NONINVARIANCE OF AN APPROXIMATION PROPERTY
FOR CLOSED SUBSETS OF RIEMANN SURFACES

BY

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Abstract. A closed subset $E$ of an open Riemann surface $M$ is said to have the approximation property $\mathcal{A}$ if each continuous function on $E$ which is analytic at all interior points of $E$ can be approximated uniformly on $E$ by functions which are everywhere analytic on $M$. It is known that $\mathcal{A}$ is a topological invariant (i.e., preserved by homeomorphisms of the pair $(M, E)$) when $M$ is of finite genus but not in general, not even for $C^\infty$ quasi-conformal automorphisms of $M$. The principal result of this paper is that $\mathcal{A}$ is not invariant even under a real-analytic isotopy of quasi-conformal automorphisms (of a certain $M$). $M$ is constructed as the two-sheeted unbranched cover of the plane minus a certain discrete subset of the real axis, and the isotopy is induced by $(x + iy, t) \mapsto x + ity$, for $t > 0$; $E$ can be taken to be that portion of $M$ which lies over a horizontal strip.

Let $M$ be an open Riemann surface and $E$ be a closed subset of $M$. Denote by $A(E)$ the collection of continuous functions on $E$ which are analytic on the interior of $E$. Say that $E$ has property $\mathcal{A}$ in $M$ if each element of $A(E)$ can be approximated uniformly on $E$ by functions which are analytic everywhere on $M$. By definition $\mathcal{A}$ is a property of the pair $(M, E)$ and is a conformal invariant; that is, if one pair is related to another by an analytic homeomorphism, the both pairs have $\mathcal{A}$ or neither one does. It is natural to ask whether $\mathcal{A}$ is invariant for other types of equivalence. The famous theorem of Bishop and Mergelyan implies that $\mathcal{A}$ is a topological invariant in case $E$ is compact, and the theorem of Arakelyan shows that $\mathcal{A}$ is a topological invariant when $M$ is planar. By "topological invariant" I mean an invariant for the equivalence defined by homeomorphism of pairs. It is known [S] that $\mathcal{A}$ is a topological invariant when $M$ is of finite genus but is not a topological invariant in general. In fact, $\mathcal{A}$ is not an invariant even for the finer partition induced by this relation: call $(M, E)$ equivalent to $(M', E')$ if and only if $M$ is conformally equivalent to $M'$ and there is a quasi-conformal homeomorphism of $M$ onto $M'$ which carries $E$ onto $E'$ [S]. When I showed him the known example illustrating this phenomenon, Dennis Sullivan asked me whether the example could be improved to one in which $\mathcal{A}$ fails to be preserved by an isotopy. The intent of this article is to provide an affirmative answer by demonstrating that a real-analytic isotopy need not preserve $\mathcal{A}$, even though it is "affine", being...
definable in local coordinates by \((x + iy, t) \mapsto x + ity\) for \(0 < t < \infty\), and all the homeomorphisms of \(M\) throughout the homotopy are quasi-conformal equivalences. The precise statement is given below as the theorem. Let us write \(\"E \in \mathcal{G}\"\) in place of \(\"E has property \mathcal{G}\"\) in \(M\).

**Theorem.** There is a connected open Riemann surface \(M\), a homotopy \(U: M \times \mathbb{R}^+ \rightarrow M\), and two closed connected subsets \(E^+\) and \(E^-\) of \(M\) such that the following hold, where \(U_t(p)\) is written in place of \(U(p, t)\).

(a) \(M\) can be given as a two-sheeted unbranched cover of a plane region by \(Z:\ M \rightarrow \mathbb{C} - \mathcal{D}\), where \(\mathcal{D}\) is a discrete subset of the real axis, and the homotopy \(U\) is induced by the affine plane homotopy \(u:\ \mathbb{C} \times \mathbb{R}^+ \rightarrow \mathbb{C}\) defined by \(u(x + iy, t) = x + ity\); that is, \(Z(U_t(p, t)) = u(Z(p), t)\) for all \(p \in M\) and \(t \in \mathbb{R}^+\).

(b) \(U\) is real-analytic on \(M \times \mathbb{R}^+\).

(c) Each \(U_t\) is a real-analytic quasi-conformal automorphism of \(M\).

(d) \(U_1\) is the identity of \(M\).

(e) As \(t \rightarrow 1\) the distortion \(|\partial U_t/\partial Z|/|\partial U_t/\partial Z|^{-1}\rightarrow 0\) uniformly on \(M\).

(f) \(\{t: U_t(E^+) \in \mathcal{G}\} = (0, 1]\).

(g) \(\{t: U_t(E^-) \in \mathcal{G}\} = (0, 1]\).

Because of the relation between \(U\) and \(u\) one may call \(U\) or \(U_t\) "affine". For \(t\) near 1 the maps \(U_t\) are as close to conformal and as close to the identity as may be desired; yet the slight movement of \(E^+\) to \(U_{1+\epsilon}(E^+)\) destroys \(\mathcal{G}\) and the slight movement of \(E^-\) to \(U_{1-\epsilon}(E^-)\) creates \(\mathcal{G}\).

The proof of the theorem will be along the following lines. A locally finite collection of closed intervals \(J\) will be selected in \(\mathbb{R}\) according to certain technical requirements. \(M\) will be formed as follows: two copies of \(\mathbb{C}\) minus all the \(J\)'s will be joined in the standard way along corresponding slits \(J\), leaving out the set \(\mathcal{D}\) consisting of all the endpoints of the \(J\)'s. The natural projection \(Z:\ M \rightarrow \mathbb{C} - \mathcal{D}\) exhibits \(M\) as a two-sheeted unbranched covering of a plane region. The homotopy defined on \(\mathbb{C} \times \mathbb{R}^+\) by \(u(x + iy, t) = x + ity\) transfers via \(Z\) to a homotopy defined on each component of \(Z^{-1}(V) \times \mathbb{R}^+\), where \(V\) is any vertical strip in \(\mathbb{C} - \mathcal{D}\) whose projection on the \(y\)-axis is \(\mathbb{R}, \mathbb{R}^+,\) or \(\mathbb{R}^-\). These homotopies agree whenever there is an overlap; so they define a homotopy \(U\) on \(M\). Assertions (a)–(e) will follow immediately.

Define \(S^+_\lambda\) to be the strip \(\{x + iy: |y| < \lambda\}\) and \(S^-_\lambda\) (resp., \(S^+_\lambda\)) to be the intersection of \(S^+_\lambda\) with the closed right (resp., left) half-plane. Define \(E^+_\lambda = Z^{-1}(S^+_\lambda)\). The intervals \(J\) will be arranged in \(\mathbb{R}\) in such a fashion that for a certain \(\lambda_0 > 0\) none of the sets \(S^+_\lambda\) for \(\lambda > \lambda_0\) (resp., \(S^-_\lambda\) for \(\lambda > \lambda_0\)) supports a nontrivial bounded analytic function which vanishes on \(S^+_\lambda \cap \mathcal{D}\) (resp., \(S^-_\lambda \cap \mathcal{D}\)). This will imply that \(\mathcal{G}\) fails for the corresponding \(E^+_\lambda\) (resp., \(E^-_\lambda\)). Because \(U_t(E^+_\lambda) = E^+_\lambda\), this will mean that one-half of (f) and (g), namely with "\(\subseteq\)" in place of "\(=\)" will be true for \(E^+ = E^+_\lambda\) and \(E^- = E^-_\lambda\).

