

APPLICATIONS OF THE FOURIER-WIENER TRANSFORM TO DIFFERENTIAL EQUATIONS ON INFINITE DIMENSIONAL SPACES. I

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ABSTRACT. Let (H, i, B) be an abstract Wiener space and p_t be the Wiener measure on B with variance t . Let $[B]$ be the complexification of B and \mathcal{E}_a be the class of exponential type analytic functions defined on $[B]$. We define the Fourier-Wiener c -transform for any f in \mathcal{E}_a by

$$\mathcal{F}_c f(y) = \int_B f(x + iy) p_c(dx)$$

and the inverse transform by $\mathcal{F}_c^{-1} f(y) = \mathcal{F}_c f(-y)$. Then the inversion formula holds and \mathcal{F}_2 extends to $L^2(B, p_1)$ as a unitary operator. Next, we apply the above transform to investigate the existence, uniqueness and regularity of solutions for Cauchy problems associated with the following two equations: (1) $u_t = -\mathcal{R}^k u$, (2) $u_{tt} = -\mathcal{R}^k u$; and the elliptic type equation (3) $-\mathcal{R}^k u = f$ ($k > 1$), where Δ is the Laplacian and $\mathcal{R}u(x) = -\Delta u(x) + (x, Du(x))$.

1. Introduction. Gross [6] initiated the theory of differential equations on infinite dimensional abstract Wiener spaces (H, i, B) [5]. H is a given real separable Hilbert space with inner product \langle, \rangle and norm $\| \cdot \| = \sqrt{\langle, \rangle}$; B is the completion of H with respect to a measurable norm; i is the canonical embedding of H into B . In [6], Gross showed that the solution of the Cauchy problem associated with the infinite dimensional heat equation

$$u_t(x, t) = \frac{1}{2} \Delta u(x, t)$$

can be represented by an integral with respect to the Wiener measure p_t , where the Laplacian Δ of a real-valued function f on B is defined as the trace (on H) of the second Fréchet-derivative $D^2 f$ of f when the latter exists. Other second order parabolic type differential equations also have this property. For example: (1) Piech [13] represented the solution of the Cauchy problem associated with the second order differential equation with variable coefficients

$$u_t(x, t) = \text{trace} [A(x) D^2 u(x, t)]$$

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as an integral with respect to a measure which is absolutely continuous with respect to the Wiener measure; (2) in [14], Piech investigated the existence and regularity of the Cauchy problem associated with the differential equation $u_t(x, t) = -\mathcal{N}u(x, t)$, where $\mathcal{N}f(x) = -\Delta f(x) + (x, Df(x))$ (if it exists) and (\cdot, \cdot) denotes the natural pairing of (B, B^*) . It turns out that the solution can be written as an integral with respect to the transition measure $p_{1-e^{-2t}}(e^{-t}x, d\psi)$.

In the case of differential equations of order greater than two, less work has been done. An approach has been considered recently by Uglanov [19]. There the fundamental solutions of higher order elliptic and parabolic type differential operators of the form $(A\mathcal{D}, \mathcal{D})^k$ (A is a constant matrix) are obtained in the sense of generalized measures. However, both localization properties and regularity of solutions are lacking.

In this paper, we shall apply the Fourier-Wiener transform [1] (see 3.1 for definition) and integration by parts [10] to examine Cauchy problems associated with the following types of differential equations:

$$(1) u_t(x, t) = -\mathcal{N}^k u(x, t),$$

$$(2) u_{tt}(x, t) = -\mathcal{N}^k u(x, t),$$

and also the elliptic type differential equation

$$(3) -\mathcal{N}^k u(x) = g(x)$$

where $k \geq 1$.

We owe the definition of the Fourier-Wiener transform to Cameron and Martin [1]. There the Fourier-Wiener transform was defined on a class of mean exponential type analytic functionals on the classical Wiener space \mathcal{C} (i.e. the Banach space of continuous functions on $[0, 1]$ vanishing at 0) and extended to $L^2(\mathcal{C})$ as a unitary operator (see [2]). Later, Segal [16] defined the Fourier-Wiener transform on the class of polynomial cylinder functions on Hilbert space. Hida [7], following [2], developed a more satisfactory Fourier analysis on the dual space of nuclear spaces. However, the natural domain of definition of the Fourier-Wiener transform in [7] is somewhat the same as that of [16]. As far as the applications of the Fourier-Wiener transform are concerned we find that it is desirable to enlarge the (natural) domain of definition of the Fourier-Wiener transform as much as possible (so that it at least also contains a large class of functions other than cylinder functions).

For this purpose, we abstract [1] to define the Fourier-Wiener transform on the class of exponential type analytic functions \mathcal{E}_a on $[B]$ —the complexification of B . Consequently, if the initial functions are assumed in \mathcal{E}_a , all the solutions can be represented by series; (1) and (2) have their solutions in \mathcal{E}_a , (3) has its solution in \mathcal{E}_a provided $g \in \mathcal{E}_a$ and $\int_B g(x)p_1(dx) = 0$. The nonhomogeneous cases of (1) and (2) are also considered.

2. Preliminaries. Let H be a real separable Hilbert space with norm $|\cdot| = \sqrt{\langle \cdot, \cdot \rangle}$. Let $\|\cdot\|$ be a fixed measurable norm on H (for definition, see, for example, [5] or [10]) and B be the completion of H with respect to $\|\cdot\|$. (H, i, B) is called an abstract Wiener space [5], where i is the canonical embedding mapping from H into B . If μ_t ($t > 0$) is the Gauss cylinder set measure on H with variance t , then μ_t induces a cylinder set measure $\tilde{\mu}_t$ on B which in turn extends to a σ -additive

measure p_t on $(B, \mathfrak{B}(B))$, where $\mathfrak{B}(B)$ is the σ -algebra of Borel sets in B . p_t is called the Wiener measure with variance t and the following properties hold:

$$p_{st}(E) = p_t(s^{-1/2}E) \quad \text{for } s > 0; \quad p_t(-E) = p_t(E), \tag{1}$$

$$p_s * p_t(E) = p_{s+t}(E). \tag{2}$$

We need the following theorem to prove Theorem 3.7 (integration by parts) which is important in this paper.

PROPOSITION 2.1 (KUO). *Let (H, i, B) be an abstract Wiener space. Then there exist another abstract Wiener space (H, i_0, B_0) and an increasing sequence of orthogonal projections $\{P_n\}$ converging strongly to the identity in H such that (1) the B_0 -norm is stronger than the B -norm (hence $B_0 \subset B$), (2) each P_n extends by continuity to a projection \tilde{P}_n of B_0 , and (3) \tilde{P}_n converges strongly to the identity in B_0 (w.r.t. B_0 -norm). (For a proof, see [11].)*

REMARK 1. It is easy to see that if p_t, \tilde{p}_t are the Wiener measures on B and B_0 respectively, then

$$\int_B f(x)p_t(dx) = \int_{B_0} f(x)\tilde{p}_t(dx) \tag{3}$$

for all nonnegative measurable or integrable functions f on B . \square

Notation. If A is an operator on H , we denote by $\|\cdot\|_{tr}$ the trace norm of A and by $\|\cdot\|_{H-S}$ the Hilbert Schmidt norm of A . We also denote by $\|\cdot\|_{X,Y}$ the operator norm in $L(X, Y)$.

REMARK 2. We note that $B^* \subseteq H^* \approx H \subseteq B$ and $(x, y) = \langle x, y \rangle$ for $x \in H, y \in B^*$. \square

THEOREM 2.2 (GOODMAN). *Let A be a bounded linear operator with range in B^* (hence $A \in L(B, B^*)$). Then A is a trace class operator on H . Moreover,*

$$\|A\|_{tr} \leq \int_B \|x\|^2 p_1(dx) \cdot \|A\|_{B, B^*}. \tag{4}$$

(We shall see by the next theorem that $\int_B \|x\|^2 p_1(dx) < \infty$.)

THEOREM 2.3 (FERNIQUE). *There exists $\beta > 0$ such that*

$$\int_B \exp(\beta\|x\|^2) p_1(dx) < \infty. \tag{5}$$

COROLLARY 2.4. $\int_B \exp(\beta\|x\|) p_1(dx) < \infty$ for all $\beta > 0$.

REMARK 3. Skorohod showed that the above corollary is true for some $\beta > 0$, at approximately the same time as Fernique did in 1970. However, this is obviously an easy consequence of Fernique's result.

For more details about abstract Wiener space we refer to Kuo [11] or the original papers of Gross [3], [4], [5] and Segal [17]. For integration theory on Hilbert space, we refer to Skorohod [19]. \square

We shall denote by $([X], \|\cdot\|_{[X]})$ the complexification of a real Banach space X , where $[X] = \{x_1 + ix_2; x_1, x_2 \in X\}$, and $\|x_1 + x_2\|_{[X]} = (\|x_1\|_X^2 + \|x_2\|_X^2)^{1/2}$.

Let X, Y be real Banach spaces, and $[X], [Y]$ be their respective complexifications. We then have

(i) $L(X, Y) \hookrightarrow L([X], [Y])$ by extension and $\|T\|_{X,Y} = \|T\|_{[X],[Y]}$ for $T \in L(X, Y)$.

(ii) If $T \in L([X], [Y])$, then there exists T_j ($j = 1, 2$) in $L(X, Y)$ such that $T = T_1 + iT_2$ (where we consider T_j as an operator in $L([X], [Y])$ in the sense of (i)).

(iii) $(\|T_1\|_{X,Y}^2 + \|T_2\|_{X,Y}^2)^{1/2} \leq \|T\|_{[X],[Y]} \leq \sqrt{2} (\|T_1\|_{X,Y}^2 + \|T_2\|_{X,Y}^2)^{1/2}$.

Therefore, we may consider $L([X], [Y])$ as the complexification of $L(X, Y)$.

REMARK 4. If B is a real Banach space, we define $W_0 = \mathbf{R}$ and $W_n = L(B, W_{n-1})$, $n = 1, 2, \dots$. Then we have $[W_n] = L([B], [W_{n-1}])$.

Now suppose f is an analytic function on some open set $D \subset [B]$ (for more details, we refer to [8], [9]). Then $D^n f(x)$ is realized as an operator in $[W_n]$ and $D^n f(x)$ is of the form $T_1(x) + iT_2(x)$ with $T_1(x), T_2(x) \in W_n$ and $T_1(x), T_2(x)$ extend to $[B]$ in the sense of (i).

3. The Fourier-Wiener transform on abstract Wiener space. Cameron and Martin [1], [2] have defined the Fourier-Wiener transform on the classical Wiener space \mathcal{C} . We shall generalize their results to abstract Wiener space. However, the proofs below are quite different from those of [1].

DEFINITION 3.1. Let (H, i, B) be an abstract Wiener space, and $[B]$ the complexification of B . If f is a function defined on $[B]$, then for each $y \in [B]$, $x \in B$, $f(x + iy)$ is a function of x defined on B . Assuming $f(\cdot + iy) \in L^1(B, p_c)$, we define

$$(\mathcal{F}_c f)(y) = \int_B f(x + iy) p_c(dx)$$

and

$$(\mathcal{F}_c^{-1} f)(y) = \int_B f(x - iy) p_c(dx) \quad \text{where } c > 0.$$

We will call $\mathcal{F}_c f$ the Fourier-Wiener c -transform of f , and \mathcal{F}_c^{-1} the inverse Fourier-Wiener c -transform of f . When $c = 2$, $\mathcal{F}_2 f$ is customarily called the Fourier-Wiener transform of f .

