A SPECTRAL SEQUENCE FOR GROUP PRESENTATIONS
WITH APPLICATIONS TO LINKS

BY

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Abstract. A spectral sequence is associated with any presentation of a group G. It turns out that this spectral sequence is independent of the chosen presentation. In particular if G is the fundamental group of a link L in $\mathbb{R}^3$ the spectral sequence leads to invariants that compare to the Milnor invariants of L.

0. Introduction. Recently Stallings used the cobar construction of a resolution to associate to each group G a 2nd quadrant spectral sequence $E_{s,t}^r$ which is 0 for $s > t$ and which satisfies $E_{s,t}^{\infty} = I^s G / I^{s+1} G$ where IG is the fundamental ideal of G [9]. Here we present a different construction with all the properties mentioned above but with some advantages. First, it can be read off from any group presentation. Second, $E_{s,t}^r = 0$ for $t > s + 2$. In Stallings’ sequence one has no information on those terms (and they are definitely not zero). Third, and most important, the $E_{s,s}$ and $E_{s,s+1}$ terms are related to the Baer invariants of G [1]. This is better than the results in [5] which do depend upon the presentation while ours do not.

We describe our sequence in §1; in §2 we show that the sequence is intrinsically defined by using the results of [4]. In §3 we apply our results to the theory of links in $\mathbb{R}^3$.

[The referee would like to thank J. Ratcliffe for pointing out to him the existence of [4]. Thanks to it, the referee was able to improve some results and prove a conjecture of the author’s (the main theorem).]

1. The spectral sequence of a presentation.

(1.0) We shall consider complexes of algebras. A normal short complex is one

$$\cdots \rightarrow A_2 \rightarrow A_1 \partial \rightarrow A_0,$$

for which $A_n = 0$, $n > 2$, and $\partial$ is a normal monomorphism (see [4, p. 225]). Then we have an exact sequence

$$0 \rightarrow A_1 \partial \rightarrow A_0 \varepsilon \rightarrow H_0(A) \rightarrow 0.$$
If $A_0$ is a projective (resp. free) algebra, then $A$ is called a projective (resp. free) presentation of $H_0(A)$. We apply this to the case where our algebra is an integral group ring.

(1.1) Let $G$ be a group; then $G_n$ stands for the $n$th member of its lower central series [6, Chapter V, §9]. In particular $G_2$ is the commutator subgroup and $G = G/G_2$ is the abelianization of $G$.

The ring $ZG$ is the integral group ring of $G$ with augmentation $\varepsilon: ZG \to Z$. Let $IG = \ker \varepsilon$; then $I^nG$ stands for the $n$th power of $IG$.

(1.2) Let now

$$\langle x_j : r_j \rangle \quad (P)$$

be the presentation [6, p. 205] for $G$. This means that we have a free group $F$ in the $x_j$ and that $G \cong F/R$, where $R$ is the smallest normal subgroup of $F$ generated by $(r_j) \subseteq F$. We write $R = \langle r_j \rangle^F$.

Consider the 2-sided ideal $N = (r_j - 1)$ of $ZF$ generated by the $r_j - 1$. Then we have a free presentation

$$0 \to N \to ZF \to ZG \to 0$$

of $ZG$. Since $N \subseteq IF$ we may take the short complex [4, §2]

$$0 \to N \to IF \quad J$$

(here $J_q = 0$, $q > 2$, $J_1 = N$ and $J_0 = IF$), which is a free presentation of $IG$, via the isomorphism $H_0(J) \cong IG$, since $IF$ is $F$-free [6, Chapter VI, Theorem 5.5]. By Lemma 5.2 of [4], $IF$ is a projective algebra.

$J$ can be considered to be the augmentation kernel of the complex

$$0 \to N \to ZF \quad C$$

and the powers $J^p$ of $J$ define a filtration $F_p C = J^p$ on $C$. Notice that if we define

$$N(0) = N(1) = N \quad \text{and} \quad N(p) = N(1)I^{p-1}F + IFN(p-1) = N(p-1)IF + IFN(p-1) \quad (1)$$

then $J^p = N(p) \oplus I^pF$.

(1.3) The filtration $F$ induces a spectral sequence in the usual manner [6, Chapter VIII, §2]. Since our filtration degree is negative, our sequence lies in the 2nd quadrant and since $C_q = 0$, $q > 2$, then $E_{r,s+k}^r = 0$ for $k > 2$, whereas

$$E_{r,s}^r = I^rF / (I^{s+1}F + N(s-r+1) \cap I^rF), \quad s > 0, \quad (2)$$

and

$$E_{r,s+1}^r = (N(s) \cap I^{s+1}F) / (N(s+1) \cap I^{s+1}F), \quad s > 1. \quad (3)$$

**Definition (1.4)** The spectral sequence $E$ is called the spectral sequence of $G$ (associated to the presentation $(P)$).

2. **The main theorem.** Our main goal is to show that $E$ depends only on $G$.

(2.1) Let then $(P)$ be the presentation in (1.2) and let

$$\langle x_k : r_l \rangle \quad (Q)$$
be another presentation. Put $F' = \langle x_k' \rangle$ and $R' = \langle r'_i \rangle^F$; let $E'$ be the spectral sequence associated to $(Q)$.

**Lemma (2.2)** If there exists an epimorphism $\phi: F \rightarrow F'$ with $\phi(R) = R'$ then $\phi$ induces an isomorphism $\Phi: E \rightarrow E'$ of spectral sequences.

**Main Theorem (2.3)** If $(P)$ and $(Q)$ are any two presentations of the group $G$ then the corresponding spectral sequences are isomorphic.

This allows us to drop the parenthetical remark in Definition (1.4).

The theorem follows from (2.2) for there exists a presentation of $G$, $(S)$: $\langle y_a : s_p \rangle$ where $L = \langle y_a : \rangle$ and $S = \langle s_p \rangle^L$ and epimorphisms $\psi: L \rightarrow F$ and $\psi': L \rightarrow F'$ with $\psi(S) = R$ and $\psi(S') = R'$.