The proof of the other half of (f) and (g) is technically more difficult and will require the bulk of the work. These ideas will be involved. Fix one \(E^+_\lambda\), \(\lambda < \lambda_0\), or one \(E^-_\lambda\), \(\lambda < \lambda_0\), and call it \(E\). Because of the spatial arrangement of \(\mathcal{D}\) there will
exist a sequence of meromorphic functions $H_n$ on $C$ each of which has a zero or a pole at each point of $\mathcal{D}$ and nowhere else and each of which is small on a large bounded set $X_n$, is large on a co-compact subset $Y_n$ of $S^\infty_x$, and is nearly 1 on a vertical strip $\sigma_n$ which separates $X_n$ from $Y_n$. Multiplying $H_n$ by an exponential function and extracting a square root, we obtain an analytic function $\pi_n$ on $M$ which separates these four sets: $Z^{-1}(X_n)$, the two components of $Z^{-1}(\sigma_n)$, and $Z^{-1}(Y_n)$. $E - \cup Z^{-1}(\sigma_n)$ consists of separated pieces of finite genus; so if $f \in A(E)$, there is an analytic $\Phi$ on $M$ such that $g = f - \Phi$ is very small on $E - \cup Z^{-1}(\sigma_n)$. $g$ can be approximated by $\sum g_n$, where $g_n$ is $C^1$ on $E^o \cup Z^{-1}(X_n)$, is supported in $Z^{-1}(\sigma_n)$, and has small $\bar{Z}$-derivative. Because $\pi_n$ separates the four sets indicated above, $g_n$ can be written $g_n = \tilde{g}_n \circ \pi_n$, where $\tilde{g}_n$ is (essentially) a smooth function of compact support in $C$ having small $\bar{Z}$-derivative. Because of this property of $\tilde{g}_n$ and of the nature of $\pi_n(Z^{-1}(X_n \cup \sigma_n \cup Y_n))$, $\tilde{g}_n$ can be approximated reasonably well by a rational function $k_n$. Thus, $k_n \circ \pi_n$ approximates $g_n$ on $E \cup Z^{-1}(X_n)$. The poles of $k_n \circ \pi_n$ in $M$ can be removed by a multiplier without essentially altering the goodness of the approximation to $g_n$. The resulting analytic functions $\psi_n$ on $M$ can be summed to an analytic function $\Psi$, because $Z^{-1}(X_n) \cap M$, and thus $\Phi + \Psi$ will approximate $f$.

The reader acquainted with [S] will recognize a similarity in spirit between the above scheme for showing $E \in \mathcal{D}$ and the method used in §11 of [S]. However, the method of [S] is technically simpler in several respects. For example, the projection $\pi$ of the surface in [S] served to separate all the components of all the $\pi^{-1}(\sigma_n)$ at once, and it was not necessary to allow singularities to arise in intermediate steps for later removal. The punctures in the surface of [S] allowed the existence of enough globally analytic functions but prevented the existence of an isotopy. The surface $M$ of this article is punctured in order to allow construction of the separating functions $\pi_n$; since the punctures lie over the real axis, they do not interfere with the isotopy. I thank Ted Gamelin for asking me whether a related result utilized an iteration of some sort. This remark prompted me to look for an approximation of $g = f - \Phi$ by means of separate approximations of “pieces” of $g$; this in turn opened up the possibility of individual separating functions $\pi_n$ for each “piece”. And I thank Dennis Sullivan for raising the question which this paper answers.

The rest of the paper is devoted to a proof of the theorem. Most of the technical aspects are gathered into manageable or convenient aggregates and termed lemmas or corollaries.

**Lemma 1 (a).** If $|z| < \frac{1}{2}$, then $|\log(1 + z) - z| < |z|/2$.

(b) If $|w| < 1$, then $|e^w - 1| < (e - 1)|w| < 2|w|$.

(c) If $\sum |a_n| < \frac{1}{2}$, then $|\Pi(1 + a_n) - 1| < 3 \sum |a_n|$.

**Proof.** (a) and (b) are well known and follow readily from easy manipulations of Taylor series. Assume $\sum |a_n| < \frac{1}{2}$. From (a) it follows that $|\log \Pi(1 + a_n) - \sum a_n| < \frac{1}{2} \sum |a_n|$; so $|\log \Pi(1 + a_n)| < \frac{\log 3}{4} \sum |a_n| < \frac{3}{4}$. Now an application of (b) yields (c).
Lemma 2. For $0 < \lambda < \infty$ and complex $z_0$ and $z$ define

$$\tau(z; z_0, \lambda) = \frac{(a^z - a^{z_0})}{(a^z + a^{z_0})},$$

where $a = \exp(\pi/2\lambda)$. Suppose $0 < \lambda < \pi(2 \log 2)^{-1}$ and $|x - x_0| > 1$, where $x = \Re z$ and $x_0 = \Re z_0$. Let $\omega = \text{signum}(x - x_0)$. Then

$$|1 - \omega \tau(z, z_0, \lambda)^{\pm 1}| < 4a^{-|x - x_0|} = 4 \min(a^x \cdot a^{-x_0}, a^{x_0} \cdot a^{-x}),$$

Proof. In case $x > x_0 + 1$ compute

$$|1 - \omega \tau(z, z_0, \lambda)^{\pm 1}| = \frac{|a^{z_0} + a^z|}{|a^z + a^{z_0}|} = a^{x_0 - x} \left| \frac{1 + a^{z_0 - z}}{1 + a^{z_0 - z}} \right| < a^{x_0 - x} \frac{2}{1 - a^{x_0 - x}} < 4a^{x_0 - x},$$

because $|a^{z_0 - z}| = 1$ and $a^{x_0 - x} < a^{-1} < \frac{1}{2}$. The computations for $\tau^{-1}$ and for the case $x < x_0 - 1$ are very similar.

Lemma 3. If $x_n$ is a sequence of distinct real numbers such that $|x_n| \to \infty$ and $0 < \lambda < \infty$, the following are equivalent.

(a) There exists a nonzero bounded analytic function on $S_\lambda$ which vanishes on $X = \{x_n: n \geq 1\}$.

(b) On $S_\lambda^+$ there is a nonzero bounded analytic function which vanishes on $X \cap S_\lambda^+$, and the analogous statement holds for $S_\lambda^-$.

(c) $\Sigma a^{-|x_n|} < \infty$, where $a = \exp(\pi/2\lambda)$.

(d) $\prod_n \text{sgn}(x_n)(a^{x_n} - a^x)/(a^{x_n} + a^x)$ converges normally on the plane to a function which is analytic on $S_{2\lambda}^+$ (the interior of $S_{2\lambda}$), is bounded by 1 on $S_\lambda$, and vanishes in $S_{2\lambda}$ precisely on $X$.

Proof. The conformal map $w = \phi(z) = \exp(\pi z/2\lambda)$ carries $S_\lambda$ to the closed right half-plane minus 0 and takes $X$ to a positive sequence $\{w_n\}$ which clusters only at 0 and/or $\infty$. The Blaschke condition for a real sequence $w_n$ tending to 0 (resp. $\infty$) in the right half-plane is easily seen to be $\Sigma w_n < \infty$ (resp. $\Sigma w_n^{-1} < \infty$). Because a convergent Blaschke product on the open half-plane is analytic on the closed half-plane minus the cluster set of its zeros, (a) is equivalent to (c). Trivially, (a) implies (b). The above mapping $w = \phi(z)$ (resp. $w = 1/\phi(z)$) sends $S_\lambda^-$ (resp. $S_\lambda^+$) onto the half-disc $\{w: |w| < 1, \Re w > 0\} - \{0\}$, which contains the disc $\Delta = \{w: |w - \frac{1}{2}| < \frac{1}{2}\}$. So (b) implies the Blaschke condition for the real sequence $w_n = \phi(x_n) \to 0$ (resp. $w_n = 1/\phi(x_n) \to 0$) in $\Delta$, which implies (c). Recall that a sequence, series, or product of meromorphic functions $f_n$ is said to converge normally on a region $R$ if for each compact subset $K$ of $R$ all the functions $f_n$ are analytic on $K$ for $n > N(K)$ and the sequence, series, or product of the $f_n$ for $n > N(K)$ converges uniformly on $K$. Lemma 2 and simple observations about $\tau(z; x_n, \lambda)$ reveal that (c) implies (d). Finally, (d) trivially implies (a).