Note that $\mathcal{F}_c f(y) = \int_B f(\sqrt{cx} + iy) p_1(dx)$ and $\mathcal{F}_c^{-1} f(y) = \mathcal{F}_c f(-y)$.

Notation. For notational convenience we use $\| \cdot \|$ as the norm for both B and $[B]$. (Since if $x \in B$, $\|x\|_B = \|x\|_{[B]}$.)

DEFINITION 3.2. Denote by \mathcal{E}_a the collection of functions defined on $[B]$ satisfying the following two conditions:

- (i) $|f(z)| \leq K_1 \exp(K_2 \|z\|)$ for some constants $K_1, K_2 > 0$.
- (ii) For $x, y \in [B]$, $f(x + \lambda y)$ is an entire function of $\lambda \in \mathbf{C}$.

We denote by $\mathcal{E}_a(B)$ those functions which are defined on B and have an extension to \mathcal{E}_a . (Note that if the extension exists, it is unique.)

REMARK 1. We call \mathcal{E}_a the class of exponential type analytic functions. When $B = \mathcal{C}$, the classical Wiener space, \mathcal{E}_a contains the space E_m of mean exponential type analytic functionals defined and used by Cameron and Martin [1]. We shall

identify $\mathfrak{E}_a(B)$ and \mathfrak{E}_a if the domain of definition (i.e. B or $[B]$) does not play an important role.

PROPOSITION 3.3. (i) $\mathfrak{F}_c f$ and $\mathfrak{F}_c^{-1} f$ exist for $f \in \mathfrak{E}_a$.

(ii) $\mathfrak{F}_c(\mathfrak{E}_a) \subset \mathfrak{E}_a$ and similarly, $\mathfrak{F}_c^{-1}(\mathfrak{E}_a) \subset \mathfrak{E}_a$.

(We shall see later that $\mathfrak{F}_c(\mathfrak{E}_a) = \mathfrak{E}_a$.)

PROOF. (i) As a consequence of Fernique's theorem (see Corollary 2.4) $\exp(K_2\|x\|) \in L^1(B, p_c)$. Hence

$$|f(x + iy)| \leq K_1 \exp(K_2\|y\|)\exp(K_2\|x\|) \in L^1(B, p_c)$$

for every $y \in [B]$ and (i) follows.

(ii) It suffices to show that $\mathfrak{F}_c(\mathfrak{E}_a) \subset \mathfrak{E}_a$.

In fact,

$$\begin{aligned} |\mathfrak{F}_c f(y)| &\leq \int_B |f(x + iy)| p_c(dx) \\ &\leq K_1 \int_B \exp(K_2\|x + iy\|) p_c(dx) \\ &\leq K_1 \left(\int_B \exp(K_2\|x\|) p_c(dx) \right) \exp(K_2\|y\|). \end{aligned}$$

So 3.2(i) is fulfilled.

Now observe that for any closed curve γ in \mathbb{C} ,

$$\begin{aligned} \int_\gamma (\mathfrak{F}_c f)(x + \lambda y) d\lambda &= \int_\gamma \int_B f(u + ix + i\lambda y) p_c(du) d\lambda \\ &= \int_B \left[\int_\gamma f((u + ix) + \lambda(iy)) d\lambda \right] p_c(du) \\ &= \int_B 0 \cdot p_c(du) = 0. \end{aligned}$$

3.2(ii) follows by Morera's Theorem. Hence, $\mathfrak{F}_c(\mathfrak{E}_a) \subset \mathfrak{E}_a$. \square

PROPOSITION 3.4. (a) \mathfrak{E}_a is a vector space and is closed under multiplication.

If $f \in \mathfrak{E}_a$, we also have

(b) f is infinitely F -differentiable (hence continuous).

(c) $D^n f(z) h_1 \dots h_n \in \mathfrak{E}_a$, where $h_1, \dots, h_n \in [B]$, $n = 1, 2, \dots$

(d) $\|D^n f(z)\|_{[w_n]} \leq K_1 \exp(nK_2)\exp(K_2\|z\|)$ for some constants K_1, K_2 .

(e) For any $c_1 > 0$, there exist constants $K(f, c_1)$, which depends only on f and c_1 , and M_f which depends only on f such that $\|D^n f(z)\|_{[w_n]} \leq K(f, c_1) M_f^n$ for all $\|z\| \leq c_1$.

(f) If $x, y \in [B]$, the Taylor's series (in y) $f(x + y) = \sum_{n=0}^\infty D^n f(x) y^n / n!$ converges absolutely and uniformly with respect to both x and y on every bounded set in $[B]$.

PROOF. By the definition of \mathfrak{E}_a and the Cauchy formula, the proofs are straightforward. \square

Moreover, we have

COROLLARY 3.5. $f \in \mathcal{G}_a(B)$ iff

- (a) f is infinitely Fréchet-differentiable in B , and
- (b) for any $c_1 > 0$, there exist constants K and M such that $\|D^n f(x)\|_{W_n} < KM^n$ for all $\|x\| \leq c_1$.

The following lemma is a consequence of [10, Theorem 1]. We state it without proof.

LEMMA 3.6. Let f be a continuously Fréchet-differentiable function from B into B^* . Assume that there is an $r > 0$ such that

- (i) $\int_B [\sup_{\|h\| < r} \|Df(x+h)\|_{B, B^*}] p_t(dx) < \infty$,
- (ii) $\int_B \|f(x)\|_{B^*}^2 p_t(dx) < \infty$.

Then $\int_B Df(x) h p_t(dx) = t^{-1} \int_B \langle x, h \rangle f(x) p_t(dx)$ for $h \in H$, where we regard $\langle \cdot, h \rangle$ as a measurable function on B (see, for example, [11]).

Referring to [10, Corollary 1], we have

THEOREM 3.7. Let $f(x)$ be a continuously Fréchet-differentiable function from B into B^* . If in addition to the assumptions (i) and (ii) of Lemma 3.6, we assume

- (iii) $\int_B \|Df(x)\|_{tr} p_t(dx) < \infty$,

then $\int_B \langle x, f(x) \rangle p_t(dx) = t \int_B \text{trace}_H(Df(x)) p_t(dx)$.

PROOF. Let (H, i_0, B_0) , $\{P_n\}$ be the Wiener space and projections respectively in Proposition 2.1 and \tilde{P}_n be the extension of P_n to B_0 .

By Remark 1 in §2, we have

$$\int_B \langle x, f(x) \rangle p_t(dx) = \int_{B_0} \langle x, f(x) \rangle \tilde{p}_t(dx),$$

where (\cdot, \cdot) on the right-hand side is the (B_0, B_0^*) pairing (note that $B_0^* \supset B^*$).

Recall that \tilde{P}_n is extended from P_n by continuity, so $\tilde{P}_n(B_0)$ is the $\|\cdot\|_0$ -closure of $P_n(H)$. Since $P_n(H)$ is finite dimensional, $\tilde{P}_n(B_0) = P_n(H)$ (for $\|\cdot\|_0$ and $\|\cdot\|$ are equivalent in $P_n(H)$).

Let $\{e_j: j = 1, 2, \dots\}$ be an orthonormal basis of H . Then

$$\begin{aligned} \int_{B_0} (\tilde{P}_n x, f(x)) \tilde{p}_t(dx) &= \int_{B_0} \langle \tilde{P}_n x, f(x) \rangle \tilde{p}_t(dx) \\ &= \sum_{j=1}^{\infty} \int_{B_0} \langle e_j, f(x) \rangle \langle \tilde{P}_n x, e_j \rangle \tilde{p}_t(dx) \\ &\quad \text{(by the dominated convergence theorem)} \\ &= \sum_{j=1}^{\infty} \int_{B_0} \langle e_j, f(x) \rangle \langle x, P_n^* e_j \rangle \tilde{p}_t(dx) \\ &= t \sum_{j=1}^{\infty} \int_{B_0} \langle Df(x) P_n e_j, e_j \rangle \tilde{p}_t(dx), \quad \text{by Lemma 3.6.} \end{aligned}$$

Whence by (iii) and the dominated convergence theorem we have

$$\int_{B_0} (\tilde{P}_n x, f(x)) \tilde{p}_t(dx) = t \int_{B_0} \text{trace}[Df(x) P_n] \tilde{p}_t(dx).$$

(Note that if f is Fréchet-differentiable in B , then $f|_{B_0}$ is also Fréchet-differentiable in B_0 and $Df(x) = D(f|_{B_0})(x)$ for all $x \in B_0$.)

Recall that $\tilde{P}_n \rightarrow I_{B_0}$ strongly, we have $(\tilde{P}_n x, f(x)) \rightarrow (x, f(x))$. Also, since $P_n \rightarrow I_H$ strongly, we have $\text{trace}[Df(x)P_n] \rightarrow \text{trace}[Df(x)]$. Moreover,

$$|(\tilde{P}_n x, f(x))| \leq \|f(x)\|_{B_0} \|x\|_{B_0} \leq \text{const} \|f(x)\|_{B^*} \|x\|_{B_0} \in L^1(\tilde{p}_1),$$

by assumption (ii), §2, Remark 1 and Hölder's inequality. Besides, $|\text{trace } Df(x)P_n| \leq \|Df(x)\|_{tr}$.

Therefore by the dominated convergence theorem

$$\begin{aligned} \int_B (x, f(x)) p_t(dx) &= \lim_{n \rightarrow \infty} \int_{B_0} \langle \tilde{P}_n x, f(x) \rangle \tilde{p}_t(dx) \\ &= \lim_{n \rightarrow \infty} t \int_{B_0} \text{trace}_H [Df(x)P_n] \tilde{p}_t(dx) \\ &= t \int_{B_0} \text{trace}_H [Df(x)] \tilde{p}_t(dx) \\ &= t \int_B \text{trace}_H [Df(x)] p_t(dx). \quad \square \end{aligned}$$

REMARK 2. It is not hard to see that if $g \in \mathfrak{E}_a(B)$, then $Dg: B \rightarrow B^*$ satisfies (i), (ii) and (iii) of Theorem 3.7. In fact (ii) follows by 3.2(1) and Corollary 2.4, and (iii) follows by Goodman's theorem (2.2). For (i), we let $r = 1$. Then

$$\begin{aligned} \sup_{0 < |h| < 1} \|D^2 g(x+h)\|_{B, B^*} &\leq \sup_{0 < |h| < 1} c_g \exp(2c'_g + c'_g \|x+h\|) \\ &\leq c_g \exp((2 + c_1)c'_g) \exp(c'_g \|x\|) \in L^1(p_t), \end{aligned}$$

where c_1 is the constant such that $\|h\| \leq c_1|h|$. \square

LEMMA 3.8. Let T be a symmetric linear operator in $[W_n]$. Then for $x, y \in B$,

$$\int_B \int_B T(x+iy)^n p_c(dx) p_c(dy) = 0 \quad \text{for } n = 1, 2, \dots \tag{1}$$

PROOF. For $n = 1$,

$$\begin{aligned} \int_B \int_B T(x+iy) p_c(dx) p_c(dy) &= \int_B \int_B T(-x-iy) p_c(dx) p_c(dy) \\ &= - \int_B \int_B T(x+iy) p_c(dx) p_c(dy), \end{aligned}$$

since p_c is even; hence $\int_B \int_B T(x+iy) p_c(dx) p_c(dy) = 0$.