(2.4) Now we proceed to prove (2.2). Let $(IG)^{(s)}$ be the s-fold tensor product of $IG$ over $G$. By [4, Lemma 5.2], $(IG)^{(s)}$ has a structure of $G$-module. We contend that

$$E_{r,s}^1 = H_0(G, (IG)^{(s)})$$

and

$$E_{r,s+1}^1 = H_1(G, (IG)^{(s)}).$$

To show this we employ [4, Theorem 7.1]: $J$ is a normal short complex (cf. [4, §2]) and $H_0(J) \cong IG$ is a projective presentation of $IG$. Notice that $J_0 = IF$ and by formula (6.3) of [4], $V^r(J)$ is defined by (1) and so $V^r(J) = N(s)$. Then by formulæ (6.6) (loc. cit.),

$$\text{Tor}^G_0((IG)^{(s)}, Z) = I^s F / (I^{s+1} F + N(s))$$

and

$$\text{Tor}^G_1((IG)^{(s)}, Z) = (I^{s+1} F \cap N(s)) / N(s + 1)$$

which in view of (2) and (3) prove our claim. Now, if $(P)$ and $(Q)$ are presentations and $\phi: F \rightarrow F'$ the epimorphism of the hypothesis, it induces an automorphism $\phi'$ of $G$ and by [4, Lemma 5.2] an automorphism $\phi^{(s)}$ of $(IG)^{(s)}$. Then $\Phi_{r,s,t}: E_{r,s,t}^1 \rightarrow E_{r,s,t}^1$ is an isomorphism for all $t$: for $t = s$ and $s + 1$ by (4) and (5) and for $t > s + 2$ because both sides are trivial. The induced map is natural by definition and it commutes with the differentials. By construction $E^2 = H(E^1, d')$ so that $\Phi: E^2 \rightarrow E'^2$ is an isomorphism as well. By induction $E^r = E'^r$ for all $r$. Q.E.D.

(2.5) We proceed to describe the terms $E_{r,s,t}^1$ and $E_{r,1,2}^1$: for $E_{r,s,t}^1$ we employ (4)

$$H_0(G, (IG)^{(s)}) = [IG \otimes G \cdots \otimes G IG] \otimes G Z$$

$$= [IG \otimes G \cdots \otimes G IG] \otimes G (IG \otimes G Z)$$

where the first brackets enclose an $s$-fold product and the second enclose an $(s - 1)$-fold product. By [6, Chapter VI, Lemma 4.1] $IG \otimes G Z = \widetilde{G}$ which is a
trivial $G$-module. Thus
\[
[ IG \otimes_g \cdots \otimes_g IG ] \otimes_g \overline{G} = [ IG \otimes_g \cdots \otimes_g IG ] \otimes_g \left( IG \otimes_g \overline{G} \right)
\]
\[
= [ IG \otimes_g \cdots \otimes_g IG ] \otimes_g \left( IG \otimes_g \left( Z \otimes Z \overline{G} \right) \right)
\]
\[
= [ IG \otimes_g \cdots \otimes_g \left( \left( IG \otimes_g Z \right) \otimes Z \overline{G} \right) \right]
\]
\[
= [ IG \otimes_g \cdots \otimes_g \left( \overline{G} \otimes Z \overline{G} \right) ].
\]

By successive applications of this we get

**Lemma (2.6)** $E^1_{-s, s} = \text{the s-fold tensor product of } \overline{G} \text{ over } Z$.

**Remark.** In the notation of [4, §5], $E^1_{-s, s} = \overline{G}^{(s)}$.

**Lemma (2.7)** $E^1_{-1, 2} = H_2(G; Z)$.

**Proof.** $E^1_{-1, 2} = H_1(G, IG) = H_2(G; Z)$ by [6, Chapter VI, Theorem 12.1].

(2.8) In our thesis we worked out an explicit isomorphism $E^1_{-s, s} \rightarrow \overline{G}^{(s)}$ as follows: $\overline{G}$ is naturally isomorphic to $IF/(N + I^2F)$. Consider
\[
\gamma: (IF/ (N + I^2F))^{(s)} \rightarrow I^sF/ (N(s) + I^{s+1}F)
\]
defined by
\[
\gamma\left( \prod (x_i - 1) \otimes \cdots \otimes (x_i - 1) \right) = \prod (x_j - 1) + (N(s) + I^{s+1}F).
\]
If $\Phi^{(s)}_{-1, 1}: (IF/N + I^2F) \rightarrow (IF'/N' + I^2F')$ is the isomorphism defined by $\phi$ (and $N' = (r'_i - 1) \subseteq ZF'$) then
\[
\Phi^{(s)}_{-s, s} \gamma = (\Phi^{(s)}_{-1, 1})^{(s)} \gamma'.
\]

Similarly, if $h: F \rightarrow ZF$ is a map $x \mapsto x - 1$ then $h$ induces an isomorphism $(R \cap F_2)/[F, R] \rightarrow (N \cap I^2F)/N(2)$ and the former quotient is the well-known Hopf formula for $H_2(G; Z)$ [6, p. 204]. We omit the proofs.

**Proposition (2.9)** $E^1_{-s, s} = E^\infty_{-s, s} = I^sG/I^{s+1}G$.

**Proof.** Since $ZG \cong ZF/N$, $IG = IF/N$. Consider
\[
\begin{array}{cccc}
0 & \rightarrow & N & \rightarrow & IF & \rightarrow & IG & \rightarrow & 0 \\
\uparrow & & \uparrow & & \uparrow & & \uparrow & & \uparrow \\
0 & \rightarrow & \ker h & \rightarrow & I^sF & \rightarrow & I^sG & \rightarrow & 0
\end{array}
\]
where $h = f|I^sF$ then $\ker h = \ker f \cap I^sF = N \cap I^sF$. Hence $I^s(G) \cong I^sF/ (N \cap I^s) \cong (N + I^sF)/N$.

By the Noetherian isomorphism theorem,
\[
I^{s+1}F \subset I^sF, E^1_{-s, s} \cong I^sF/ (I^sF \cap N + I^{s+1}F)
\]
\[
\cong \frac{I^sF/ (I^sF \cap N)}{(I^sF \cap N + I^{s+1}F)/ (I^sF \cap N)} \cong \frac{I^sF/ (I^sF \cap N)}{I^{s+1}F/ (I^sF \cap N)} = I^sG/I^{s+1}G.
\]


**Lemma (2.10)** If $g$ is an element of $G_n$ then $g - 1$ is an element in $I^nG$ for all $n > 1$.

**Theorem (2.11)** Let $E$ be the spectral sequence of the group $\Gamma = G/G_{q+1}$; $q$ is any integer $> 1$. Let $E$ be the spectral sequence of $G$. Then

$$E'_{r,s} \simeq \bar{E}'_{r,s} \quad \text{for} \quad 1 < r < q,$$

$$E'_{r,s} \simeq \bar{E}'_{r,s} \quad \text{for} \quad 1 < s < r < q.$$  

**Proof.** Statement (7) follows from (6) because

$$E'_{r,s} \simeq \cdots \simeq E'_{s,s} \simeq \bar{E}'_{s,s} \simeq \bar{E}'_{s,s} \simeq \cdots \simeq \bar{E}'_{s,s}.$$  