Lemma 4. There exist sequences $s_n$, $t_n$, $K_n$, and $N_n$ of positive integers such that

1. $1 < s_1 < t_1 < s_2 < t_2 < \cdots$ and $1 < K_1 < N_1 < K_2 < N_2 < \cdots$,
2. $2^{-s_n/n} \Sigma_{1}^{2^n} N_j \to 0$ as $n \to \infty$; in particular, $t_n - s_n \to \infty$,
3. $16^{s_n} \Sigma_{n+1}^{\infty} 4^{-s_{n}}(K_j + 4^{-s_{n}}/N_j) \to 0$ as $n \to \infty$; in particular, $s_{n+1} - t_n \to \infty$ and $\Sigma_{1}^{\infty} 4^{-s_{n}}(K_n + 4^{-s_{n}}/N_n) < \infty$, and
4. $\Sigma 4^{-s_{n}} N_n = \infty = \Sigma 4^{-s_{n}} 4^{\epsilon s_{n}} K_n$ for every $\epsilon > 0$. 

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Proof. Define three sequences of integers by the recursion: $r_1 = 1$, $s_n = nr_n$, $t_n = n(3s_n + n)$, $r_{n+1} = 2t_n + n$. It is easy to see that $r_n$ and $s_n$ are strictly increasing with $n$ and that $s_{n+1} > nr_{n+1} > t_n > s_n$. Put $N_n = 4^s$ and $K_n = N_n^{-s/n} = N_n^{-4^{-s/n}} = 4^{s/n-4^{-s/n}} = 4^{(n-1)/n}$. (1) and (4) are immediate.

Each of the sums $\Sigma_i^s$ in (2) and $\Sigma_{i+1}^s$ in (3) is readily seen to be a sum of distinct powers of $2$. Therefore, each sum is at most twice its largest term. For the sum $\Sigma_1^s$ in (2) this estimate is $2 \cdot 2^s N_n = 2 \cdot 2^s = 2 \cdot 2^s / 2^{-n}$, and (2) follows. For the sum $\Sigma_{i+1}^s$ in (3) the estimate is $2 \cdot 2^{-s} \cdot 2 K_n^{-1} = 4 \cdot 4^{-s} = 4 \cdot 4^{-s} = 4 \cdot 4^{-s} = 4 \cdot 4^{-s}$, and (3) follows.

Let us turn now to the specification of $M$, $U$, $E^+$, and $E^-$. Put $\lambda_0 = \pi(4 \log 2)^{-1}$; any value could serve for $\lambda_0$, but this one is convenient because $\exp(\pi/2\lambda_0) = 4$. Select $s_n$, $t_n$, $K_n$, and $N_n$ according to Lemma 4. For each $n > 0$ select $K_n$ disjoint closed subintervals of the real interval $(s_n, s_{n+1})$ and $N_n$ disjoint closed subintervals of $(-s_n - 1, -s_n)$; refer to each of these tiny intervals as $J$. Consider the intervals $J$ as slits in the plane and join two copies of the plane slit by all the $J$’s in the standard fashion by joining the upper edge of each $J$ in each plane with the lower edge of the same $J$ in the other plane. Call the resulting surface $\overline{M}$ and let $\overline{\pi}$: $\overline{M} \to C$ be the natural projection of $\overline{M}$ onto $C$. $\overline{M}$ is exhibited as a branched two-sheeted cover of $C$. Let $\beta \subseteq \overline{M}$ be the set of branch points of this covering; that is $\beta \in B$ if and only if $\overline{\pi}(\beta) = 0$ if and only if $\overline{\pi}(\beta)$ is an endpoint of one of the intervals $J$. Now define $M = \overline{M} - B$, $Z = \overline{\pi}|_M$, and $\overline{\beta} = \overline{\pi}(B)$ is the set of endpoints of the $J$’s. $Z : M \to C - \overline{\beta}$ realizes $M$ as a two-sheeted unbranched cover of the region $C - \overline{\beta}$. Denote by $E_x^\pm$ the sets $Z^{-1}(S_x^\pm)$ in $M$ and put $E^+ = E_x^+$ and $E^- = E_x^-$. As previously indicated, $U$ will be induced by the plane isotopy $u(x + iy, t) = x + ity$, $t > 0$, in this manner. $M$ is covered by open sets $T$ for which $Z|_T$ is a homeomorphism and $Z(T) = I \times I'$, where $I$ is an open interval of $R$ and $I' = R$ or $R^+$ or $R^-$. $I'$ can equal $R$ precisely when $I$ is disjoint from $\overline{\beta}$. For each such $T$ define a homotopy $U_T$: $T \times R^+ \to T$ by $U_T(p, t) = (Z|_T)^{-1}u(Z(p), t)$. It is clear that if $T_1 \cap T_2 \neq \emptyset$, then for all $p \in T_1 \cap T_2$ and all $t > 0$, $U_T(p, t) = U_{T_1}(p, t) = U_{T_2}(p, t)$. Therefore, we may define $U(p, t) = U_T(p, t)$ for any $T$ which contains $p$ and for all $t > 0$. Because $Z$ is a local coordinate at every point of $M$ and is an analytic homeomorphism on each $T$, it is immediate from the definition of $U$ and elementary properties of $u$ that (a)–(e) hold.

Because $u_l(S_x^+ = S_{\lambda_0}^+$ it is immediate that $U_l(E^+_{\lambda}) = E^+_{\lambda_0}$. So (f) and (g) are equivalent to the following statements:

(f) $E^+_{\lambda} \in \mathcal{E}$ if and only if $\lambda < \lambda_0$.

(g) $E^-_{\lambda} \in \mathcal{E}$ if and only if $\lambda < \lambda_0$.

The "only if" parts of (f) and (g) are proved as follows. Fix $\lambda > \lambda_0$ and define $a = a(\lambda) = \exp(\pi/2\lambda)$, $a(\lambda_0) = 4$, and $a(\lambda) = 4^{1-\varepsilon}$ for some $\varepsilon > 0$ when $\lambda > \lambda_0$. For $0 < s < x < s + 1$ and $a < 4$ we find $a^{-s} > 4^{-1}a^{-s}$ and $4^{-x} > 4^{-1}4^{-x}$. From Lemma 3 and from part (4) of Lemma 4 it then follows that if $\varphi$ is a bounded analytic function on $S_\alpha^-$ for $\lambda > \lambda_0$ (or on $S_\alpha^+$ for $\lambda > \lambda_0$) which vanishes at every point of $\overline{\beta} \cap S_\alpha^-$ (or $\overline{\beta} \cap S_\alpha^+$), then $\varphi$ vanishes identically. The following lemma
then yields that the corresponding sets $E_\lambda^\pm = Z^{-1}(S_\lambda^\pm)$ in $M$ do not belong to $\mathcal{C}$.

**Lemma 5.** Suppose $\pi_1: M_1 \to R$ is a two-sheeted branched cover of a connected open subset $R$ of $C$ with branch set $B_1 \subseteq M_1$. Let $B_0 \subseteq B_1$; put $M_2 = M_1 - B_0$ and $\pi_2 = \pi_1|_{M_2}$. If $S$ is a subregion of $R$ such that $S \neq R$ and $0$ is the only bounded analytic function on $S$ which vanishes at all points of $\pi_1(B_1) \cap S$, then $\pi_2^{-1}(S)$, which is $\pi_1^{-1}(S) - B_0$, does not have property $\mathcal{C}$ in $M_2$.