For $n = 2$,

$$\begin{aligned} \int_B \int_B T(x+iy)^2 p_c(dx) p_c(dy) &= \int_B \int_B (Tx^2 - Ty^2) p_c(dx) p_c(dy) \\ &\quad \left(\text{since } \int_B Txy p_c(dx) = 0 \right) \\ &= \int_B \int_B Tx^2 p_c(dx) - \int_B \int_B Ty^2 p_c(dy) = 0. \end{aligned}$$

For $n > 2$, we write

$$\int_B \int_B T(x + iy)^n p_c(dx) p_c(dy) = \int_B \int_B (x, T(x + iy)^{n-1}) p_c(dx) p_c(dy) + i \int_B \int_B (y, T(x + iy)^{n-1}) p_c(dy) p_c(dx). \quad (**)$$

For the first term, we let $f(x) = T(x + iy)^{n-1}$. For the second term, we let $g(y) = T(x + iy)^{n-1}$. Then by Theorem 3.7, we get

$$(**) = c(n - 1) \int_B \int_B \text{trace}_H T(x + iy)^{n-2} p_c(dx) p_c(dy) - c(n - 1) \int_B \int_B \text{trace}_H T(x + iy)^{n-2} p_c(dy) p_c(dx) = 0,$$

where $Df(x) = (n - 1)T(x + iy)^{n-2}$ and $Dg(y) = i(n - 1)T(x + iy)^{n-2}$ are interpreted as operators of the form $A_1 + iA_2$ with A_1 and A_2 in $L(B, B^*)$, whence $\text{trace}_H(A_1 + iA_2) = \text{trace}_H A_1 + i \text{trace}_H A_2$ makes sense. \square

THEOREM 3.9. For $f \in \mathcal{E}_a$, we have

$$\mathfrak{F}_c^{-1} \mathfrak{F}_c f(z) = f(z), \quad z \in [B]. \quad (2)$$

PROOF. (Step 1) We claim that (2) holds for $f(z) = Tz^n, n = 1, 2, \dots$, with T a symmetric operator in $[W_n]$.

In fact,

$$\begin{aligned} \mathfrak{F}_c^{-1} \mathfrak{F}_c(T(\cdot)^n)(z) &= \int_B \int_B T(x + iy + z)^n p_c(dx) p_c(dy) \\ &= Tz^n + \int_B \int_B \sum_{j=0}^{n-1} Tz^j (x + iy)^{n-j} p_c(dx) p_c(dy) \\ &= Tz^n \quad \text{by Lemma 3.8.} \end{aligned}$$

(Step 2) By Proposition 3.4, we have

$$f(z) = \sum_{n=0}^{\infty} (1/n!) D^n f(0) z^n.$$

Consequently,

$$\mathfrak{F}_c f(z) = \sum_{n=0}^{\infty} (1/n!) \mathfrak{F}_c(D^n f(0)(\cdot)^n)(z),$$

by Lebesgue's dominated convergence theorem.

Finally, by Step 1 and the dominated convergence theorem again,

$$\begin{aligned} \mathfrak{F}_c^{-1} \mathfrak{F}_c f(z) &= \sum_{n=0}^{\infty} (1/n!) \mathfrak{F}_c^{-1} \mathfrak{F}_c(D^n f(0)(\cdot)^n)(z) \\ &= \sum_{n=0}^{\infty} (1/n!) D^n f(0) z^n = f(z). \quad \square \end{aligned}$$

REMARK 3. Combining 3.9 and 3.3, we see that $\mathfrak{F}_c(\mathcal{E}_a) = \mathcal{E}_a$. \square

THEOREM 3.10. $\mathfrak{E}_a(B)$ is dense in $L^2(B, p_c)$.

REMARK 4. It is easy to see that if $f \in \mathfrak{E}_a(B)$, then

$$\begin{aligned} f_y: x &\rightarrow f(x + y), & y \in B, \\ f^d: x &\rightarrow f(dx), & d \in R, \end{aligned}$$

are also in $\mathfrak{E}_a(B)$. \square

PROOF OF 3.10. By Remark 4, we may assume $c = 1$.

Let $\{e_j: j = 1, 2, \dots, n\}$ be a set $\subset B^*$. Define $P_n^a(x) = \prod_{j=1}^n(x, e_j)^{a_j}$ where $a = (a_1, a_2, \dots, a_n)$ and a_j 's are positive integers. Obviously, $P_n^a \in \mathfrak{E}_a(B)$ for all a and n . Since the algebra generated by P_n^a (i.e. the linear combination of products of P_n^a 's) is dense in $L^2(B, p_1)$ (see, for example [7], [15]), our conclusion follows immediately. \square

THEOREM 3.11 (PARSEVAL'S FORMULA). For all $f, g \in \mathfrak{E}_a(B)$, we have

$$\int_B (\mathfrak{F}_2 f)(y)g(y)p_1(dy) = \int_B f(y)\mathfrak{F}_2 g(y)p_1(dy). \tag{3}$$

PROOF. It is not hard to see by direct computations and induction that (3) is true for $f(x) = Ax^m$ and $g(x) = Tx^n$ (where A and T are symmetric members in W_m and W_n , respectively). The theorem then follows by Proposition 3.4 and Lebesgue's dominated convergence theorem. (This provides a more direct proof for the Parseval's formula without using the knowledge of the Hermite polynomials. See also Remark 6.) \square

COROLLARY 3.12. (i) $\int_B \mathfrak{F}_2 f(y) \cdot \mathfrak{F}_2 g(y)p_1(dy) = \int_B f(x)g(-x)p_1(dx)$,

(ii) $\int_B \mathfrak{F}_2 f(y) \cdot \mathfrak{F}_2 g(y)p_1(dy) = \int_B f(x)g(x)p_1(dx)$,

(iii) $\int_B |\mathfrak{F}_2 f(y)|^2 p_1(dy) = \int_B |f(x)|^2 p_1(dx)$.

PROOF. (i) follows by Theorem 3.11. (ii) follows by the fact that $\overline{\mathfrak{F}_2 g(y)} = \mathfrak{F}_2^{-1} \bar{g}(y)$ and (iii) follows from (ii). \square

Notation. We customarily use \mathfrak{F} to stand for \mathfrak{F}_2 .

REMARK 5. Following [2], we may also extend the Fourier-Wiener transform to $L^2(B, p_1)$ in the following way.

If $f \in L^2(B, p_1)$ and $\{f_n\}$ is a sequence of functions in $\mathfrak{E}_a(B)$ such that $f_n \rightarrow f$ in $L^2(B, p_1)$ (by 3.10), then we define the Fourier-Wiener transform of f by $\mathfrak{F}f = L^2\text{-lim } \mathfrak{F}f_n$.

It follows by Corollary 3.12(iii) that \mathfrak{F} extends to a unitary operator on $L^2(B, p_1)$, i.e. $\|\mathfrak{F}f\|_{L^2} = \|f\|_{L^2}$ for all $f \in L^2(B, p_1)$. \square

REMARK 6. Let $h_n = (-1)^n(n!)^{-1/2}e^{x^2/2}(d^n/dx^n)e^{-x^2/2}$ be the one-dimensional normalized Hermite polynomial of degree n . Let e_i be an orthonormal basis of H which lies in B^* . For any multiple indices $I = (i_1, i_2, \dots, i_n)$, we define $h_I(x) = \prod_{k=1}^n h_{i_k}((x, e_k))$. Then $\{h_I\}$ forms an orthonormal basis in $L^2(B, p_1)$ (see [15]). Employing the computation of Hida [7], we get

$$\mathfrak{F}h_I(y) = i^{|I|}h_I(y), \quad y \in B. \tag{4}$$

(This identity obviously also implies Theorem 3.11.) Furthermore, for any $f \in L^2(B, p_1)$, we have $f = \sum_I a_I h_I(L^2)$ and by (4) we get

$$\mathfrak{F}f = \sum_I a_I i^{|I|} h_I(L^2). \tag{5}$$

Since $h_I \in \mathfrak{E}_a(B)$, if we take (5) as the definition of the Fourier-Wiener transform on $L^2(B, p_1)$, (5) is obviously equivalent to the definition given in Remark 5. \square

4. Application I. Let f be a function defined on B . We may regard f as a function g defined in a neighborhood of the origin of H by restricting f to the coset $x + H$ and defining $g(h) = f(x + h)$. If g is k -times Fréchet-differentiable at 0 then we say that f is k -times H -differentiable at x . We denote the k -times H -differentiation of f at x by $f^{(k)}(x)$. After Gross [6], if f is a twice H -differentiable function on B such that $f''(x)$ is a trace class operator on H for each $x \in B$ then we define the Laplacian $\Delta f(x)$ by $\text{trace}_H f''(x)$.

If f is twice Fréchet-differentiable in B then the second Fréchet-derivative $D^2f(x) \in L(B, B^*)$ and f is automatically twice H -differentiable. Restricting to H , $D^2f(x)|_H = f''(x)$, where $f''(x)$ is a symmetric member of $L(H, H)$ and therefore of trace class by Goodman's theorem (see [11]).

In this section, we apply the Fourier-Wiener transform to the Cauchy problem below.

$$u_t(x, t) = -\mathcal{N}^k u(x, t); \quad u(x, 0) = f(x) \quad (t > 0, x \in B) \tag{1}$$

where k is any integer ≥ 1 , and

$$\mathcal{N}u(x) = -\Delta u(x) + (x, Du(x)) \quad (\text{if it exists}). \tag{2}$$

If u is twice Fréchet-differentiable in B , we may write

$$\Delta u(x) = \text{trace}_H [D^2u(x)]. \tag{3}$$

Our goal is to show that for $f \in \mathfrak{E}_a(B)$, there exists a unique solution for (1) which is also in $\mathfrak{E}_a(B)$.

REMARK 1. When $k = 1$, Piech [14] has shown the existence of a solution for equation (1) (with f a Lip-1 function) such that $u(x, t) \rightarrow_{t \rightarrow 0} f(x)$ uniformly on bounded sets. In [15], she also studied the semigroup associated with $u_t = -\mathcal{N}u$ on $L^2(B, p_1)$. A more general definition of \mathcal{N} which does not require the existence of both Δu and $(x, Du(x))$ is also given in [15]. \square

We shall separate our proof into two parts.

Part I (Existence). For $c > 0$, we let

$$\mathcal{N}_c u(x) = -\Delta u(x) + c(x, Du(x)) \tag{4}$$

and

$$\tilde{\mathcal{N}}u(x) = (x, Du(x)). \tag{5}$$

LEMMA 4.1. For $u \in \mathfrak{E}_a$, we have

- (i) $\Delta(\mathfrak{E}_a) \subset \mathfrak{E}_a$, $\tilde{\mathcal{N}}(\mathfrak{E}_a) \subset \mathfrak{E}_a$ and consequently, $\mathcal{N}_c(\mathfrak{E}_a) \subset \mathfrak{E}_a$.
- (ii) $(\mathfrak{F}_{1/c} \mathcal{N}_c u)(y) = c \tilde{\mathcal{N}}(\mathfrak{F}_{1/c} u)(y)$ for $y \in B$ (or $[B]$).

PROOF. (i) Trivial.