To prove (6) it is enough to show that

$$IG/I^{r+1}G \simeq \Gamma/I^{r+1}\Gamma \quad \text{for} \quad r < q.$$  

The canonical epimorphism $G \to G/G_{q+1}$ induces the ring epimorphism $ZG \to ZG$. Define

$$\phi: IG \to IG/I^{r+1}G \quad \text{by} \quad g - 1 \to g' - 1 + I^{r+1}G,$$

where $g' = gG_{q+1}$. Since $\phi(I^{r+1}G) = I^{r+1}G$, $\phi$ induces the epimorphism

$$\Phi: IG/I^{r+1}G \to IG/I^{r+1}G$$

given by

$$g - 1 + I^{r+1}G \to g' - 1 + I^{r+1}G.$$  

But $g - 1 + I^{r+1}G$ generates $IG/I^{r+1}G$. Finally, we define an inverse to $\Phi$. Define

$$\psi: IG/I^{r+1}G \to IG, \quad g' - 1 \to g - 1 + I^{r+1}G,$$

where $g' = gG_{q+1}$. The map $\psi$ is well defined, for if $g' = h'$, then $h = gw$, where $w \in G_{q+1}$, but $gw - 1 = (g - 1)(w - 1) + (g - 1) + (w - 1)$ and by Lemma (2.10), $w - 1 \in I^{r+1}G \subset I^{r+1}G$, since $r < q$ and $(g - 1)(w - 1) \in I^{r+2}G \subset I^{r+1}G$. Therefore, $\psi(h' - 1) = (gw - 1) + I^{r+1}G = (g - 1) + I^{r+1}G$. Consider the composite map, $IG \to IG \to IG/I^{r+1}G$, this is a ring homomorphism, and it carries $I^{r+1}G \to I^{r+1}G \to 0$. Therefore $\psi$ induces

$$\psi: IG/I^{r+1}G \to IG/I^{r+1}G.$$  

But $\psi \circ \Phi = 1$ and $\Phi \circ \psi = 1$; hence the result.

**Remarks.** (1) In the course of the proof of Theorem (2.11) we have shown that

$$\Phi: IG/I^nG \simeq IG/I^nG$$

where $\Gamma = G/G_n$ (see (8)).

(2) Let $E$ be the sequence of $G$ associated to the presentation (P) as defined in (1.4), and let $K$ be an Eilenberg-Mac Lane space of type $(G, 1)$. If $\Lambda^pK$ denotes the $p$-fold smash product $[9]$ of $K$ with itself, then the formula $\bar{E}^1_{p,q} = H_p(\Lambda^pK)$ describes a spectral sequence $\bar{E}$ whose $1$-skeleton is described in [5, §1] and [9, §3]. Since $\bar{E}^1_{p,q}$ is isomorphic to $E^1_{p,q}$ and since $\bar{E}^\infty$ is isomorphic to $E^\infty$, there is a natural map $\bar{E}^1 \to E^1$. This map, however, is not monic because the terms $\bar{E}^1_{p,q+k}$ ($k > 2$) are not zero while the corresponding terms in $E$ are. The map, on the other hand, is onto.
3. Applications to links. Let \( S^{(n)} \) be the space consisting of \( n \)-disjoint circles \( S_1, \ldots, S_n \). Assume that fixed orientations have been chosen for \( S^{(n)} \) and \( R^3 \). By an oriented \( n \)-link \( l \) in \( R^3 \) is meant a homeomorphic image of \( S^{(n)} \) in \( R^3 \). Thus \( l \) can be thought of as an ordered collection \((l_1, l_2, \ldots, l_n)\) of homeomorphisms \( l_i: S_i \to R^3 \); where the images \( L_1, L_2, \ldots, L_n \) of the \( l_i \)'s are to be disjoint. Denote \( R^3 \setminus \bigcup L \) by \( L \), and the fundamental group of the complement of \( L \) in \( R^3 \), \( \pi(R^3 \setminus L, x_0) \) by \( G(L) \), where \( x_0 \in R^3 \setminus L \) is a fixed chosen base point. Let \( N_1, N_2, \ldots, N_n \) be solid tori chosen to be regular, disjoint neighborhoods of \( L_1, L_2, \ldots, L_n \) respectively. Let \( p_i(t) (0 < t < 1) \) be a path from the point \( x_0 \) to a point on the boundary of \( N_i \). A meridian-longitude pair \((\alpha_i, \beta_i)\) for \( L \) is a pair of elements of \( G(L) \) where:

(i) \( \alpha_i \) is represented by a closed loop in \( R^3 \setminus L \) described as follows: traverse \( p_i \), then traverse a closed loop on the boundary of \( N_i \setminus L_i \) which has linking number +1 with \( l_i \) and finally return to \( x_0 \) along \( p_i \);

(ii) \( \beta_i \) is represented by a closed loop in \( R^3 \setminus L \) described as follows: traverse \( p_i \), then traverse a simple closed curve on the boundary of \( N_i \) which has linking number 0 with \( l_i \) and which is nullhomologous in \( R^3 \setminus L_i \), and finally return to \( x_0 \) along \( p_i \).

The elements \( \alpha_i, \beta_i \) of \( G(L) \) are well defined in \( G(L) \) up to the choice of \( p_i \) and the orientations chosen for \( S^{(n)} \) and \( R^3 \). Any other \( i \)-th meridian-longitude pair \((\alpha'_i, \beta'_i)\) for \( L \) is obtained from \((\alpha_i, \beta_i)\) by simultaneous conjugation, that is, \( \alpha'_i = g\alpha_ig^{-1} \) and \( \beta'_i = g\beta_ig^{-1} \) for some \( g \in G(L) \).

Two links \( l \) and \( l' \) are said to be isotopic if there exists a continuous family \( h_t: S^{(n)} \to R^3 \) of homeomorphisms, for \( 0 < t < 1 \), with \( h_0 = l \) and \( h_1 = l' \). The fundamental group \( G(L) \) of the complement of \( L \) in \( R^3 \) is not invariant under isotopy of the link. In 1952, K. T. Chen proved [2] that \( G(L)/G_q(L) \), where \( G_q(L) \) is the \( q \)-th lower central subgroup of \( G(L) \), is invariant under isotopy of the link for any arbitrary positive integer \( q \). In 1957, Milnor gave [7] a presentation describing the group \( G(L)/G_q(L) \) and defined the so-called Milnor invariants for a link.

It is known that: if \( G \) is the fundamental group of the complement of an \( n \)-link \( l \) in \( R^3 \) then \( G/G_2 \) is free abelian of rank \( n \).

In Theorem (2.11) we found that if \( E \) is the spectral sequence of \( G/G_{s+1} \), \( s > 1 \), and \( E \) is the spectral sequence of \( G \) that then \( E^{s,s}_{s,s} \simeq E^{s,s}_{s,s} \). In the light of the above stated result of Chen we can conclude:

**Theorem (3.1)** Let \( G \) be the group of a certain \( n \)-link \( l \). Let \( E \) be the spectral sequence of \( G/G_{s+1} \). Then \( E^{s,s}_{s,s} \) is an isotopy invariant of the link \( l \).