**Proof.** For any function $g$ on $\pi_2^{-1}(S)$, define $\Delta g$ on $S - \pi_1(B_1)$ by $\Delta g(z) = (g(p_1) - g(p_2))^2$, where $(p_1, p_2) = \pi_2^{-1}(z)$. (See [RS], where this idea is used.) $\Delta g$ is analytic on $S - \pi_1(B_1)$ whenever $g$ is analytic on $\pi_2^{-1}(S)$. If furthermore $g$ is bounded on $\pi_2^{-1}(S)$, Riemann's theorem on removable singularities implies that $g$ extends to a bounded analytic function on $\pi_1^{-1}(S) \subseteq M_1$ and that $\Delta g$ extends similarly to $S$. The extended $\Delta g$ vanishes at every point $z_1 \in \pi_1(B_1) \cap S$, because as $z$ tends to $z_1$ the two points $p_1$ and $p_2$ coalesce to the single point of $B_1$ lying over $z_1$. By hypothesis $\Delta g$ must therefore vanish identically on $S$ whenever $g$ is a bounded analytic function on $\pi_2^{-1}(S)$.

Now choose $z_0 \in R - (\overline{S} \cup \pi_1(B_1))$ and let $(p_1, p_2) = \pi_2^{-1}(z_0)$. Select a meromorphic function $f$ on $M_2$ which has a pole at $p_1$, as its only singularity [BS]. Then $f$ is analytic on $\pi_2^{-1}(S)$ and yet $f$ cannot be approximated uniformly on $\pi_2^{-1}(S)$ by an analytic function $F$ on $M_2$. For if $F$ were analytic on $M_2$ and $|f - F| < 1$ on $\pi_2^{-1}(S)$, the foregoing paragraph shows that $\Delta(f - F) \equiv 0$ on $S$. By uniqueness of analytic functions $\Delta(f - F) \equiv 0$ on $R - (\pi_1(B_1) \cup \{z_0\})$. However, this is contradicted by the fact that $f - F$ is bounded near $p_2$ and unbounded near $p_1$. So such an $F$ does not exist, and $\pi_2^{-1}(S)$ does not have property $\mathcal{C}$ in $M_2$.

The proof of the "if" parts of (f') and (g'), namely, that $E_\lambda^+ \in \mathcal{C}$ for $0 < \lambda < \lambda_0$ and that $E_\lambda^- \in \mathcal{C}$ for $0 < \lambda < \lambda_0$ will require the construction of certain auxiliary functions on $C$ and on $M$. Henceforth let $b$ be a variable ranging over $\mathcal{B} = \pi(B) = \{\text{endpoints of the intervals } J\}$, and let $s_n, t_n, K_n$, and $N_n$ be as in Lemma 4. Define functions $A_n, B_n, C_n, and D_n$ as follows, where for $b < 0$ we let $k = k(b)$ be the unique integer such that $s_k < -b < s_k + 1$, $\lambda_b = k\lambda_0(1 + k)^{-1}$, and $a_b = \exp(\pi/2\lambda_b) = 4 \cdot 4^{1/k}$.

$$A_n(z) = \prod_{b < -t_n} \tau(z; b, \lambda_b)^{-1} = \prod_{a_b} \frac{a_b^z + a_b^b}{a_b^z - a_b^b},$$

$$B_n(z) = \prod_{-t_n < b < t_n} \tau(z; b, n\lambda_b) = \prod_{4^{z/n} - 4^{b/n}}\frac{4^{z/n} - 4^{b/n}}{4^{z/n} + 4^{b/n}},$$

$$C_n(z) = \prod_{b > t_n} (-\tau(z; b, \lambda_b)^{-1}) = \prod_{4^b + 4^z}\frac{4^b + 4^z}{4^b - 4^z},$$

$$D_n(z) = A_n(z)B_n(z)C_n(z).$$

**Lemma 6.** (a) The product for $D_n$ converges normally on the plane to a meromorphic function all of whose zeros and poles are simple.

(b) $\{z: D_n(z) = 0 \text{ or } \infty \text{ and } |\text{Im } z| < 4\lambda_0/3\} \subseteq \mathcal{B}$. 

(c) If $D_n(z) = \infty$ and $|\text{Re } z| < t_n$, then either $z \in \mathbb{D}$ or else $|\text{Im } z| > 2n\lambda_0$.

For every $\delta > 0$ and $t > 0$ the following hold for all large enough $n$.

(d) $|D_n(z) - 1| < \delta$ whenever $t_n - t < |\text{Re } z| < t_n + t$.

(e) $|D_n(z)| < 1 + \delta$ on $S_{\lambda_0} \cap \{z: |\text{Re } z| < t_n + t\}$.

(f) $|D_n(z)| > 1 - \delta$ on $(S_{\lambda_0} \cap \{z: \text{Re } z > t_n - t\}) \cup (S_{\lambda_0 - \delta} \cap \{z: \text{Re } z < -t_n + t\})$.

**Proof.** Let $\varepsilon$ be very small, let $x = \text{Re } z$, and let $-s_k - 1 < b < -s_k < x - 1$. By Lemma 2

$$|1 - \tau(z; b, \lambda_b)^{-1}| < 4 \cdot 4^{(k+1)x/k} \cdot 4^{-(k+1)\lambda_b/k} \leq 4 \cdot 4^{2x} \cdot 4^{-s_k} \cdot 4^{-s_k/k}.$$ 

From Lemma 1 and part (3) of Lemma 4 it follows that $A_n$ is normally convergent on $\mathbb{C}$ and that $|A_n(z) - 1| < \varepsilon$ for $-t_n - t < \text{Re } z$, for all large $n$. In a similar manner we find that $C_n$ is normally convergent and $|C_n(z) - 1| < \varepsilon$ for $\text{Re } z < t_n + t$, for large $n$. $B_n$ is convergent, being a finite product, and Lemma 1, Lemma 2, and part (2) of Lemma 4 give $|B_n(z) - 1| < \varepsilon$ for $\text{Re } z > t_n - t$, for large $n$. Because there are an even number of $b$'s in $(-t_n, t_n)$, $B_n(z) = \prod \tau = \prod(-\tau)$, and the same argument as above shows that $|B_n(z) - 1| < \varepsilon$ for $|\text{Re } z| > t_n - t$, for large $n$. Each $\tau(z; b, \lambda)$ is periodic with period $4\lambda_i$, and $z_0$ is a zero of $\tau$ if and only if $z_0 + 2\lambda i$ is a pole. The smallest $\lambda$ involved in the product for $D_n$ is $\lambda = k\lambda_0(1 + k)^{-1}$ for $k = n + 1$; so $\lambda > 2\lambda_0/3$ and $2\lambda > 4\lambda_0/3$. (a), (b), and (c) are clear, and if $\varepsilon$ is small enough (d) follows from the estimates above. Because $|A_n|$ and $|C_n|$ are each bounded by $1 + \varepsilon$ in $|\text{Re } z| < t_n + t$ and $|B_n| < 1$ in $S_{\lambda_0}$, we have (e) for small $\varepsilon$ and large $n$. Finally, because $|B_n(z) - 1| < \varepsilon$ for $|\text{Re } z| > t_n - t$, $|C_n| > 1$ in $S_{\lambda_0}$, and $|A_n| > 1$ in $S_{(n+1)\lambda_0/(n+2)}$, we have (f) for small $\varepsilon$ and large $n$.

**Lemma 7.** If $z_0 \in \mathbb{C}$, $\delta > 0$, and $V$ is an open neighborhood of an arc which connects $z_0$ to $\infty$, then there exists an entire function $h$ having a simple zero at $z_0$ with no other zeros such that $|h - 1|_{C - V} < \delta$.