(ii) Recall Remark 2 of §3. Let $f(x) = Du(x + iy)$ and $t = 1/c$ and apply Theorem 3.7. We get

$$\int_B (x, Du(x + iy))p_{1/c}(dx) = c^{-1} \int_B \text{trace}_H [D^2u(x + iy)]p_{1/c}(dx).$$

On the other hand,

$$\int_B (x, Du(x + iy))p_{1/c}(dx)$$

(where the differentiation is with respect to x)

$$\begin{aligned} &= \int_B \tilde{\mathcal{U}}u(x + iy)p_{1/c}(dx) - i \int_B (y, Du(x + iy))p_{1/c}(dx) \\ &= \int_B \tilde{\mathcal{U}}u(x + iy)p_{1/c}(dx) - \left(y, \int_B iDu(x + iy)p_{1/c}(dx) \right) \end{aligned}$$

(by the definition of vector valued integral)

$$\begin{aligned} &= \int_B \tilde{\mathcal{U}}u(x + iy)p_{1/c}(dx) - \left(y, \int_B D_y u(x + iy)p_{1/c}(dx) \right) \\ &= \int_B \tilde{\mathcal{U}}u(x + iy)p_{1/c}(dx) - \left(y, D \left[\int_B u(x + iy)p_{1/c}(dx) \right] \right) \end{aligned}$$

(by the fact that $u \in \mathfrak{E}_a$ and the dominated convergence theorem)

$$= \mathfrak{F}_{1/c} \tilde{\mathcal{U}}u(y) - \tilde{\mathcal{U}}(\mathfrak{F}_{1/c}u)(y).$$

Thus, $(\mathfrak{F}_{1/c} \mathcal{U}_c u)(y) = c \tilde{\mathcal{U}}(\mathfrak{F}_{1/c}u)(y)$ (see also [10]). \square

Now, by taking the Fourier-Wiener transform and letting $v(y, t) = (\mathfrak{F}_1 u)(y, t)$, (1) becomes

$$v_t(y, t) = -\tilde{\mathcal{U}}^k v(y, t); \quad v(y, 0) = \mathfrak{F}_1 f(y) \quad (y \in B, t > 0) \tag{6}$$

provided that

$$v_t(y, t) = (\mathfrak{F}_1 u_t)(y, t). \tag{7}$$

By Theorem 3.9, if we assume $f \in \mathfrak{E}_a$, then it suffices to consider the Cauchy problem (6) and the condition (7).

LEMMA 4.2. Let $T \in W_n$ (T is not necessarily symmetric), and $f(x) = Tx^n$. Then

$$\tilde{\mathcal{U}}f(x) = nf(x), \quad x \in B \text{ (or } [B]).$$

REMARK 2. The above lemma says that $\{1, 2, \dots\}$ are eigenvalues of $\tilde{\mathcal{U}}$ with the associated eigenvectors in \mathfrak{E}_a . It is well known that $\{1, 2, \dots\}$ is the spectrum of \mathcal{U} when we regard \mathcal{U} as a (selfadjoint) operator on $L^2(B, p_1)$ (see, for example, [12], [14], [15]). Since $\mathfrak{E}_a(B) \subset L^2(B, p_1)$, hence there is no eigenvector of $\tilde{\mathcal{U}}$ in $\mathfrak{E}_a(B)$ with eigenvalue other than $\{1, 2, \dots\}$ (by Theorem 3.9). \square

PROOF OF 4.2. It is easy to see that

$$Df(x)h = Tx^{n-1}h + Tx^{n-2}hx + \dots + Thx^{n-1}, \quad x \in B.$$

Consequently,

$$\tilde{\mathcal{U}}f(x) = Df(x)(x) = nTx^n = nf(x). \quad \square$$

LEMMA 4.3. Let T be as in Lemma 4.2, and $u(x, t) = e^{-tn^k}Tx^n$. Then

- (a) $u(x, t)$ satisfies $u_t = -\tilde{\mathcal{U}}^k u$ and $u(x, 0) = Tx^n$.
- (b) $u(x, t) \rightarrow_{t \rightarrow 0} Tx^n$ uniformly on bounded subsets of $[B]$.

PROOF. By Lemma 4.2, $-\tilde{\mathcal{U}}^k u(x, t) = -n^k e^{-tn^k}Tx^n = -n^k u(x, t) = u_t(x, t)$. This proves (a).

(b) is trivial. \square

Notation. We define for $f \in \mathcal{E}_a$

$$\|f\|_N = \sup\{|f(x)| : \|x\| < N\}. \quad (8)$$

Obviously, $\|\cdot\|_N$ is a norm on \mathcal{E}_a .

THEOREM 4.4. Suppose $\{T_n\}$ is a sequence of bounded linear operators such that $T_n \in W_n, n = 0, 1, 2, \dots$. Assume that there exist constants K and M such that

$$\|T_n\|_{W_n} \leq KM^n, \quad n = 1, 2, \dots \quad (9)$$

Then we have

(a) The series $\sum_{j=0}^{\infty} (1/j!)T_j x^j$ converges absolutely and uniformly on every bounded subset of $[B]$ to a function f in \mathcal{E}_a .

(b) If for $t \geq 0$, we let

$$u(x, t) = \sum_{n=0}^{\infty} (e^{-tn^k}/n!)T_n x^n, \quad (10)$$

then this series converges absolutely, x -uniformly on bounded subsets and t -uniformly on $[0, \infty)$. Also $u(\cdot, t) \in \mathcal{E}_a$.

- (c) For each $x \in [B], u(x, t) \in C^\infty([0, \infty))$.
- (d) $u(x, t)$ satisfies $u_t(x, t) = -\tilde{\mathcal{U}}^k u(x, t)$ and $u(x, 0) = f(x)$.
- (e) $u(x, t) \rightarrow f(x)$ uniformly on bounded sets as $t \rightarrow 0$.

PROOF. (a) Since

$$\|T_j x^j\|_N \leq \|T_j\|_{W_j} N^j \leq K(MN)^j$$

and

$$\sum_{j=0}^{\infty} (1/j!)K(MN)^j = K \exp(MN) < \infty,$$

hence, by the M -test $\sum_{j=0}^{\infty} (1/j!)T_j x^j$ converges absolutely and uniformly on every bounded set. Consequently, for each pair of $x, y \in [B], \sum_{j=0}^{\infty} (1/j!)T_j(x + \lambda y)^j$ converges λ -uniformly on compact subsets of C . Thus $f(x + \lambda y)$ is an entire function of λ . Moreover, it is easy to see that $|f(x)| < K \exp(M\|x\|)$ and (a) follows.

(b) Observe that $e^{-tn^k} < 1$, and so (b) follows immediately from (a).

(c) Since $|\sum_{n=0}^{\infty} (-n^k)^m (e^{-tn^k}/n!)T_n x^n| \leq \sum_{n=0}^{\infty} n^{km} (KM^n/n!) \|x\|^n$ for every $m \geq 1$, hence $u(x, t) \in C^\infty([0, \infty))$ for each $x \in [B]$ and

$$(\partial^m / \partial t^m)u(x, t) = \sum_{n=0}^{\infty} (-n^k)^m (e^{-tn^k} / n!) T_n x^n.$$

(d) Follows immediately by Lemma 4.3, (c) and the fact that $\sum_{n=0}^{\infty} (n^k / n!) (KM)^n < \infty$.

(e) Follows from Lemma 4.3 and (b).

COROLLARY 4.5. *If $g \in \mathcal{E}_a$, we define*

$$u(x, t) = \sum_{n=0}^{\infty} (e^{-tn^k} / n!) D^n g(0) x^n \quad \text{for } t \geq 0. \tag{11}$$

Then (i) $u(x, t)$ satisfies $u_t(x, t) = -\mathfrak{H}^k u(x, t)$; $u(x, 0) = g(x)$.

(ii) $u(x, t) \rightarrow_{t \rightarrow 0} g(x)$ uniformly on bounded sets in $[B]$.

Next we establish a class of functions satisfying condition (7).

LEMMA 4.6. *Suppose $u(x, t) \in \mathcal{E}_a$ for every $t \in [t_1, t_2] \subset \mathbf{R}$, and for every $x \in [B]$, $u(x, t)$ is a C^1 -function of t in $[t_1, t_2]$. Assume that for each $y \in [B]$, there is a function $W^*(x) \in L^1(B, p_c)$ such that if $x \in B$*

$$|(\partial / \partial t)u(x + iy, t)| \leq W^*(x) \quad \text{for all } t \text{ in } [t_1, t_2]. \tag{12}$$

Then we have

$$(\partial / \partial t)(\mathfrak{F}_c u)(y, t) = \mathfrak{F}_c((\partial / \partial t)u)(y, t).$$

PROOF. Suppose $t_0, t_0 + h \in [t_1, t_2]$. Then

$$\begin{aligned} |u(x + iy, t_0 + h) - u(x + iy, t_0)| / |h| & \\ & < |(\partial / \partial t)u(x + iy, t_0 + \theta h)|, \quad \text{where } 0 < \theta < 1, \\ & < W^*(x) \in L^1(B, p_c), \quad \text{by (12)}. \end{aligned}$$

It follows by Lebesgue's dominated convergence theorem that

$$\begin{aligned} \lim_{h \rightarrow 0} h^{-1}(\mathfrak{F}_c u(y, t_0 + h) - \mathfrak{F}_c u(y, t_0)) & \\ &= \lim_{h \rightarrow 0} \int_B h^{-1}(u(x + iy, t_0 + h) - u(x + iy, t_0)) p_c(dx) \\ &= \int_B \lim_{h \rightarrow 0} h^{-1}(u(x + iy, t_0 + h) - u(x + iy, t_0)) p_c(dx) \\ &= \int_B (\partial / \partial t)u(x + iy, t_0) p_c(dx) \\ &= \mathfrak{F}_c((\partial / \partial t)u)(y, t_0). \quad \square \end{aligned}$$

THEOREM 4.7. *Suppose $f \in \mathcal{E}_a$ and $g(y) = \mathfrak{F}_1 f(y)$. Define*

$$u(y, t) = \sum_{n=0}^{\infty} (e^{-tn^k} / n!) D^n g(0) y^n$$

and

$$v(x, t) = \mathfrak{F}_1^{-1} u(x, t). \tag{13}$$

Then (i) $v(x, t)$ solves the Cauchy problem (1) i.e. $v_t(x, t) = -\mathcal{U}^k v(x, t)$; $v(x, 0) = f(x)$.

(ii) $v(x, t) \rightarrow_{t \rightarrow 0} f(x)$ uniformly on bounded sets.

PROOF. Obviously, $v(x, 0) = \mathcal{F}_1^{-1}u(x, 0) = f(x)$. By Corollary 4.5 and Proposition 3.3, we have $g \in \mathcal{E}_a$ and $u_t(y, t) = -\tilde{\mathcal{U}}^k u(y, t)$. Employing Lemma 4.1, we get $(\mathcal{F}_1^{-1}\tilde{\mathcal{U}}^k u)(x, t) = \mathcal{U}^k v(x, t)$. Now, to prove (i), it remains to show that

$$(\mathcal{F}_1^{-1}u_t)(x, t) = v_t(x, t). \tag{*}$$

In fact, by 4.4(b), (c), we have

$$\begin{aligned} u_t(y, t) &= \sum_{n=0}^{\infty} e^{-tn^k} (-n^k/n!) D^n g(0) y^n \\ &= \frac{1}{2\pi i} \int_{|\lambda|=2} \left(\sum_{n=0}^{\infty} e^{-n^k t} (-n^k/\lambda^{n+1}) \right) g(\lambda y) d\lambda. \end{aligned}$$

It follows that $|u_t(y, t)| < (\sum_{n=0}^{\infty} n^k/2^n) K_1 \exp(2K_2\|y\|)$, where $K_1, K_2 > 0$ are those constants such that $|g(z)| < K_1 \exp(K_2\|z\|)$. In particular, for each $y \in [B]$, say $\|y\| \leq c_1$, we have for $x \in B$

$$|u_t(x + iy, t)| \leq \left(\sum_{n=0}^{\infty} n^k/2^n \right) K_1 \exp(2K_2c_1) \exp(2K_2\|x\|) \in L^1(B, p_1).$$

Therefore condition (12) is fulfilled and (*) follows by Lemma 4.6. This proves (i).