Let \( \langle a_{ij} : r_{ij} \rangle (i = 1, 2, \ldots, n; j = 1, \ldots, k_i) \) be a Wirtinger presentation for \( G(L) \) (henceforth we shall write \( G \) for \( G(L) \)) where to each crossing point \( Q_{ij} \) of the projection corresponds a relation \( r_{ij} = 1 \), \( r_{ij} = [b_{ij}, a_{ij}]a_{ij}^{-1}b_{ij}^{-1} b_{ij} = a_{ij}^{-1}b_{ij}^{-1} a_{ij}^{-1} (\lambda(ij), \mu(ij)) \) are given by the segment of \( L \) which crosses over at \( Q_{ij} \), and \( \epsilon_{ij} = \pm 1 \) is the signature of the crossing. Let \( v_{ij} = [b_{ij}, a_{ij}] \) and \( a_{11} = a_1 \). Define

\[ u_{i1} = 1 \quad \text{and} \quad u_{ij} = v_{i-1,j} v_{i-2,j} \cdots v_{i1} \quad (j = 2, 3, \ldots, k_i) \]
and

\[ w_{ik} = b_{i1}^{-1}b_{i2}^{-1} \cdots b_{ik}^{-1}. \]  

Then \( G \) may be presented by

\[ \langle a_j : h_{ij}, s_i \rangle \quad (i = 1, \ldots, n; j = 1, 2, \ldots, k_i), \]

\[ h_{ii} = 1, \quad h_{ij} = u_j a_j a_j^{-1} \quad (j = 2, \ldots, k_i), \]

\[ s_i = [a_i, w_{ik}]. \]  

Note, \( w_{ik} \) is an \( i \)th longitude of \( L \) in \( G \). Thus \( ZG \simeq ZF/N \) where \( F \) is the free group on the \( a_j \)'s and \( N \) is the ideal of \( ZF \) generated by \( h_{ij} - 1, s_i - 1 \) \((i = 1, \ldots, n; j = 2, \ldots, k_i)\). Since \( h_{ij} - 1 = (u_j - a_j a_j^{-1})a_j a_j^{-1} \) and \( a_j a_j^{-1} \) is a unit of \( ZF, N \) is generated as an ideal of \( ZF \) by

\[ \{u_j - a_j a_j^{-1}, s_i - 1\} \quad (i = 1, \ldots, n; \ j = 2, \ldots, k_i). \]  

**Lemma (3.2)** Let \( N_1 \) be the ideal of \( ZF \) generated by \( \{u_j - a_j a_j^{-1}\} \) \((i = 1, \ldots, n; j = 2, \ldots, k_i)\). Then

\[ N_1 \cap I^2F = N_1(s), \]

where \( IF = \ker(ZF \to Z), N_1(1) = N_1, \) and \( N_1(s) = IFN_1(s-1) + N_1(s-1)IF \) \((s > 1)\).

**Proof.** The elements \( \{u_j - a_j a_j^{-1} + N_1(2)\} \) generate the \( Z \)-module \( N_1/N_1(2) \). Moreover we shall show that \( \{u_j - a_j a_j^{-1} + N_1(2)\} \) forms a basis for \( N_1/N_1(2) \). Indeed, if for some integers \( n_j, \Sigma n_j(u_j - a_j a_j^{-1}) = 0 + N_1(2) \), where the summation is over \( i = 1, \ldots, n \) and \( j = 2, \ldots, k_i \). Then \( \Sigma n_j(u_j - a_j a_j^{-1}) \in N_1(2) \), hence \( \Sigma n_j(u_j - a_j a_j^{-1}) \in I^2F \). Thus

\[ (\partial/\partial a_{ij}) \Sigma n_j(u_j - a_j a_j^{-1})(1) = 0 \]  

(cf. [3]).

But \( u_j \in F_2, \) (9), hence \( u_j - 1 \in I^2 \), so, \( (\partial/\partial a_{ii})(u_j - 1)(1) = 0 \) and \( \partial a_{ij}/\partial a_{ii} = 0 \) if \((i, j) \neq (s, t) \) and \( \partial a_{ij}/\partial a_{ii} = 1 \). Therefore,

\[ (\partial/\partial a_{ij}) \Sigma n_j(u_j - a_j a_j^{-1})(1) = (\partial/\partial a_{ii}) \Sigma n_j(u_j - 1 - a_j a_j^{-1} + 1)(1) \]

\[ = \Sigma -n_j(\partial/\partial a_{ii})(a_j a_j^{-1})(1) \]

\[ = \Sigma -n_j((\partial/\partial a_{ii})a_j(1) + a_j(1) + a_j(\partial/\partial a_{ii})a_j^{-1}(1)) \]

\[ = -n_{st}. \]

Hence \( n_{st} = 0 \) (see (14)). Thus the sequence of \( Z \)-modules

\[ 0 \to N_1(2) \to N_1 \to N_1/N_1(2) \to 0, \]

is split exact. Let \( M \) be the \( Z \)-submodule of \( N_1 \) generated by \( \{u_j - a_j a_j^{-1}\} \) \((i = 1, \ldots, n; j = 2, \ldots, k_i)\). Then \( N_1 = M + N_1(2) \). Since

\[ (\partial/\partial a_{ij})(u_j - a_j a_j^{-1})(1) = -1, \quad u_j - a_j a_j^{-1} \in IF, \]

but not in \( I^2F \). So \( M \cap I^2F = \{0\} \), and

\[ N_1 \cap I^2F = N_1(2). \]  

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But
\[ N_i(s + 1) = \sum_{i=1}^{s-1} I^i F N_i(2) I^{s-i-1} F \]
and
\[ N_i(s) \cap I^{s+1} F = \sum_{i=0}^{s-1} I^i F (N_i \cap I F) I^{s-i-1} F. \]

Therefore
\[ N_i(s + 1) = \sum_{i=0}^{s-1} I^i F (N_i \cap I^2 F) I^{s-i-1} F = N_i(s) \cap I^{s+1} F. \quad (16) \]

The proof of (13) follows from (15) by induction on \( s \).

Let \( N_2 \) be the ideal of ZF generated by \{ \( s_i - 1 \) \( i = 1, \ldots, n \) \}, then one can write \( N = N_1 + N_2 \), \( N_1 \) is the ZF-ideal generated by \{ \( u_{ij} - a_{ij} a_{i}^{-1} \) \( i = 1, \ldots, n; \)
\( j = 2, \ldots, k_i \) \} (see Lemma (3.2)).