**Proof.** $|e^a - 1| < 2|a|$ for $|a| < 1$, by Lemma 1(b); so $|a - \beta| < 1$ implies $|e^a - e^\beta| < 2(|a| - |\beta|)|e^\beta|$. Choose a branch of $\log(z - z_0)^{-1}$ in the complement of the given arc $\gamma$ which joins $z_0$ to $\infty$. In $V$ choose a connected simply connected neighborhood $V_1$ of $\gamma$ which is the interior of a locally polygonal set. Then $V_1 \cup \{\infty\}$ is connected and locally connected; so Arakelyan's Theorem [A1], [A2] can be applied to the function $\log(z - z_0)^{-1}$ on $\mathbb{C} - V_1$ with $\varepsilon = \min(1, \delta/2)$ to yield an entire function $g$ so that $|g(z) - \log(z - z_0)^{-1}| < \varepsilon$ for $z \in \mathbb{C} - V_1 \supset \mathbb{C} - V$. Put $a = g(z)$ and $\beta = \log(z - z_0)^{-1}$ in the opening sentence of this proof, and we obtain $|\exp(g(z)) - (z - z_0)^{-1}| < 2\varepsilon|z - z_0|^{-1} < \delta/|z - z_0|$ for $z \in \mathbb{C} - V$. Thus, $|z - z_0\exp(g(z)) - 1| < \delta$ for $z \in \mathbb{C} - V$. The function $h(z) = (z - z_0)\exp(g(z))$ has the desired behavior.

**Corollary 8.** There is a sequence $H_n$ of meromorphic functions on $\mathbb{C}$ which have these properties. Let $t > 0$ and $\delta > 0$ be arbitrary.

1. $H_n(z) = H_n(z)$.
2. The zeros of $H_n$ are simple and comprise the set $\{b: |b| < t_n\}$.
3. The poles of $H_n$ are simple and comprise the set $\{b: |b| > t_n\}$.
(4) \( \sup \{ |H_n(z) - 1| : t_n - t < |\text{Re } z| < t_n + t \} \to 0 \) as \( n \to \infty \).

(5) \( \sup \{ |H_n(z)| : |\text{Re } z| < t_n + t \text{ and } |\text{Im } z| < n\lambda_0 \} \to 1 \) as \( n \to \infty \).

(6) \( \inf \{ |H_n(z)| : \text{Re } z > t_n - t \text{ and } |\text{Im } z| < \lambda_0 \} \to 1 \) as \( n \to \infty \).

(7) \( \inf \{ |H_n(z)| : \text{Re } z < -t_n + t \text{ and } |\text{Im } z| < \lambda_0 - \delta \} \to 1 \) as \( n \to \infty \).

**Proof.** Start with the meromorphic functions \( D_n \) of Lemma 6; note that \( D_n(z) = \overline{D_n(z)} \). Enumerate the zeros and poles of \( D_n \) in \( \{ z : \text{Im } z > 0 \} \) as \( z_1, z_2, \ldots \); then \( \overline{z}_1, \overline{z}_2, \ldots \) are the zeros and poles in the lower half-plane. For each \( j \) let \( V_j \) be a small neighborhood of the line segment \( L_j = \{ z = x + iy : x = \text{Re } z_j \text{ and } y > \text{Im } z_j \} \) which joins \( z_j \) to \( \infty \). Because \( L_j \) is disjoint from the set \( T_n = S_{\lambda_0} \cup \{ z : |\text{Re } z| < t_n \text{ and } |\text{Im } z| < n\lambda_0 \} \cup \{ z : |\text{Re } z| \in \bigcup [s_k, s_k + 1] \} \), we may assume that \( V_j \cap T_n = \emptyset \), as well. By Lemma 7 there is an entire function \( h_j \) which is zero only at \( z_j \) and which satisfies \( |h_j - 1| < 2^{-n-j} \) outside \( V_j \). Put \( k_j(z) = h_j(\overline{z}) \). Let \( F_n = \prod (h_j k_j)^{-1} \), where the exponent is chosen to be +1 in case \( D_n(z_j) = \infty = \overline{D_n(\overline{z_j})} \) and is chosen to be -1 in case \( D_n(z_j) = \overline{D_n(\overline{z_j})} = 0 \). Because \( \sum 2\delta_j = 2 \cdot 2^{n} < \infty \) and each compact set meets only finitely many of the \( V_j \) or their conjugates, the product for \( F_n \) converges normally on the plane to a meromorphic function which by Lemma 1 satisfies \( |F_n - 1| < 2^{-n} \) on \( T_n \), for \( n > 2 \). From Lemma 6 it is clear that \( H_n = F_n D_n \) has all the required properties.

Define \( R(x_0; t, \lambda) \) to be the rectangle \( \{ z = x + iy : |x - x_0| < t \text{ and } |y| < \lambda \} \). For \( n > 0 \) define \( t_n = -t_n \), \( G_n(z) = 2z^{-1}H_n(z) \), and \( G_{-n}(z) = 2^{-n}zH_n(z) \), where \( H_n \) is as in Lemma 8. Fix \( \lambda_+ < \lambda_0 \) and \( \lambda_- < \lambda_0 \), define \( \lambda_+ = \lambda_+ \) for \( n > 0 \) and \( \lambda_+ = \lambda_- \) for \( n < 0 \), and put \( S(n) = S_{\lambda_+} \) for \( n > 0 \) and \( S(n) = S_{\lambda_-} \) for \( n < 0 \). Define these sets:

\[
\sigma_n = \{ z \in S(n) : 16^{-1} < |G_n(z)| < 16 \},
\]
\[
X_n = \{ z : |\text{Re } z| < s_{n|n|} + 1 \text{ and } |\text{Im } z| < n\lambda_0 \},
\]
\[
\cup (S(n) \cap \{ z : |\text{Re } z| < t_n \} < 1) - \sigma_n,
\]
\[
Y_n = (S(n) \cap \{ z : |\text{Re } z| > t_n \} > 1) - \sigma_n.
\]

**Lemma 9.** The following statements hold for \( |n| \) sufficiently large.

1. \( G_n \) is meromorphic on \( \mathbb{C} \) with a simple zero or pole at each \( b \in \mathcal{D} \) and no other zeros nor poles; \( G_n(z) = \overline{G_n(z)} \).

2. \( G_n \) is an analytic homeomorphism on a neighborhood of \( R(t_n; 5, \lambda_0) \).

3. \( R(t_n; 3, \lambda_n) \subseteq \sigma_n \subseteq R(t_n; 5, \lambda_n) \).

4. \( |G_n| < 16^{-1} \) on \( X_n \), \( |G_n| > 16 \) on \( Y_n \), \( 16^{-1} < |G_n| < 16 \) on \( \sigma_n \), and \( G_n \) is an analytic homeomorphism on \( \sigma_n \).

5. There is a smooth function \( \Theta_n : [16^{-1}, 16] \to (0, \pi) \) such that \( G_n(\sigma_n) = \{ w : 16^{-1} < |w| < 16 \text{ and } |\text{arg } w| < \Theta_n(|w|) \} \).

**Proof.** For definiteness let us consider the case \( n < 0 \); the case \( n > 0 \) is treated in a very similar manner. (1) is clear because the same thing is true for \( H_n \). By Corollary 8 \( H_{-n}(z + t_n) \to 1 \) uniformly on \( R(0; 7, 3\lambda_0) \) as \( n \to -\infty \). Because \( 2^{-z} \) is an analytic homeomorphism on \( S_{3\lambda_0} \) and \( G_n(z + t_n) = 2^{-n}H_{-n}(z + t_n) \to 2^{-z} \) uniformly on \( R(0; 7, 3\lambda_0) \), \( G_n(z + t_n) \) is an analytic homeomorphism on \( R(0; 6, 2\lambda_0) \) for large \( n < 0 \), and (2) follows.
From parts (7) and (5) of Corollary 8 we obtain \( \frac{1}{2} < |H_n(z)| \) for \( \text{Re} \ z < t_n + 5 \) and \( |H_n(z)| < 2 \) for \( \text{Re} \ z > t_n - 5 \) for large \( n < 0 \). If \( \text{Re} \ z < t_n - 5 \), we have \( |2^{n \cdot \text{re}^z}| > 32 \); so \( |G_n(z)| > 16 \). If \( \text{Re} \ z > t_n + 5 \), we have \( |2^{n \cdot \text{re}^z}| < 32^{-1} \); so \( |G_n(z)| < 16^{-1} \). Thus, \( \sigma_n \subseteq R(t_n; 5, \lambda_n) \). Because \( \frac{1}{2} < |H_n(z)| < 2 \) in \( R(t_n; 5, \lambda_n) \), which contains \( R(t_n; 3, \lambda_n) \), and \( 8^{-1} < |2^{n \cdot \text{re}^z}| < 8 \) on \( R(t_n; 3, \lambda_n) \), we see that \( 16^{-1} < |G_n(z)| < 16 \) on \( R(t_n; 3, \lambda_n) \). Thus, \( R(t_n; 3, \lambda_n) \subseteq \sigma_n \) and (3) is proved.