(ii) Let $\|x\| \leq c_1$; we have

$$\begin{aligned} |v(x, t) - f(x)| &= |\mathcal{F}_1^{-1}u(x, t) - \mathcal{F}_1^{-1}g(x)| \\ &\leq \frac{1}{2\pi} \left| \int_B \int_{|\lambda|=2} \left(\sum_{n=0}^{\infty} (e^{-n^k t} - 1)/\lambda^{n+1} \right) g(\lambda(y - ix)) d\lambda p_1(dy) \right| \\ &\leq K_1 t \left(\sum_{n=0}^{\infty} n^k/2^n \right) \exp(2K_2c_1) \left(\int_B \exp(2K_2\|y\|) p_1(dy) \right) \\ &\rightarrow 0 \text{ uniformly on } \{\|x\| \leq c_1\}. \quad \square \end{aligned}$$

REMARK 3. We note that the formula (11) in Corollary 4.5 can also be written as

$$u(x, t) = (1/2\pi i) \int_{|\lambda|=2} \left(\sum_{n=0}^{\infty} e^{-n^k t} / \lambda^{n+1} \right) g(\lambda x) d\lambda$$

and (13) also has the following form

$$\begin{aligned} v(x, t) &= \mathcal{F}_1^{-1}u(x, t) \\ &= \frac{1}{2\pi i} \int_B \int_B \left[\int_{|\lambda|=2} \left(\sum_{n=0}^{\infty} e^{-n^k t} / \lambda^{n+1} \right) f(z + \lambda x + i\lambda y) d\lambda \right] p_1(dz) p_1(dy). \end{aligned} \tag{14}$$

When $k = 1$, the solution of (1) is also given by

$$\begin{aligned} v(x, t) &= \mathcal{F}_1^{-1}(g(e^{-t}y))(x) \\ &= \int_B \int_B f(z + e^{-t}x + ie^{-t}y) p_1(dy) p_1(dz). \end{aligned} \tag{15}$$

It can be shown that the integral (15) can also be written as

$$\int_B f(e^{-t}x + \sqrt{1 - e^{-2t}} y) p_1(dy) = \int_B f(y) o_t(x, dy), \tag{15'}$$

where $o_t(x, E) = p_{1-e^{-2t}}(e^{-t}x, E)$.

We note that this is a well-known formula (see [14], [15]), and that the measures $o_t(x, dy)$ are the transition measures of the infinite dimensional Ornstein-Uhlenbeck process. \square

Part II (Semigroup theory and uniqueness). First of all, we shall study the semigroup induced by the solution of the Cauchy problem (6). Using this as a tool, we are able to show that there is only one solution in $\tilde{\mathcal{E}}_a$ (see Definition 4.8) for (6) under some t -smoothness assumption (see 4.14). Then the uniqueness of (1) follows by applying the Fourier-Wiener transform.

DEFINITION 4.8. Denote by $\tilde{\mathcal{E}}_a$ the space of analytic functions f defined on $[B]$ such that $\|f\|_N = \sup\{|f(z)|: \|z\| \leq N\}$ is finite for every $N = 1, 2, 3, \dots$

Obviously, $\mathcal{E}_a \subset \tilde{\mathcal{E}}_a$.

PROPOSITION 4.9. (i) $\|f\|_N$ is a norm on $\tilde{\mathcal{E}}_a$ and $\|f\|_N \leq \|f\|_{N+1}$ for $N = 1, 2, 3, \dots$

(ii) With the topology induced by $\|\cdot\|_N$, $\tilde{\mathcal{E}}_a$ forms a (sequentially) complete metrizable locally convex space.

PROOF. (i) Trivial.

(ii) Since $\|\cdot\|_N$ is a sequence of norms which separate “points” of $\tilde{\mathcal{E}}_a$, hence $\tilde{\mathcal{E}}_a$ forms a metrizable locally convex space. It remains to show that $\tilde{\mathcal{E}}_a$ is complete. Suppose $\{f_n\}$ is a Cauchy sequence in $\tilde{\mathcal{E}}_a$. Then for each N , $\{f_n\}$ is also a Cauchy sequence on $\{\|z\| \leq N\}$; hence $\{f_n\}$ converges uniformly on $\{\|z\| \leq N\}$ to a function, say f^N . We thereby define a function f such that $f = f^N$ on $\{\|z\| \leq N\}$. Since $\|\cdot\|_N \leq \|\cdot\|_{N+1}$, f is well defined and $f_n \rightarrow f$ uniformly on every bounded set. It follows that f is analytic. Moreover, for each fixed N , $\|f_n\|_N < \text{some constant } C_1$ for all n . We conclude that $\|f\|_N < C_1$, i.e. $f \in \tilde{\mathcal{E}}_a$. \square

LEMMA 4.10. Every $f \in \tilde{\mathcal{E}}_a$ has a Maclaurin expansion which converges absolutely and uniformly on bounded sets.

PROOF. Since $f(\lambda x)$ has a Maclaurin expansion in λ for each $x \in [B]$ we have $f(x) = \sum_{n=0}^{\infty} (1/n!) D^n f(0) x^n$. Furthermore,

$$(1/n!) D^n f(0) x^n = (1/2\pi i) \int_{|z|=2} f(zx) / z^{n+1} dz.$$

Consequently, for $\|x\| \leq N$ we have

$$(1/n!) |D^n f(0) x^n| \leq 2^{-n} \|f\|_{2N} \tag{16}$$

and

$$\sum_{n=0}^{\infty} (1/n!) |D^n f(0) x^n| \leq \left(\sum_{n=0}^{\infty} 2^{-n} \right) \|f\|_{2N} = 2 \|f\|_{2N}.$$

Thus we complete the proof. \square

PROPOSITION 4.11. $\tilde{\mathcal{E}}_a$ is the completion of \mathcal{E}_a .

PROOF. It suffices to show that \mathcal{E}_a is dense in $\tilde{\mathcal{E}}_a$. In fact, given any function $f \in \tilde{\mathcal{E}}_a$, Lemma 4.10 implies that as $p \rightarrow \infty$, $\sum_{n=0}^p (1/n!)D^n f(0)x^n \in \mathcal{E}_a$ converges to f uniformly on every bounded set. Thus \mathcal{E}_a is dense in $\tilde{\mathcal{E}}_a$. \square

Now we define a family of operators $\{T_t\}$ on $\tilde{\mathcal{E}}_a$ in the following way.

Let k be the integer in the Cauchy problem (1) and let $f \in \tilde{\mathcal{E}}_a$. For $t > 0$, we define

$$T_t f(x) = \sum_{n=0}^{\infty} (e^{-n^k t} / n!) D^n f(0) x^n \tag{17}$$

i.e. we multiply the ‘‘coefficient’’ of ‘‘ x^n ’’ by $e^{-n^k t}$ for $n = 0, 1, 2, \dots$

Obviously, $T_t f \in \tilde{\mathcal{E}}_a$ and by (16) we also have

$$\|T_t f\|_N \leq \sup_{\|x\| < N} \left\{ \sum_{n=0}^{\infty} (1/n!) D^n f(0) x^n \right\} < 2\|f\|_{2N}.$$

Hence T_t 's are continuous, i.e. $T_t \in L(\tilde{\mathcal{E}}_a, \tilde{\mathcal{E}}_a)$.

PROPOSITION 4.12. $\{T_t\}$ is an equicontinuous semigroup of class (C_0) [20, Chapter 9].

PROOF. Obviously,

$$T_{t+s} = T_t T_s \quad \text{and} \quad T_0 = I. \tag{18}$$

By the mean value theorem, we have (for $t_0 > 0$)

$$\begin{aligned} \|T_t f - T_{t_0} f\|_N &< \sum_{n=0}^{\infty} (1/n!) |e^{-n^k t} - e^{-n^k t_0}| \|D^n f(0) x^n\|_N \\ &< |t - t_0| \|f\|_{2N} \left(\sum_{n=1}^{\infty} n^k / 2^n \right), \end{aligned}$$

by (16). Thus

$$\lim_{t \rightarrow t_0} T_t f = T_{t_0} f \quad \text{for } t_0 > 0. \tag{19}$$

Finally, suppose q is any continuous seminorm on $\tilde{\mathcal{E}}_a$. Then there exist N and a constant C_1 such that $q(f) \leq C_1 \|f\|_N$. Consequently, $q(T_t f) \leq C_1 \|T_t f\|_N < 2C_1 \|f\|_{2N}$, i.e. $\{T_t\}$ is equicontinuous. \square

PROPOSITION 4.13. $-\tilde{\mathcal{U}}^k$ is the infinitesimal generator of $\{T_t\}$ and the domain of $-\tilde{\mathcal{U}}^k = \tilde{\mathcal{E}}_a$.

PROOF. Let $f \in \tilde{\mathcal{E}}_a$ and $|h| < 1$. Then for each integer N we have

$$\begin{aligned} \|h^{-1}(T_h - I)f + \tilde{\mathcal{U}}^k f\|_N &\leq \sup_{\|x\| < N} \left[\sum_{n=1}^{\infty} |(e^{-n^k h} - 1)/h + n^k| |D^n f(0) x^n / n!| \right] \\ &< (|h|/2) \left(\sum_{n=1}^{\infty} n^{2k} / 2^n \right) \|f\|_{2N} \rightarrow 0 \quad \text{as } h \rightarrow 0. \end{aligned}$$

Therefore, $\lim_{h \rightarrow 0} h^{-1}(T_h - I)f$ exists and equals $-\tilde{\mathcal{U}}^k f$. \square

REMARK 4. As a consequence (see Yosida [20]), $\tilde{\mathcal{U}}^k$ is a closed operator on $\tilde{\mathcal{E}}_a$. Furthermore, T_t can be uniquely represented by

$$T_t f = \lim_{n \rightarrow \infty} \exp(t \tilde{\mathcal{U}}^k (I - n^{-1} \tilde{\mathcal{U}}^k)^{-1}) f. \quad \square$$

THEOREM 4.14. *There is a unique solution $v(y, t)$ for the Cauchy problem (6) such that (i) $v(\cdot, t) \in \tilde{\mathcal{E}}_a$ for each $t \geq 0$, (ii) $v(y, t)$ is (strongly) differentiable in t (i.e. $(\partial/\partial t)v(y, t)$ exists uniformly on bounded sets). Moreover, the solution is given by $v(y, t) = T_t(\mathcal{F}_1 f)(y)$.*

PROOF. Let $g(y, s)$ be any solution for (6) such that (i) and (ii) hold.