**Lemma (3.3)** If \( s_i - 1 \) is in \( I^2 F \) for \( i = 1, \ldots, n \), then
\[ E_{r} \simeq E_{r-1} \simeq \cdots \simeq E_{s-r} \simeq \bigotimes^s I F / (N + I^2 F). \]

**Proof.** By (2) and (3),
\[ E_{s-r} = I^r F / (N(s - r + 1) \cap I^r F + I^{s+r} F). \]

Let \( t = s - r + 1 \), then \( 2 < t < s \). Now \( N(t) = N_1(t) + N_2(t) \), where \( N_2 \) is the ideal of ZF generated by \( s_i - 1 \), hence \( N_2 \subset I^r F \). So, \( N(t) \cap I^r F = N_2(t) \cap I^r F \) \( \cap I^r F \). But \( N_1(t) = N_1 \cap I^r F \) (see (13)). Therefore \( N_1(t) \cap I^r F = N_1 \cap I^r F \). Since \( N_2 \subset I^r F \), \( N_2(t) \subset I^{s+r} F \). Hence for \( 1 < r < s - 1 \), \( N(s - r + 1) \cap I^r F + I^{s+r} F = N_1 \cap I^r F + I^{s+r} F = N_1(s) + I^{s+r} F \); the last equality follows from (13). Therefore
\[ I^r F / (N_1(s) + I^{s+r} F) \simeq E_{s-r-1} \simeq E_{s-r-2} \simeq \cdots \simeq E_{s-r} \simeq \bigotimes^s IF / (N + I^2 F). \]

**Corollary (3.4)** If \( s_i - 1 \) is in \( I^2 F \) for \( i = 1, \ldots, n \) then \( E_{s-r-1} \) \( (1 < r < s - 1) \) is free abelian of rank \( n^2 \).

**Proof.** This follows from the fact that \( G / G_2 \) is free abelian of rank \( n \), Lemma (3.3) and the isomorphism \( I/N + I^2 \simeq G/G_2 \).

Next we shall describe a basis for \( E_{s-r} = I^r F / (N_1(s) + I^{s+r} F) \) \( (1 < r < s - 1) \).

Here again we assume that \( s_i - 1 \in I^r F, i = 1, \ldots, n \).

Recall that \( N_1 \) is the ideal of ZF generated by \{ \( u_{ij} - a_{ij} a_{i}^{-1} \) \( i = 1, 2, \ldots, n; \)
\( j = 2, 3, \ldots, k_i \) \}. Let \( \eta_{ij} = a_{ij} a_{i}^{-1} \) and \( \chi_i = a_i - 1. \) Then
\[ \eta_{ij} = u_{ij} - 1 - a_{ij} a_{i}^{-1} + 1 \]
\[ = (a_{i} - 1) - (a_{ij} - 1) + (u_{ij} - 1) - (a_{ij} - 1)(a_{i}^{-1} - 1) + (a_{i}^{-1} - 1). \]

Let \( W_{ij} = (u_{ij} - 1) - (a_{ij} - 1)(a_{i}^{-1} - 1) + (a_{i} - 1)(a_{i}^{-1} - 1) \). Then \( W_{ij} \in I^2 F \).

Hence
\[ \begin{cases} a_{ij} = 1 + \chi_i + W_{ij} + \eta_{ij}, \\ a_{ij}^{-1} = 1 - \chi_i - W_{ij} - \eta_{ij}, \end{cases} \quad (17) \]
where $W_{ij} = W_{ij} + (a_j - 1)(a_j^{-1} - 1) \in I^2F$ and $\eta_{ij} \in N_1$.

Since $G \simeq F/R$, where $F$ is the free group on $\{a_j : i = 1, \ldots, n; j = 1, \ldots, k_i\}$, the set $\{(a_j - 1)(a_{j_1} - 1) \cdots (a_{j_k} - 1) + N_1(s) + I^{*+1}F \} (i_1, i_2, \ldots, i_s = 1, \ldots, n)$ and $j_1, j_2, \ldots, j_s = 1, 2, \ldots, k_i$ generates $I^*F/(N_1(s) + I^{*+1}F)$. Using the equalities (17) one can write

$$\prod_{i=1}^s (a_{j_i} - 1) + N_1(s) + I^{*+1}F = \prod_{i=1}^s \chi_i + N_1(s) + I^{*+1}F.$$ 

Thus,

$$\{(a_1 - 1)(a_2 - 1), \ldots, (a_s - 1) + N_1(s) + I^{*+1}F\}$$

(18)

forms a generating set of $I^*/(N_1(s) + I^{*+1})$. But there are $n^s$ elements in the set (18); hence (18) forms a $Z$-basis for $I^*/(N_1(s) + I^{*+1})$ (see Corollary 3.4).

Consider the $E_{s-1}^{*+1}$ term of the spectral sequence $E_i$, where all terms of degree $\neq 0, 1$ of $F_{s-1}$ are zero. Therefore we have

$$0 \rightarrow E_{s-1}^{*+1} \rightarrow E_{s-1}^{*+1} \rightarrow E_{s-1}^{*+1} \rightarrow 0.$$ 

Explicitly, we have

$$\rightarrow 0 \rightarrow (N \cap I^F) / (N(2) \cap I^F) \rightarrow I^F / (I^{*+1}F + N(2) \cap I^F) \rightarrow 0,$$

where $d_{s-1}^{*+1}$ is induced from the inclusion $N \cap I^F \rightarrow I^F$. But

$$E_{s+1}^{*+1} \simeq H(E_{s+1}^{*+1}) \simeq \ker d_{s-1}^{*+1} / d_{s-2}^{*+1} (E_{s-2}^{*+1}) \simeq \frac{I^F / (N(2) \cap I^F + I^{*+1}F)}{(N \cap I^F + N(2) \cap I^F + I^{*+1}F) / (N(2) \cap I^F + I^{*+1}F)},$$

(19)

since $N(2) \cap I^F \subset N \cap I^F$.

**THEOREM (3.5)** If $s_i - 1 \in I^F (i = 1, 2, \ldots, n)$, then

$$E_{s+1}^{*+1} \simeq \frac{I^F / (N_1(s) + I^{*+1}F)}{(N_2 + N_1(s) + I^{*+1}F) / (N_1(s) + I^{*+1}F)},$$

(20)

where the set $\{(a_1 - 1)(a_2 - 1) \cdots (a_s - 1) + N_1(s) + I^{*+1}F\} (i_1, i_2, \ldots, i_s = 1, \ldots, n)$, gives a basis for $I^F / (N_1(s) + I^{*+1}F)$, and where the set $(s_i - 1) + N_1(s) + I^{*+1} (i = 1, \ldots, n)$, gives a basis for $(N_2 + N_1(s) + I^{*+1}F) / (N_1(s) + I^{*+1}F)$.