Because \( t_n - s_n \gg 0 \) for large \( |n| \), we have from Corollary 8 that \( |H_n(z)| < 2 \) for \( z \in X_n \) for large \( n < 0 \). By (3) and the definition of \( X_n \), \( \text{Re} \ z > t_n + 1 \) for \( z \in X_n \); so \( |2^{n \cdot \text{re}^z}| < \frac{1}{2} \) and \( |G_n(z)| < 2 \cdot \frac{1}{2} = 1 \) for \( z \in X_n \). From (3) and the definition of \( \sigma_n \), this means that \( |G_n| < 16^{-1} \) on \( X_n \). In a similar manner we obtain \( |G_n| > 16 \) on \( Y_n \). By (2) and (3), \( G_n \) is an analytic homeomorphism on \( \sigma_n \), and (4) is proved.

Because \( G_n \) is a diffeomorphism and does not vanish on a neighborhood of \( R(t_n; 5, \lambda_n) \), the functions \( r = |G_n(z)| \) and \( \theta = \text{arg}(G_n(z)) \), where \( -\pi < \theta < \pi \), constitute a global differentiable coordinate pair on a neighborhood of \( R(t_n; 5, \lambda_n) \). We can therefore parametrize \( G_n(x - i\lambda_n) \) as \( G_n(x - i\lambda_n) = r \exp(i\Theta_n(r)) \) for a smooth function \( \Theta_n \). Because \( \text{arg} \ 2^{n \cdot \text{re}^z} > 0 \) for \( \text{Im} \ z = -\lambda_n \) and \( H_n \) is nearly 1, \( \Theta_n \) takes values in \( (0, \pi) \) for large \( n < 0 \). The set \( \gamma = \{ r = r_0 \} \cap R(t_n; 5, \lambda_n) \) consists of regular arcs which have no endpoints in \( R(t_n; 5, \lambda_n) \). If we knew that \( \gamma \) consisted of just one arc which meets the boundary of \( R(t_n; 5, \lambda_n) \) in just two points, we could complete the argument as follows. For \( r_0 \in [16^{-1}, 16) \), \gamma does not meet \( \{ x - t_n = \pm 5 \} \), for on the latter set \( |G_n| \) is approximately 32 or 32^{-1}, which is not close to 16 or 16^{-1}. So \( \gamma \) connects \( \text{Im} \ z = -\lambda_n \) to \( \text{Im} \ z = +\lambda_n \). Because \( G_n(z) = G_n(z) \) and \( G_n > 0 \) on \( [t_n - 5, t_n + 5] \), \( \gamma \) is parametrized by a single symmetric interval \( -\theta_0 < \theta < \theta_0 \). Evidently \( (r_0, \theta_0) \) corresponds to a point of \( \{ \text{Im} \ z = -\lambda_n \} \) or a point of \( \{ \text{Im} \ z = -\lambda_n \} \). As we have previously observed, \( \{ \text{Im} \ z = -\lambda_n \} \) corresponds to positive \( \theta \). Since \( G(x - i\lambda_n) = r \exp(i\Theta_n(r)) \), this shows that (5) holds.

Finally, to see that \( \gamma = \{ r = r_0 \} \cap R(t_n; 5, \lambda_n) \) consists of a single arc, consider the following elementary calculation for an analytic nonvanishing \( f \):

\[
2|f| \frac{\partial |f|}{\partial x} = \frac{\partial |f|^2}{\partial x} = \frac{\partial (ff')}{\partial x} = \frac{\partial f}{\partial x} f' + f \frac{\partial f}{\partial x} = f' \frac{\partial f}{\partial x} + f \frac{\partial (f')}{\partial x} = 2 \text{Re} f f';
\]

so \( \partial |f|/\partial x = |f|^{-1} \text{Re}(f f') \). Apply this formula to \( G_n \), taking into account the fact that \( H_n \) is very close to 1 on a neighborhood of \( R(t_n; 5, \lambda_n) \) and hence \( H_n' \) is very close to 0. The result is

\[
\frac{\partial |G_n|}{\partial x} = |G_n|^{-1} \text{Re}(\overline{G_n} G_n') \approx |2^{n \cdot \text{re}^z}|^{-1} \text{Re}((2^{n \cdot \text{re}^z})^{-1} (2^{n \cdot \text{re}^z}'))
\]

\[
= \frac{\partial}{\partial x} |2^{n \cdot \text{re}^z}| = (\log 2)|2^{n \cdot \text{re}^z}| < (\log 2)32^{-1}
\]

on \( R(t_n; 5, \lambda_n) \). So for large negative \( n \), \( \partial r/\partial x = \partial |G_n|/\partial x < 0 \) on \( R(t_n; 5, \lambda_n) \). This means that each horizontal line \( \{ \text{Im} \ z = \text{constant} \} \) meets \( \gamma \) in at most one point;
and γ consists of at most one arc, since it has no endpoints in \( R(t_n; 5, \lambda_n)^\circ \).

Define for large \(|n|\) the following subsets of \( \mathbb{C} \). [See Figure 1.]

\[
V_n^+ = \{ z : 4^{-1} < |z| < 4 \text{ and } \frac{1}{2} \Theta_n(|z|) < \arg z < \pi - \frac{1}{2} \Theta_n(|z|) \},
\]

\[
V_n^- = \{ z : \bar{z} \in V_n^+ \},
\]

\[
W = \{ z : 4^{-1} < |z| < 4 \text{ and } |\arg z| < \frac{1}{2} \Theta_n(|z|) \},
\]

\[
W' = \{ z : -z \in W \}.
\]

**Figure 1**

**Lemma 10.** For large \(|n|\) there is an analytic function \( \pi_n \) on \( M \) which enjoys these properties [see Figure 2].

1. \( \pi_n^2 = G_n \circ Z \);
2. \( \pi_n \) maps \( Z^{-1}(X_n \cup \sigma_n \cup Y_n) \) into \( \mathbb{C} - (V_n^+ \cup V_n^-) \);
3. \( \pi_n(Z^{-1}(X_n)) \subseteq \{ z : |z| < 4^{-1} \}, \pi_n(Z^{-1}(Y_n)) \subseteq \{ z : |z| > 4 \} \), and \( \pi_n \) is an analytic homeomorphism of \( Z^{-1}(\sigma_n) \) onto \( W \cup W' \).

**Proof.** Recall the surface \( \bar{M} \) and its projection \( \bar{\pi} \) onto \( \mathbb{C} \), which has branch set \( B \subseteq \bar{M} \). \( G_n \) has a simple zero or pole at each point \( b \) of \( \mathcal{D} = \bar{\pi}(B) \), and no other zeros nor poles. From this fact and the definition of \( \bar{M} \) and \( \bar{\pi} \) it follows that the function \( G_n \circ \bar{\pi} \) has a single-valued square root, call it \( \bar{\pi}_n \), on \( \bar{M} \). Indeed, \( \bar{M} \) can be
thought of as the classical Riemann surface constructed from the multiple-valued function $\sqrt{G_n}$ on the plane. Restricting $\pi_n$ to $M$ we have $\sqrt{G_n} \circ Z$ as a single-valued analytic function, call it $\pi_n$, on $M$ having no zeros nor poles. Properties (2) and (3) are immediate from Lemma 9. Note that $\pi_n(p^+) = -\pi_n(p^-)$ if $\{p^+, p^-\} = Z^{-1}(z)$ for $z \in \sigma_n$.