Fix $t > 0$. Set $F(y, s) = T_{t-s}g(y, s)$ for $t > s > 0, y \in [B]$. Since $g(\cdot, s) \in \tilde{\mathcal{E}}_a$, the domain of $-\tilde{\mathcal{U}}^k$, it follows by [20, Chapter 9, Theorem 2] that $F(y, s)$ is strongly differentiable and

$$\begin{aligned} (\partial/\partial s)F(y, s) &= \tilde{\mathcal{U}}^k T_{t-s}g(y, s) + T_{t-s}(\partial/\partial s)g(y, s) \\ &\quad \text{(since } T_t \text{ is an equicontinuous semigroup and } g \text{ satisfies (ii))} \\ &= T_{t-s} \tilde{\mathcal{U}}^k g(y, s) - T_{t-s} \tilde{\mathcal{U}}^k g(y, s) \\ &= 0, \quad y\text{-uniformly on bounded sets.} \end{aligned}$$

This implies that $F(y, s) = C$ (a constant) on $[B]$ for $0 < s < t$. In particular, $F(y, t) = F(y, 0) = T_t g(y, 0) = T_t(\mathcal{F}_1 f)(y)$. On the other hand, $F(y, t) = T_0 g(y, t) = g(y, t)$. We conclude that $g(y, t) = T_t(\mathcal{F}_1 f)(y)$. \square

COROLLARY 4.15. *The Cauchy problem (1) has a unique solution $u(x, t)$ such that (i) $u(\cdot, t) \in \mathcal{E}_a$ for each $t \geq 0$, (ii) $u(x, t)$ is a (strongly) C^1 -function of t with the property that for $b, N > 0, \sup_{\|y\| < N} \sup_{0 < t < b} |u_t(x + iy, t)|$, as a function of x , belongs to $L^1(B, p_1)$.*

PROOF. Recalling Theorem 3.9, it suffices to show that $\mathcal{F}_1 u(y, t)$ satisfies the conditions (i) and (ii) of Theorem 4.14. Obviously, by Proposition 3.3 and (i), $\mathcal{F}_1 u(y, t)$ satisfies condition (i) of Theorem 4.14. To show that $\mathcal{F}_1 u(y, t)$ satisfies condition (ii) of Theorem 4.14, we let $v(y, t) = \mathcal{F}_1 u(y, t)$ and write

$$\varepsilon^{-1}(v(y, t + \varepsilon) - v(y, t)) = \int_B \varepsilon^{-1}(u(x + iy, t + \varepsilon) - u(x + iy, t)) p_1(dx).$$

Since

$$|\varepsilon^{-1}(u(x + iy, t + \varepsilon) - u(x + iy, t))| = |u_t(x + iy, t + \theta(x, y, t)\varepsilon)|$$

where $0 < \theta(x, y, t) < 1$, it follows that if $\varepsilon < 1$

$$\sup_{\|y\| < N} \varepsilon^{-1} |u(x + iy, t + \varepsilon) - u(x + iy, t)| < \sup_{\|y\| < N} \sup_{0 < s < 1 + t} |u_s(x + iy, s)|.$$

Hence, by the property (ii) and Lebesgue’s dominated convergence theorem, we get

$$\begin{aligned} & \lim_{\epsilon \rightarrow 0} \|\epsilon^{-1}(v(y, t + \epsilon) - v(y, t)) - v_t(y, t)\|_N \\ & \leq \lim_{\epsilon \rightarrow 0} \int_B \sup_{\|y\| < N} |\epsilon^{-1}(u(x + iy, t + \epsilon) - u(x + iy, t)) - u_t(x + iy, t)| p_1(dx) \\ & = \int_B \lim_{\epsilon \rightarrow 0} \sup_{\|y\| < N} |\epsilon^{-1}(u(x + iy, t + \epsilon) - u(x + iy, t)) - u_t(x + iy, t)| p_1(dx) \\ & = 0. \end{aligned}$$

Moreover,

$$\begin{aligned} & \lim_{\epsilon \rightarrow 0} \left\| \int_B (u_t(x + iy, t + \epsilon) - u_t(x + iy, t)) p_1(dx) \right\|_N \\ & \leq \lim_{\epsilon \rightarrow 0} \int_B \sup_{\|y\| < N} |u_t(x + iy, t + \epsilon) - u_t(x + iy, t)| p_1(dx) \\ & = \int_B \lim_{\epsilon \rightarrow 0} \sup_{\|y\| < N} |u_t(x + iy, t + \epsilon) - u_t(x + iy, t)| p_1(dx) \\ & = 0, \end{aligned}$$

by the condition (ii) and the dominated convergence theorem.

Thus $v(y, t) = \mathfrak{F}_1 u(y, t)$ satisfies (ii) of Theorem 4.14. \square

As a further application of the Fourier-Wiener transform, Corollaries 4.5 and 4.15, we may consider the following nonhomogeneous Cauchy problem.

$$w_t(x, t) + \mathfrak{U}^k w(x, t) = h(x, t); \quad w(x, 0) = f(x), \tag{20}$$

where $f(x) \in \mathfrak{E}_a$ and $h(x, t)$ is a sufficiently smooth function of t such that $h(\cdot, t) \in \mathfrak{E}_a$. More precisely, we have

THEOREM 4.16. *Suppose $h(x, t): [B] \times [0, \infty) \rightarrow \mathbb{C}$ has the following three properties: (i) $h(x + \lambda y, t)$ is an entire function of λ for each $t \in [0, \infty]$ and $x, y \in [B]$. (ii) Let $b > 0$. There exist constants K_1, K_2 such that $\sup_{0 < t < b} |h(x, t)| < K_1 \exp(K_2 \|x\|)$ for every $x \in [B]$. (iii) $h(x, t)$ satisfies property (ii) of Corollary 4.15.*

Then there exists a unique solution $w(x, t)$ for the Cauchy problem (20) such that $w(x, t)$ satisfies (i), (ii) of Corollary 4.15. Moreover, $w(x, t)$ is given by

$$w(x, t) = \mathfrak{F}_1^{-1}(T_t(\mathfrak{F}_1 f))(x) + \int_0^t \mathfrak{F}_1^{-1}(T_{t-s}(\mathfrak{F}_1 h))(x) ds. \tag{21}$$

PROOF. By similar arguments as in the proof of Proposition 3.3 we have

- (a) $\mathfrak{F}_1 h(y, t), \mathfrak{F}_1^{-1} h(y, t)$ also have the properties (i) and (ii).
- (b) For every $t \geq 0, h(x, t) \in \mathfrak{E}_a$ (by (i), (ii)).
- (c) $\sup_{0 < t < b} \|D^n h(x, t)\|_{[w_n]} < K_1 \exp(K_2 n) \exp(K_2 \|x\|)$.

Next, by the same argument as in the proof of Corollary 4.15, we have

- (d) $\mathfrak{F}_1 h(y, t)$ is a (strongly) C^1 -function of t (by all three properties (i)–(iii)).

To finish the proof, we consider the following Cauchy problem (which is the Fourier-Wiener 1-transform of (20)):

$$\tilde{w}_t(y, t) + \tilde{\mathfrak{U}}^k \tilde{w}(y, t) = \tilde{h}(y, t); \quad \tilde{w}(y, 0) = \tilde{f}(y), \tag{22}$$

where “ $\tilde{}$ ” means the Fourier-Wiener 1-transform.

We claim that $\tilde{w}(y, t) = T_t \tilde{f}(y) + \int_0^t T_{t-s} \tilde{h}(y, s) ds$ is the unique solution of (22) satisfying conditions (i), (ii) of Theorem 4.14. In fact, if we let $G(y, t) = \int_0^t T_{t-s} \tilde{h}(y, s) ds$, then

$$\begin{aligned} \frac{1}{\varepsilon}(G(y, t + \varepsilon) - G(y, t)) &= \frac{1}{\varepsilon} \left(\int_0^{t+\varepsilon} T_{t+\varepsilon-s} \tilde{h}(y, s) ds - \int_0^t T_{t-s} \tilde{h}(y, s) ds \right) \\ &= \varepsilon^{-1} \int_t^{t+\varepsilon} T_{t+\varepsilon-s} \tilde{h}(y, s) ds \\ &\quad + \varepsilon^{-1}(T_\varepsilon - I) \left(\int_0^t T_{t-s} \tilde{h}(y, s) ds \right). \end{aligned}$$

Since $\tilde{h}(y, s)$ is (strongly) continuous,

$$\lim_{\varepsilon \rightarrow 0} \varepsilon^{-1} \int_t^{t+\varepsilon} T_{t+\varepsilon-s} \tilde{h}(y, s) ds = T_0 \tilde{h}(y, t) = \tilde{h}(y, t)$$

(y -uniformly on bounded sets). Since $\int_0^t T_{t-s} \tilde{h}(y, s) ds \in \tilde{\mathcal{E}}_a$, we have

$$\lim_{\varepsilon \rightarrow 0} \varepsilon^{-1}(T_\varepsilon - I) \left(\int_0^t T_{t-s} \tilde{h}(y, s) ds \right) = -\tilde{\mathcal{U}}^k \left(\int_0^t T_{t-s} \tilde{h}(y, s) ds \right).$$

Thus $G(y, t)$ is (strongly) differentiable and

$$(\partial/\partial t)G(y, t) = -\tilde{\mathcal{U}}^k \left(\int_0^t T_{t-s} \tilde{h}(y, s) ds \right) + \tilde{h}(y, t). \tag{23}$$

As a result,

$$(\partial/\partial t)\tilde{w}(y, t) = -\tilde{\mathcal{U}}^k(T_t \tilde{f}(y)) - \tilde{\mathcal{U}}^k \left(\int_0^t T_{t-s} \tilde{h}(y, s) ds \right) + \tilde{h}(y, t),$$

or $\tilde{w}_t(y, t) + \tilde{\mathcal{U}}^k \tilde{w}(y, t) = \tilde{h}(y, t)$. Obviously, $\tilde{w}(y, 0) = \tilde{f}(y)$.

By (23) and Theorem 4.14, the uniqueness follows. Finally, by taking the inverse Fourier-Wiener transform, the solution of the Cauchy problem is represented by

$$w(x, t) = \mathcal{F}_1^{-1} T_t \tilde{f}(x) + \mathcal{F}_1^{-1} \left(\int_0^t T_{t-s} \tilde{h}(y, s) ds \right)(x, t).$$

Note that

$$\begin{aligned} |T_{t-s} \tilde{h}(y - ix, s)| &\leq \sum_{n=0}^{\infty} |(1/n!) D^n \tilde{h}(0, s)(y - ix)^n| \\ &\leq K_1 \sum_{n=0}^{\infty} (1/n!) \exp(K_2 n)(\|y\| + \|x\|)^n \\ &\hspace{15em} \text{for } 0 \leq s \leq t, \text{ (by (c))} \\ &= K_1 \exp(e^{K_2} \|x\|) \exp(e^{K_2} \|y\|) \in L^1(B, p_1(dy)). \end{aligned}$$

It follows by Fubini's theorem that

$$\mathcal{F}_1^{-1} \left(\int_0^t T_{t-s} \tilde{h}(y, s) ds \right)(x, t) = \int_0^t \mathcal{F}_1^{-1} (T_{t-s} \tilde{h})(x, s) ds.$$

Hence, $w(x, t) = \mathcal{F}_1^{-1} (T_t \mathcal{F}_1 f)(x) + \int_0^t \mathcal{F}_1^{-1} (T_{t-s} \mathcal{F}_1 h)(x, s) ds$. Of course, $w(x, 0) = \mathcal{F}_1^{-1} (T_0 \mathcal{F}_1 f)(x) = f(x)$. Thus we complete the proof of Theorem 4.16. \square

REMARK 5. We note that all functions of the form $h(x, t) = P(t)f(x)$ where $P(t)$ is C^1 and $f \in \tilde{\mathcal{E}}_a$, satisfying the hypotheses of Theorem 4.16. Recall that $\tilde{\mathcal{E}}_a(B)$ is

dense in $L^2(B, p_1)$, so we have a rich class of functions $h(x, t)$ satisfying (i)–(iii) of Theorem 4.16. \square

REMARK 6. Since $\mathcal{F}_1^{-1}(T_{t-s}\mathcal{F}_1h)(x, s)$ also has the property (ii), we see that

$$\lim_{t \rightarrow 0} \int_0^t \mathcal{F}_1^{-1}(T_{t-s}\mathcal{F}_1h)(x, s) ds = 0$$

uniformly on bounded sets. Therefore, $w(x, t) \rightarrow f(x)$ also uniformly on bounded sets. \square

5. Application II. We shall consider in the following two other types of equations associated with the operator \mathcal{U} . The first one is

$$-\mathcal{U}^k u(x) = f(x), \tag{1}$$

and the second one is the Cauchy problem

$$u_{tt}(x, t) = -\mathcal{U}^k u(x, t); \quad u(x, 0) = f_1(x), \quad u_t(x, 0) = f_2(x). \tag{2}$$

By means of the Fourier-Wiener transform, our approach is very much the same as that of §4. Uniqueness of solutions of (1) and (2) and the nonhomogeneous problem associated with (2) will also be considered.