**PROOF.** Since $s_i - 1 \in I^F$, $N_2 \subset I^F$. Hence $N \cap I^F = N_1 \cap I^F + N_2 = N_1(s) + N_2$ (see (13)). Also since $N(2) = N_1(2) + N_2(2)$ and $N_2(2) \subset I^{*+1}F$, it follows that

$$N(2) \cap I^F = N_1(2) \cap I^F = N_1 \cap I^F \cap I^F = N_1 \cap I^F = N_1(s).$$
Substituting these equalities in (19) we get (20). The rest of Theorem (3.5) is clear. Since
\[ s_i - 1 = [a_i, w_{ik}] - 1 = (a_i w_{ik} - w_{ik} a_i) a_i^{-1} w_{ik}^{-1} \]
\[ = ((a_i - 1)(w_{ik} - 1) - (w_{ik} - 1)(a_i - 1)) a_i^{-1} w_{ik}^{-1}. \]
Hence \( N_2 \) may be thought of as being generated by \( \chi_i(w_{ik} - 1) - (w_{ik} - 1) \chi_i; \)
\( i = 1, \ldots, n \). Thus, as a \( \mathbb{Z} \)-module, \( (N_2 + N_1(s) + I^{s+1}F)/(N_1(s) + I^{s+1}F) \) is generated by \( \chi_i(w_{ik} - 1) - (w_{ik} - 1) \chi_i + N_1(s) + I^{s+1}F \).

A simple computation shows that for any \( n \)-link,
\[ s_i - 1 = \sum_{j=1}^{n} \mu(i, j)(\chi_j \chi_i - \chi_i \chi_j) + N_1(2) + I^3F \]
where \( \mu(i, j) \) is the linking number of the \( i \)th and \( j \)th components of \( L \). Hence \( E_{2,2}^3 \) gives very little information about \( L \).

Next, we give an example where we compute \( E_{3,3}^3 \) for a link whose \( s_i \)'s belong to \( I^3F \). The link is shown in the figure and one has
\[
\begin{align*}
b_{1,2i-1} &= a_{3j-3}, & b_{1,2i} &= a_{2j}^{-1}, \\
b_{2,2j-1} &= a_{3j-4}, & b_{2,2j} &= a_{3j-4}^{-1}, \\
b_{3,3j-3} &= a_{1j}, & b_{3,3j-2} &= a_{2j}, \\
b_{3,3j-1} &= a_{1j}^{-1}, & b_{3,3j} &= a_{2j}^{-1}. 
\end{align*}
\]
Computing \( w_{1,2m} \), \( w_{2,2m} \) and \( w_{3,4m} \), we get
\[
\begin{align*}
w_{1,2m} &= a_{31}^{-1}([a_{34}, a_{24}][a_{38}, a_{26}] \cdots [a_{3,2m}, a_{22}^{-1}])a_{31}, \\
w_{2,2m} &= a_{32}^{-1}([a_{33}, a_{12}] [a_{37}, a_{14}] \cdots [a_{3,4m}, a_{1,2m}])a_{32}^{-1}, \\
\text{and} \quad w_{3,4m} &= a_{23}^{-1}([a_{22}, a_{12}] \cdots [a_{2,2j}, a_{1,2j}^{-1}] \cdots [a_{2,2m}, a_{1,2m}])a_{32}^{-1}.
\end{align*}
\]
Hence,
\( i) s_1 = [a_1, a_3^{-1}([a_{3,4j}, a_{2,2j+2}])a_3], \)
\( ii) s_2 = [a_2, a_{3,2m}([a_{3,4j-1}, a_{1,2j}])a_{3,2m}^{-1}], \)
\( iii) s_3 = [a_3, a_{2,2m}([a_{2,2j}, a_{1,2j}])a_{2,2m}]. \)

Upon making use of the substitutions (17) for the different \( a_{ij} \) and \( a_{ij}^{-1} \) we obtain
\[
\begin{align*}
s_1 - 1 &= m[X_1, [X_2, X_3]] + N_1(3) + I^4F, \\
s_2 - 1 &= m[X_2, [X_3, X_1]] + N_1(3) + I^4F, \\
s_3 - 1 &= m[X_3, [X_1, X_2]] + N_1(3) + I^4F,
\end{align*}
\] where by \([X, Z]\) we mean the usual Lie bracket, \([X, Z] = XZ - ZX\). Thus \( (N_2 + N_1(3) + I^4F)/(N_1(3) + I^4F) \) is generated by \( s_1 - 1, s_2 - 1 \) and \( s_3 - 1 \) as in Theorem (3.5). But \([X_1, [X_2, X_3]] + [X_2, [X_3, X_1]] + [X_3, [X_1, X_2]] = 0 \) so that \( s_1 - 1 \) and \( s_2 - 1 \) form a basis for \( (N_2 + N_1(3) + I^4F)/(N_1(3) + I^4F) \). Therefore
\[ E_{3,3}^3 \simeq \mathbb{Z} \oplus \cdots \oplus \mathbb{Z} \oplus \mathbb{Z}_m \oplus \mathbb{Z}_m, \]
where there are twenty-five copies of $\mathbb{Z}$ in the above sum; since

$$(I^IF)/(N_i(s) + I^{s+1}) \simeq \mathbb{Z} \oplus \cdots \oplus \mathbb{Z},$$

there are twenty-seven copies of $\mathbb{Z}$. Thus 3-links of the type shown in the figure whose $m$'s differ are distinguishable links.

Finally, we point out how some of the Milnor invariants show up in computing the $E_{s,s}$ terms. Here then is a brief account of Milnor's work.

In [7] Milnor showed that the group $G/G_{s+1}$, for any nonnegative integer $s$, may be presented by $\langle a_1, \ldots, a_n; [\alpha_i, \omega_j], F_{s+1} \rangle$ ($i = 1, \ldots, n$), where $a_i = a_{i1} = a_i$ represents an $i$th meridian of $L$, $\omega_j$ is a word in $\alpha_{i1}, \ldots, \alpha_n$ that represents an $i$th longitude of $L$ in $G/G_{s+1}$ and $F$ is the free group on $\{\alpha_i; i = 1, \ldots, n\}$. 

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The Magnus expansion of $\omega_i$ is obtained by substituting $\alpha_j = 1 + X_j$, $\alpha_j^{-1} = 1 - X_j + X_j^2 - X_j^3 + \cdots$ in the word $\omega_i$. Thus $\omega_i$ can be expressed as a formal, noncommutative power series in the indeterminants $X_1, \ldots, X_n$. Namely,

$$
\omega_i = 1 + \sum_{j_1=1}^{n} \mu(j_1, i)X_{j_1} + \sum_{j_1, j_2=1}^{n} \mu(j_1, j_2, i)X_{j_1}X_{j_2} + \cdots
$$

$$
+ \sum_{j_1, j_2, \ldots, j_t=1}^{n} \mu(j_1, j_2, \ldots, j_t, i)X_{j_1}X_{j_2} \cdots X_{j_t} + \cdots
$$

Thus a coefficient is defined for each sequence $j_1, j_2, \ldots, j_t, i$ ($t > 1$) of integers between 1 and $n$.