Lemma 11. Let $|n|$ be large. For every $\delta > 0$ and every $C^1$ function $g$ on $Z^{-1}(X_n \cup \sigma_n \cup Y_n)$ such that $g = 0$ on the closure of $Z^{-1}(X_n \cup Y_n)$ and $|\partial g/\partial Z| < \delta$ on $Z^{-1}(\sigma_n)$ there exists a meromorphic function $\psi$ on $M$ such that $|\psi - g| < 17\delta$ on $Z^{-1}(X_n \cup \sigma_n \cup Y_n)$. The poles of $\psi$ lie outside $Z^{-1}(X_n \cup \sigma_n \cup Y_n)$.

Proof. Let $|n|$ be so large that Lemma 10 holds. By Lemma 10 and the hypothesis on $g$ we can find a $C^1$ function $\tilde{g}$ on $C$ such that $\tilde{g} = 0$ on $\{z: |z| < 4^{-1} + \epsilon\}$ or $|z| > 4 - \epsilon\}$ for some $\epsilon > 0$ and $\tilde{g} \circ \pi_n = g$ on $Z^{-1}(X_n \cup \sigma_n \cup Y_n)$. Calculate

$$\frac{\partial \tilde{g}}{\partial Z} = \frac{\partial (\tilde{g} \circ \pi_n)}{\partial Z} = \frac{\partial (\tilde{g} \circ \pi_n \circ Z^{-1})}{\partial \tilde{z}} \circ Z = \frac{\partial \tilde{g}}{\partial \tilde{z}} \circ \pi_n \circ Z^{-1} \circ Z = \frac{\partial \tilde{g}}{\partial \tilde{z}} \circ \pi_n.$$

So $|\partial \tilde{g}/\partial \tilde{z}| < \delta$ on $W \cup W'$. Now we apply a method of Mergelyan [M, §3, Chapter I] to approximate $\tilde{g}$ by a rational function $k$. Then $k \circ \pi_n$ will approximate $\tilde{g} \circ \pi_n = g$. Let $\Gamma^\pm$ be curves oriented positively (counterclockwise) inside $V^\pm_n$ which are within $\epsilon$ of the boundary of $V^\pm_n$. See Figure 3.

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where $\Sigma_R$ is the set of points inside $\Gamma_R$ and outside both $\Gamma^+$ and $\Gamma^-$. Because $\tilde{g}$ vanishes on $\Gamma_R$, the line integral reduces to

$$I(z_0) = - (2\pi i)^{-1} \int_{\Gamma^+ \cup \Gamma^-} \tilde{g}(z) (z - z_0)^{-1} \, dz.$$  

Because $\tilde{g}$ vanishes on most of $\Sigma_R$, the integral over $\Sigma_R$ reduces to an integral over $T_n = \{ z \mid 4 < |z| < 4 - \epsilon \}$. See Figure 4, in which $T_n$ is shaded.


Figure 4

Because $\tilde{g}$ is $C^1$ and $|\partial \tilde{g} / \partial \bar{z}| < \delta$ on $C - (V^+_n \cup V^-_n)$, we can take $\epsilon$ so small that $|\partial \tilde{g} / \partial \bar{z}| < 2\delta$ on $T_n$. Then

$$\left| \pi^{-1} \int \int_{\Sigma_R} \right| = \pi^{-1} \left| \int \int_{T_n} \right| < 2\delta \pi^{-1} \int \int_{T_n} |z - z_0|^{-1} \, dx \, dy$$

$$< 2\delta \pi^{-1} \int_{|z| < 4} |z - z_0|^{-1} \, dx \, dy < 2\delta \pi^{-1} \sqrt{4\pi \cdot \pi \cdot 4^2} = 16\delta.$$  

The last inequality is Lemma 3.1.1 of [B]. Now for $z_0 \in C - (V^+_n \cup V^-_n)$ and $z \in \Gamma^+ \cup \Gamma^-$, the distance $|z - z_0|$ is bounded away from 0; thus, $I(z_0)$ can be uniformly approximated for $z_0 \in C - (V^+_n \cup V^-_n)$ by a finite Riemann sum for this integral, which is manifestly a rational function $k(z_0)$ having its poles in $V^+_n \cup V^-_n$. Choosing such a $k$ for which $|I(z_0) - k(z_0)| < \delta$ for all $z_0 \in C - (V^+_n \cup V^-_n)$, we have $|\tilde{g} - k| < 17\delta$ on $C - (V^+_n \cup V^-_n)$ and so $|k \circ \pi_n \circ \tilde{g} - \pi_n| < 17\delta$ on $\pi_n^{-1}(C - (V^+_n \cup V^-_n)) \supseteq Z^{-1}(X_n \cup \sigma_n \cup Y_n)$. Thus, $\psi = k \circ \pi_n$ does what is required. The poles of $\psi$ are not in $Z^{-1}(X_n \cup \sigma_n \cup Y_n)$, because $\psi$ is bounded there.

**Lemma 12.** Let $S$ be a closed subset of $C$ which is star-shaped with respect to 0, let $z_n$ be a sequence of points of $C - S$ tending to $\infty$, and let $k_n$ be a sequence of nonnegative integers. For each $\epsilon > 0$ there is an entire function $\varphi$ such that $|1 - \varphi| < \epsilon$ on $S$ and for each $n$, $\varphi$ has a zero of order at least $k_n$ at $z_n$.

**Proof.** Given $\epsilon > 0$, let $\epsilon_n > 0$ be chosen so that $\Sigma k_n \epsilon_n < \min(\frac{1}{2}, \epsilon/3)$. Put $\gamma_n = \{ rz_n : r > 1 \}$; $\gamma_n$ is an arc joining $z_n$ to $\infty$ in $C - S$, because $S$ is star-shaped.
Each compact set meets at most finitely many $\gamma_n$. Choose a neighborhood $V_n$ of each $\gamma_n$ so that $V_n \subseteq C - S$ and so that each compact meets only finitely many $V_n$. Using Lemma 7 choose an entire function $h_n$ so that $h_n(z_n) = 0$ and $|1 - h_n| < \varepsilon_n$ on $C - V_n$. By Lemma 1 the product $\prod h_n^n$ converges normally on the plane to a function $\varphi$ having the desired properties.

**Corollary 13.** In Lemma 11 we can require $\psi$ to be analytic on $M$ if we relax the approximation to $|\psi - g| < 18\delta$.

**Proof.** Write $Z_n = Z^{-1}(X_n \cup \sigma_n \cup Y_n)$. Given $g$ and $\delta$ satisfying the hypothesis of Lemma 11, let $\psi$ be meromorphic on $M$ and satisfy $|\psi - g|_{Z_n} < 17\delta$. Let $P$ be the pole set of $\psi$ and enumerate $Z(P) = \{z_1, z_2, \ldots \}$. Note that $P \cap Z_n = \emptyset$; so $Z(P) \cap (X_n \cup \sigma_n \cup Y_n) = \emptyset$. For each $z_j$ let $k_j$ be the larger of the orders of the poles of $\psi_1$ at the two points of $Z^{-1}(z_j)$. In $Z_n$ $g$ vanishes off a compact set; so $|g|_{Z_n} < \infty$. Thus $|\psi_1|_{Z_n} = K < \infty$. Apply Lemma 12 to the star-shaped $S = X_n \cup \sigma_n \cup Y_n$, the sequence $\{z_j\}$, the integers $\{k_j\}$, and $\varepsilon = \delta/K$ to find an entire function $\varphi$ so that $|\varphi - 1|_{S} < \delta/K$ and $\varphi$ has a zero of order at least $k_j$ at each $z_j$. Put $\psi = (\varphi \circ Z)\psi_1$, which has no poles on $M$ by construction.

