in $V \cup fV$ such that $g_0 = \text{id}$ and also, $g_1 = f$ on $W$. (For instance, one might take $g_t = f p_t^{-1} f^{-1} p_t$, where $p_t$ is an isotopy with support in $V$ which shrinks $W$ so close to $\partial C$ that $f = \text{id}$ on $p_t(W)$.) Since $(V \cup fV) - W$ is connected, it follows from Lemma A that we may actually choose $g_t$ to preserve volume. Then $g_1^{-1} f$ preserves volume and, because it equals the identity on $W$ and near $\partial C$, it has support in a cell of volume less than $\frac{2}{3} \text{vol}_n C$. It will be one of the factors of $f$. We will complete the proof by expressing $g_1$ as a product of 2 factors of the required type.

By construction, $g_1$ has support in $V \cup fV$. Since $\text{vol}_n (V \cup fV) < \frac{2}{3} \text{vol}_n C$ there are disjoint cells $B_1, \ldots, B_m$ in $(\text{Int} \, C) - (V \cup fV)$ such that $\frac{1}{3} \text{vol}_n C = \sum_{j=1}^m \text{vol}_n B_j$. (See Figures 6, 7.) Join $\partial C$ to $B_1$ by a smooth arc $\gamma_1$ and, for $1 < j < m$, join $\partial B_{j-1}$ to $\partial B_j$ by a smooth arc $\gamma_j$. We may suppose that these arcs are disjoint and do not meet $V$ or the boundaries $\partial C$ and $\partial B_j$ except possibly at their endpoints. Thus the complement in $C$ of a suitable neighbourhood of $\bigcup_j B_j \cup \gamma_j$ will be a cell. Clearly, $g_1 = \text{id}$ near $\bigcup_j B_j \cup \gamma_j$. The idea now is to modify $g_1$ so that it equals the identity near $\bigcup_j B_j \cup \gamma_j$ and hence has support in a cell.

As a first step, notice that we may assume that the arcs $g_1 \gamma_j$ all lie outside $W$. For, if they do not, we may find an isotopy $k_t$ (which we may assume to be volume preserving because $n > 3$) which pushes the arcs $g_1 \gamma_j$ outside $W$, and has support in $V$. Then we may factor $k_1 g_1 k_t^{-1}$ instead of $g_1$.

Now, let $h$ be a (non-volume-preserving) diffeomorphism which is the identity near $\bigcup_j (B_j \cup \gamma_j \cup g_1 \gamma_j)$ and near $\partial C$, and which pushes the support of $g_t$ outside $W$. Then $h g_1 h^{-1}$ is an isotopy with support in $\text{Int}(C - W)$ such that $h g_1 h^{-1} = g_1$ near $\bigcup_j B_j \cup \gamma_j$. By Lemma A there is an element $q \in \text{Diff}_0 \mathbb{R}^n$ with support in $\text{Int}(C - W)$ which equals $g_1$ near $\bigcup_j B_j \cup \gamma_j$. Then
supp $q^{-1}g_1 \subset \text{Int}[C - (\bigcup_j B_j \cup \gamma_j)]$

so that both $q$ and $q^{-1}g_1$ are supported by cells in $\text{Int} C$ with volume less than $\frac{2}{3} \text{vol}_\Omega C$. Thus $f = q(q^{-1}g_1)(g_1^{-1}f)$ has the required factorization. \[\square\]

**Figure 7**

3.

**Proof of Lemma 3.** There is a (non-volume-preserving) diffeomorphism $h$ of $\mathbb{R}^n$ which takes each cell $C_i$ onto the disc $C_i'$ of radius $\frac{1}{4}$ centred at $x_i = (i, 0, \ldots, 0)$. (To see this, first choose a diffeomorphism which takes the centre of each $C_i$ to $x_i$ and then shrink the images of the $C_i$ down.) Therefore, the problem reduces to finding a locally finite collection of disjoint discs $D_i'$ which contain the $C_i'$ and whose $(h_* \Omega)$-volumes satisfy the given conditions. Care is needed when $\text{vol}_\Omega(\mathbb{R}^n - \bigcup_i C_i) < \infty$ since in this case the $D_i'$ must be chosen so that

$$\text{vol}_{h_* \Omega}(\mathbb{R}^n - \bigcup_i D_i') = \text{vol}_{\Omega}(\mathbb{R}^n - \bigcup_i C_i) - \sum_i (w_i - \text{vol}_{\Omega} C_i).$$

However, because we can choose the $D_i'$ to have nice geometric shapes, we can make them fill up as much or as little of $\mathbb{R}^n$ as necessary. For instance, we may take them to be annuli $\mu < ||x|| < \lambda$, with a hole drilled along the negative $x_1$-axis so that they are cells, and distorted along the positive $x_1$-axis so that $C_i' \subseteq D_i'$ for all $i$. (See Figure 8.)

**Figure 8**
Proof of Lemma 4. As in the proof of Lemma 3, we may identify the cells $C_i$ with the set of discs centred at the points $x_i$ and with radius $\frac{1}{4}$. It is now not difficult to construct an increasing sequence of cells $C_i''$ with $C_i'' \subset \text{Int} C_i'' + 1$ for all $i$, whose union is $\mathbb{R}^n$ and which are such that $C_i \subset \text{Int}(C_i'' + 1 - C_i'')$ for all $i$. Moreover, if the $w_i$ are any positive numbers such that $w_i \to \text{vol}_q \mathbb{R}^n$ as $i \to \infty$ and $\text{vol}_q C_i < w_{i+1} - w_i$ for all $i$, then we may clearly choose the $C_i''$ so that $\text{vol}_q C_i'' = w_i$. Let $D_i''$ be a similar sequence for the cells $D_i$ with $\text{vol}_q D_i'' = w_i$ also.

Now use Lemma A to construct an $\Omega$-isotopy $f_i$ such that $f_i(C_i') = D_i''$ for all $i$. Construct an isotopy $f_i$ in $\text{Diff}_q \mathbb{R}^n$ so that $f_i(C_i') = D_i''$. Then extend $f_i|C_i''$ to an isotopy $f_i$ in such a way that $f_i(C_i') = D_i''$, and so on.) By Lemma A again, there is an $\Omega$-isotopy $h_i$ with support in $\bigcup_{i \geq 0} \text{Int}(D_i'' + 1 - D_i'')$ which moves each cell $f_i(C_i)$ onto $D_i$. Clearly, $g = h_1 f_1$ satisfies the required conditions. □

Appendix. The following lemma gives all the information we need about extending volume-preserving isotopies. It is proved by Krygin in [5].

Lemma A. Let $W$ be an $(n-1)$-dimensional compact submanifold of $\mathbb{R}^n$ and suppose that $g_t$, $0 < t < 1$, is a smooth family of embeddings of $W$ into some open subset $U$ of $\mathbb{R}^n$, with $g_0$ equal to the inclusion. If each component of $\mathbb{R}^n - g_t(W)$ has the same $\Omega$-volume as the corresponding component of $\mathbb{R}^n - W$, there is an isotopy $f_t$ in $\text{Diff}_q \mathbb{R}^n$ with support in $U$ and such that $f_0 = \text{id}$, and $f_1 = g_1$ on $W$. Moreover, if $g_t$ is defined on some components $V$ of $\mathbb{R}^n - W$, and if it preserves $\Omega$ on $V$ and preserves the total volume of the other components of $\mathbb{R}^n - W$ either for all $t$ or when $t = 1$, then we may suppose that $f_t = g_t$ on $V$ for those values of $t$.

References

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