THEOREM 5.1. (i) *In order that equation (1) has a solution in \mathcal{E}_a , it is necessary and sufficient that $f \in \mathcal{E}_a$ and $\mathcal{F}_1 f(0) = 0$.* (ii) *The solution of (1) is unique in the sense that if $W_1(x), W_2(x) \in \mathcal{E}_a$ and each solves (1) then $W_1 - W_2$ is a constant.*

PROOF. (i) a. *Necessity.* Since $\mathcal{U}(\mathcal{E}_a) \subset \mathcal{E}_a$ (Lemma 4.1), if $u(x)$ is a solution in \mathcal{E}_a then $\mathcal{U}^k u(x) \in \mathcal{E}_a$ so does f .

Next, applying the Fourier-Wiener transform to (1), we get

$$-\tilde{\mathcal{U}}^k(\mathcal{F}_1 u)(y) = \mathcal{F}_1 f(y). \tag{3}$$

Therefore $\mathcal{F}_1 f(0) = -\tilde{\mathcal{U}}^k(\mathcal{F}_1 u)(0) = 0$.

b. *Sufficiency (existence).* Suppose $f \in \mathcal{E}_a$ and $\mathcal{F}_1 f(0) = 0$. We let $g(y) = \mathcal{F}_1 f(y)$. Since $g \in \mathcal{E}_a$, $g(y) = \sum_{n=1}^{\infty} (1/n!) D^n g(0) y^n$ (since $g(0) = 0$). It is easy to see that

$$v(y) = \mathcal{F}_1 u(y) = v(0) - \sum_{n=1}^{\infty} (1/n^k n!) D^n g(0) y^n$$

solves equation (3). Consequently,

$$u(x) = \mathcal{F}_1^{-1} v(x) = v(0) - \mathcal{F}_1^{-1} \left(\sum_{n=1}^{\infty} (1/n^k n!) D^n g(0) y^n \right) (x) \tag{4}$$

solves (1).

(ii) *Uniqueness.* Observe that if $h \in \mathcal{E}_a$, then $\tilde{\mathcal{U}}h = 0$ iff $h \equiv$ a const. It follows that if W_1, W_2 are two solutions for (1) such that $W_1, W_2 \in \mathcal{E}_a$ and we let $H = W_1 - W_2$, then we must have $\mathcal{U}H = 0$; hence $\tilde{\mathcal{U}}(\mathcal{F}_1 H) \equiv 0$.

According to our observation $\mathcal{F}_1 H \equiv C_1$, a constant. Therefore $H \equiv \mathcal{F}_1^{-1}(C_1) \equiv C_1$. \square

REMARK 1. Note that if $f \in \mathcal{E}_a$ with $\mathcal{F}_1 f(0) = 0$, then for $x \in [B]$ we have $|T_t(\mathcal{F}_1 f)(y - ix)| \leq (K_1 e^{K_2 \|x\|}) e^{-t e^{K_2 \|y\|}}$ for some constants K_1, K_2 and $y \in B$. Observing that $\int_B \int_0^\infty e^{-t} \exp(K_2 \|y\|) p_1(dy) dt < \infty$, we may also write (4) (by Fubini's theorem) as

$$u(x) = \int_B u(y)p_1(dy) + \int_0^\infty \mathfrak{F}_1^{-1}(T_t(\mathfrak{F}_1 f))(x) dt, \tag{5}$$

where T_t is defined by (17) of §4. \square

In the remainder of this section, we first show the existence of a solution for (2). Then by means of semigroup theory, we are able to prove uniqueness and also obtain the solution of the nonhomogeneous equation

$$u_{tt}(x, t) + \mathfrak{U}^k u(x, t) = h(x, t); \quad u(x, 0) = f_1(x), \quad u_t(x, 0) = f_2(x).$$

The following theorem is straightforward; we omit the proof.

THEOREM 5.2. *Assume g_1, g_2 are functions in $\tilde{\mathcal{E}}_a$. Define*

$$v(y, t) = \sum_{n=0}^\infty (P_n(t)/n!) D^n g_1(0) y^n + \sum_{n=0}^\infty (Q_n(t)/n!) D^n g_2(0) y^n, \tag{6}$$

where $P_n(t) = \cos(\sqrt{n})^k t$, $Q_0(t) = t$, $Q_n(t) = (\sqrt{n})^{-k} \sin(\sqrt{n})^k t$ for $n = 1, 2, 3, \dots$. Then for $t \in R$,

- (i) $v_{tt}(y, t) = -\mathfrak{U}^k v(y, t)$; $v(y, 0) = g_1(y)$ and $v_t(y, 0) = g_2(y)$.
- (ii) As $t \rightarrow 0$, $v(y, t) \rightarrow g_1(y)$, $v_t(y, t) \rightarrow g_2(y)$ uniformly on bounded sets in $[B]$.

THEOREM 5.3. *Assuming $f_1, f_2 \in \mathcal{E}_a$ and $g_j = \mathfrak{F}_1 f_j$ ($j = 1, 2$), we define $u(x, t) = \mathfrak{F}_1^{-1} v(x, t)$, where v is given by (6). Then*

- (i) $u(x, t)$ solves the Cauchy problem (2).
- (ii) As $t \rightarrow 0$, $u(x, t) \rightarrow f_1(x)$, $u_t(x, t) \rightarrow f_2(x)$ uniformly on bounded subsets of $[B]$.

PROOF. (i) It is sufficient to show that

$$(\mathfrak{F}_1^{-1} v_{tt})(x, t) = (\mathfrak{F}_1^{-1} v)_{tt}(x, t). \tag{7}$$

To see this, we let $w(y, t) = v_t(y, t)$ i.e.

$$w(y, t) = - \sum_{n=1}^\infty (n^{k/2}/n!) (\sin(n^{k/2}t)) D^n g_1(0) y^n + \sum_{n=1}^\infty ((\cos(n^{k/2}t))/n!) D^n g_2(0) y^n.$$

It follows

$$|w_t(y, t)| \leq (1/2\pi) \int_{|\lambda|=2} \left(\sum_{n=1}^\infty n^{k/2}/2^{n+1} \right) |g_1(\lambda y)| |d\lambda| + (1/2\pi) \int_{|\lambda|=2} \left(\sum_{n=1}^\infty 1/2^{n+1} \right) |g_2(\lambda y)| |d\lambda|.$$

Employing arguments similar to the proof of Theorem 4.7(i), we see that condition (12) of Lemma 4.6 is satisfied by $w(y, t)$, and so $\mathfrak{F}_1^{-1} w_t(x, t) = (\mathfrak{F}_1^{-1} w)_{tt}(x, t)$.

Applying the same argument once more, we obtain

$$(\mathfrak{F}_1^{-1} w) = (\mathfrak{F}_1^{-1} v_t) = (\mathfrak{F}_1^{-1} v)_t.$$

(7) follows immediately from the preceding two identities.

(ii) follows by the same techniques in the proof of Theorem 4.7(ii). \square

Next, we define the semigroup associated with the problem $v_t(y, t) = -\tilde{\mathcal{U}}^k v(y, t)$; $v(y, 0) = g_1(y)$ and $v_t(y, 0) = g_2(y)$ as follows. Let

$$A_n(t) = \begin{bmatrix} P_n(t) & Q_n(t) \\ P'_n(t) & Q'_n(t) \end{bmatrix}.$$

Define for $t \geq 0$ a linear operator T_t on $\tilde{\mathcal{E}}_a \times \tilde{\mathcal{E}}_a$ by

$$T_t \begin{bmatrix} g_1 \\ g_2 \end{bmatrix} = \sum_{n=0}^{\infty} \frac{1}{n!} A_n(t) \begin{bmatrix} D^n g_1(0) x^n \\ D^n g_2(0) x^n \end{bmatrix}. \tag{8}$$

Let $\| (g_1, g_2) \|_N = (\|g_1\|_N^2 + \|g_2\|_N^2)^{1/2}$, $N = 1, 2, 3, \dots$. Then $\{ \|(\cdot, \cdot) \|_N \}$ defines a sequence of norms on $\tilde{\mathcal{E}}_a \times \tilde{\mathcal{E}}_a$. With the topology induced by $\{ \|(\cdot, \cdot) \|_N \}$, $\tilde{\mathcal{E}}_a \times \tilde{\mathcal{E}}_a$ obviously also forms a (sequentially) complete metrizable topological convex space (see 4.9).

Obviously,

$$T_t \begin{bmatrix} g_1 \\ g_2 \end{bmatrix} \in \tilde{\mathcal{E}}_a \times \tilde{\mathcal{E}}_a.$$

It follows by (16) of §4 that

$$\begin{aligned} \left\| T_t \begin{bmatrix} g_1 \\ g_2 \end{bmatrix} \right\|_N &< \left[(2\|g_1\|_{2N} + 2t\|g_2\|_{2N})^2 + \left(\sum_{n=1}^{\infty} n^{k/2}/2^n \right) \|g_1\|_{2N} + 2\|g_2\|_{2N} \right]^{1/2} \\ &< 2 \left(\max \left(2t, \sum_{n=1}^{\infty} (n^{k/2}/2^n) \right) \right) \| (g_1, g_2) \|_{2N} \end{aligned} \tag{9}$$

i.e., T_t is continuous. Moreover, we have

PROPOSITION 5.4. (i) $T_{t+s} = T_t T_s$ for $t, s \geq 0$; $T_0 = I$.

(ii) $\lim_{t \rightarrow t_0} T_t \begin{bmatrix} g_1 \\ g_2 \end{bmatrix} = T_{t_0} \begin{bmatrix} g_1 \\ g_2 \end{bmatrix}$.

(iii) If S is a compact set in $[0, \infty)$, then the family $\{T_s; s \in S\}$ of operators is equicontinuous. That is, given any continuous seminorm q on $\tilde{\mathcal{E}}_a \times \tilde{\mathcal{E}}_a$, there is a constant C_1 and a seminorm $\| \cdot \|_N$ such that

$$q \left(T_t \begin{bmatrix} g_1 \\ g_2 \end{bmatrix} \right) \leq C_1 \left\| \begin{bmatrix} g_1 \\ g_2 \end{bmatrix} \right\|_N.$$

REMARK 2. T_t is not an equicontinuous family of operators. To see this, let $g_1(x) = 0, g_2(x) = 1$ in (8). We obtain

$$T_t \begin{bmatrix} 0 \\ 1 \end{bmatrix} = t \rightarrow \infty \quad \text{as } t \rightarrow \infty. \quad \square$$

By standard computations, the following two propositions are obvious.