Let $\Delta(i_1, \ldots, i_r) = \text{g.c.d.} \mu(j_1, \ldots, j_r)$, where $j_1, \ldots, j_r$ $(2 \leq t \leq r - 1)$ is to range over all sequences obtained by cancelling at least one of the indices $i_1, \ldots, i_r$ and permuting the remaining indices cyclically. Then Milnor proved that: the residue classes

$$
\bar{\mu}(j_1, \ldots, j_t, k) \equiv \mu(j_1, \ldots, j_t, k) \mod \Delta(j_1, \ldots, j_t, k)
$$

are isotopy invariants of $L$ provided that $t < s$.

If we restrict ourselves to links whose $\omega_i$'s belong to $F_{s-1}$ for $(i = 1, \ldots, n)$, then $\mu(j_1, \ldots, j_t, i) = 0$ for $1 < t < s - 2$. But then $\bar{\mu}(j_1, \ldots, j_{s-1}, i) = \mu(j_1, \ldots, j_{s-1}, i)$, and hence $\mu(j_1, \ldots, j_{s-1}, i)$ are isotopy invariants for such links.

Let $\bar{F}$ be the kernel of $ZF \to Z$. Let $\bar{N}$ be the ideal of $ZF$ generated by $[\alpha, \omega] - 1$ $(i = 1, \ldots, n)$, and $\bar{F}_{s+1} = 1$. Let $\bar{E}$ be the spectral sequence associated with the presentation given by Milnor for the group $G/G_{s+1}$. Now

$$
\bar{E}_{s,s} = I^s\bar{F}/(\bar{N} + I^{s+1}\bar{F}).
$$

If $\omega_i \in \bar{F}_{s-1}$, then $[\alpha, \omega_i] - 1 \in I^s\bar{F}$ $(i = 1, \ldots, n)$ and $\bar{N} \cap I^s\bar{F} = \bar{N}$. Hence for this case,

$$
\bar{E}_{s,s} = I^s\bar{F}/(\bar{N} + I^{s+1}\bar{F}) \approx \frac{I^s\bar{F}/I^{s+1}\bar{F}}{(\bar{N} + I^{s+1}\bar{F})/I^{s+1}\bar{F}}.
$$

Where $I^s\bar{F}/I^{s+1}\bar{F}$ is a free $Z$-module write

$$
\{ X_{i_1}X_{i_2} \cdots X_{i_s} + I^{s+1}\bar{F}: i_1, \ldots, i_s = 1, \ldots, n \}
$$

as a basis, and where ($\bar{N} + I^{s+1}\bar{F})/I^{s+1}\bar{F}$ is a free $Z$-module generated by $\{[\alpha, \omega] - 1 + I^{s+1}\bar{F}: i = 1, \ldots, n \}$. Thus

$$
[\alpha, \omega] - 1 = \sum_{j_1, \ldots, j_{s-1}=1}^{n} [X_{i_1}, \mu(j_1, \ldots, j_{s-1}, i)X_{j_1}X_{j_2} \cdots X_{j_{s-1}}] + I^{s+1}\bar{F}.
$$

Therefore we can replace the set of generators above of the $Z$-module $(\bar{N} + I^{s+1}\bar{F})/I^{s+1}\bar{F}$ by the set

$$
\left\{ \sum_{j_1, \ldots, j_{s-1}=1}^{n} [X_{i_1}, \mu(j_1, \ldots, j_{s-1}, i)X_{j_1} \cdots X_{j_{s-1}}] + I^{s+1}\bar{F}: i = 1, \ldots, n \right\}.
$$

(21)

We already proved $E_{s,s} \approx \bar{E}_{s,s}$ (see, Theorem (2.11)). We shall describe a precise isomorphism for the case at hand (see, Theorem (3.5)).
\[ E_{s,s}^2 = \frac{I^s F / (N_i(s) + I^{s+1} F)}{(N_2 + N_1(s) + I^{s+1} F) / (N_1(s) + I^{s+1} F)} \]

\[ \rightarrow \frac{I^s \overline{F} / I^{s+1} \overline{F}}{(N + I^{s+1} \overline{F}) / I^{s+1} \overline{F}} = \overline{E}_{s,s}^1 \]

is an isomorphism. From the Wirtinger presentation of \( G \) we have \( r_{ij} = b_j a_j b_i^{-1} a_{j+1}^{-1} \). Thus \( a_{ij+1} = b_j a_j b_i^{-1} = b_j b_{i-1} \cdots b_{i} a_i b_{i-1} \cdots b_{j-1} b_j^{-1} \). Let \( z_{ij} = b_j b_{i-1} \cdots b_{j} \).

Define a sequence of homomorphisms \( M_k : F \rightarrow \overline{F} \) as follows, by induction on \( k \):

\[ M_1(a_{ij}) = a_{il}, \quad M_{k+1}(a_{ij+1}) = M_k(z_{ij} a_i z_{ij}^{-1}), \quad M_{k+1}(a_{il}) = a_{il}. \]

Then it can be proved by induction on \( k \) that

\[ M_k(a_{ij}) = a_{ij} \mod(F_k R), \quad M_k(a_{ij}) = M_{k+1}(a_{ij}) \mod(\overline{F}_k). \]

We claim that \( \overline{M}_{s+1} : IF \rightarrow \overline{F} \), where \( \overline{M}_{s+1} \) is the map induced from \( M_{s+1} : F \rightarrow \overline{F} \), induces the required isomorphism. Because

\[ M_{s+1}(a_{ij}) = M_{s+1}(z_{ij-1} a_i z_{ij-1}^{-1}) = M_{s+1}(z_{ij-1}) a_i M_{s+1}(z_{ij-1}^{-1}) \equiv a_{il} \mod F_2; \]

it follows that \( \overline{M}_{s+1}(u_{ij} - a_j a_{i-1}^{-1}) \in \overline{I^2 \overline{F}} \). Hence \( M_{s+1}(N_1(s)) \subset I^{s+1} \overline{F} \), moreover, because of \( M_{s+1}(w_{ik}) = M_{s+1}(z_{ik}) = w_{ik} \mod(F_{s+1} R) \). Since \( w_{ik} \) represents an \( i \)th longitude of \( G \), \( M_{s+1}(w_{ik}) \) represents an \( i \)th longitude in \( G/G_{s+1} \). Let \( M_{s+1} : (I^s F) / (N_1(s) + I^{s+1} F) \rightarrow I^s \overline{F} / I^{s+1} \overline{F} \) be the canonical homomorphism induced from \( M_{s+1} \). Then \( M_{s+1} \) is an isomorphism, since \( I^s F / (N_1(s) + I^{s+1} F) \) and \( I^s \overline{F} / I^{s+1} \overline{F} \) both have rank \( n^2 \).