Therefore, $|\psi - g|_{Z_n} < |\psi - \varphi|_{Z_n} + |\varphi - g|_{Z_n} < \delta + 17\delta = 18\delta$.

Equipped with the foregoing technical tools we can now detail the proof that $E_{\lambda}^+ \in \mathcal{R}$ for $\lambda < \lambda_0$ and $E_{\lambda}^- \in \mathcal{R}$ for $\lambda < \lambda_0$. Fix one such $E_{\lambda}^+$ or $E_{\lambda}^-$ and call it $E$. Let $f \in A(E)$ and $\varepsilon > 0$. By Corollary 13 there is a set of integers $\mathcal{R} = Z \cap [N, \infty)$, in case $E = E_{\lambda}^+$, or $\mathcal{R} = Z \cap (-\infty, -N)$, in case $E = E_{\lambda}^-$, such that for every $n \in \mathcal{R}$ and every $C^1$ function $g$ on $Z_n = Z^{-1}(X_n \cup \sigma_n \cup Y_n)$ which is supported in the relative (to $Z_n$) interior of $Z^{-1}(\sigma_n)$ there is an analytic function $\psi$ on $M$ such that $|g - \psi|_{Z_n} < 18|\partial g / \partial Z|_{Z_n}$. (If $\varepsilon = |\partial g / \partial Z|_{Z_n} > 0$, this is Corollary 13; if $\varepsilon = 0$, then $g$ is analytic on the connected set $Z_n$ and since it vanishes on $Z^{-1}(X_n)$, it vanishes identically and we can take $\psi = 0$.) For every $n \in \mathcal{R}$ put $W_n = E \cap Z^{-1}\{z : |\text{Re } z - t_n| < 1\}$. $E - \bigcup_n \in \mathcal{R} W_n$ consists of a sequence of separated closed connected subsets, which we may number $E_n, n \in \mathcal{R}$. Do this numbering so that $W_n$ sits between $E_n$ and $E_n$, where $|n| = |n + 1$. Each $E_n$ has a neighborhood of finite genus; indeed, the closure of $E_n$ in $M$ is compact. We may select these neighborhoods to be disjoint from each other. By Theorem 1.5 of [S] there is an analytic function $\Phi$ on $M$ such that $|f - \Phi|_{E_n} < 2^{-|n|} \varepsilon$, where $\varepsilon = \varepsilon/40$. Put $g = f - \Phi$. Note that $|g|_{\bigcup E_n} < \varepsilon/2$.

Select a $C^1$ function $\chi : C \to [0, 1]$ which depends only on $x = \text{Re } z$ and which has these properties: $\chi \equiv 1$ on $[-1, 1]$, $\chi \equiv 0$ outside $(-2, 2)$, and $|\partial \chi / \partial x| < 2$. Then $|\partial \chi / \partial z| < 1$. Put $\chi_n = \chi \circ (Z - t_n)$; then $|\partial \chi_n / \partial Z| < 1$ on $M$, and we may assume that $\chi_n \chi_m \equiv 0$ for $m$ and $n$ in $\mathcal{R}$, $m \neq n$, because of the large gaps between $t_m$ and $t_n$ for large $|n|$. Define $g_n = \chi_n g$ for $n \in \mathcal{R}$. $|\Sigma g_n - g|_{E_n} < \varepsilon/2$ because of the following. For each $p$ there is an integer $m$ such that $(\Sigma g_n)(p) = g_m(p)$, because at most one term in $\Sigma g_n(p)$ is nonzero. Thus, $|\Sigma g_n(p) - g(p)| = |g_m(p) - g(p)| = |\chi_m(p) - 1| |g(p)|$. If $\chi_m(p) - 1 \neq 0$, then $p \in \bigcup E_n$ and $|g(p)| < \varepsilon/2$, while $|\chi_m(p) - 1| < 1$. 

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In $E^0$ we calculate
\[ \frac{\partial g_n}{\partial \bar{Z}} = \frac{\partial (\alpha_n g)}{\partial \bar{Z}} = \frac{\partial \alpha_n}{\partial \bar{Z}} g + \alpha_n \frac{\partial g}{\partial \bar{Z}} = \frac{\partial \alpha_n}{\partial \bar{Z}} g, \]
because $g \in A(E)$. Now \((\frac{\partial \alpha_n}{\partial \bar{Z}})g(p) = 0\) unless \(1 < |\Re Z(p) - t_n| < 2\), in which case \((\frac{\partial \alpha_n}{\partial \bar{Z}})g(p) < \max(2^{-|\ln \theta|}, m \in \mathcal{U}, m = n - 1, n, \text{or } n + 1) < 2 \cdot 2^{-|\ln \theta|}\). Thus, the support of $g_n$ in $Z_n$ is contained in $R(t_n; 2, \lambda)$, which belongs to the relative (to $Z_n$) interior of $\alpha_n$, by Lemma 9, and \(|\frac{\partial g_n}{\partial \bar{Z}}|_{Z_n} < 2 \cdot 2^{-|\ln \theta|}\).

Next we approximate $g_n$ on $Z_n$ by a $C^1$ function $h_n$ on $Z_n$. Specifically, let $s_n$ be the map of $\alpha_n$ into $C$ given by $s_n(z) = t_n + r_n(z - t_n)$, where $r_n < 1$ and $r_n$ is close to 1, and define $h_n$ by $h_n = g_n$ on $Z^{-1}(X_n \cup Y_n)$ and $h_n = (Z|_{Z_n}^{-1} \circ s_n \circ (Z|_{Z_n}^{-1})$ for each component $\Sigma$ of $Z^{-1}(\alpha_n)$. Because $g_n$ vanishes on a neighborhood of $Z^{-1}(X_n \cup Y_n)$, the same will be true for $h_n$ if $r_n$ is close enough to 1. In this case $h_n$ will be $C^1$ on $Z_n$ and \(|\frac{\partial h}{\partial \bar{Z}}|_{Z_n} = r_n|\frac{\partial g}{\partial \bar{Z}}|_{Z_n} < |\frac{\partial g}{\partial \bar{Z}}|_{Z_n} < 2 \cdot 2^{-|\ln \theta|}\). Because $Z^{-1}(\alpha_n)$ is compact we may take $r_n$ so close to 1 that \(|g_n - h_n|_{Z_n} < 2^{-|\ln \theta|}\).

By Corollary 13 there is an analytic function $\psi_n$ on $M$ such that \(|\psi_n - h_n|_{Z_n} < 18 \cdot 2 \cdot 2^{-|\ln \theta|} = 36 \cdot 2^{-|\ln \theta|}\). Because $\sum_{n \in \aleph} 2^{-|\ln \theta|} < 1$ and every compact set in $M$ is contained in all but perhaps finitely many of the $Z_n$, the sum $\sum \psi_n$ converges normally on $M$ to an analytic function $\Psi$. Let $F$ be $\Phi + \Psi$, which is analytic on $M$, and estimate
\[
|F - f| = \left| \sum \psi_n + \Phi - f \right| \\
= \left| \sum (\psi_n - h_n) + \sum (h_n - g_n) + \left( \sum g_n - g \right) + (g + \Phi - f) \right| \\
< \sum |\psi_n - h_n| + \sum |h_n - g_n| + \sum g_n - g + |g + \Phi - f|.
\]
On $E$ the first sum is at most $\sum_{n \in \aleph} 36 \cdot 2^{-|\ln \theta|} < 36 \theta$. The second sum has at most one nonzero term at any point of $E$; so it is dominated by $\max(2^{-|\ln \theta|}, n \in \mathcal{U}) < \theta/2$. The third term $|\sum g_n - g|$ is at most $\theta/2$, as estimated earlier, and the last expression \(|g + \Phi - f| \) is identically zero by definition of $g$. Therefore, we have $|F - f| < 37 \theta = 37\varepsilon/40 < \varepsilon$ on $E$, and the proof is complete.

**REFERENCES**