PROPOSITION 5.5. Let

$$A = \begin{bmatrix} 0 & I \\ -\tilde{\mathcal{U}}^k & 0 \end{bmatrix}.$$

Then A is the infinitesimal generator of T_t and the domain of A is $\tilde{\mathcal{E}}_a \times \tilde{\mathcal{E}}_a$.

PROPOSITION 5.6. Let $U_t = e^{-tT}$, $t \geq 0$. Then the family of operators $\{U_t\}$ forms an equicontinuous semigroup of class (C_0) on $\tilde{\mathcal{E}}_a \times \tilde{\mathcal{E}}_a$ and its infinitesimal generator is given by $A - I$.

COROLLARY 5.7. There is a unique solution $W(x, t)$ such that both $W(x, t)$ and $W_t(x, t)$ satisfy the conditions (i), (ii) of Theorem 4.14 and

$$\begin{aligned} W_{tt}(x, t) + 2W_t(x, t) &= -\tilde{\mathcal{U}}^k W(x, t) - W(x, t), \\ W(x, 0) &= g_1(x); W_t(x, 0) = g_2(x) - g_1(x) \end{aligned} \tag{10}$$

where g_1, g_2 are in $\tilde{\mathcal{E}}_a$. Moreover, the solution is given by $W(x, t) = e^{-t}v(x, t)$ where $v(x, t)$ is the function given by (6) in Theorem 5.2.

PROOF. Let $H_1 = W, H_2 = W_t + W$; (10) becomes

$$\begin{bmatrix} H_1 \\ H_2 \end{bmatrix}_t = (A - I) \begin{bmatrix} H_1 \\ H_2 \end{bmatrix} \quad \text{where } A = \begin{pmatrix} 0 & I \\ -\tilde{\mathcal{U}}^k & 0 \end{pmatrix}.$$

By Proposition 5.6 and a technique similar to the proof of Theorem 4.14, uniqueness follows immediately.

Next, we show that if $v(x, t)$ is any solution for the Cauchy problem (i) of Theorem 5.2, then $W(x, t) = e^{-t}v(x, t)$ satisfies (10). Indeed,

$$W_t = -e^{-t}v + e^{-t}v_t, \quad W_{tt} = e^{-t}v - 2e^{-t}v_t + e^{-t}v_{tt};$$

hence $W_{tt} + 2W_t = -e^{-t}v + e^{-t}(-\tilde{\mathcal{U}}^k v) = -\tilde{\mathcal{U}}^k W - W$. Also,

$$W_t(x, 0) = -v(x, 0) + v_t(x, 0) = g_2(x) - g_1(x)$$

and

$$W(x, 0) = v(x, 0) = g_1(x). \quad \square$$

COROLLARY 5.8. The Cauchy problem (i) of Theorem 5.2 has a unique solution $v(x, t)$ such that $v(x, t)$ and $v_t(x, t)$ satisfy the conditions (i), (ii) of Theorem 4.14.

PROOF. Let $v(x, t)$ be the solution given by (6) and $v_1(x, t)$ be any solution. Then as we have seen in the proof of Corollary 5.7, $W(x, t) = e^{-t}v_1(x, t)$ must be a solution of (10). By the hypotheses on v_1 , $W(x, t)$ satisfies the hypotheses in Corollary 5.7. It follows by the uniqueness, $e^{-t}v(x, t) = e^{-t}v_1(x, t)$ for $t > 0$; thus $v = v_1$. \square

Finally, we conclude that

COROLLARY 5.9. Assume f_1, f_2 are in \mathcal{E}_a . Then the Cauchy problem (2) has a unique solution $u(x, t)$ such that $u(x, t)$ and $u_t(x, t)$ satisfy the conditions (i), (ii) of Corollary 4.15.

PROOF. The uniqueness follows by taking the Fourier-Wiener 1-transform of (2) and by Corollary 5.8. \square

As an implication of Proposition 5.6, we have

COROLLARY 5.10. For each pair $g_1, g_2 \in \tilde{\mathcal{G}}_a$, we have

$$\frac{\partial}{\partial t} T_t \begin{bmatrix} g_1 \\ g_2 \end{bmatrix} = AT_t \begin{bmatrix} g_1 \\ g_2 \end{bmatrix} = T_t A \begin{bmatrix} g_1 \\ g_2 \end{bmatrix}. \tag{11}$$

PROOF. It follows from Proposition 5.6 and [20, Chapter 9, Theorem 2] that

$$\frac{\partial}{\partial t} U_t \begin{bmatrix} g_1 \\ g_2 \end{bmatrix} = (A - I)U_t \begin{bmatrix} g_1 \\ g_2 \end{bmatrix} = U_t(A - I) \begin{bmatrix} g_1 \\ g_2 \end{bmatrix}.$$

Since $T_t = e^t U_t$, we get

$$\begin{aligned} \frac{\partial}{\partial t} T_t \begin{bmatrix} g_1 \\ g_2 \end{bmatrix} &= (e^t U_t + e^t(A - I)U_t) \begin{bmatrix} g_1 \\ g_2 \end{bmatrix} \\ &= (e^t U_t + e^t U_t(A - I)) \begin{bmatrix} g_1 \\ g_2 \end{bmatrix} \\ &= AT_t \begin{bmatrix} g_1 \\ g_2 \end{bmatrix} = T_t A \begin{bmatrix} g_1 \\ g_2 \end{bmatrix}. \quad \square \end{aligned}$$

As a further application of Theorems 5.2, 5.3 and Corollaries 5.8, 5.9, 5.10, we may consider the following Cauchy problem associated with (2).

$$\begin{aligned} u_{tt}(x, t) &= -\mathcal{U}^k u(x, t) + h(x, t) \quad \text{with } u(x, 0) = f_1(x), \\ u_t(x, 0) &= f_2(x), \quad f_1, f_2 \in \mathcal{G}_a. \end{aligned} \tag{12}$$

Analogous to Theorem 4.16, we have

THEOREM 5.11. Assuming that $h(x, t)$ satisfies the three conditions (i)–(iii) in Theorem 4.16, then there is a unique solution $u(x, t)$ for the Cauchy problem (12) so that $u(x, t)$ and $u_t(x, t)$ satisfy the conditions (i), (ii) of Corollary 4.15.

PROOF. Reduction. We may assume $f_1 = 0$ by replacing u by $u - f_1$. Then the Cauchy problem becomes $u_{tt}(x, t) = -\mathcal{U}^k u(x, t) + h(x, t) - \mathcal{U}^k f_1(x)$ with $u(x, 0) = 0$, $u_t(x, 0) = f_2$. (Note that the three conditions (i)–(iii) remain valid for $h(x, t) - \mathcal{U}^k f_1(x)$, by Remark 5 of §4.)

We see that (12) is now equivalent to the Cauchy problem below

$$\begin{bmatrix} v_1 \\ v_2 \end{bmatrix}_t = \begin{bmatrix} 0 & I \\ -\mathcal{U}^k & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} + \begin{bmatrix} 0 \\ h(x, t) \end{bmatrix}; \quad \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}_{t=0} = \begin{bmatrix} 0 \\ f_2 \end{bmatrix} \tag{13}$$

with $v_1 = u$, $v_2 = u_t$.

Imitating the proof of Theorem 4.16, we find that

$$\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \mathcal{F}_1^{-1} T_t \mathcal{F}_1 \begin{bmatrix} 0 \\ f_2 \end{bmatrix} + \int_0^t \mathcal{F}_1^{-1} T_{t-s} \mathcal{F}_1 \begin{bmatrix} 0 \\ h(x, t) \end{bmatrix} ds$$

is the unique solution of (13). It follows that the solution of (12) (with $f_1 = 0$) is given by

$$\begin{aligned} u(x, t) &= \mathcal{F}_1^{-1} \left(\sum_{n=0}^{\infty} (Q_n(t)/n!) D^n \tilde{f}_2(0) y^n \right) (x) \\ &\quad + \int_0^t \mathcal{F}_1^{-1} \left(\sum_{n=0}^{\infty} (Q_n(t-s)/n!) D^n \tilde{h}(0, s) y^n \right) (x) ds, \end{aligned} \tag{14}$$

where “ \sim ” is the Fourier-Wiener 1-transform. \square

REMARK 3. In the next paper we shall continue to apply the Fourier-Wiener transform to investigate the existence and regularity of solutions of the following two types of differential equations:

$$(i) u_t(x, t) = (-1)^{k+1} \Delta^k u(x, t); u(x, 0) = f(x).$$

$$(ii) u_{tt}(x, t) = (-1)^{k+1} \Delta^k u(x, t); u(x, 0) = f(x), u_t(x, 0) = g(x).$$

REFERENCES

1. R. H. Cameron and W. T. Martin, *Fourier-Wiener transforms of analytic functionals*, Duke Math. J. **12** (1945), 489–507.
2. _____, *Fourier-Wiener transforms of functionals belonging to L_2 over the space \mathcal{C}* , Duke Math. J. **14** (1946), 99–107.
3. L. Gross, *Integration and nonlinear transformations in Hilbert space*, Trans. Amer. Math. Soc. **94** (1960), 404–440.
4. _____, *Harmonic analysis on Hilbert space*, Mem. Amer. Math. Soc., No. 46, 1963.
5. _____, *Abstract Wiener spaces*, Proc. Fifth Berkeley Sympos. Math. Stat. and Prob. (Berkeley, Calif., 1965/66), Vol. II: Contributions to Probability Theory, Part I, Univ. California Press, Berkeley, Calif., 1967, pp. 31–42.
6. _____, *Potential theory on Hilbert space*, J. Functional Analysis **1** (1967), 123–181.
7. T. Hida, *Stationary stochastic processes*, Princeton Univ. Press, Princeton, N. J., 1970.
8. E. Hille, *Functional analysis and semi-groups*, Amer. Math. Soc. Colloq. Publ., vol. 31, Amer. Math. Soc., Providence, R. I., 1948.
9. E. Hille and R. S. Phillips, *Functional analysis and semi-groups*, rev. ed., Amer. Math. Soc. Colloq. Publ., vol. 31, Amer. Math. Soc., Providence, R. I.; reprinted 1974.
10. H.-H. Kuo, *Integration by parts for abstract Wiener measures*, Duke Math. J. **41** (1974), 373–379.
11. _____, *Gaussian measures in Banach spaces*, Lecture Notes in Math., vol. 463, Springer-Verlag, Berlin and New York, 1975.
12. E. Nelson, *Dynamical theories of Brownian motion*, Princeton Univ. Press, Princeton, N. J., 1967.
13. M. A. Piech, *A fundamental solution of the parabolic equation on Hilbert space*, J. Functional Analysis **3** (1969), 85–114.
14. _____, *Parabolic equations associated with the number operator*, Trans. Amer. Math. Soc. **194** (1974), 213–222.
15. _____, *The Ornstein-Uhlenbeck semi-group in an infinite dimensional L^2 setting*, J. Functional Analysis **18** (1975), 271–285.
16. I. E. Segal, *Tensor algebra over Hilbert space. I*, Trans. Amer. Math. Soc. **81** (1956), 106–134.
17. _____, *Distributions in Hilbert space and canonical systems of operators*, Trans. Amer. Math. Soc. **88** (1958), 12–41.
18. A. V. Skorohod, *Integration in Hilbert space*, Springer-Verlag, Berlin and New York, 1974.
19. A. V. Uglanov, *Differential equations with constant coefficients for generalized measures on Hilbert space*, Math. USSR Izv. **9** (1975), 414–440.
20. K. Yosida, *Functional analysis*, rev. ed., Springer-Verlag, Berlin and New York, 1970.

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