Also,

\[ \overline{M}_{s+1} : (N_2 + N_1(s) + I^{s+1} F) / (N_1(s) + I^{s+1} F) \rightarrow (N + I^{s+1} \overline{F}) / I^{s+1} \overline{F} \]

is an isomorphism.

So if one can extract a basis from the generating set (21) of the free \( Z \)-module \((N + I^{s+1} \overline{F}) / I^{s+1} \overline{F} \), one can then express \( E_{s,s}^2 \simeq \overline{E}_{s,s}^2 \) as a direct sum of a finite number of infinite cyclic groups and cyclic groups of finite order; hence obtaining an explicit demonstration of how the \( \mu \)'s appear in \( E_{s,s}^2 \). For example, for the link described in the figure we have

\[ E_{s,s}^2 \simeq \overline{E}_{s,s}^2 \simeq Z \oplus \cdots \oplus Z \oplus Z_m \oplus Z_m, \]

where \( m = \mu(1, 2, 3) = \mu(3, 2, 1) = \mu(2, 3, 1) \).

Here are some properties of the Milnor invariants that we will need (see [7]).

(A) The \( \bar{\mu} \) satisfy a cyclic symmetry, that is, \( \bar{\mu}(i_1, i_2, \ldots, i_s) = \bar{\mu}(\sigma(i_1), \sigma(i_2), \ldots, \sigma(i_s)) \), where \( \sigma \) is a cyclic permutation of \( 1, 2, \ldots, s \). By an invariant \( \bar{\mu}(i_1, \ldots, i_{r+s}) \) of type \([r, s]\) will be meant one which involves the index \('1' \text{\ r-times and the index \('2' \text{\ s-times. Then}

(B) (i) All invariants of type \([r, 0]\) and \([r, 1]\) \((r > 2)\) are zero. The invariants of type \([1, 1]\) are the linking numbers and these are not necessarily zero.

(ii) All invariants of type \([2m + 1, 2]\) are also zero.
(iii) For the invariants of type \([2m, 2]\) we have

\[
\bar{\mu}(1, \ldots, 1, 2, 1, 2) = - \binom{2m}{1} \mu(1, \ldots, 1, 1, 2, 2),
\]

\[
\bar{\mu}(1, \ldots, 1, 2, 1, 1, 2) = \binom{2m}{2} \mu(1, \ldots, 1, 1, 2, 2),
\]

\[
\bar{\mu}(1, \ldots, 1, 2, 1, 1, 1, 2) = - \binom{2m}{3} \mu(1, \ldots, 1, 1, 2, 2), \quad \text{etc.}
\]

In view of cyclic symmetry (see (A)) this means that all of the invariants of type \([2m, 2]\) are completely determined by \(\bar{\mu}(1, \ldots, 1, 1, 2, 2)\).

Let \(L\) be a two-link. Then

\[
E^2_{-2,2} \simeq \frac{I^2 F/I^3 F}{(\bar{N} + I^2 F)/I^3 F}.
\]

The set \(\{X_1^2 + I^3 F, X_2^2 + I^3 F, X_1 X_2 + I^3 F, X_2 X_1 + I^3 F\}\) is a basis for \(I^2 F/I^3 F\); while \((\bar{N} + I^2 F)/I^3 F\) is generated by

\[
[\alpha_1, \omega_1] - 1 = [X_1, \mu(2, 1)X_2] + I^3 F, \quad [\alpha_2, \omega_2] - 1 = [X_2, \mu(1, 2)X_1] + I^3 F.
\]

But \([X_1, \mu(2, 1)X_2] = - [X_2, \mu(1, 2)X_1]\). Therefore \((\bar{N} + I^2 F)/I^3 F\) is a free \(\mathbb{Z}\)-module with basis \(\{\mu(1, 2)(X_1 X_2 - X_2 X_1) + I^3 F\}\). But the set \(\{X_1^2 + I^3 F, X_2^2 + I^3 F, X_1 X_2 + I^3 F, X_2 X_1 + I^3 F\}\) may be taken as a basis for \(I^2 F/I^3 F\); it follows that

\[
E^3_{-2,2} \simeq E^2_{-2,2} \simeq \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}_{\mu(1,2)}.
\]

Next assume \([\alpha_1, \omega_1] - 1 \in I^3 F\) \((i = 1, 2)\). Then \((\bar{N} + I^3 F)/I^4 F\) is generated by

\[
[\alpha_1, \omega_1] - 1 = \sum_{j_1, j_2 = 1, 2} [X_1, \mu(j_1, j_2, 1)X_{j_1}X_{j_2}] + I^4 F,
\]

\[
[\alpha_2, \omega_2] - 1 = \sum_{j_1, j_2 = 1, 2} [X_2, \mu(j_1, j_2, 2)X_{j_1}X_{j_2}] + I^4 F.
\]

But, all the \(\mu(j_1, j_2, i), i = 1, 2\), appearing above are zero, due to properties (B) (i) and (B) (ii). Hence nothing could be said about such a link by looking at \(E^3 F_{-3,3}\). So we consider the case \([\alpha_i, \omega_i] - 1 \in I^4 F\) \((i = 1, 2)\). Then \((\bar{N} + I^4 F)/I^5 F\) is generated by

\[
[\alpha_1, \omega_1] - 1 = \mu(1, 1, 2, 2)(X_1^2 X_2 + 2X_1X_2X_2X_1 - 2X_1X_2X_2X_2 - X_2^2 X_1^2) + I^5 F,
\]

\[
[\alpha_2, \omega_2] - 1 = \mu(1, 1, 2, 2)(X_2^2 X_1 + 2X_1X_2X_2X_1 - 2X_1X_2X_2X_1 - X_1^2 X_2^2) + I^5 F.
\]

Hence \((\bar{N} + I^4 F)/I^5 F\) is a free \(\mathbb{Z}\)-module with basis the vector

\[
\mu(1, 1, 2, 2)(X_1^2 X_2 + 2X_1X_2X_2X_1 - 2X_1X_2X_2X_2 - X_2^2 X_1^2) + I^5 F.
\]

The free \(\mathbb{Z}\)-module \(I^4 F/I^5 F\) has rank 16. Hence the spectral sequence term

\[
E^4_{-4,4} \simeq E^4_{-4,4} \simeq \mathbb{Z} \oplus \cdots \oplus \mathbb{Z} \oplus \mathbb{Z}_{\mu(1,1,2,2)},
\]

where there are fifteen copies of \(\mathbb{Z}\) in the summand.
Thus, for the special links whose longitudes belong to $I^*F$ the term $E^3_{-s,2}$ sheds light on the Milnor invariants. Naturally one would like to do this study for more general links. The calculations are similar to those in [8].

REFERENCES


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